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RECURSIVE COBOUNDARY FORMULA FOR CYCLES IN ACYCLIC CHAIN COMPLEXES

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Abstract. Given an $(m-1)$-dimensional cycle z in a finitely generated acyclic chain complex, we want to explicitly construct an m-dimensional chain $\mathrm{Cob}(z)$ whose algebraic boundary is z. The acyclicity of the chain complex implies that a solution exists (it is not unique) but the traditional linear algebra methods of finding it lead to a high complexity of computation. We are searching for more efficient algorithms based on geometric considerations. The main motivation for studying this problem comes from the topological and computational dynamics, namely, from designing general algorithms computing the homomorphism induced in homology by a continuous map. This, for turn, is an essential step in computing such invariants of dynamical properties of nonlinear systems as Conley index or Lefschetz number. Another potential motivation is in the relationship of our problem to the problem of finding minimal surfaces of closed curves.

1. Introduction

The following problem is addressed in this paper: Given a $(m-1)$-dimensional cycle z in a finitely generated acyclic chain complex (C, ∂) explicitly construct a m-dimensional chain $c = \mathrm{Cob}(z)$ called here *coboundary* of z such that

$$\partial c = z. \tag{1}$$

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The hypothesis that C is acyclic guarantees the existence of a solution which, in general, is not unique. In applications from which the problem came, C is a chain complex associated with a convex or star-shape polyhedron via triangulations or cubical subdivisions. Thus we know that C is acyclic without having explicitly computed its homology. The problem of solving equation (1) for c may however lead to heavy calculations due to a possibly very large number of cells (e.g. simplexes or cubes) generating the space C_m.

The main motivation for studying this problem comes from the topological and computational dynamics, namely, the need for designing general algorithms computing the homomorphism induced in homology by a continuous map $f: X \to Y$, where X and Y are polyhedral subsets of $\mathbb{R}^n$. The homomorphism in homology serves, among others, for finding Conley Index or Lefschetz Number which are invariants of certain dynamical properties of nonlinear systems. This problem was addressed, among others, in [1], [10], [15], [16].

Numerical analysis of f leads to point-to-set maps $\mathcal{F}: X \rightrightarrows Y$ with the property $f(x) \in \mathcal{F}(x)$ for all x. These maps, called *multivalued representations* of f belong to the class of maps extensively studied by [8], [9], and their followers. The values $\mathcal{F}(x)$ of those maps are acyclic polyhedral subsets of X, they are constant on open cells (open simplexes or open cubes) and their values on faces of an open cell are contained in the value on that cell. Those properties imply that $\mathcal{F}$ is a geometric realization of an abstract scheme called *acyclic career* ([17]). The proof of the Acyclic Career Theorem ([17]) permits an algorithmic construction of a so called *chain selector* $\varphi: C(X) \to C(Y)$ of $\mathcal{F}$ in [1] under the condition that we have an algorithm for solving equation (1). Once the chain map on chain complexes is constructed, we may use a very efficient program for computing the homomorphism in homology, which was designed by [18], [19] and based on the algorithm given in [12].

Another potential motivation for studying this problem is in the fact that it is related to the problem of minimal surfaces of closed curves. This problem arising from differential geometry and calculus of variations (c.f. [7]) became now one of central problems of computational geometry. A particular case of a minimal surface is a *Seifert surface* (c.f. [4]) which is a surface (possibly with self-intersections) of minimal area enclosing a given knot in the space. A *knot* $\mathcal{K}$ is a diffeomorphic embedding of the circle S^1 to $\mathbb{R}^3$ (at least, in the classical real space formulation). This is a very hard problem intensely studied and we do not claim that our coboundary construction is an important step in this direction. However, there is a similarity which gives some hope that what we do may be eventually useful here when combined with other tools. This is how the two problems can be related:

Let $\mathcal{C}$ be a polygonal closed curve with consecutive vertices

$$v_0, v_1, \ldots, v_{n-1}, v_n = v_0$$

which approximates a given knot $\mathcal{K}$. The first rough attempt would be to take a convex hull of all vertices of $\mathcal{C}$ and triangulate it. Then the support of the cycle

$$z := [v_0, v_1] + \ldots + [v_{n-1}, v_0]$$

is $\mathcal{C}$. If we solve equation (1), the support $\mathcal{S}$ of c will be a two-dimensional surface whose one-dimensional skeleton contains $\mathcal{C}$. Obviously we have no guarantee that our surface is close to a minimal surface of $\mathcal{K}$ but maybe the search for an optimal triangulation and the best solution of (1) can be somewhat refined.

In [1] two possible methods of constructing coboundaries were indicated for the case of cubical complexes. One, referred to as the *algebraic construction* is by solving systems of over-determined linear equations with integer coefficients by matrix algebra methods. That method was implemented in [14]. Another one, referred to as the *geometric construction*, was already indicated in [1] and completed in [2]. It is designed for cubical complexes only because it is based on the structure of cells as products of unitary intervals. The first method is potentially universal but the matrix algebra makes it not nearly as efficient as the geometric algorithm in [2] whose complexity is linear with respect to number of generators in the support of z.

In this paper, we develop a new universal method which combines ideas coming from [1], [2] together with the algorithm for computing homology of chain complexes presented in [12] which we shall call here the *KMS reduction algorithm*. The work on improvement of the data structure for our prototype algorithm is in progress so we cannot yet analyze the computational complexity. We expect that, in general, it could be as high as the complexity of the KMS reduction algorithm in [12] and the matrix algebra algorithm of [14] which should be close to $\mathcal{O}(n^3)$ where n is the number of generators involved. In fact, what we do here could be presented as a successive row reduction of large matrices. We believe however, that the ability of expressing the construction in the geometric language is very important for improving the data structure in the future. We already noticed some particular cases as *exterior face collapses* in [13] where the procedure is much more efficient than in general. Thus we emphasize the importance of seeing the geometry behind the algebra.

Although there is an extensive literature on the geometric problem of finding minimal surfaces, the algebraic coboundary statement (1) in arbitrary dimension addressed in this paper is very young: To the author's best knowledge, it has not been studied from the computational point of view prior to already cited [1], [2]. If the homology computation in low dimensions (polyhedra in $\mathbb{R}^3$) is of concern,

the most efficient known method is due to [5]. More discussion on the problems and application of computational topology can be found in [6].

The contents is organized as follows. We wish to make this paper accessible to a reader with no extensive abstract algebra or algebraic topology background. Thus Section 2 contains an overview of basic definitions and necessary results on chain complexes and their homology. In Section 3 simple illustrative examples of the main construction are given. We believe that presenting them before we give the general construction will make the rest more readable. In Section 4 we present a modified version of the KMS reduction procedure given in [12]. The modification is aimed at applying it to our problem. In Section 5 we present the theorems and prototype algorithm for solving equation (1). We finish Section 5 with several remarks on related questions.

2. Preliminaries from algebraic topology

The nature of algebraic operations in chain complexes performed in this paper is that if they are valid for integer scalars then they are also valid for scalars in any field $\mathcal{F}$. Hence the first case is the most general but some of the presented constructions require assuming field coefficients. When those constructions are generalized for integer scalars, their computational complexity increases since we are not allowed to divide. For those reasons we shall consider coefficients in a commutative ring $\mathcal{R}$ with unity. A reader who is unfamiliar with the ring theory may assume that $\mathcal{R}$ is either $\mathbb{Z}$ or a field $\mathcal{F}$ since these are the only cases of our interest.

DEFINITION 2.1. A sequence $(C, \partial) = (\{C_q\}_{q\in\mathbb{Z}}, \{\partial_q : C_q \to C_{q-1}\}_{q\in\mathbb{Z}})$ is a *finitely generated free chain complex with coefficients in* $\mathcal{R}$ if

(a) each C_q is a finitely generated free module over $\mathcal{R}$ (a finitely dimensional vector space if $\mathcal{R}$ is a field and a finitely generated free Abelian group if $\mathcal{R} = \mathbb{Z}$),
(b) $C_q = 0$ for all but finitely many $q \in \mathbb{Z}$,
(c) the map $\partial_q : C_q \to C_{q-1}$ is a linear map (a group homomorphism if $\mathcal{R} = \mathbb{Z}$) satisfying the property

$$\partial_q \partial_{q+1} = 0 \tag{2}$$

for each $q \in \mathbb{Z}$.

The elements of C_q are called *chains* or, more precisely, *q-chains* and the map ∂_q is called the *boundary map*.

For simplicity of notation we shall write C for (C, ∂) understanding that we know what is the boundary map and we shall avoid indices q whenever they are

clear from the context. For example, equation (2) may be written shorter as $\partial^2 = 0$.

We should mention that the formal Homological Algebra definition of chain complexes with coefficients in $\mathcal{R}$ is given via tensor products. If C is a chain complex with coefficients in $\mathbb{Z}$ then the corresponding chain complex with coefficients in $\mathcal{R}$ is $(C \otimes \mathcal{R}, \partial \otimes \mathrm{id}_{\mathcal{R}})$. Since we want our paper be accessible to readers without the abstract algebra background, we shall avoid referring to tensor products. Our definition is equivalent in the context of this paper. The advantage of the formal definition is that it permits studying what happens with chain complexes when a coefficient ring is changed.

The best known example of a finitely generated chain complex is the *simplicial complex* of a triangulated polyhedron P. The exact definition of a simplicial complex can be found in any standard Algebraic Topology textbook, e.g. in [17]. Another example of a finitely generated chain complex is the *cubical complex* $C(X)$ lectured in [11], also briefly presented in [1], [2]. A cubical polyhedron $X \in \mathbb{R}^n$ is a finite union of so called *elementary cubes* $Q \in \mathbb{R}^n$ defined by

$$Q = I_1 \times \ldots \times I_n$$

where I_j is an interval with integer endpoints of length 0 or 1. More precisely, I_j either $\{k\}$ or $[k, k+1]$, for some $k \in \mathbb{Z}$. In the cubical complex $C(X)$, $C_q(X)$ is generated by all q-dimensional elementary cubes of X. The boundary operator is defined on each elementary q-dimensional cube as an alternating sum of their $(q-1)$-dimensional faces with any two opposite parallel faces appearing with opposite signs. The cubical complexes naturally arise from the problem of numerical approximation of continuous maps [16], [15] and they have nice properties which simplicial complexes do not have. The problem of orientation of generators does not appear since the linear order on each coordinate axis and the orientation of the canonical basis implicitly impose the choice of Q as the positive generator. The Cartesian product structure is very convenient too. For example a product of elementary cubes is an elementary cube in the product space but a product of simplexes is not a simplex. Finally, any polyhedron does not need to be a cubical set but it is homeomorphic to a cubical set.

Let's now go back to the general chain complex C. A chain $z \in C_q$ is called a *cycle* or, more precisely, a *q-cycle* if $\partial z = 0$. A chain $z \in C_q$ is called a *boundary* if there exists $c \in C_{q+1}$ such that $\partial c = z$. The set of all q-cycles is the submodule $Z_q := \ker \partial_q$ of C_q while the set of all boundaries is the submodule $B_q := \mathrm{im}\, \partial_{q+1}$. Equation (2) implies that

$$\mathrm{im}\, \partial_{q+1} \subset \ker \partial_q$$

and hence the quotient

$$H_q(C) := \ker \partial_q / \mathrm{im}\, \partial_{q+1}$$

called the q-th *homology module* of the chain complex C is well defined. The *homology* of C is the sequence

$$H_*(C) := \{H_q\}_{q\in\mathcal{R}}.$$

The equation (2) implies that each ∂_q induces the trivial map

$$\partial'_q: H_q(C) \to H_{q-1}(C), \quad \partial'_q = 0$$

on quotient groups. Thus the homology sequence $H_*(C)$ may also be viewed as a chain complex called the *homology complex* of C with the trivial boundary operators. Conversely, if in some chain complex C we have $\partial_q = 0$ for all q, then $H_*(C) = C$. This simple observation is crucial for the algorithm we are going to present in the next section.

For a reader who never heard about homology, we should just mention what is the geometric interpretation of the homology groups of a polyhedron P. A lot of information can be extracted just from the dimension β_q of the free component of the module H_q. This number is called the *q'th Betti number* of P. Thus β_0 is the number of connected components of P, β_1 the number of nontrivial fundamental (i.e. generic) loops in P and, if $P \subset R^3$, β_2 is the number of "holes" in P (think about holes in a brick of Ementaler cheese). A non-free component, called the *torsion* would appear in higher dimensions only (e.g. in the *projective plane*) when a cycle is not a boundary but its multiplicity is a boundary.

A chain complex C is *acyclic* if $H_*(C) = 0$. Note that there is a subtlety in relating acyclic topological spaces to acyclic chain complexes. A polyhedron is called *acyclic* if it has the same homology as a point. The homology of a point p, however is not all trivial:

$$H_q(\{p\}) = \begin{cases} 0 & \text{if } q \neq 0, \\ \mathcal{R} & \text{if } q = 0. \end{cases}$$

A triangulated polyhedron P is called *acyclic* if its reduced homology $\widetilde{H}_*(P)$ is equal to 0. If C is a simplicial complex of P, this is equivalent to saying that the *augmented* chain complex $(\widetilde{C}, \widetilde{\partial})$ is acyclic where

$$\widetilde{C}_q = \begin{cases} C_q & \text{if } q \neq -1, \\ \mathcal{R} & \text{if } q = -1, \end{cases}$$

and

$$\widetilde{\partial}_q = \begin{cases} \partial_q & \text{if } q \neq 0, \\ \varepsilon & \text{if } q = 0, \end{cases}$$

where the map $\varepsilon: C_0 \to \mathcal{R}$ called the *augmentation map* is given on vertices by $\varepsilon v := 1$ for all vertex v. It is easily seen that the 0-cycles of $\widetilde{C}$ are generated by chains $v - u$ where u, v are vertices. The *reduced homology* of P is defined

by $\widetilde{H}_*(P) = H_*(\widetilde{C})$. The acyclicity of $\widetilde{C}$ implies, in particular, that any two vertices can be connected by a path of edges.

The same applies to cubical chain complexes.

A tool of comparing different chain complexes and their homologies is a chain map. A linear map $\varphi\colon C \to C' = \{\varphi_q : C_q \to C'_q\}_{q\in\mathbb{Z}}$ is a *chain map* if, for every $q \in \mathbb{Z}$,

$$\partial_q \varphi_q = \varphi_{q-1}\partial_q. \tag{3}$$

Any chain map φ maps cycles to cycles and boundaries to boundaries. That implies that the quotient map $\varphi_*\colon H_*(C) \to H_*(C')$ is well defined. Moreover, if $\varphi\colon C \to C'$ and $\psi\colon C' \to C''$, then

$$(\psi\varphi)_* = (\psi)_*(\varphi)_*. \tag{4}$$

Studying topological properties of spaces we only need to know its homology complex up to an isomorphism. A way of proving that two complexes have isomorphic homologies is by chain homotopies and chain equivalences. Let $\varphi, \psi\colon (C, \partial) \to (C', \partial')$ be chain maps. A collection of linear maps $D_q\colon C_q \to C'_{q+1}$ is a *chain homotopy* between φ and ψ if for all q

$$\partial'_{q+1}D_q + D_{q-1}\partial_q = \psi - \varphi.$$

We will need the following well known result

Lemma 2.2. *If there exists a chain homotopy between φ and ψ, then*

$$\varphi_* = \psi_*.$$

A chain map $\varphi\colon C \to C'$ is called a *chain equivalence* if there exists a chain map $\psi\colon C' \to C$ such that $\psi\varphi = \mathrm{id}_C$ and $\varphi\psi = \mathrm{id}_{C'}$.

We shall use the following consequence of Lemma 2.2 and (4):

Theorem 2.3. *If $\varphi\colon C \to C'$ is a chain equivalence then $\varphi_*\colon H_*(C) \to H_*(C')$ is an isomorphism.*

3. Simple examples

The aim of this section is to develop geometric intuitions which will help to understand the general procedures introduced in the next section.

In all examples the ring of scalars is $\mathbb{Z}$. We base the examples on the simplicial chain complex $\mathcal{C}$ of the square on Figure 1(i). Thus C_0 is generated by the set of vertices $E_0 := \{A, B, C, D\}$, C_1 by edges $E_1 := \{a, b, c, d\}$ oriented as on

Figure 1(i), and C_2 by triangles $E_2 := \{S, T\}$ with clockwise orientation. The only nontrivial boundary maps are ∂_1 and ∂_2 defined as follows.

$$\begin{aligned} \partial_1\colon a &\mapsto B - A, & \partial_2\colon S &\mapsto e - b - a, \\ b &\mapsto C - B, & T &\mapsto d + c - e. \\ c &\mapsto C - D, \\ d &\mapsto D - A, \\ e &\mapsto C - A, \end{aligned}$$

Note that a, b, c, d are *exterior* or *free* edges, in the sense any one each of them is an edge of an exactly one triangle. The edge e is an *interior* edge in the sense that it is a common edge of more then one triangle.

EXAMPLE 3.1 (Exterior face collapses). Imagine that we push a from outside so that S collapses onto the remaining part of its boundary formed of e and b. Then a is projected to the chain $e - b$ and S disappears. No new homologically nontrivial cycle is created by the fact that S disappears since the image of the cycle $\partial S = -a + e - b$ is $-(e - b) + e - b = 0$. This indicates that the homology of the complex $\overline{\mathcal{C}}$ obtained by projections of generators $a \mapsto e - b$, $S \mapsto 0$ should be the same as that of $\mathcal{C}$. The new complex $\overline{\mathcal{C}}$ illustrated on Figure 1(ii) is a subcomplex of $\mathcal{C}$ with bases $\overline{E}_0 := E_0$, $\overline{E}_1 := \{b, c, d, e\}$ and $\overline{E}_2 := \{T\}$. By repeating the same with the edge e and triangle T, we get a one-dimensional complex $\overline{\overline{\mathcal{C}}}$ illustrated on Figure 1(iii). The same procedure can be repeated for a free vertex and its unique edge, so $\overline{\overline{\mathcal{C}}}$ can be reduced in next three collapsed to a single vertex.

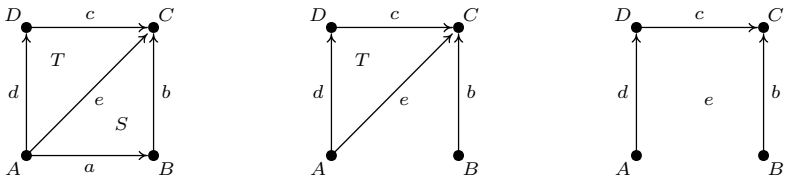

FIGURE 1. Exterior edge collapses

EXAMPLE 3.2 (Interior face reduction). Let's go back to the initial complex $\mathcal{C}$. The edge e is common to S and T so the previous procedure does not apply to it. However, we may eliminate it from the boundaries by adding equations for ∂S and ∂T:

$$\partial S + \partial T = \partial(S + T) = e - b - a + d + c - e = d + c - b - a.$$

This may be visualized as follows: Imagine that e is collapsed as before through T until it is projected onto $d + c$. Then e drags S along with it so S takes the whole space inside the square. Thus we have the projections

$$e \mapsto d + c, \quad S \mapsto \overline{S} := S + T, \quad T \mapsto 0.$$

The new complex $\overline{\mathcal{C}}$ is generated by $\overline{E}_0 = E_0$, $\overline{E}_1 = \{a, b, c, d\}$ and $\overline{E}_2 = \{\overline{S}\}$. It still is a subcomplex of $\mathcal{C}$ since $\overline{S} = S+T$ but it is no longer a simplicial complex. If we want to continue reductions in the same dimension, we can now collapse the exterior edge a through the square $\overline{S}$ until it is projected to $d + c - b$. That gives the same complex $\overline{\overline{\mathcal{C}}}$ as in the previous example, pictured on Figure 1(iii). An alternative is to continue interior reduction in the lower dimension to remove vertices common to two edges. Another example of this kind is given in [12].

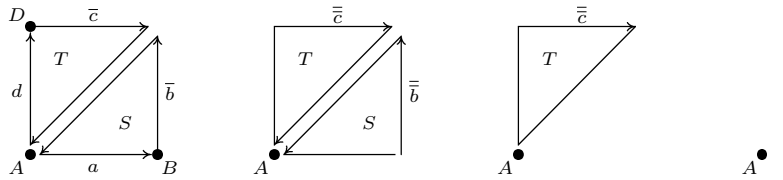

FIGURE 2. Starting from a vertex reduction

EXAMPLE 3.3 (Interior vertex reduction). Face reductions may also be performed starting from the lowest dimension. Imagine that we push the vertex C along the edge e so it projects to the vertex A. The edge e disappears. If we want to do that without moving interiors of S and T, we must drag the edges b and c with the vertex C along $-e$, so we get projections

$$C \mapsto A, \quad e \mapsto 0, \quad b \mapsto \overline{b} := b - e, \quad c \mapsto \overline{c} := c - e.$$

This is a purely algebraic operation but Figure 2(i) shows a geometric interpretation of it. Note that the new edges $\overline{b}$ and $\overline{c}$ follow the same path for a while. Project in a similar way B to A along a and D to A along d. Thus

$$B \mapsto A, \quad a \mapsto 0, \quad \overline{b} \mapsto \overline{\overline{b}} := a + \overline{b} = a + b - e,$$

and

$$D \mapsto A, \quad d \mapsto 0, \quad \overline{c} \mapsto \overline{\overline{c}} := d + \overline{c} = d + c - e.$$

This is illustrated on Figure 2(ii). Next, $\overline{\overline{b}}$ is the free edge of S and so the external collapse applies to this pair and we get what is pictured on Figure 2(iii). We do the same with $\overline{\overline{c}}$ which is the free edge of T and we get the single point A.

The next examples are aimed at illustrating the construction of a coboundary $\mathrm{Cob}(z)$ of a 1-cycle z given in the same complex $\mathcal{C}$ as above. Thus $\mathrm{Cob}(z)$ should satisfy $\partial\,\mathrm{Cob}(z) = z$. It will be easy to guess what $\mathrm{Cob}(z)$ is by looking at Figure 1(iii) but we shall follow the algebraic formula which will be given in the next section and give its geometrical interpretation.

EXAMPLE 3.4. Let $z = a+b-c-d$. We follow the path of z while collapsing edges as in Example 3.1. The first collapse projects a to $\overline{a} = e - b$, thus z is projected through S to $\overline{z} = (e-b)+b-c-d = e-c-d$. Observe that

$$z - \overline{z} = a + b - e = -\partial S = \partial(-S).$$

We next follow the path of $\overline{z}$ when e is collapsed through T to $\overline{\overline{e}} = d + c$. Then $\overline{z}$ is projected to

$$\overline{\overline{z}} = (d+c) - c - d = 0.$$

Let's ignore this for instance and observe that $\overline{z} - \overline{\overline{z}} = -\partial T = \partial(-T)$. Thus

$$z = z - \overline{\overline{z}} = z - \overline{z} + \overline{z} - \overline{\overline{z}} = \partial(-S) + \partial(-T) = \partial(-S-T).$$

Thus $\mathrm{Cob}(z) = -S - T$ is a solution of $\partial\,\mathrm{Cob}(z) = z$.

EXAMPLE 3.5. Exterior collapses are not always possible so we will see what happens to a cycle when interior face reduction is performed as in Example 3.2. The previously considered cycle is not very interesting since reducing e doesn't change z and the second operation is a collapse already described.

Let $z = a+b-e$. When the first reduction in Example 3.2 is applied (which is a bad choice but we pretend that we do not see the picture), e is projected to $\overline{e} = d + c$ so z is projected to $\overline{z} = a + b - d - c$. We have

$$z - \overline{z} = a + b - e - a - b + d + c = d + c - e = \partial T.$$

We now follow the path of $\overline{z}$ while collapsing a through $\overline{S} = S + T$. Then $a \mapsto \overline{\overline{a}} = d + c - b$ and $\overline{z} \mapsto \overline{\overline{z}} = d + c - b + b - d - c = 0$. Again, $\overline{z} - \overline{\overline{z}} = -\partial\overline{S}$ so

$$\begin{aligned} z &= z - 0 = z - \overline{z} + \overline{z} - \overline{\overline{z}} \\ &= \partial T + \partial(-\overline{S}) = \partial T + \partial(-S-T) = \partial(T-S-T) = \partial(-S), \end{aligned}$$

hence $\mathrm{Cob}(z) = -S$ is a solution of $\partial\,\mathrm{Cob}(z) = z$.

It is visible from Example 3.5 that computing coboundaries by interior face reductions can get more complicated than by exterior face collapses since there are changes of bases. Moreover, we had to express new $\overline{S}$ back in terms of S and T hence those two generators cannot be removed from the data structure after performing the reduction. Fortunately, one can prove that, in cases to which we shall apply our results, the external face collapses are sufficient.

Finally, we note that a coboundary of a cycle is not always unique:

Example 3.6. Let $z = C - A$, Then $z \in \widetilde{\mathcal{C}}$ is an augmented cycle, i.e. a solution of the equation $\varepsilon(z) = 0$ (see the comment on reduced homology in the previous section). Then $z = \partial(d + c) = \partial(a + b)$.

We refer the reader to [2] for an example of a coboundary construction of a 1-cycle in a three dimensional cube.

4. Modified KMS reduction procedure

Let now $(C, \partial) = (\{C_q\}_{q\in\mathbb{Z}}, \{\partial_q\}_{q\in\mathbb{Z}})$ be a finitely generated free chain complex with coefficients in a given commutative ring $\mathcal{R}$ with unity. Let n_0 and N be, respectively, the least and the greatest value of q such that $C_q \neq 0$. Most commonly, $n_0 = 0$ but, for example, in an augmented simplicial or cubical chain complex we have $n_o = -1$. Let $d_q := \dim C_q := \mathrm{card}(E_q)$ for each q.

Assume that a fixed base E_q of C_q is given for each q such that $C_q \neq 0$, we shall call it the *canonical basis* of C_q. Let $\langle \cdot\,, \cdot \rangle$ denote the associated scalar product, i.e. the bilinear form $C_q \times C_q \to \mathcal{R}$ defined on generators by

$$\langle e, e' \rangle := \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{otherwise,} \end{cases}$$

where $e, e' \in E_q$. Given $c \in E_q$, the generators $e \in E_{q-1}$ such that $\langle \partial c, e\rangle \neq 0$ are called *faces* of c. The number $\langle \partial c, e\rangle$ is called the *incidence number* of e in ∂c.

Let $m \in \mathbb{Z}$ be a fixed number. Assume that $a \in E_{m-1}$ and $b \in E_m$ are two fixed elements such that $\lambda := \langle \partial b, a\rangle$ is invertible in $\mathcal{R}$. Thus ∂b can be written as

$$\partial b = \lambda a + r,$$

where $\langle a, r\rangle = 0$. Let

$$E_{m-1} = \{a_1, \dots a_{d_{m-1}-1}, a\}, \quad E_m = \{b_1, \dots b_{d_m-1}, b\}.$$

We shall write r as

$$r = \sum_{i=1}^{d_{m-1}-1} \alpha_i a_i.$$

For any $q \in \mathbb{Z}$, define a map $p_q \colon C_q \to C_q$ by the formula

$$p_q v := \begin{cases} v - \lambda^{-1}\langle v, a\rangle \partial b & \text{if } q = m-1, \\ v - \lambda^{-1}\langle \partial v, a\rangle b & \text{if } q = m, \\ v & \text{otherwise,} \end{cases} \tag{7}$$

where $v \in C_q$. We let $\overline{C}_q := \mathrm{im}\, p_q$ be the image of C_q in p_q.

PROPOSITION 4.1. *The map p_q is a linear projection of C_q onto its image $\overline{C}_q$. More precisely, it is a linear map with the property $p_q w = w$ for all $w \in \overline{C}_q$.*

PROOF. The linearity of p_q is obvious since the maps ∂ and $\langle \cdot\,, a\rangle$ are linear. The second conclusion is equivalent to the identity $p_q^2 = p_q$. This is trivial if $q \notin \{m-1, m\}$.

Let $q = m-1$ and $v \in C_{m-1}$. Put $\alpha := \langle v, a\rangle$. Then

$$\begin{aligned} p_{m-1}^2 v &= p_{m-1}(v - \lambda^{-1}\alpha\partial b) = v - \lambda^{-1}\alpha\partial b - \lambda^{-1}\langle v - \lambda^{-1}\alpha\partial b, a\rangle\partial b \\ &= v - \lambda^{-1}\alpha\partial b - \lambda^{-1}\alpha\partial b + \lambda^{-2}\alpha\lambda\partial b = p_{m-1}v. \end{aligned}$$

Let $q = m$ and $v \in C_m$. Put $\beta := \langle \partial v, b\rangle$. Then

$$\begin{aligned} p_m^2 v &= p_m(v - \lambda^{-1}\beta b) = v - \lambda^{-1}\beta b - \lambda^{-1}\langle v - \lambda^{-1}\beta b, a\rangle b \\ &= v - \lambda^{-1}\beta b - \lambda^{-1}\beta b + \lambda^{-2}\beta\lambda b = p_m v. \end{aligned}$$

□

Note that the above projection is not an orthogonal projection. We should explicitly identify the images of the basic elements under p_{m-1} and p_m. For simplicity of notation we put $\overline{v} := p_q v$ and $\gamma_i := \lambda^{-1}\langle \partial b_i, a\rangle$. It is easily seen that

$$\overline{a_i} = a_i, \qquad \overline{a} = -\lambda^{-1} r, \tag{8}$$

$$\overline{b_i} = b_i - \gamma_i b, \qquad \overline{b} = 0. \tag{9}$$

Note that the vectors listed in (8) are columns of the matrix of p_{m-1} with respect to the canonical basis and those listed in (9) are columns of p_m. In matrix notation,

$$p_{m-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & -\lambda^{-1}\alpha_1 \\ 0 & 1 & 0 & \dots & 0 & -\lambda^{-1}\alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\lambda^{-1}\alpha_{d_{m-1}-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{10}$$

and

$$p_m = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & \dots & -\gamma_{d_m-1} & 0 \end{pmatrix}. \tag{11}$$

We put

$$\overline{E}_q = \begin{cases} \{\overline{b_1}, \dots \overline{b}_{d_m-1}\} & \text{if } q = m, \\ \{a_1, \dots a_{d_{m-1}-1}\} & \text{if } q = m-1, \\ E_q & \text{otherwise.} \end{cases} \tag{12}$$

PROPOSITION 4.2. *$\overline{E}_q$ is a basis for $\overline{C}_q$ for all $q \in \mathbb{Z}$ and $\overline{E}_m \cup \{b\}$ is a basis for C_m.*

PROOF. The conclusion is trivial for $q \notin \{m-1, m\}$. For $q = m-1$ and $q = m$, it is enough to apply a standard matrix algebra argument to matrices in (10) and (11). □

THEOREM 4.3. *The sequence of projections $p := \{p_q\}_{q\in\mathbb{Z}} : C \to C$ is a chain map.*

PROOF. We need to show that $\partial_q p_q = p_{q-1}\partial_q$ for all q. This is obvious for $q \notin \{m-1, m, m+1\}$.

Let $q = m-1$ and $v \in C_{m-1}$. Then

$$\partial p_{m-1} v = \partial(v - \lambda^{-1}\langle v, a\rangle \partial b) = \partial v - \lambda^{-1}\langle v, a\rangle \partial^2 v = \partial v$$

because $\partial^2 = 0$. On the other hand $p_{m-2}\partial v = \partial v$ since $p_{m-2} = \mathrm{id}$.

Let $q = m$ and $v \in C_m$. Then

$$\partial p_m v = \partial(v - \lambda_{-1}\langle \partial v, a\rangle b) = \partial v - \lambda_{-1}\langle \partial v, a\rangle \partial b = p_{m-1}\partial v.$$

Let $q = m+1$ and $v \in C_{m+1}$. Then

$$p_m \partial v = \partial v - \lambda^{-1}\langle \partial^2 v, a\rangle b = \partial v$$

because $\partial^2 = 0$. On the other hand $\partial p_{m+1} v = \partial v$ since $p_{m+1} = \mathrm{id}$. □

COROLLARY 4.4. *The sequence $(\overline{C}, \partial) := \{(\overline{C}_q, \partial_q)_{q\in\mathbb{Z}}\}$ is a chain subcomplex of (C, ∂) and the restriction $\overline{p} : C \to \overline{C}$ of the codomain of p to its image is a chain map.*

PROOF. The second conclusion follows from the first one and from Theorem 4.3. We have to show that ∂ is well defined on $\overline{C}$ i.e. that $\partial(\overline{C}_q) \subset \overline{C}_{q-1}$ for all q. Indeed, by Theorem 4.3 we have

$$\partial(\overline{C}_q) = \partial p_q(C_q) = p_{q-1}\partial(C_q) \subset p_{q-1}(C_{q-1}) = \overline{C}_{q-1}.$$ □

THEOREM 4.5. $H_*(\overline{C}) \cong H_*(C)$.

PROOF. We will show that $\overline{p} : C \to \overline{C}$ is a chain equivalence with the inclusion $i : \overline{C} \hookrightarrow C$ as a homotopical inverse. Indeed, by Proposition 4.1, $\overline{p}$ is a projection so $\overline{p}i = \mathrm{id}_{\overline{C}}$. Hence it is sufficient to find a chain homotopy between $i\overline{p} = p$ and id_C. Let $D_q : C_q \to C_{q+1}$ be given by

$$D_q = \begin{cases} \lambda^{-1}\langle v, a\rangle b & \text{if } q = m-1, \\ 0 & \text{otherwise,} \end{cases}$$

for any $v \in C_q$. We need to show the identity

$$\mathrm{id}_{C_q} - i_q \overline{p}_q = \partial_{q+1} D_q + D_{q-1}\partial_q. \tag{13}$$

This is obvious if $q \notin \{m-1, m\}$ since, in that case, both sides are 0. Let $q = m-1$ and $v \in C_{m-1}$. Then

$$v - p_{m-1}v = v - (v - \lambda^{-1}\langle v, a\rangle \partial b) = \lambda^{-1}\langle v, a\rangle \partial b.$$

On the other hand

$$\partial D_{m-1}v + D_{m-2}\partial v = \partial D_{m-1}v = \partial(\lambda^{-1}\langle v, a\rangle b) = \lambda^{-1}\langle v, a\rangle \partial b,$$

so the identity holds.

Let $q = m$ and $v \in C_m$. Then

$$v - p_m v = v - (v - \lambda^{-1}\langle \partial v, a\rangle b) = \lambda^{-1}\langle \partial v, a\rangle b. \tag{14}$$

On the other hand

$$\partial D_m v + D_{m-1}\partial v = D_{m-1}\partial v = \lambda_{-1}\langle \partial v, a\rangle b.$$

so the identity holds. □

Let now

$$C^0 \xrightarrow{\overline{p}^1} C^1 \xrightarrow{\overline{p}^2} C^2 \longrightarrow \cdots$$

be a sequence of chain subcomplexes and projections obtained from (C, ∂) by iterating the above construction as long as it is possible to choose $m \in \{n_0, n_0+1, \ldots, N\}, a \in C_{m-1}$ and $b \in C_m$ such that $\lambda := \langle \partial b, a\rangle$ is invertible. Thus $C^0 := C$, $C^{k+1} := \overline{C}^k$ and $E_q^{k+1} := \overline{E}_q^k$ for all $q \in \mathbb{Z}$. We denote by ∂^k the restriction of ∂ to C^k.

Denote by $M(k) = \sum_q \operatorname{card}(E_q^k)$, for $k = 0, 1, 2, \ldots$ Since C is finitely generated, $M(k) < \infty$ and $M(k+1) = M(k) - 2$, therefore there exists a final element of that sequence denoted by (C^f, ∂^f), beyond which the construction cannot be extended.

THEOREM 4.6. *If $\mathcal{R}$ is a field, then $\partial^f = 0$ and $H(C) \cong H(C^f) = C^f$.*

PROOF. The identity $H(C) = H(C^f)$ follows from Theorem 4.5 by induction. If $\mathcal{R}$ is a field then λ is invertible if and only if and only if it is non-zero. Hence the construction can be iterated as long as there exist m and two elements $a \in E_{m-1}^k$ and $b \in E_m$ satisfying $\langle \partial b, a\rangle \neq 0$ i.e. as long as $\partial \neq 0$. Therefore $\partial^f = 0$. But this means that $H(C^f) = C^f$. □

REMARK 4.7. Our construction is very similar to the one presented in [12]. An important difference is that, in [12], a new chain complex $(\overline{C}, \overline{\partial})$ is not a subcomplex of (C, ∂) as it is here: The basic elements b_i complementing b are left the same but the boundary operator is redefined. Thus a completely new complex is created. That permits removing unnecessary data from the data structure and decides about the effciency of the KMS algorithm. However, from the point of view of computing the coboundary of a cycle, this imposes a problem. Our new

version of the KMS construction permits easily expressing new generators in terms of the original basis and this is exactly what we shall need in the next section. When the computation of homology up to an isomorphism is the only goal, a data structure can be found which will give the same algorithm for our construction as for the previous KMS construction.

Remark 4.8. Another difference between the presented construction and the one in [12] is that the KMS algorithm performs the reductions starting from the highest dimension $q = N$ down to the lowest $q = n_0$ whereas we do not impose that order here. The choice of the order in [12] was motivated by the belif that this would give the lowest complexity in the general case. It is uncertain, however, whether or not a different order could be more efficient in particular cases. If we try to simultanously use this procedure with the procedure of finding a chain selector of an acyclic-valued map, it is the order from the lowest to the highest dimension which is imposed.

Remark 4.9. The presented procedure produces the homology complex when $\mathcal{R}$ is a field. The case of coefficients in a ring with unity where the Euclidean algorithm for division with reminders is valid ($\mathbb{Z}$ is an example of such a ring), the KMS algorithm has been generalized by [18] and the same applies to the modified version presented here. The computation complexity is much higher in that case. In some special cases, even if the (C, ∂) is a chain complex with coefficients in $\mathbb{Z}$, there are abstract theorems which imply that the homologies, whatever they are, must be free abelian groups, i.e. $H_q \cong \mathbb{Z}^{\beta_q}$ for all q. The exponent β_q is called the q'th *Betti number* of C. This is the case, for example, when C is a complex of a graph, of a polyhedron in $\mathbb{R}^2$, or of a manifold in $\mathbb{R}^3$. Thus when homology is free, it is enough to determine the Betti numbers. In that case, there is a theorem which says that

$$H(C) \otimes \mathcal{R} \cong H(C \otimes \mathcal{R}),$$

and we have $Z \otimes \mathcal{R} \cong \mathcal{R}$. We purpousely avoided talking about tensor product but an elementary way of saying the same is the following: If we know that the homology is free, we do the calculation over a field Z_2, where the computing time is the lowest, and the dimension of the vector space $H_q(C, Z_2)$ must be the same Betti number β_q. Of course, Z_2 might not give us an appropriate information about the homology of a map, so the computation for scalars in $\mathbb{Q}$ or $\mathbb{R}$ might be preferable for those purposes.

5. Coboundary formula

In this section (C, ∂) will be an acyclic finitely generated chain complex with coefficients in a field $\mathcal{F}$. Let $z \in Z_{m-1}$ be a $(m-1)$-cycle. By the acyclicity assumption, there exists a chain $c \in C_m$ such that $\partial c = z$. In general, c is not

unique. The purpose is to provide an explicit algorithm for finding at least one such a chain which will be denoted by $\mathrm{Cob}(z)$.

LEMMA 5.1. *Let $a \in C_{m-1}$, $b \in C_m$ and $p\colon C \to C$ be as in the elementary reduction step of the previous section. Let $z \in Z_{m-1}$ be a given cycle. Put $\alpha := \langle z, a\rangle$. Then*

(a) $\overline{z} := p_{m-1}z \in \overline{Z}_{m-1}$.
(b) $\partial(\alpha\lambda^{-1}b) = z - \overline{z}$.

PROOF. (a) is obvious since $\overline{p}$ is a chain map. For (b), let

$$z = \alpha a + \sum \beta_i a_i$$

where a_i are as in the previous section. Then

$$\overline{z} = \alpha\overline{a} + \sum \beta_i \overline{a_i} = -\alpha\lambda^{-1}r + \sum \beta_i a_i.$$

Hence $z - \overline{z} = \alpha a + \alpha\lambda^{-1}r = \alpha\lambda^{-1}(\lambda a + r) = \alpha\lambda^{-1}\partial b = \partial(\alpha\lambda^{-1}b)$. □

Note that the conlusion of Lemma 5.1 holds true even if $\alpha = 0$ but in that case $\overline{z} = z$. We may want to avoid repetitions of (a, b)-reductions which are not necessary for computing $\mathrm{Cob}(z)$. We introduce the following notation. Given any $c \in C_{m-1}$ let

$$E_{m-1}(c) := \{a \in E_{m-1} : \langle c, a\rangle \neq 0\}, \quad E_m(c) := \{b \in E_m : \langle \partial b, c\rangle \neq 0\}.$$

LEMMA 5.2. *Let $z \in Z_{m-1}$, $z \neq 0$. Then $E_{m-1}(z) \neq \emptyset$ and, for any $a \in E_{m-1}(z)$, $E_m(a) \neq \emptyset$.*

PROOF. The first conlusion is obvious. For the second one, suppose the contrary. Then $\langle \partial b, a\rangle = 0$ for all $b \in E_m$ hence a is orthogonal to $\operatorname{im}\partial_m = B_{m-1}$. But C is acyclic so $B_{m-1} = Z_{m-1}$ thus $\langle z, a\rangle = 0$, a contradiction. □

We shall now head towards a prototype algorithm computing $\mathrm{Cob}(z)$. The idea is very simple. We start from $c = 0$. Then use recursively (a, b)-reductions simultanously adding the term $\alpha\lambda^{-1}b$ which appears in Lemma 5.1 to the previous value of c, and replacing the previous value of z by $\overline{z}$. This procedure must end since, each time we repeat it, the cardinality of $\overline{E}_{m-1}$ and $\overline{E}_m$ decreases by one and those sets are finite. When it ends, the final value of $\overline{z}$ is 0, and the final value of c satisfies $\partial c = z - 0 = z$.

To formalize this dicussion let us go back to the sequence of projections in (14). Let

$$C^0 \xrightarrow{\overline{p}^1} C^1 \xrightarrow{\overline{p}^2} C^2 \longrightarrow \cdots$$

be the sequence of projections and let $(a^k, b^k) \in (E^k_{m-1}, E^k_m)$ be the reduced pairs of generators for a fixed m. Let z^k be defined by

$$z^0 := z, \quad z^k := \overline{p}^k z^{k-1}, \quad k = 1, 2, \ldots$$

Since each projection is a chain map, each z^k is a cycle. Thus, by Lemma 5.1,

$$z^{k-1} - z^k = \partial(\alpha_k \lambda_k^{-1} b^{k-1}) \tag{15}$$

for some $\lambda_k \neq 0$ and some α_k (possibly equal to 0 but then $z^{k-1} = z^k$). We construct a sequence of chains $c^k \in C_m$, $k = 0, 1, \ldots$ as follows.

$$c^0 := 0, \quad c^k := c^{k-1} + \alpha_k \lambda_k^{-1} b^k, \quad k = 1, 2, \ldots \tag{16}$$

By induction it follows from (15) and (16) that

$$z - z^k = \partial c^k. \tag{17}$$

By the same argument as in discussion following (14), there is a finite element C^{f_m} in the sequence beyond which the construction in this dimension cannot be extended and then $\partial_m^{f_m} = 0$ where $\partial^k = 0$ is the restriction of ∂ to C^k.

Theorem 5.3. $\partial c^{f_m} = z$.

Proof. Most of the proof is already done in the construction of c^k. Since C is acyclic, by the proof of Theorem 4.6, C^{f_m} is acyclic too. Since $\partial_m^{f_m} = 0$, we have $\ker \partial_{m-1}^{f_m} = \operatorname{im} \partial_m^{f_m} = 0$ hence $z^{f_m} = 0$. The conclusion follows from (17).□

In the practical computation of $c := c^{f_m}$ we notice a subtlety in the fact that $\overline{p}_q^k$ is expressed in terms of the bases E_q^{k-1} and E_q^k and we want to find the coordinates of c with respect to the canonical basis $E_m = E_m^0$. We already saw that in Example 3.2. The problem does not appear when it is possible to only use exterior face collapses as in Example 3.1 since then $\overline{b_i} = b_i$ and always $\overline{a_i} = a_i$ in (12).

In general, we need a formula for a projection $p^k \colon C \to C$ in canonical coordinates such that $\overline{p}^k$ is obtained by restriction of the domain of p^k to C^{k-1} and its codomain to C^k. This can be formalized by considering the sequence of inclusions

$$C^0 \overset{i^1}{\hookleftarrow} C^1 \overset{i^2}{\hookleftarrow} C^2 \hookleftarrow \cdots$$

We define the inclusion $j \colon C \to C^k$ by induction as

$$j^0 := \mathrm{id}_C, \quad j^k := j^{k-1} i^k, \quad k = 1, 2, \ldots \tag{18}$$

Thus $p^k \colon C \to C$ is given by

$$p^k := j^{k-1} \overline{p}^k p^{k-1}. \tag{19}$$

Similarly, ∂^k, the restriction of ∂ to C^k, is expressed by a matrix in local basis $E_m^k \times E_{m-1}^k$ and the relation between these two maps is

$$\partial^k = p^k \partial j^k. \tag{20}$$

The data structure which is easiest to describe but probably not the most economic, is the following:

Chains are vectors. We assume that the bases $\overline{E}_q$ ere linearly ordered so they are arrays of vectors. By $\overline{\langle z, a\rangle}$ we shall denote the incidence number with respect to the local basis $\overline{E}_{m-1}$. Maps are defined on elements of a basis so they can be written as matrices. Note that the only elements of the sequence $(\{C_q\}, \{\partial_q\})$ appearing in the calculations are C_{m-1}, C_m and ∂_m. Thus we will assume that $E = (E_{m-1}, E_m)$, $\partial = \partial_m$, $p = (p_{m-1}, p_m)$ and so on.

The algorithm calls certain algebraic procedures which we shall not describe in detail: their meaning will be transparent from the terminology.

ALGORITHM 5.4 (Compute a coboundary c of a cycle z).

input
  E: pair of arrays;
  ∂: matrix;
  z: vector;
output c: vector;
function Cob(z): vector;
variables
  $\overline{E}$: pair of arrays;
  $i, j, \overline{p}$: pairs of matrices;
  a, b, c: vectors;
  α, λ: scalars;
begin
  $c := 0$;
  $\overline{E} := E$;
  $\overline{p} := \mathrm{id}_C$;
  $i := \mathrm{id}_C$;
  $j = \mathrm{id}_C$;
  while $z \neq 0$ **do**
    $(a, b, \lambda) :=$ Reduction Triple for $\overline{E}$;
    $\alpha := \overline{\langle z, a\rangle}$;
    $c := c + \alpha\lambda^{-1} jb$;
    $(\overline{p}, i) :=$ Equivalence Pair for $(a, b; \overline{E})$;
    $\overline{E} := \overline{p}(\overline{E})$;
    $z := \overline{p}_{m-1} z$;
    $j := ji$;
  end while;
return c;
end;

By the construction which lead to Theorem 5.3 we get the following

Theorem 5.5. *Algorithm* 5.4 *ends after running through the* **while** *loop at most* d_{m-1} *times. The returned output* c *satisfies*

$$\partial c = z.$$

Proof. The loop while is never entered if $z = 0$. In this case the algorithm returns the initial value $c = 0$ and, obviously, $\partial 0 = 0$. If the loop is entered, the sequences of projections are created exactly as in the construction leading to Theorem 5.3. We proved that the final values are $z^{f_m} = 0$ and $c = c^{f_m}$ so the algorithm stops here. □

Remark 5.6. If z is a small cycle in a big complex, it would be useless to do reductions of pairs (a, b) such that a is not a face of z. For such computations, the procedure of finding Reduction Triple for $\overline{E}$ can replaced by the following:

Find $\overline{E}_{m-1}(z)$;
$a :=$ last entry of $\overline{E}_{m-1}(z)$;
Find $\overline{E}_m(a)$;
$b :=$ last entry of $\overline{E}_m(a)$;
$\lambda : \overline{\langle \overline{\partial} b, a \rangle}$;

Lemma 5.2 guarantees that the lists $\overline{E}_{m-1}(z)$ and $\overline{E}_m(a)$ are nonempty so the procedure is successful.

Remark 5.7. When R is a *rectangle* in $\mathbb{R}^n$, i.e. a product of intervals with integer coordinates and $C := C(R)$ is a cubical complex of X, a fast geometric algorithm of finding a coboundary of an $(m-1)$-cycle z is given in [2]. The algorithm is based on the recurrence with respect to the dimension d of R. Here is the main idea of the recurrence step. Let R' be a $(d-1)$-dimensional face of X. The orthogonal projection of R onto R' induces, in an evident way, a projection $p\colon C(R) \to C(R')$ which is orthognal with respect to the canonical basis of $C(R)$ consisting of unitary cubes. We let $z' := pz$ and compute $\mathrm{Cob}(z')$. Then $\mathrm{Cob}(z)$ is obtained by adding to $\mathrm{Cob}(z')$ all m-dimensional unitary cubes through which z is projected, with appropriety chosen coeffcients. It is visible that this construction may be viewed as a major shortcut of what we presented in our paper. Instead of projecting in small steps through one elementary cube per time, we get the whole cyclinder enclosed between a face $e \in E_{m-1}(z)$ and a corresponding face of z', in a single operation of replacing an endpoint by an interval in the expression of e as a product of intervals. Due to this simplification, the complexity of the algorithm in [2] is linear with respect to the number of elements of $E_{m-1}(z)$. Cubical complexes are very particular but we belive that those type of shortcuts in our universal algorithm will be possible in many situations when a subdivision of a space to cells is chosen in a "custom-fit" way.

Let us introduce some more definitions. A generator $a \in E_{m-1}$ is a *free face* if there exists a unique $b \in E_m$ such that $\langle \partial b, a \rangle \neq 0$. If so, the projection $\overline{p}$ of the pair (a, b) is called an *elementary collapse*. We say that C *collapses* to zero if there exists a sequence (14) of elementary collapses bringing it to zero.

REMARK 5.8. As we previously noticed, the coboundary formula becomes much more simple if C collapses to 0. Indeed, in that case, the numbers γ_i in (9) are 0 so every new basis is a selection from a previous one. Thus it shoud be possible to avoid most of matrix algebra computation and efficiently organize data structure in the form of a tree as in [13]. The problem is that not every acyclic complex collapses: Geometric Topology provides examples of spaces such as the *House with Two Rooms* [3] which are acyclic but have no free faces. Fortunately, such examples are very unusual. Any convex polyhedron collapses. The author is convinced that every star-shape polyhedron, as arises in 1, collapses too but we haven't seen the proof yet.

REMARK 5.9. We assumed field coefficients because the reduction procedure in the previous section assumes the invertibility of λ. However, the construction generalizes to ring coefficients as in [18]. The problem is the higher complexity in that case. However, unlike in the previous section, it is assumed here that C is acyclic, so no torsions would appear. Therefore it should be possible to extend the construction for any ring without adding new costly operations.

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