Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 18, 2001, 321–235

ON H. FRIEDRICH'S FORMULATION OF THE EINSTEIN EQUATIONS WITH FLUID SOURCES

Yvonne Choquet-Bruhat — James W. York

Dedicated to Andrzej Granas

Abstract. We establish a variant of the symmetric quasi linear first order system given by H. Friedrich for the evolution equations of gravitating fluid bodies in General Relativity which can be important to solve realistic problems. Our version has the advantage of introducing only physical characteristics. We state explicitly the conditions under which the system is hyperbolic and admits a well posed Cauchy problem.

1. Introduction

Cattaneo [4] and Ferrarese [8], have stressed long ago that the fundamental physical data are those defined on time lines instead of those defined on spacelike hypersurfaces. In the case of a fluid, timelike world lines have a material reality given by the flow, the unknowns associated to them correspond to a lagrangian picture, in contrast with the usual eulerian one. In many situations, particularly in cosmology, the lagrangian description seems more fundamental. However, one cannot be too confident because General Relativity holds many surprises. For example, consider a fluid in internal equilibrium and in equilibrium with a stationary black hole. Under some conditions (cf. [3]) the thermodynamic temperature is found by certain eulerian observers whose world lines have vanishing vorticity. This corresponds to our "well posedness" result given in Section 11.4. We believe that eulerian and lagrangian descriptions are both usefull.

2000 *Mathematics Subject Classification.* 35F25, 35L20.
Key words and phrases. Einstein relativity, fluids.

©2001 Juliusz Schauder Center for Nonlinear Studies

Ferrarese [9], [10] has given a complete 3+1 decomposition of the connection and Riemann tensor in a frame whose time axis is tangent to the time lines and the space axes orthogonal to it. Using this decomposition he has written the Einstein-dust equations as a (infinite dimensional) dynamical system. Recently H. Friedrich [11] has used a lagangian description of the flow together with a tetrad formalism to write the Einstein equations coupled with a perfect fluid as a symmetric first order system, formally hyperbolic. His treatment includes dust as a particular case.

The Einstein-dust system, as well as the Einstein-Euler system for a perfect fluid, have been proved long ago [5] to form a well posed hyperbolic system in the sense of Leray. The relativistic Euler system has been put in first order symmetric hyperbolic form (FOSH) first by K. O. Friedrichs himself [12] , then by general methods by Anile, Boillat and Ruggeri (see references in [1]). Rendall [13] has obtained, for well chosen equations of state, FOSH systems for the Euler equations which extend to vacuum, but do not yet permit, as he points out himself, the solution of realistic problems of motion of compact fluid bodies in otherwise empty space. The system obtained by H. Friedrich can be important to solve such realistic problems.

In this article we establish a variant of Friedrich's system for a lagrangian and tetrad description of the flow and the spacetime geometry. Contrary to Friedrich we will not work with the Weyl tensor but with the Riemann tensor, as in [2]. It avoids the introduction of unphysical characteristics and seems to us more natural in this non vacuum case as well as somewhat simpler. The systems we obtain for vacuum, dust or perfect fluid are symmetric and hyperbolic in the same sense as Friedrich's system. They are hyperbolic in K. O. Friedrichs's sense, which leads to existence theorems for solutions in Sobolev spaces, only if the time lines admit global spacelike sections, and the initial data are given on such a section. These existence theorems are needed even if one wants to consider the evolution equations as a well posed dynamical system.

2. Metric and coframe

We write the metric in an orthonormal frame, i.e.

$$g = -(\theta^0)^2 + \sum_{i=1}^{3} (\theta^i)^2. \tag{2.1}$$

We choose the time axis to be *tangent to the time lines*, i.e. the cobasis θ is such that θ^i does not contain dx^0. We set

$$\theta^i = a^i_j dx^j, \quad \theta^0 = U dx^0 + b_i dx^i. \tag{2.2}$$

We will call such a frame a C.F. (Cattaneo–Ferrarese) frame. We denote by A_i^j the matrix inverse of a_i^j. It holds that

$$dx^i = A_j^i \theta^j, \quad dx^0 = U^{-1}(\theta^0 - A_j^i b_i \theta^j)\ .$$

The Pfaff derivatives ∂_α in the C.F. frame are linked to the partial derivatives $\partial/\partial x^\alpha$ by the relations

$$\begin{aligned} \frac{\partial}{\partial x^0} &= U\partial_0, & \frac{\partial}{\partial x^i} &= a_i^j \partial_j + b_i \partial_0, \\ \partial_0 &= U^{-1}\frac{\partial}{\partial x^0}, & \partial_i &= A_i^j \left[\frac{\partial}{\partial x^j} - U^{-1} b_j \frac{\partial}{\partial x^0}\right]. \end{aligned}$$

3. Bianchi equations

Whatever the coframe, the components of the Riemann tensor satisfy the identities

$$\nabla_\alpha R_{\beta\gamma,\lambda\mu} + \nabla_\beta R_{\gamma\alpha,\lambda\mu} + \nabla_\gamma R_{\alpha\beta,\lambda\mu} \equiv 0, \tag{3.1}$$

hence, if the Ricci tensor $R_{\alpha\beta}$ satisfies the Einstein equations

$$R_{\alpha\beta} = \rho_{\alpha\beta}, \tag{3.2}$$

it holds that

$$\nabla_\alpha R^{\cdot\cdot\cdot\alpha}_{\beta\gamma::\mu,} = \nabla_\beta \rho_{\gamma\mu} - \nabla_\gamma \rho_{\beta\mu}. \tag{3.3}$$

Equations (3.1) and (3.3) imply the following ones (cf. an analogous system in [2])

$$\nabla_0 R_{hk,0j} + \nabla_k R_{0h,0j} - \nabla_h R_{0k,0j} = 0, \tag{3.4}$$

and, in using a symmetry of the Riemann tensor,

$$\nabla_0 R^0_{:::i,0j} + \nabla_h R^h_{:::i,0j} = J_{i,0j} \equiv \nabla_0 \rho_{ji} - \nabla_j \rho_{0i}. \tag{3.5}$$

The equations (3.4) and (3.5) are for each given pair $(0j)$ a first order system for the components $R_{hk,0j}$ and $R_{0h,0j}$. The principal operator is a symmetric 6 by 6 matrix:

$$\begin{pmatrix} \partial_0 & 0 & 0 & \partial_2 & -\partial_1 & 0 \\ 0 & \partial_0 & 0 & 0 & \partial_3 & -\partial_2 \\ 0 & 0 & \partial_0 & -\partial_3 & 0 & \partial_1 \\ \partial_2 & 0 & -\partial_3 & \partial_0 & 0 & 0 \\ -\partial_1 & \partial_3 & 0 & 0 & \partial_0 & 0 \\ 0 & -\partial_2 & \partial_1 & 0 & 0 & \partial_0 \end{pmatrix}.$$

Analogous equations and result hold with the pair $(0j)$ replaced by (lm), $l < m$. The system finally obtained has a principal matrix consisting of 6 identical 6 by 6 blocks around the diagonal, it is symmetric.

However, the system obtained for the components of the Riemann tensor in the coframe with time axis θ^0 and space axis θ^i, with principal operator the matrix $M^\alpha \partial_\alpha$, cannot be said to be hyperbolic in the usual sense: the principal matrix M^0 is the unit matrix hence positive definite, but the operators ∂_α are not partial derivatives. We say that the system is a quasi FOSH (First Order Symmetric Hyperbolic) system (cf. Section 11).

REMARK 1. One can associate with the Riemann tensor, as in the usual case of the $3+1$ splitting, two pairs of "electric" and "magnetic" 2-tensors whose non zero components in the frame are:

$$E_{ij} \equiv R_{0i,0j}, \qquad D_{ij} \equiv \frac{1}{4}\varepsilon_{ihk}\varepsilon_{jlm}R^{hk,lm},$$
$$H_{ij} \equiv \frac{1}{2}\varepsilon_{ihk}R^{hk}_{:::::,0j}, \qquad B_{ji} \equiv \frac{1}{2}\varepsilon_{ihk}R_{0j}^{:::::hk},$$

where ε_{ijk} is the totally antisymmetric Kronecker tensor. These definitions imply here

$$R_{hk,0j} \equiv \varepsilon^{i}_{::hk}H_{ij}, \quad R_{hk,lm} \equiv \varepsilon^{i}_{::hk}\varepsilon^{j}_{::lm}D_{ij}, \quad R_{0j,hk} = \varepsilon^{i}_{::hk}B_{ji}.$$

The previous system is a quasi FOSH system for these pairs of tensors.

REMARK 2. The system (3.1), (3.2) contains also the equations, which we do not use for evolution,

$$\nabla_h R_{ij,\lambda\mu} + \nabla_j R_{hi,\lambda\mu} + \nabla_i R_{jh,\lambda\mu} = 0 \tag{3.6}$$

and

$$\nabla_h R^{h}_{\cdot\cdot 0,\lambda\mu} = \nabla_\lambda \rho_{\mu 0} - \nabla_\mu \rho_{\lambda 0}. \tag{3.7}$$

These equations are not usual constraints on a submanifold $t = \operatorname{cons}\tan t$ because ∂_i contains the transversal derivative $\partial/\partial t$. We call them Bianchi quasi-constraints.

4. Coframe structure coefficients

The structure coefficients c of the coframe are defined by

$$d\theta^\alpha \equiv -\frac{1}{2}c^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma.$$

We have

$$d\theta^0 = \partial_i U \theta^i \wedge dx^0 + \partial_\alpha b \theta^\alpha \wedge dx^i$$

that is,

$$d\theta^0 = U^{-1}\partial_i U \theta^i \wedge (\theta^0 - A^h_j b_h \theta^j) + A^h_j \partial_\alpha b_h \theta^\alpha \wedge \theta^j,$$

therefore

$$c^0_{0i} = U^{-1}\partial_i U - A^j_i \partial_0 b_j = -c^0_{i0}$$

and, with $f_{[ij]} \equiv f_{ij} - f_{ji}$,

$$c^0_{ij} = U^{-1} b_h A^h_{[j} \partial_{i]} U - A^h_{[j} \partial_{i]} b_h$$

that is

$$c^0_{ij} = U A^h_{[i} \partial_{j]} (U^{-1} b_h)$$

also

$$d\theta^i = A^j_k (\partial_h a^i_j \theta^h \wedge \theta^k + \partial_O a^i_j \theta^0 \wedge \theta^k)$$

hence

$$c^i_{k0} = A^j_k \partial_0 a^i_j, \quad c^i_{hk} = A^j_{[h} \partial_{k]} a^i_j.$$

Remark 3. If the time lines are not hypersurface orthogonal (i.e. if $b_i \neq 0$) the coefficients c^h_{ij} are different from the structure coefficients of the space frame θ^i.

5. Connection

The connection is such that

$$\omega^\alpha_{\beta\gamma} - \omega^\alpha_{\gamma\beta} = c^\alpha_{\beta\gamma} \quad \text{and} \quad \omega_{\gamma\beta,\lambda} + \omega_{\gamma\lambda,\beta} = 0$$

with (η is the Minkowski metric) $\omega_{\gamma\beta,\lambda} \equiv \eta_{\alpha\lambda} \omega^\alpha_{\gamma\beta}$. We set $c_{\gamma\beta,\lambda} \equiv \eta_{\alpha\lambda} c^\alpha_{\gamma\beta}$, and we have (cf. Choquet-Bruhat–DeWitt I, p. 308)

$$\omega_{\alpha\lambda,\mu} = \frac{1}{2}(c_{\alpha\lambda,\mu} - c_{\alpha\mu,\lambda} - c_{\lambda\mu,\alpha}).$$

We find that, as foreseen from antisymmetry, $\omega^0_{00} = \omega^0_{i0}$. We set $Y_i \equiv \omega_{00,i}$, and we have

$$Y_i \equiv \omega_{00,i} = \omega^0_{0i} = \omega^i_{00} = -c_{0i,0} = c^0_{0i} = U^{-1} \partial_i U - A^j_i \partial_0 b_j.$$

We know that $\omega^j_{0i} = \omega_{0i,j}$ is antisymmetric in i and j. We set

$$\omega_{0i,j} \equiv f_{ij} = \frac{1}{2}\{A^h_j \partial_0 a^i_h - A^h_i \partial_0 a^j_i + A^h_{[i} \partial_{j]} b_h\}.$$

Remark 4. Let $e_{(\alpha)} \equiv \partial_\alpha$ be the frame dual to θ^α, i.e. with components $\delta^\lambda_{(\alpha)}$. Then

$$\nabla_\beta e^\lambda_{(\alpha)} = \omega^\lambda_{\beta\alpha},$$

in particular $\omega_{0i,j}$ is the projection on $e_{(j)}$ of the derivative of $e_{(i)}$ in the direction of $e_{(0)}$. We have $f_{ij} = 0$ if the frame is Fermi transported along the time line.

The connection coefficient $\omega_{i0,j}$ is the sum of a term symmetric in i and j and an antisymmetric one, we have

$$X_{ij} \equiv \omega_{i0,j} = \omega^j_{i0} = \frac{1}{2}\{A^h_j \partial_0 a^i_h + A^h_i \partial_0 a^j_i + A^h_{[i} \partial_{j]} b_h\}.$$

The antisymmetric term vanishes if the time lines are hypersurface orthogonal ($b_i = 0$).

The coefficients ω^h_{ij} are also linear expressions in terms of the first derivatives of the frame coefficients, they are identical to the connection constructed with the a^j_i at fixed x^0 if $b_i = 0$.

6. Frame evolution

The relations between the connection coefficients ω^h_{ij} and the frame coefficients a^j_i give equations for the dragging of these coefficients along the time lines when the connection is known. Indeed we have found

$$\partial_0 a^i_j = a^k_j c^i_{0k} = 2a^k_j(\omega^i_{0k} - \omega^i_{k0}) \equiv 2a^k_j(f^i_k - X^i_k), \tag{6.1}$$

$$\partial_0 b_i = a^h_i(-Y_h + U^{-1}\partial_h U. \tag{6.2}$$

No equation gives the evolution of U. It can be considered as a gauge variable fixing the time parameter. H. Friedrich chooses the coordinate x^0 to be the proper time along the given time lines, i.e. $U = 1$

The quantity f^i_j is also a gauge variable fixing the evolution of the space frame. If we choose it to be Fermi transported like Friedrich, then $f^i_j = 0$.

7. Curvature

7.1. Evolution of connection. Using the general formula (cf. Choquet-Bruhat–DeWitt, I p. 306, with opposite sign convention)

$$R^{\cdot\cdot\alpha}_{\lambda\mu\ldots\beta} \equiv \partial_\lambda\omega^\alpha_{\mu\beta} - \partial_\mu\omega^\alpha_{\lambda\beta} - \omega^\alpha_{\rho\beta}(\omega^\rho_{\lambda\mu} - \omega^\rho_{\mu\lambda}) + \omega^\alpha_{\lambda\rho}\omega^\rho_{\mu\beta} - \omega^\alpha_{\mu\rho}\omega^\rho_{\lambda\beta}, \tag{7.1}$$

$$R^{\cdot\cdot\cdot i}_{0h\ldots j} \equiv \partial_0\omega^i_{hj} - \tilde{\nabla}_h f^i_j - Y_h f^i_j - (f^k_h - X_h^{\cdot\cdot k})\omega^i_{kj} + Y^i X^{\cdot\cdot\cdot}_{hj} - Y_j X_h{}^i, \tag{7.2}$$

where $\tilde{\nabla}$ is the pseudo covariant derivative constructed with ∂_i and ω^h_{ij} (Cattaneo–Ferrarese transversal derivative). In the Fermi gauge $R^{\cdot\cdot\cdot i}_{0h\ldots j}$ reduces to

$$R^{\cdot\cdot\cdot i}_{0h\ldots j} \equiv \partial_0\omega^i_{hj} + X_h^{\cdot\cdot k}\omega^i_{kj} + Y_i X_h^{\cdot\cdot j} - Y_j X_h{}^i. \tag{7.3}$$

The principal operator in these equations is the dragging along the time lines of ω^h_{ij}. We have

$$R^{\cdot\cdot\cdot i}_{h0\ldots0} \equiv \tilde{\nabla}_h Y_i - \partial_0 X_h^{\cdot\cdot i} + Y^i Y_h - X_h^{\cdot\cdot j} X_j^{\cdot\cdot i} + f_h^{\cdot\cdot j} X_j^{\cdot\cdot i} - f_j^{\cdot\cdot i} X_h^{\cdot\cdot j}. \tag{7.4}$$

We deduce from this identity, in the Fermi gauge,

$$R_{00} \equiv \tilde{\nabla}_i Y^i - \partial_0 X^i_i + Y^i Y_i - X_h^{\cdot\cdot j} X_j^{\cdot\cdot h}. \tag{7.5}$$

In the next sections we will write other equations to achieve the determination of both Y and X.

7.2. Quasi-constraints. The other components of the Riemann tensor are given in the C-F frame by

$$R_{hk\ldots j}^{\cdot\cdot\cdot i} \equiv \widetilde{R}_{hk\ldots j}^{\cdot\cdot\cdot i} - f_j^i(X_{hk} - X_{kh}) + X_{\cdot\cdot k}^{\cdot\cdot i} X_{jh} - X_{jk} X_{\cdot\cdot h}^{i}, \tag{7.6}$$

where $\widetilde{R}_{hk\ldots j}^{\cdot\cdot\cdot i}$ denotes the expression formally constructed as a Riemann tensor with the coefficients ω_{ij}^h.

$$R_{kh\ldots j}^{\cdot\cdot\cdot 0} \equiv \partial_k \omega_{hj}^0 - \partial_h \omega_{kj}^0 - \omega_{\rho j}^0(\omega_{kh}^\rho - \omega_{hk}^\rho) + \omega_{k\rho}^0 \omega_{hj}^\rho - \omega_{h\rho}^0 \omega_{kj}^\rho, \tag{7.7}$$

that is, with previous notations,

$$R_{kh\ldots j}^{\cdot\cdot\cdot 0} \equiv \widetilde{\nabla}_k X_{hj} - \widetilde{\nabla}_h X_{kj} - Y_j(X_{kh} - X_{hk}). \tag{7.8}$$

These equations do not enter in the evolution system for the connection that we are considering, they do not contain the derivative ∂_0, we call them connection quasi-constraints.

Remark 5. Modulo the expression of the connection in terms of frame coefficients one has the well known symmetry:

$$R_{kh,0j} = R_{0j,kh}.$$

One deduces from the identity (7.8)

$$R_{h0} \equiv \widetilde{\nabla}_j X_{h}^{\cdot\cdot j} - \widetilde{\nabla}_h X_j^j - Y^j(X_{jh} - X_{hj}).$$

8. Vacuum case

In the vacuum case there are no a priori given time lines. We can give arbitrarily on the spacetime the projection Y_i of $\nabla_{e_{(0)}} e_{(0)}$ on $e_{(i)}$. The quantities U and f_i^j being also arbitrarily given the Bianchi equations together with (6.1), (6.2), (7.3), (7.5) constitute a quasi FOSH system for the unknown E_{ij}, D_{ij}, H_{ij}, B_{ij}, a_i^j, b_i, ω_{ij}^h, X_{ij}. Its solution determines the spacetime metric.

9. Perfect fluid

9.1. Fluid equations. The stress energy tensor of a perfect fluid is

$$T_{\alpha\beta} = (\mu + p) u_\alpha u_\beta + p g_{\alpha\beta}.$$

Then

$$\rho_{\alpha\beta} = (\mu + p) u_\alpha u_\beta + \frac{1}{2}(\mu - p) g_{\alpha\beta}.$$

One supposes that the matter energy density μ is a given function of the pressure p and entropy S; this last function is conserved along the flow lines

$$u^\alpha \partial_\alpha S = 0. \tag{9.1}$$

The Euler equations of the fluid express the generalized conservation law

$$\nabla_\alpha T^{\alpha\beta} = 0,$$

and are equivalent to the equations

$$(\mu + p)u^\alpha \nabla_\alpha u^\beta + (u^\alpha u^\beta + g^{\alpha\beta})\partial_\alpha p = 0, \quad \text{with } u^\alpha u_\alpha = -1$$

and

$$(\mu + p)\nabla_\alpha u^\alpha + u^\alpha \partial_\alpha \mu = 0.$$

In our coframe they reduce to:

$$\partial_0 S = 0, \tag{9.2}$$

$$(\mu + p)Y_i + \partial_i p = 0, \quad Y_i \equiv \omega^i_{00}, \tag{9.3}$$

$$\partial_0 \mu + (\mu + p)X^i_i = 0. \tag{9.4}$$

Using the index F of the fluid defined by

$$F(p, S) = \int \frac{dp}{\mu(p, S) + p},$$

the equation (9.3) reads

$$Y_i = -\partial_i F, \tag{9.5}$$

while (9.4) gives, modulo (9.2),

$$\mu'_p \partial_0 F + X^i_i = 0. \tag{9.6}$$

The commutation relation between Pfaff derivatives $(\partial_0\partial_i - \partial_i\partial_0)F = c^\alpha_{0i}\partial_\alpha F$ implies therefore

$$\mu'_p[\partial_0 Y_i + Y_i \partial_0 F + (f^j_i - X^j_i)\partial_j F] - \partial_i \mu'_p \partial_0 F - \partial_i X^h_h = 0. \tag{9.7}$$

The use of (9.5) and (9.6) replaces $\partial_\alpha F$ by functions of Y, X and p, S. The derivatives $\partial_i \mu'_p$ are functions of Y, p, S and $\partial_k S$, since

$$\partial_i \mu'_p = \mu"{}_{p^2}\partial_i p + \mu''_{pS}\partial_i S.$$

We introduce $S_k = \partial_k S$ as new unknowns. They are dragged along the flow lines by the following equation deduced from (9.2):

$$\partial_0 S_k = c^j_{0k} S_j \equiv (f^j_k - X^k_j)S_j. \tag{9.8}$$

Following H. Friedrich we replace in (9.7) $\partial_i X^h_h$ by its expression deduced from the equation

$$R_{i0} \equiv \widetilde{\nabla}_h X_i^{\cdot\cdot h} - \widetilde{\nabla}_i X^h_h - Y^h(X_{hi} - X_{ih}) = 0,$$

and we obtain, changing names of indices

$$\text{(9.9)}\quad \mu'_p\partial_0 Y_h - \tilde{\nabla}_j X^j_h - Y^h(X_{hi} - X_{ih}) + \mu'_p[Y_h\partial_0 F + (f^j_h - X^j_h)\partial_j F] + \partial_h\mu'_p\partial_0 F = 0.$$

In (7.4) we replace $\tilde{\nabla}_h Y_i$ by $\tilde{\nabla}_i Y_h + c^0_{hi}Y_0$, with

$$Y_0 \equiv -\partial_0 F \equiv -(\mu'_p)^{-1}X^i_i, \quad c^0_{hi} \equiv X_{ih} - X_{hi}$$

and we obtain

$$\text{(9.10)}\quad \partial_0 X^{\cdot i}_h - \tilde{\nabla}^i Y_h - c^0_{hi}Y_0 - Y_h Y^i + (X^{\cdot\cdot j}_h - f^{\cdot\cdot j}_h)X^{\cdot\cdot i}_j + f^{\cdot\cdot i}_j X^{\cdot\cdot j}_h = -R^{\cdot\cdot i}_{h0\ldots 0}.$$

The principal operator on the unknowns Y and X in the equations (9.9), (9.10) is diagonal by blocks. Each block, corresponding to a given index h, is symmetric, it reads:

$$\begin{pmatrix} \mu'_p\partial_0 & -\partial_1 & -\partial_2 & -\partial_3 \\ -\partial_1 & \partial_0 & 0 & 0 \\ -\partial_2 & 0 & \partial_0 & 0 \\ -\partial_3 & 0 & 0 & \partial_0 \end{pmatrix}.$$

If $\mu'_p > 0$ the matrix M^0 is positive definite in the C.F. frame. The system (9.9), (9.10) is a quasi FOSH system for the pairs Y_h, X^j_h.

Remark 6. The characteristic determinant associated with the system (9.9), (9.10), obtained by replacing ∂_α with a covariant vector ξ is

$$\left\{\xi_0^2\left(\mu'_p\xi_0^2 - \sum_{i=1,2,3}\xi_i^2\right)\right\}^3.$$

The roots of $\mu'_p\xi_0^2 - \sum_{i=1,2,3}\xi_i^2 = 0$ correspond to sound waves. Their speed is at most 1 (speed of light) if and only if $\mu'_p \geq 1$.

The full system of fluid equations is quasi diagonal by blocks. The matrices $M^\alpha\partial_\alpha$ corresponding to (9.2), (9.4), (9.8) reduce to ∂_0.

9.2. Sources of the Bianchi equations. In our frame the source tensor $\rho_{\alpha\beta}$ reduces to

$$\rho_{00} = \frac{1}{2}(\mu + 3p), \quad \rho_{0i} = 0, \quad \rho_{ij} = \frac{1}{2}(\mu - p)\delta_{ij}.$$

We have seen that $\partial_\alpha p$ and $\partial_\alpha\mu$ are smooth functions of p, S, S_i, Y and X. The same property holds for $J_{i,0j}$ and $J_{i,hk}$.

9.3. Conclusion. Assembling the results of the previous sections we find the following theorem.

Theorem 7. *The* EEF (*Einstein–Euler–Friedrich*) *system for a gravitating perfect fluid, with flow lines taken as time lines, U and $f^i_j \equiv \omega^i_{0j}$ given arbitrarily, is a quasi* FOSH *system for the Riemann curvature tensor, the frame and connection coefficients, the density of matter, the entropy and its space derivatives.*

As remarked in Section 3 the system is not a usual FOSH system: the time lines are by choice timelike but the hypersurfaces $x^0 = \mathrm{cons}\tan t$ are not necessarily spacelike for the characteristic cone.

10. Case of dust

10.1. Matter equations. The stress energy tensor of a dust source is

$$T_{\alpha\beta} \equiv \mu u_\alpha u_\beta.$$

The flow lines, tangent to the unit vector u^α, are then geodesics, it holds that

$$u^\alpha \nabla_\alpha u^\beta = 0, \tag{10.1}$$

while the conservation of matter reads:

$$u^\alpha \partial_\alpha \mu + \mu \nabla_\alpha u^\alpha = 0. \tag{10.2}$$

We take the flow lines as time lines, i.e. $u^\alpha = \delta^\alpha_0$. Then (10.2) reads

$$\partial_0 \mu + \mu X^i_i = 0, \tag{10.3}$$

while (10.1) gives

$$\omega^i_{00} \equiv Y_i = 0. \tag{10.4}$$

Using this equation we see that, given arbitrarily on the spacetime the gauge variables a^0_0 the equations (6.1), (6.2) reduce to a quasi-diagonal system with principal operator ∂_0 for the frame coefficients when the connection (which appears undifferentiated) is known.

10.2. Bianchi equations. For a dust stress energy tensor it holds that

$$\rho_{\alpha\beta} = \mu\left(u_\alpha u_\beta + \frac{1}{2} g_{\alpha\beta}\right),$$

hence in the chosen frame

$$\rho_{00} = \frac{1}{2}\mu, \quad \rho_{0i} = 0, \quad \rho_{ij} = \frac{1}{2}\mu\delta_{ij}.$$

A straightforward computation gives then, choosing the Fermi transported frame for simplicity

$$J_{i,0j} \equiv \nabla_0 \rho_{ji} - \nabla_j \rho_{0i} = \mu\Big(-\frac{1}{2}\delta_{ij} X^h_h + X_{ji}\Big),$$

$$J_{i,hj} \equiv \nabla_h \rho_{ji} - \nabla_j \rho_{hi} = \frac{1}{2}\{\partial_h \mu \delta_{ji} - \partial_j \mu \delta_{hi} + \mu(\omega^i_{hj} - \omega^i_{jh})\}.$$

There appears a difficulty which is the presence of the space derivatives $\partial_h \mu$ in $J_{i,hj}$. We get rid of this problem, inspired (though with a different choice) by H. Friedrich's treatment, by replacing the unknown $R^{::\cdot i}_{hk\dots j}$ by another unknown with the same symmetries, namely

$$F_{ij,hk.} = R_{ij,hk} + \delta_{ik} R_{jh} - \delta_{ih} R_{jk} - \delta_{jk} R_{ih} + \delta_{ih} R_{ik} + D_{ij,hk} R$$

with

$$D_{ij,hk} \equiv \frac{1}{2}(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}).$$

Using this unknown, the contracted Bianchi identities become

$$\nabla_0 R^0_{:::i,hj} + \nabla_k F^k_{:::i,hj} \equiv 0, \tag{10.5}$$

while the original ones become

$$\nabla_0 F_{ij,hk} + \nabla_j R_{0i,hk} - \nabla_i R_{0j,hk} \equiv \nabla_0 \{\delta_{ik} R_{jh} - \delta_{ih} R_{jk} - \delta_{jk} R_{ih} + \delta_{ih} R_{ik} + X_{ijhk} R\} \tag{10.6}$$

One obtains a quasi FOSH system for $R_{0i,hk}$ and $F_{ij,hk}$ by replacing on the right hand side of this equation $R_{\alpha\beta}$ by $\rho_{\alpha\beta}$: now there appears only the time derivative $\partial_0 \mu$ which, satisfying (10.3), can be eliminated in favor of undifferentiated terms.

Note that $R_{ij,hk}$ does not appear in other equations than the Bianchi equations.

10.3. Conclusion. We have proved the following theorem

Theorem 8. *The* EDFF (*Einstein–Dust–Ferrarese–Friedrich) system, with flow lines taken as time lines, U and $f^i_j \equiv \omega^i_{0j}$ given arbitrarily, is a quasi* FOSH *system for the Riemann curvature tensor, the frame and connection coefficients, the density of matter and its space derivatives.*

11. Cauchy problem

11.1. Hyperbolicity. The fact that for the systems that we have obtained the hypersurfaces $x^0 = \text{cons}\tan t$ are not necessarily spacelike for the characteristic cones poses a difficulty for the Cauchy problem, since energy estimates used in proving existence theorems rely on this property.

We have, using standard definitions

PROPOSITION 9. *A quasi* FOSH *system, with principal operator the matrix* $M^\alpha \partial_\alpha$*, is a* FOSH *system with* x^0 *as a time variable if the matrix of coefficients of* $\partial/\partial x^0$ *is positive definite, namely if*

$$\widetilde{M}^0 \equiv M^\alpha A^0_\alpha \equiv U^{-1}(M^0 - A^i_j b_i M^j)$$

is positive definite.

11.1.1. Case of dust.

LEMMA 10. *The* EDFF (*Einstein–Dust–Ferrarese–Friedrich*) *system is a* FOSH *system relative to* $x^0 = \text{cons}\tan t$ *slices as long as the quadratic form*

$$\overline{g}_{jh} = \sum_{i=1,2,3} a^i_j a^i_h - b_j b_h \tag{11.1}$$

is positive definite and $U > 0$*.*

PROOF. The matrix $\widetilde{M}^0_{(\text{bian})}$ of coefficients of $\partial/\partial x^0$ corresponding to the Bianchi equations is after multiplication by U, setting

$$B_i \equiv -A^j_i b_i,$$

$$\begin{pmatrix} 1 & 0 & 0 & B_2 & -B_1 & 0 \\ 0 & 1 & 0 & 0 & B_3 & -B_2 \\ 0 & 0 & 1 & -A_3 & 0 & B_1 \\ B_2 & 0 & -B_3 & 1 & 0 & 0 \\ -B_1 & B_3 & 0 & 0 & 1 & 0 \\ 0 & -B_2 & B_1 & 0 & 0 & 1 \end{pmatrix}.$$

Its eigenvalues are, each with multiplicity 2:

$$1, \quad 1 + \sqrt{(B_2^2 + B_3^2 + B_1^2)}, \quad 1 - \sqrt{(B_2^2 + B_3^2 + B_1^2)}.$$

They are positive, because the given condition implies $B_2^2 + B_3^2 + B_1^2 < 1$.

The other matrices $\widetilde{M}^0_{(\text{conn})}$ and $\widetilde{M}^0_{(\text{matter})}$ are unit matrices. □

11.1.2. Fluid case.

LEMMA 11. *The* EEF (*Einstein–Euler–Friedrich*) *system is a* FOSH *system relatively to* $x^0 = \text{cons}^t$ *slices as long as the quadratic form* $\overline{g}_{jh}$ *given in* (11.1) *is positive definite,* $U > 0$ *and* $\mu'_p \geq 1$*.*

PROOF. The principal operator is symmetric and composed of blocks around the diagonal.

The matrix $\widetilde{M}^0_{(\text{bian})}$ is the same as in the case of dust.

The matrix $\widetilde{M}^0_{\text{con}}$ corresponding to the connection evolution is, after multiplication by U

$$\begin{pmatrix} \mu'_p & -B_1 & -B_2 & -B_3 \\ -B_1 & 1 & 0 & 0 \\ -B_2 & 0 & 1 & 0 \\ -B_3 & 0 & 0 & 1 \end{pmatrix}.$$

Its eigenvalues are 1, with multiplicity two, and:

$$\frac{1}{2}\left\{1+\mu'_p \pm 2\sqrt{\frac{1}{4}(\mu'_p-1)^2+\sum B_i^2)}\right\}. \tag{11.2}$$

These eigenvalues are positive under the hypothesis (11.1) if $\mu'_p \geq 1$. □

REMARK 12. The condition $\mu'_p \geq 1$ expresses that the sound speed is at most equal to the speed of light.

11.2. Cauchy data. The Cauchy problem is less natural in the C.F. formulation than in the usual $3+1$ decomposition.

11.2.1. Initial data. An initial data set on a coordinate patch of a 3 dimensional submanifold is composed of the following elements:

(1) A field of coframes $\overline{a}^{i\,j}_j$ and a field of covariant vectors $\overline{a}^0_i$.

The quadratic form on M with coefficients

$$\overline{g}_{jh} = \sum_{i=1,2,3} \overline{a}^i_j \overline{a}^i_h - \overline{b}_j \overline{b}_h \tag{11.3}$$

is supposed to be positive definite. The submanifold $M_0 \equiv M \times \{0\}$ of $M \times R$ is then spacelike for any lorentzian metric on $M \times R$ which reduces on M to

$$-(\overline{U}dx^0 + \overline{b}_i dx^i)^2 + \sum_{i=1,2,3} (\overline{\theta}^i)^2, \quad \overline{U} > 0.$$

(2) Quantities $\overline{\omega}^h_{ij}$, $\overline{X}^j_i$, $\overline{Y}_i$.

(3) Pairs of 2-tensors $\overline{E}^{ij}$, $\overline{H}_{ij}$, $\overline{D}_{ij}$, $\overline{B}_{ij}$.

(4.a) (case of dust) A scalar function $\overline{\mu} \geq 0$.

(4.b) (perfect fluid case) Two scalar functions $\overline{p} \geq 0$ and $\overline{S} \geq 0$, and equation of state $\mu(p,S)$ such that $\mu'_p \geq 1$ for all p in a neighbourhood of $(\overline{p}, \overline{S})$.

11.2.2. Constraints. The initial data on M do not determine on M the derivatives $\partial_i \equiv \overline{\partial}_i - A^j_i \overline{b}_j \partial_0$. To satisfy the quasi-constraints on M we must first deduce from the given data and the considered evolution system the values on M of the derivatives with respect to x^0 of the relevant unknowns. The quasiconstraints give then relations on the initial data which we call constraints and suppose to be satisfied on M.

11.3. Existence theorem for the reduced system. Local existence theorems for a solution of the Cauchy problem are a direct consequence of known results for quasilinear FOSH systems. We enunciate the result in the perfect fluid case. The spaces $H_s^{u.\mathrm{loc}}(M)$ are the usual uniformly local Sobolev spaces on M.

THEOREM 13. *Let there be given an initial data set in $H_s^{u.\mathrm{loc}}(M)$, $s > 3/2+1$. Then for any choice on $M \times I$ of the gauge variables in $C^0(I, H_s^{u.\mathrm{loc}}(M))$, with I an interval of R containing 0, there exists a neighbourhood I' of 0 in R such that the* EEF *equations admit a solution in $C^0(I', H_s^{u.\mathrm{loc}}(M)) \cap C^1(I', H_{s-1}^{u.\mathrm{loc}}(M))$.*

To prove that the solution satisfies, modulo the initial constraints, the original Einstein–Euler equations would be a messy task, which we will not endeavour. The uniqueness of the obtained solution gives an indirect proof, modulo the existence theorem already known [5], [6].

11.4. Irrotational flows. We recall the equation (6.2)

$$\partial_0 b_i = a_i^h(-Y_h + \partial_h \log U), \tag{11.4}$$

with, in the case of a perfect fluid, $Y_h = -\partial_h F$. We choose then (gauge choice, the geometrical result is independent of it) $U = \exp(-F)$. The equation reduces then to

$$\partial_0 b_i = 0. \tag{11.5}$$

Therefore it holds that $b_i = 0$ througout the motion if it is so initially. It is the well known property of conservation of zero vorticity (irrotationality) of a perfect fluid flow.

For an irrotational flow it holds that X_{ij} is a symmetric 2-tensor, equal up to the sign to the extrinsic curvature of the space slices $x^0 = \mathrm{cons}\tan t$ (components in the frame θ^i), and $Y_i = U^{-1}\partial_i U$, while $\omega_{ij}^h = \overline{\omega}_{ij}^h$ are the connection coefficients of the space metric $\overline{g} = \sum_{i=1,2,3}(\theta^i)^2$, $\theta^i \equiv a_h^i dx^h$. The pseudo covariant derivatives $\widetilde{\nabla}_i$ are covariant derivatives $\overline{\nabla}_i$ in the space metric.

For an irrotational flow the EFF system is a FOSH system.

The same result holds in the case of dust the appropriate gauge choice is then $U = \mathrm{cons}\tan t$.

REMARK 14. The relations between frame and connection, as well as connection and curvature, for hypersurface orthogonal time lines are a special case of those found in the $3+1$ decomposition [7]. In particular, in the present notations, the Einstein constraints take the familiar form:

$$R_{i0} \equiv \overline{\nabla}_j X_i^j - \partial_i X_j^j = 0, \tag{11.6}$$

$$S_{00} \equiv \frac{1}{2}(R_{00} + \delta^{ij}R_{ij}) \equiv \frac{1}{2}(\overline{R} - X_i^{\,j} X_j^{\,i} + (X_i^i)^2) = \rho_{00} = \frac{1}{2}(\mu + 3p). \tag{11.7}$$

References

[1] M. Anile, *Relativistic Fluids and Magneto-Fluids*, Cambridge Univ. Press., 1989.

[2] A. Anderson, Y. Choquet-Bruhat and J. W. York, *Einstein–Bianchi hyperbolic system for general relativity*, Topol. Methods Nonlinear Anal. **10** (1997), 353–373.

[3] D. Brown D. and J. York, *Physical rigins of time asymmetry* (1994), Cambridge Univ. Press, 465–474.

[4] C. Cattaneo, *Proiezioni naturali e derivate trasversa in una varieta a metrica iperbolica normale*, Ann. Math. Pura Appl. (4) **48** (1959), 361–386.

[5] Y. (Choquet) Foures-Bruhat, *Théorèmes d'existence en mécanique des fluides relativistes*, Bull. Soc. Math. France **86** (1958), 155–175.

[6] ______, *Etude des équations des fluides chargés relativistes inductifs et conducteurs*, Comm. Math. Phys. **3** (1966), 334–357.

[7] ______, *Sur l'intégration des équations de la Relativité Générale*, Arch. Rational Mech. Anal. **5** (1956), 951–966.

[8] G. Ferrarese, *Contributi a la tecnica delle proiezioni in una varietà a metrica iperbolica normale*, Rend. Mat. Appl. **22** (1963), 147–168.

[9] ______, *Introduzione a la Dinamica Riemanniana dei Sistemi Continui* (Pitagora, ed.), Bologna, 1979.

[10] G. Ferrarese and C. Catani, *Generalized frames of references and intrinsic Cauchy problem in General Relativity*, Physics on Manifolds (M. Flato, R. Kerner and A. Lichnerowicz, eds.), Reidel, 1994, pp. 93–108.

[11] H. Friedrich, *On the evolution equations for gravitating ideal fluid bodies in General Relativity*, Phys. Rev. D **57** (1998), 2317–2322.

[12] K. O. Friedrichs, *On the laws of relativistic electro-magneto-fluid dynamics*, Comm. Pure Appl. Math. **27** (1974), 749–808.

[13] A. Rendall, *The initial value problem for a class of general relativistic fluid bodies*, J. Math. Phys. **33** (1992), 1047–1053.

Manuscript received September 6, 2001

Yvonne Choquet-Bruhat
Université Paris 6
tour 22-12, 4 place Jussieu
75230 Paris, FRANCE

E-mail address: ycb@ccr.jusieu.fr

James W. York
Department of Physics UNC
Chapel Hill, N.C, USA

E-mail address: jaswyork@earthlink.net