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REACTION-DIFFUSION EQUATIONS ON UNBOUNDED THIN DOMAINS

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Abstract. We prove existence and upper semicontinuity of attractors for a reaction-diffusion equation on a family of thin unbounded domains collapsing onto a lower dimensional subspace.

1. Introduction

In their paper [7] J. Hale and G. Raugel posed the following problem. Consider an evolution equation on a spatial domain Ω and assume that Ω is small in some direction: to what extent is it possible to approximate the model by mean of an equation on a lower dimensional spatial domain? Is it possible to determine the approximant?

This problem is particularly interesting in the case of equations generating dissipative dynamical systems. In fact, if such systems satisfy some additional compactness properties, they possess compact global attractors, which retain most of the dynamical information. It is then possible to express the concept of closeness of two semiflows in terms of the Hausdorff distance of their attractors.

A typical example is given by reaction-diffusion equations of the form

$$\begin{cases} u_t = \Delta u - \lambda u + f(u) + g & \text{in }]0,\infty[\, \times \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial \nu_\varepsilon} = 0 & \text{in }]0,\infty[\, \times \partial\Omega_\varepsilon, \end{cases} \tag{1.1}$$

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where $(\Omega_\varepsilon)_{\varepsilon>0}$ is a family of open *bounded* domains collapsing onto some lower dimensional subspace. In [7], Hale and Raugel treated in detail the case of domains of the form

$$\Omega_\varepsilon = \{(x,y) \mid x \in \omega \text{ and } 0 < y < \varepsilon h(x)\}, \tag{1.2}$$

where ω is an open bounded domain and h is a smooth positive function defined on ω. They identified a limit equation and proved convergence of the semiflows and upper-semicontinuity of attractors. Also, if ω is an interval in $\mathbb{R}$, they constructed a family of inertial manifolds for equations (1.1).

Domains of the form (1.2) are very special: in particular, they are not allowed to exibit holes or other horizontal branches. A much more general class of thin domains, namely domains of the form

$$\Omega_\varepsilon = \{(x,\varepsilon y) \in \mathbb{R}^{N+P} \mid (x,y) \in \Omega\}, \tag{1.3}$$

where Ω is an open bounded domain, was investigated by Prizzi and K. Rybakowski in [13]. They developed an abstract framework for the analysis of such problems, based on a property of strong spectral convergence (i.e. convergence of eigenvalues and eigenfunctions) satisfied by the linear part of the equation. In [14], under some additional conditions on Ω, they established also the existence and the persistence of large gaps in the spectra of the corresponding linear operators and they used this property to construct inertial manifolds for equation (1.1). Some applications of the Conley index to thin domain problems are contained in the recent paper [4] of M. Carbinatto and K. Rybakowski. For more references, the reader is referred to the Montecatini lecture notes [15] by G. Raugel.

If the domains Ω_ε are *unbounded*, the semiflows generated by (1.1) might loose their compactness properties. Establishing the existence of compact global attractors becomes then itself an interesting task. In [2] Babin and Vishik overcame the difficulties arising from the lack of compactness by introducing weighted Sobolev spaces. The choice of weighted spaces, however, imposes some severe conditions on the forcing term g and on the initial data. Very recently, Wang ([18]) established the asymptotic L^2-compactness of the semiflows and consequently the existence of global ($L^2 - L^2$) attractors for reaction-diffusion equations on $\mathbb{R}^N$ (or, more generally, on unbounded subdomains of $\mathbb{R}^N$) avoiding the use of weighted spaces. It is then natural to ask whether convergence results similar to those in [7] and [13] hold also in the case of a family of unbounded thin domains. However, since the techniques developed in [7] and [13] rely heavily on the compactness of the resolvent operator of Δ, in order to deal with general unbounded domains a different approach is needed. Spectral convergence has to be replaced by strong resolvent convergence, and a stronger version of Trotter-Kato Theorem has to be established. Moreover, following Wang's pattern, some uniform asymptotic L^2-compactness of the semiflows has to be proved. Finally, asymptotic H^1-compactness has to be recovered by a continuity argument similar to that of [12].

In the present paper, we identify a limit equation for the family (1.1) when Ω_ε, $\varepsilon > 0$, are unbounded domains of the form (1.3). We prove convergence of the semiflows and upper-semicontinuity of attractors in the H^1-strong topology. The limit problem turns out to be an abstract semilinear parabolic equation on the subspace of H^1 consisting of the functions whose partial derivatives in the y directions vanish. As in [13], under suitable conditions this abstract equation can be characterized as a system of concrete reaction-diffusion equations in N spatial variables, coupled by compatibility and balance conditions at the boundaries. We shall not treat here this aspect of the problem, and we refer the reader to [13] for further details.

The paper is organized as follows. In Section 2 we introduce notations and some necessary preliminaries. In Section 3 we deal with the linear problems associated to (1.1): in particular we establish a stronger version of the Trotter–Kato Theorem, which ensures convergence of the corresponding linear semigroups. In Section 4 we study the nonlinear problems (1.1); we prove convergence of the corresponding semiflows and we establish the existence of absorbing sets for them. In Section 5, we prove uniform asymptotic compactness of the semiflows, and finally we deduce existence and upper-semicontinuity of attractors.

2. Notation and preliminaries

Let $\Omega \subset \mathbb{R}^N \times B_{\mathbb{R}^P}(0,1)$ be a Lipschitz open, possibly unbounded, domain in $\mathbb{R}^N \times \mathbb{R}^P$, where $B_{\mathbb{R}^P}(0,1)$ is the open ball of radius one centered at zero in $\mathbb{R}^P$. We write the points of $\mathbb{R}^N \times \mathbb{R}^P$ as (x,y), with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^P$. Throughout the paper, ∇_x and ∇_y denote the gradient in $\mathbb{R}^N$ and $\mathbb{R}^P$, respectively. Analogously, Δ_x and Δ_y denote the Laplacian in $\mathbb{R}^N$ and $\mathbb{R}^P$.

For $0 < \varepsilon \le 1$, we define $\Omega_\varepsilon := T_\varepsilon(\Omega)$, where $T_\varepsilon\colon \mathbb{R}^N \times \mathbb{R}^P \to \mathbb{R}^N \times \mathbb{R}^P$ is the mapping $(x,y) \mapsto (x, \varepsilon y)$. We consider the family of reaction-diffusion Neumann problems

$$(2.1)\qquad \begin{cases} u_t = \Delta u - \lambda u + f(u) + g(x) & \text{in }]0,\infty[\, \times \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial \nu_\varepsilon} = 0 & \text{in }]0,\infty[\, \times \partial\Omega_\varepsilon, \end{cases}$$

where ν_ε is the outward normal to $\partial\Omega_\varepsilon$. We make the following assumptions:

$$(2.2)\qquad \lambda > 0, \quad g \in L^2(\mathbb{R}^N),$$

$$(2.3)\qquad f(0) = 0, \quad f(s)s \le 0, \quad f'(s) \le C \quad \text{for all } s \in \mathbb{R},$$

$$|f'(s)| \le C(1+|s|^\beta) \quad \text{for all } s \in \mathbb{R}, \tag{2.4}$$

where C is some positive constant and

$$\begin{aligned} &0 \le \beta && \text{if } (N+P)=2,\\ &0 \le \beta \le (2^*/2)-1 && \text{if } (N+P)\ge 3, \end{aligned} \tag{2.5}$$

where $2^* = 2(N+P)/(N+P-2)$. Rescaling the y variables by the factor $1/\varepsilon$, we see that (2.1) is equivalent to the family of problems

$$\begin{cases} u_t = \Delta_x u + \dfrac{1}{\varepsilon^2}\Delta_y u - \lambda u + f(u) + g(x) & \text{in }]0,\infty[\, \times \Omega,\\ \dfrac{\partial u}{\partial \nu_x} + \dfrac{1}{\varepsilon^2}\dfrac{\partial u}{\partial \nu_y} = 0 & \text{in }]0,\infty[\, \times \partial\Omega, \end{cases} \tag{2.6}$$

on the fixed domain Ω, where $\nu = (\nu_x, \nu_y)$ is the outward normal to $\partial\Omega$. We denote by $H^1_\varepsilon(\Omega)$ the Hilbert space $H^1(\Omega)$ endowed with the norm

$$\|u\|_{H^1_\varepsilon} := \bigg(\int_\Omega |\nabla_x u(x,y)|^2\,dx\,dy + \frac{1}{\varepsilon^2}\int_\Omega |\nabla_y u(x,y)|^2\,\mathrm{d}x\,dy + \int_\Omega u(x,y)^2\,dx\,dy \bigg)^{1/2}$$

and by a_ε the bilinear form

$$a_\varepsilon(u,v) := \int_\Omega \nabla_x u(x,y)\cdot\nabla_x v(x,y)\,dx\,dy + \frac{1}{\varepsilon^2}\int_\Omega \nabla_y u(x,y)\cdot\nabla_y v(x,y)\,dx\,dy, \tag{2.7}$$

defined for $u, v \in H^1(\Omega)$. Besides, we denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the standard inner product in $L^2(\Omega)$. Finally, $A_\varepsilon\colon D(A_\varepsilon) \subset H^1(\Omega) \to L^2(\Omega)$ is the linear self-adjoint operator associated to the bilinear form a_ε, defined by

$$\begin{cases} D(A_\varepsilon) := \{u \in H^1 \mid \text{there exists } w \in L^2 \text{ such that}\\ \qquad\qquad \text{for all } v \in H^1 : a_\varepsilon(u,v) = \langle w, v\rangle\},\\ A_\varepsilon u := w, \quad u \in D(A_\varepsilon). \end{cases} \tag{2.8}$$

Notice that $H^1(\Omega) = D((A_\varepsilon + I)^{1/2})$ and

$$a_\varepsilon(u,v) + \langle u, v\rangle = \langle (A_\varepsilon+I)^{1/2}u, (A_\varepsilon+I)^{1/2}v\rangle, \quad u, v \in H^1(\Omega).$$

Since the Nemitski operator $\widehat{f}$ generated by f turns out to be a locally lipschitzian map from $H^1(\Omega)$ to $L^2(\Omega)$ (see Proposition 4.1 below), equation (2.6) can be formulated as the abstract equation

$$\dot{u} + A_\varepsilon u + \lambda u = \widehat{f}(u) + g, \tag{2.9}$$

in the space $L^2(\Omega)$. By classical results on abstract semilinear parabolic equations (see [9]), equation (2.9) defines a local semiflow π_ε in the phase space $H^1(\Omega)$.

As we are interested in the behaviour of the solutions of (2.9) as $\varepsilon \to 0$, we immediately observe that, for $u \in H^1(\Omega)$, we have

$$\lim_{\varepsilon\to 0} a_\varepsilon(u,u) = \begin{cases} \displaystyle\int_\Omega |\nabla_x u(x,y)|^2\,dx\,dy & \text{if } \nabla_y u = 0 \text{ a.e.} \\ \infty & \text{otherwise.} \end{cases} \tag{2.10}$$

Thus we are lead to consider the closed subspace of $H^1(\Omega)$ consisting of all functions $u \in H^1(\Omega)$ such that $\nabla_y u = 0$. We denote this space by $H^1_s(\Omega)$ and we endow it with the norm

$$\|u\|_{H^1_s} := \left(\int_\Omega |\nabla_x u(x,y)|^2\,dx\,dy + \int_\Omega u(x,y)^2\,dx\,dy \right)^{1/2}.$$

Moreover, we denote by a_0 the bilinear form

$$a_0(u,v) := \int_\Omega \nabla_x u(x,y)\cdot\nabla_x v(x,y)\,dx\,dy, \tag{2.11}$$

defined for $u, v \in H^1_s(\Omega)$. We define $L^2_s(\Omega)$ to be the closure of $H^1_s(\Omega)$ in $L^2(\Omega)$. Finally, $A_0\colon D(A_0) \subset H^1_s(\Omega) \to L^2_s(\Omega)$ is the linear self-adjoint operator associated to the bilinear form a_0, defined by

$$\begin{cases} D(A_0) := \{u \in H^1_s \mid \text{there exists } w \in L^2_s \text{ such that} \\ \qquad\qquad \text{for all } v \in H^1_s,\ a_0(u,v) = \langle w, v\rangle\}, \\ A_0 u := w, \quad u \in D(A_0). \end{cases} \tag{2.12}$$

Again notice that $H^1_s(\Omega) = D((A_0+I)^{1/2})$ and

$$a_0(u,v) + \langle u, v\rangle = \langle (A_0+I)^{1/2}u, (A_0+I)^{1/2}v\rangle, \quad u, v \in H^1_s(\Omega).$$

As in the case of a bounded domain considered in [13], the natural candidate for being a "limit" equation for the family (2.9) is the abstract semilinear parabolic equation

$$\dot{u} + A_0 u + \lambda u = \widehat{f}(u) + g, \tag{2.13}$$

in the space $L^2_s(\Omega)$.

3. The linear problem

In this section we discuss some properties of the operators A_ε and A_0, defined by (2.8) and (2.12), and of the corresponding linear semigroups $e^{-A_\varepsilon t}$, $e^{-A_0 t}$. In particular we prove some strong convergence of $e^{-A_\varepsilon t}$ to $e^{-A_0 t}$. First, we recall that, since the operators A_ε, $\varepsilon \ge 0$, are self-adjoint and positive, there exist two positive constants α and M such that, for $u \in L^2(\Omega)$ and for $\varepsilon > 0$,

$$\begin{aligned} \|e^{-A_\varepsilon t}u\|_{L^2} &\le M e^{\alpha t}\|u\|_{L^2}, & t \ge 0, \\ \|e^{-A_\varepsilon t}u\|_{H^1_\varepsilon} &\le M t^{-1/2} e^{\alpha t}\|u\|_{L^2}, & t > 0. \end{aligned} \tag{3.1}$$

and, for $u \in L^2_s$,

$$\begin{aligned} \|e^{-A_0 t}u\|_{L^2_s} &\le M e^{\alpha t}\|u\|_{L^2_s}, && t \ge 0, \\ \|e^{-A_0 t}u\|_{H^1_s} &\le M t^{-1/2} e^{\alpha t}\|u\|_{L^2_s}, && t > 0. \end{aligned} \tag{3.2}$$

The constants α and M can be chosen independent of ε; this is a straightforward byproduct of the spectral representation of the semigroups (see e.g. [10]). The family of quadratic forms $(Q_\varepsilon)_{\varepsilon>0}$, corresponding to the bilinear forms $(a_\varepsilon)_{\varepsilon>0}$, is increasing and converges pointwise to the quadratic form

$$Q_0(u) := \begin{cases} a_0(u,u) & \text{if } u \in H^1_s(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

It is well known (see e.g. [5]) that this is enough to detect convergence of A_ε to A_0 in the strong resolvent sense in $L^2(\Omega)$. However, for our purposes, we need a more precise result:

Lemma 3.1. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $\varepsilon_n \to 0$ as $n \to \infty$. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$, let $w_0 \in L^2_s(\Omega)$ and assume that $w_n \to w_0$ in the strong topology of $L^2(\Omega)$. Let*

$$u_n := (A_{\varepsilon_n} + I)^{-1} w_n, \quad u_0 := (A_0 + I)^{-1} w_0.$$

Then

$$\|u_n - u_0\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Proof. For any $v \in H^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega \nabla_x u_n(x,y)\cdot\nabla_x v(x,y)\,dx\,dy + \frac{1}{\varepsilon_n^2}\int_\Omega \nabla_y u_n(x,y)\cdot\nabla_y v(x,y)\,dx\,dy \\ + \int_\Omega u_n(x,y)v(x,y)\,dx\,dy = \int_\Omega w_n(x,y)v(x,y)\,dx\,dy. \end{aligned} \tag{3.3}$$

Choosing $v := u_n$ in (3.4), we obtain that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. It follows that there exists $\overline{u} \in H^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup \overline{u}$ in the weak topology of $H^1(\Omega)$. Moreover, $\nabla_y u_n \to 0$ in the strong topology of $(L^2(\Omega))^M$. Then $\nabla_y \overline{u} = 0$, so $\overline{u} \in H^1_s(\Omega)$. Choosing $v \in H^1_s(\Omega)$ and passing to the limit in (3.4), we obtain

$$\begin{aligned} \int_\Omega \nabla_x \overline{u}(x,y)\cdot\nabla_x v(x,y)\,dx\,dy \\ + \int_\Omega \overline{u}(x,y)v(x,y)\,dx\,dy = \int_\Omega w(x,y)v(x,y)\,dx\,dy. \end{aligned} \tag{3.4}$$

Since $v \in H^1_s(\Omega)$ is arbitrary, we have that $\overline{u} = (A_0 + I)^{-1} w_0 = u_0$. Finally, we have

$$\begin{aligned}
\int_\Omega |\nabla_x u_0(x,y)|^2\,dx\,dy + \int_\Omega u_0(x,y)^2\,dx\,dy \\
&\le \liminf_{n\to\infty} \bigg(\int_\Omega |\nabla_x u_n(x,y)|^2\,dx\,dy \\
&\quad + \int_\Omega |\nabla_y u_n(x,y)|^2\,dx\,dy + \int_\Omega u_n(x,y)^2\,dx\,dy \bigg) \\
&\le \limsup_{n\to\infty} \bigg(\int_\Omega |\nabla_x u_n(x,y)|^2\,dx\,dy \\
&\quad + \int_\Omega |\nabla_y u_n(x,y)|^2\,dx\,dy + \int_\Omega u_n(x,y)^2\,dx\,dy \bigg) \\
&\le \lim_{n\to\infty} \bigg(\int_\Omega |\nabla_x u_n(x,y)|^2\,dx\,dy \\
&\quad + \frac{1}{\varepsilon_n^2} \int_\Omega |\nabla_y u_n(x,y)|^2\,dx\,dy + \int_\Omega u_n(x,y)^2\,dx\,dy \bigg) \\
&= \lim_{n\to\infty} \int_\Omega w_n(x,y) u_n(x,y)\,dx\,dy = \int_\Omega w_0(x,y) u_0(x,y)\,dx\,dy \\
&= \int_\Omega |\nabla_x u_0(x,y)|^2\,dx\,dy + \int_\Omega u_0(x,y)^2\,dx\,dy.
\end{aligned}$$

It follows that $u_n \to u_0$ in the strong topology of $H^1(\Omega)$. Moreover,

$$\frac{1}{\varepsilon_n^2} \int_\Omega |\nabla_y u_n(x,y)|^2\,dx\,dy \to 0 \quad \text{as } n \to \infty,$$

and hence $\|u_n - u_0\|_{H^1_{\varepsilon_n}} \to 0$ as $n \to \infty$. □

In view of Lemma 3.1, Trotter–Kato Theorem implies that, whenever $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers, $\varepsilon_n \to 0$ as $n \to \infty$, then for every $u_0 \in L^2_s(\Omega)$,

$$e^{-A_{\varepsilon_n} t} u_0 \to e^{-A_0 t} u_0 \quad \text{as } n \to \infty$$

in the strong topology of $L^2(\Omega)$. However, we need a stronger convergence result:

PROPOSITION 3.2. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $\varepsilon_n \to 0$ as $n \to \infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$, let $u_0 \in L^2_s(\Omega)$ and assume that $u_n \to u_0$ in the strong topology of $L^2(\Omega)$. Then*

$$\|e^{-A_{\varepsilon_n} t} u_n - e^{-A_0 t} u_0\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty \tag{3.5}$$

uniformly on $[t_1, t_2]$ for every $[t_1, t_2] \subset \,]0, \infty[$.

The proof of Proposition 3.2 follows essentially that of the classical Trotter–Kato Theorem (see e.g. [17]). Therefore, we give only a sketch of the proof, pointing out the necessary modifications.

Proof of Proposition 3.2. First of all, observe that

$$\begin{aligned}\|e^{-A_{\varepsilon_n}t}u_n - e^{-A_0t}u_0\|_{H^1_{\varepsilon_n}} &\\ \le \|e^{-A_{\varepsilon_n}t}u_n - e^{-A_{\varepsilon_n}t}u_0\|_{H^1_{\varepsilon_n}} &+ \|e^{-A_{\varepsilon_n}t}u_0 - e^{-A_0t}u_0\|_{H^1_{\varepsilon_n}}\\ \le Mt^{-1/2}e^{\alpha t}\|u_n - u_0\|_{L^2} &+ \|e^{-A_{\varepsilon_n}t}u_0 - e^{-A_0t}u_0\|_{H^1_{\varepsilon_n}}.\end{aligned}$$

Hence, it is sufficient to show that

$$\|e^{-A_{\varepsilon_n}t}u_0 - e^{-A_0t}u_0\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty$$

uniformly on $[t_1, t_2]$ for every $[t_1, t_2] \subset]0, \infty[$. A straightforward computation shows that, if $u_0 \in L^2_s(\Omega)$,

$$\begin{aligned}(A_{\varepsilon_n} + I)^{-1}&(e^{-A_0t} - e^{A_{\varepsilon_n}t})(A_0 + I)^{-1}u_0\\ &= \int_0^t e^{-A_{\varepsilon_n}(t-s)}((A_0 + I)^{-1} - (A_{\varepsilon_n} + I)^{-1})e^{-A_0s}u_0\, ds.\end{aligned}$$

Applying to both sides the closed operator $(A_{\varepsilon_n} + I)^{1/2}$ and taking into account (3.1), we find that

$$\begin{aligned}\|(A_{\varepsilon_n} + I)^{1/2}&(A_{\varepsilon_n} + I)^{-1}(e^{-A_0t} - e^{A_{\varepsilon_n}t})(A_0 + I)^{-1}u_0\|_{L^2}\\ &\le \int_0^t Me^{\alpha(t-s)}\|(A_{\varepsilon_n} + I)^{1/2}((A_0 + I)^{-1} - (A_{\varepsilon_n} + I)^{-1})e^{-A_0s}u_0\|_{L^2}\, ds.\end{aligned}$$

By Lemma 3.1, the integrand converges to 0 uniformly on $[0, t_1]$ for every $t_1 > 0$. Hence, we obtain that for every $y_0 \in D(A_0)$,

$$\|(A_{\varepsilon_n} + I)^{1/2}(A_{\varepsilon_n} + I)^{-1}(e^{-A_0t} - e^{A_{\varepsilon_n}t})y_0\|_{L^2} \to 0 \quad \text{as } n \to \infty \tag{3.6}$$

uniformly on $[0, t_1]$ for every $t_1 > 0$. By a standard density argument one can easily show that (3.6) holds for $y_0 \in L^2_s(\Omega)$. Now, let $u_0 \in L^2_s(\Omega)$. We have

$$\begin{aligned}\|(A_{\varepsilon_n} + I)^{1/2}&(e^{-A_0t} - e^{-A_{\varepsilon_n}t})(A_0 + I)^{-1}u_0\|_{L^2}\\ &\le \|(A_{\varepsilon_n} + I)^{1/2}(e^{-A_{\varepsilon_n}t}(A_0 + I)^{-1} - (A_{\varepsilon_n} + I)^{-1}e^{-A_{\varepsilon_n}t})u_0\|_{L^2}\\ &\quad + \|(A_{\varepsilon_n} + I)^{1/2}(A_{\varepsilon_n} + I)^{-1}(e^{-A_0t} - e^{-A_{\varepsilon_n}t})u_0\|_{L^2}\\ &\quad + \|(A_{\varepsilon_n} + I)^{1/2}((A_{\varepsilon_n} + I)^{-1}e^{-A_0t} - e^{-A_0t}(A_0 + I)^{-1})u_0\|_{L^2}.\end{aligned}$$

By Lemma 3.1 and in view of (3.6) and (3.1) we obtain that, for every $y_0 \in D(A_0)$,

$$\|(A_{\varepsilon_n} + I)^{1/2}(e^{-A_0t} - e^{-A_{\varepsilon_n}t})y_0\|_{L^2} \to 0 \quad \text{as } n \to \infty \tag{3.7}$$

uniformly on $[0, t_1]$ for every $t_1 > 0$. In order to recover (3.7) for $y_0 \in L^2_s(\Omega)$, we use again a density argument. However, the situation here is more delicate than before, so we give the details. Let $y_0 \in L^2_s(\Omega)$. For every $\eta > 0$, there exists $z_0 \in D(A_0)$ such that $\|y_0 - z_0\|_{L^2} < \eta$. Now,

$$\|(A_{\varepsilon_n} + I)^{1/2}(e^{-A_0 t} - e^{-A_{\varepsilon_n} t})y_0\|_{L^2} \leq \|(A_{\varepsilon_n} + I)^{1/2}(e^{-A_0 t} - e^{-A_{\varepsilon_n} t})z_0\|_{L^2} + \|(A_{\varepsilon_n} + I)^{1/2}(e^{-A_0 t} - e^{-A_{\varepsilon_n} t})(y_0 - z_0)\|_{L^2}.$$

The first summand in the right hand side tends to 0 as $n \to \infty$ by (3.7), so we just need to estimate the second summand.

$$\|(A_{\varepsilon_n} + I)^{1/2}(e^{-A_0 t} - e^{-A_{\varepsilon_n} t})(y_0 - z_0)\|_{L^2} \leq \|(A_{\varepsilon_n} + I)^{1/2}e^{-A_{\varepsilon_n} t}(y_0 - z_0)\|_{L^2} + \|(A_{\varepsilon_n} + I)^{1/2}e^{-A_0 t}(y_0 - z_0)\|_{L^2}.$$

By (3.1) we get

$$\|(A_{\varepsilon_n} + I)^{1/2}e^{-A_{\varepsilon_n} t}(y_0 - z_0)\|_{L^2} \leq M t^{-1/2} e^{\alpha t}\eta, \tag{3.8}$$

on the other hand,

$$\begin{aligned}\|(A_{\varepsilon_n} + I)^{1/2}&e^{-A_0 t}(y_0 - z_0)\|^2_{L^2} \\ &= \|e^{-A_0 t}(y_0 - z_0)\|^2_{L^2} + a_{\varepsilon_n}(e^{-A_0 t}(y_0 - z_0), e^{-A_0 t}(y_0 - z_0)) \\ &= \|e^{-A_0 t}(y_0 - z_0)\|^2_{L^2} + a_0(e^{-A_0 t}(y_0 - z_0), e^{-A_0 t}(y_0 - z_0)) \\ &= \|(A_0 + I)^{1/2}e^{-A_0 t}(y_0 - z_0)\|^2_{L^2}.\end{aligned}$$

By (3.2) we get

$$\|(A_{\varepsilon_n} + I)^{1/2}e^{-A_0 t}(y_0 - z_0)\|_{L^2} \leq M t^{-1/2} e^{\alpha t}\eta. \tag{3.9}$$

Since η is arbitrary, we finally obtain that

$$\|(A_{\varepsilon_n} + I)^{1/2}(e^{-A_0 t} - e^{-A_{\varepsilon_n} t})y_0\|_{L^2} \to 0 \quad \text{as } n \to \infty$$

uniformly on $[t_0, t_1]$ for every $[t_0, t_1] \subset]0, \infty[$. □

REMARK. In [13] the authors obtained a convergence result analogous to that of Proposition 3.2, by first proving a spectral convergence result for the family of operators $(A_\varepsilon)_{\varepsilon>0}$ and then using the representation of the linear semigroups on suitable bases of eigenfunctions. Here, since the operators A_ε and A_0 might not have compact resolvent, we used a different approach based on strong convergence of the resolvents. This approach seems to be even simpler and of course it could be applied as well to the problem considered in [13]. On the other hand, spectral convergence retains much more information than simple resolvent convergence. For example, in some cases, spectral convergence is very important in establishing the persistence of large gaps in the spectrum of the linear operators A_ε. This property was used in [14] to construct inertial manifolds for equations (2.9) and (2.13).

4. The nonlinear problem

In this section we consider the nonlinear equations (2.9) and (2.13). We begin by establishing some regularity of the Nemitski operator generated by f. Assume that (2.2)–(2.5) hold. The following result is well known; for a sketch of the proof see [12].

Lemma 4.1. *The assignement $u \mapsto f \circ u$ defines a map $\widehat{f}\colon H^1(\Omega) \to L^2(\Omega)$, which is Lipschitz continuous on every bounded set in $H^1(\Omega)$. Moreover, whenever u, u_1, $u_2 \in H^1(\Omega)$ and $\|u_1\|_{H^1}$, $\|u_2\|_{H^1} \le R$, the following estimates hold:*

$$\begin{aligned}\|\widehat{f}(u)\|_{L^2} &\le C_1(\|u\|_{L^2} + \|u\|_{H^1}^{(\beta+1)}),\\ \|\widehat{f}(u_1) - \widehat{f}(u_2)\|_{L^2} &\le C_1(1 + R^\beta)\|u_1 - u_2\|_{H^1}.\end{aligned}$$

Here C_1 is a positive constant.

Lemma 4.2. *Let $u \in D(A_\epsilon)$. Then $\langle \widehat{f}(u), A_\epsilon u\rangle \le C a_\epsilon(u,u)$, where C is the constant of conditions* (2.3), (2.4).

Proof. For $n \in \mathbb{N}$, choose a function $h_n \in C^\infty(\mathbb{R})$, with $0 \le h_n'(s) \le 1$ for all $s \in \mathbb{R}$, such that

$$h_n(s) = \begin{cases} s & \text{if } -n \le s \le n,\\ n+1 & \text{if } 2n \le s,\\ -(n+1) & \text{if } s \le -2n.\end{cases}$$

Define $f_n := f \circ h_n$. By (2.3), it follows that $f_n(0) = 0$, $|f_n'(s)|$ is bounded on $\mathbb{R}$ and $f_n'(s) \le C$ for all $s \in \mathbb{R}$. By Proposition IX.5 in [3], it follows that $f_n \circ u \in H^1(\Omega)$ and $\nabla(f_n \circ u) = (f_n' \circ u) \cdot \nabla u$. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\langle \widehat{f}_n(u), A_\varepsilon u\rangle &= a_\varepsilon(\widehat{f}_n(u), u)\\ &= \int_\Omega f_n'(u(x,y))\bigg(|\nabla_x u(x,y)|^2 + \frac{1}{\varepsilon^2}|\nabla_y u(x,y)|^2\bigg)\,dx\,dy\\ &\le C\int_\Omega \bigg(|\nabla_x u(x,y)|^2 + \frac{1}{\varepsilon^2}|\nabla_y u(x,y)|^2\bigg)\,dx\,dy = C a_\varepsilon(u,u).\end{aligned}$$

The proof will be complete if we show that $f_n \circ u \to f \circ u$ in $L^2(\Omega)$ as $n \to \infty$. This is true since $f_n(u(x,y)) \to f(u(x,y))$ almost everywhere in Ω as $n \to \infty$ and the estimates

$$\begin{aligned}|f_n(u(x,y))| &\le C(|u(x,y)| + |u(x,y)|^{\beta+1}),\\ |f(u(x,y))| &\le C(|u(x,y)| + |u(x,y)|^{\beta+1})\end{aligned}$$

hold. The conclusion follows from the Lebesgue dominated convergence theorem. □

By the same argument one can also prove the following

LEMMA 4.2. *If $u \in H^1_s(\Omega)$, then $\widehat{f}(u) \in L^2_s(\Omega)$. Moreover, if $u \in D(A_0)$, then $\langle \widehat{f}(u), A_0 u\rangle \le C a_0(u,u)$.*

Let $\varepsilon > 0$, let $\overline{u}_\varepsilon \in H^1(\Omega)$ and let us consider the Cauchy problem

$$\begin{cases} \dot{v} + A_\varepsilon v + \lambda v = \widehat{f}(v) + g, \\ v(0) = \overline{u}_\varepsilon. \end{cases} \tag{4.1}$$

Moreover, let $\overline{u}_0 \in H^1_s(\Omega)$ and let us consider the Cauchy problem

$$\begin{cases} \dot{v} + A_0 v + \lambda v = \widehat{f}(v) + g, \\ v(0) = \overline{u}_0. \end{cases} \tag{4.2}$$

By classical results on abstract semilinear parabolic equations (see [9]), equations (4.1) and (4.2) define local semiflows π_ε and π_0 in the phase spaces $H^1(\Omega)$ and $H^1_s(\Omega)$, respectively. We have the following

LEMMA 4.4. *Let $u_\varepsilon\colon [0,T[\to H^1(\Omega)$ be the maximal solution of the Cauchy problem* (4.1). *If $\|\overline{u}_\varepsilon\|_{L^2} \le R$, then, for $t \in [0,T[$,*

$$\|u_\varepsilon(t)\|^2_{L^2} \le e^{-\lambda t} R^2 + \frac{\|g\|^2_{L^2}}{\lambda^2}.$$

PROOF. For $t \in\,]0,T[$, we have

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2} &= \langle u_\varepsilon(t), \dot{u}_\varepsilon(t)\rangle \\ &= \langle u_\varepsilon(t), -A_\varepsilon u_\varepsilon(t) - \lambda u_\varepsilon(t) + \widehat{f}(u_\varepsilon(t)) + g\rangle \\ &\le -a_\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) - \lambda\langle u_\varepsilon(t), u_\varepsilon(t)\rangle \\ &\quad + \langle u_\varepsilon(t), \widehat{f}(u_\varepsilon(t))\rangle + \langle u_\varepsilon(t), g\rangle. \end{aligned}$$

By (2.3) we get

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2} &+ a_\varepsilon(u_\varepsilon(t), u_\varepsilon(t)) + \lambda\|u_\varepsilon(t)\|^2_{L^2} \\ &\le \langle u_\varepsilon(t), g\rangle \le \frac{\lambda}{2}\|u_\varepsilon(t)\|^2_{L^2} + \frac{1}{2\lambda}\|g\|^2_{L^2}. \end{aligned}$$

It follows that

$$\frac{d}{dt}\|u_\varepsilon(t)\|^2_{L^2} + \lambda\|u_\varepsilon(t)\|^2_{L^2} \le \frac{\|g\|^2_{L^2}}{\lambda}.$$

Multiplication by $e^{\lambda t}$ and integration yields

$$\|u_\varepsilon(t)\|^2_{L^2} \le e^{-\lambda t}\|u_\varepsilon(0)\|^2_{L^2} + \frac{\|g\|^2_{L^2}}{\lambda^2}, \tag{4.3}$$

and the conclusion follows. □

An analogous result holds for the H^1_ε-norm of the solutions:

Lemma 4.5. *Let* $u_\varepsilon\colon [0,T[\to H^1(\Omega)$ *be the maximal solution of the Cauchy problem* (4.1). *There exist two positive constants* $K_1 = K_1(C,\lambda)$, $K_2 = K_2(C,\lambda)$, *such that, if* $\|\overline{u}_\varepsilon\|_{H^1_\varepsilon} \le R$, *then, for* $t \in [0,T[$,

$$\|u_\varepsilon(t)\|^2_{H^1_\varepsilon} \le K_1 R^2 e^{-\lambda t} + K_2 \|g\|^2_{L^2}.$$

Proof. For $t \in]0,T[$ we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) &= \langle A_\varepsilon u_\varepsilon(t), \dot u_\varepsilon(t)\rangle \\ &= \langle A_\varepsilon u_\varepsilon(t), -A_\varepsilon u_\varepsilon(t) - \lambda u_\varepsilon(t) + \widehat f(u_\varepsilon(t)) + g\rangle \\ &= -\|A_\varepsilon u_\varepsilon(t)\|^2_{L^2} - \lambda a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) + \langle A_\varepsilon u_\varepsilon(t), \widehat f(u_\varepsilon(t))\rangle + \langle A_\varepsilon u_\varepsilon(t), g\rangle.\end{aligned}$$

By Lemma 4.2 and by Young inequality we obtain

$$\frac{d}{dt}a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) \le -\|A_\varepsilon u_\varepsilon(t)\|^2_{L^2} - 2(\lambda - C)a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) + \|g\|^2_{L^2}. \tag{4.4}$$

Let $\nu > 0$ and let $v \in D(A_\varepsilon)$. We have

$$a_\varepsilon(v,v) \le \frac{\nu}{2}\|A_\varepsilon v\|^2_{L^2} + \frac{1}{2\nu}\|v\|^2_{L^2},$$

whence

$$-\|A_\varepsilon v\|^2_{L^2} \le -\frac{2}{\nu}a_\varepsilon(v,v) + \frac{1}{\nu^2}\|v\|^2_{L^2}. \tag{4.5}$$

By (4.4) and (4.5), choosing $\nu := (\lambda + |\lambda - C|)^{-1}$, we obtain

$$\frac{d}{dt}a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) \le -2\lambda a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) + \frac{1}{\nu^2}\|u_\varepsilon(t)\|^2_{L^2} + \|g\|^2_{L^2}.$$

By (4.3)

$$\frac{d}{dt}a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) + 2\lambda a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) \le \frac{\|\overline{u}_\varepsilon\|^2_{L^2}}{\nu^2}e^{-\lambda t} + \left(1 + \frac{1}{\lambda^2\nu^2}\right)\|g\|^2_{L^2}.$$

Multiplication by $e^{2\lambda t}$ and integration yields

$$a_\varepsilon(u_\varepsilon(t),u_\varepsilon(t)) \le e^{-2\lambda t}a_\varepsilon(\overline{u}_\varepsilon,\overline{u}_\varepsilon) + \frac{\|\overline{u}_\varepsilon\|^2_{L^2}}{\nu^2\lambda}e^{-\lambda t} + \frac{1}{2\lambda}\left(1 + \frac{1}{\lambda^2\nu^2}\right)\|g\|^2_{L^2}$$

and the conclusion follows. □

As a consequence, we have the following result:

Proposition 4.6. *Let* $u_\varepsilon\colon [0,T[\to H^1(\Omega)$ *be the maximal solution of the Cauchy problem* (4.1). *Then*

(1) $T = \infty$,
(2) *if* $\|\overline{u}_\varepsilon\|_{H^1_\varepsilon} \le R$, *then, for every* $t \ge 0$, $\|u_\varepsilon(t)\|^2_{H^1_\varepsilon} \le K_1R^2 + K_2\|g\|^2_{L^2}$, *with* K_1 *and* K_2 *independent of* ε,

(3) *there exists a positive constant* K *and for every* $R > 0$ *there exists* $T = T(R) > 0$ *such that, whenever* $\|\overline{u}_\varepsilon\|_{H^1_\varepsilon} \le R$, $\|u_\varepsilon(t)\|_{H^1_\varepsilon} < K$ *for all* $t \ge T(R)$. *Both* K *and* $T(R)$ *are independent of* ε.

In particular, for every $\varepsilon > 0$, *the set* $\{u \in H^1(\Omega) \mid \|u\|_{H^1_\varepsilon} < K\}$ *is an absorbing set for the global semiflow* π_ε.

Analogous results hold also for the solutions of (4.2). In particular, we have:

PROPOSITION 4.7. *Let* $u_0\colon [0,T[\to H^1_s(\Omega)$ *be the maximal solution of the Cauchy problem* (4.2). *Then*

(1) $T = \infty$,
(2) *if* $\|\overline{u}_0\|_{H^1_s} \le R$, *then, for every* $t \ge 0$, $\|u_0(t)\|^2_{H^1_s} \le K_1R^2 + K_2\|g\|^2_{L^2}$,
(3) *there exists a positive constant* K *and for every* $R > 0$ *there exists* $T = T(R) > 0$ *such that, whenever* $\|\overline{u}_0\|_{H^1_s} \le R$, $\|u_0(t)\|_{H^1_s} < K$ *for all* $t \ge T(R)$.

In particular the set $\{u \in H^1_s(\Omega) \mid \|u\|_{H^1_s} < K\}$ *is an absorbing set for the global semiflow* π_0. □

We remark that the estimates in Propositions 4.6 and 4.7 are uniform with respect to ε. We are now in a position to state our first important continuous-dependence result:

THEOREM 4.8. *Let* $(\varepsilon_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative numbers, let* $\varepsilon_0 \ge 0$, *and assume that* $\varepsilon_n \to \varepsilon_0$. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $H^1(\Omega)$ ($u_n \in H^1_s(\Omega)$ *if* $\varepsilon_n = 0$) *converging in the norm of* $L^2(\Omega)$ *to some* $u_0 \in H^1(\Omega)$ ($u_0 \in H^1_s(\Omega)$ *if* $\varepsilon_0 = 0$). *Assume also that there exists a positive constant* R *such that* $\|u_n\|_{H^1_{\varepsilon_n}} \le R$ *for all* $n \in \mathbb{N}$ ($\|u_n\|_{H^1_s} \le R$ *if* $\varepsilon_n = 0$). *Let* $b \in]0,\infty[$. *Then, for every* $t \in]0,b]$ *and every sequence* $(t_n)_{n\in\mathbb{N}}$ *in* $]0,b]$ *converging to* t,

$$\|\pi_{\varepsilon_n}(t_n,u_n) - \pi_{\varepsilon_0}(t,u_0)\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

PROOF. In the case $\varepsilon_0 = 0$, Theorem 4.8 can be proved exactly like Theorem 5.1 in [13]. In fact, the proof relies only on the convergence of the linear semigroups (Proposition 3.2), on the Lipschitz continuity of $\widehat{f}$ and on the well known singular Gronwall Lemma due to D. Henry ([9, Lemma 7.1.1]). The case $\varepsilon_0 > 0$ is even easier, since it is a regular perturbation problem (for a sketch of the proof, see also [12]). □

5. Existence and upper semicontinuity of attractors

In the last section we have seen that the semiflows π_ε, $\varepsilon \ge 0$, possess absorbing sets in the H^1-topology. In order to prove existence and upper semicontinuity of attractors, we need to establish some compactness of the semiflows π_ε. Since the domain Ω is unbounded, the nonlinear map $\pi_\varepsilon(t,\,\cdot\,)$ might not be compact. However, as we shall see, it is asymptotically compact, that is, whenever $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega)$ and $t_n \to \infty$, the set $\{\pi_\varepsilon(t_n, u_n) \mid n \in \mathbb{N}\}$ is precompact in $H^1(\Omega)$. The following crucial lemma is essentially due to B. Wang (see [18]). Wang's result deals with a single equation on a fixed unbounded domain. Here we present a slightly modified version of it, which gives estimates, independent of ε, for the entire family of problems (4.1).

Lemma 5.1. *Let $u_\varepsilon\colon \mathbb{R}_+ \to H^1(\Omega)$ be the solution of the Cauchy problem (4.1), with $\|\overline{u}_\varepsilon\|_{H^1_\varepsilon} \le R$. Then, for every $\eta > 0$, there exist two positive constants $\overline{k}$ and $\overline{T}$ such that for every $t \ge \overline{T}$ and $k \ge \overline{k}$,*

$$\int_{\Omega\cap\{|x|>k\}} |u_\varepsilon(t,x,y)|^2\,dx\,dy \le \eta.$$

The constants $\overline{k}$ and $\overline{T}$ depend only on R and η and are independent of ε.

Proof. Let $\theta : \mathbb{R}_+ \to \mathbb{R}$ be a smooth function such that $0 \le \theta(s) \le 1$ for $s \in \mathbb{R}_+$, $\theta(s) = 0$ for $0 \le s \le 1$ and $\theta(s) = 1$ for $s \ge 2$. Let $D := \sup_{s\in\mathbb{R}_+} |\theta'(s)|$. For $k \in \mathbb{N}$, let us define the multiplication operator

$$\Theta_k\colon H^1(\Omega) \to H^1(\Omega), \quad (\Theta_k u)(x,y) := \theta\bigg(\frac{|x|^2}{k^2}\bigg)u(x,y).$$

We have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\Omega \theta\bigg(\frac{|x|^2}{k^2}\bigg)&|u_\varepsilon(t,x,y)|^2\,dx\,dy\\ &= \frac{d}{dt}\frac{1}{2}\langle \Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle = \langle \Theta_k u_\varepsilon(t), \dot{u}_\varepsilon(t)\rangle\\ &= \langle \Theta_k u_\varepsilon(t), -A_\varepsilon u_\varepsilon(t) - \lambda u_\varepsilon(t) + \widehat{f}(u_\varepsilon(t)) + g\rangle\\ &= -\,a_\varepsilon(\Theta_k u_\varepsilon(t), u_\varepsilon(t)) - \lambda\langle \Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle\\ &\quad + \langle \Theta_k u_\varepsilon(t), \widehat{f}(u_\varepsilon(t))\rangle + \langle \Theta_k u_\varepsilon(t), g\rangle.\end{aligned}$$

By (2.3) we get

$$\begin{aligned}\frac{d}{dt}\langle \Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle &+ 2\lambda\langle \Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle\\ &\le -2a_\varepsilon(\Theta_k u_\varepsilon(t), u_\varepsilon(t)) + 2\langle \Theta_k u_\varepsilon(t), g\rangle.\end{aligned}$$

Since

$$\begin{aligned}a_\varepsilon(\Theta_k u_\varepsilon(t), u_\varepsilon(t)) &= \int_\Omega \theta\bigg(\frac{|x|^2}{k^2}\bigg)|\nabla_x u_\varepsilon(t,x,y)|^2\,dx\,dy\\ &\quad + \int_\Omega \theta'\bigg(\frac{|x|^2}{k^2}\bigg)u_\varepsilon(t,x,y)\frac{2}{k^2}x\cdot\nabla_x u_\varepsilon(t,x,y)\,dx\,dy\\ &\quad + \frac{1}{\varepsilon^2}\int_\Omega \theta\bigg(\frac{|x|^2}{k^2}\bigg)|\nabla_y u_\varepsilon(t,x,y)|^2\,dx\,dy,\end{aligned}$$

it follows that

$$\begin{aligned}
-a_\varepsilon(\Theta_k u_\varepsilon(t), u_\varepsilon(t)) &\le -\int_\Omega \theta'\left(\frac{|x|^2}{k^2}\right) u_\varepsilon(t,x,y)\frac{2}{k^2} x\cdot\nabla_x u_\varepsilon(t,x,y)\,dx\,dy\\
&\le 2D\int_{\Omega\cap\{k\le|x|\le\sqrt{2}k\}} \frac{|x|}{k^2}|u_\varepsilon(t,x,y)||\nabla_x u_\varepsilon(t,x,y)|\,dx\,dy\\
&\le \frac{2\sqrt{2}D}{k}\int_{\Omega\cap\{k\le|x|\le\sqrt{2}k\}} |u_\varepsilon(t,x,y)||\nabla_x u_\varepsilon(t,x,y)|\,dx\,dy\\
&\le \frac{2\sqrt{2}D}{k}\|u_\varepsilon(t)\|_{L^2}\|\nabla u_\varepsilon(t)\|_{L^2}.
\end{aligned}$$

So, by Proposition (4.6), for $t \ge T(R)$, we have

$$-a_\varepsilon(\Theta_k u_\varepsilon(t), u_\varepsilon(t)) \le \frac{2\sqrt{2}DK^2}{k}.$$

Let $\eta > 0$ and choose $k = k(\eta)$ such that

$$\frac{2\sqrt{2}DK^2}{k} < \eta.$$

Then for $t > T(R)$ and $k > k(\eta)$, we obtain

$$\frac{d}{dt}\langle\Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle + 2\lambda\langle\Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle \le 2\eta + 2\langle\Theta_k u_\varepsilon(t), g\rangle.$$

By Young inequality we have

$$\langle\Theta_k u_\varepsilon(t), g\rangle \le \frac{\lambda}{2}\langle\Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle + \frac{1}{2\lambda}\int_\Omega \theta\left(\frac{|x|^2}{k^2}\right) g(x,y)^2\,dx\,dy.$$

Since $g \in L^2(\Omega)$, there exists $k' = k'(\eta)$ such that, if $k > k'(\eta)$,

$$\frac{1}{2\lambda}\int_\Omega \theta\left(\frac{|x|^2}{k^2}\right) g(x,y)^2\,dx\,dy \le \eta.$$

So we obtain that for $t > T(R)$ and for $k > \max\{k(\eta), k'(\eta)\}$,

$$\frac{d}{dt}\langle\Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle + \lambda\langle\Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle \le 3\eta.$$

Multiplication by $e^{\lambda t}$ and integration yields

$$e^{\lambda t}\langle\Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle - e^{\lambda T(R)}\langle\Theta_k u_\varepsilon(T(R)), u_\varepsilon(T(R))\rangle \le \frac{4\eta}{\lambda}e^{\lambda t}$$

for $t > T(R)$. It follows that for $t > T(R)$

$$\begin{aligned}
\langle\Theta_k u_\varepsilon(t), u_\varepsilon(t)\rangle &\le e^{-\lambda(t-T(R))}\langle\Theta_k u_\varepsilon(T(R)), u_\varepsilon(T(R))\rangle + \frac{4\eta}{\lambda}\\
&\le e^{-\lambda(t-T(R))}K^2 + \frac{4\eta}{\lambda}.
\end{aligned}$$

Finally, for $t \geq T(R) + \lambda^{-1}\log(\eta^{-1})$ and for $k > \max\{k(\eta), k'(\eta)\}$, we get

$$\int_{\Omega\cap\{|x|>\sqrt{2}k\}} |u_\varepsilon(t,x,y)|^2\,dx\,dy \leq \int_\Omega \theta\bigg(\frac{|x|^2}{k^2}\bigg)|u_\varepsilon(t,x,y)|^2\,dx\,dy \leq \bigg(K^2 + \frac{4}{\lambda}\bigg)\eta,$$

and the proof is complete. □

Similarly, one can prove

Lemma 5.2. *Let $u_0\colon \mathbb{R}_+ \to H^1_s(\Omega)$ be the solution of the Cauchy problem (4.2), with $\|\overline{u}_0\|_{H^1_s} \leq R$. Then, for every $\eta > 0$, there exist two positive constants $\overline{k}$ and $\overline{T}$ such that for every $t \geq \overline{T}$ and $k \geq \overline{k}$,*

$$\int_{\Omega\cap\{|x|>k\}} |u_0(t,x,y)|^2\,dx\,dy \leq \eta.$$

The constants $\overline{k}$ and $\overline{T}$ depend only on R and η.

Now we are able to state and prove our first compactness result:

Theorem 5.3. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers, let $\varepsilon_0 \geq 0$, and assume that $\varepsilon_n \to \varepsilon_0$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ ($u_n \in H^1_s(\Omega)$ if $\varepsilon_n = 0$). Assume also that there exists a positive constant R such that $\|u_n\|_{H^1_{\varepsilon_n}} \leq R$ for all $n \in \mathbb{N}$ ($\|u_n\|_{H^1_s} \leq R$ if $\varepsilon_n = 0$). Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to ∞. Then there exists $u_0 \in H^1(\Omega)$ ($u_0 \in H^1_s(\Omega)$ if $\varepsilon_0 = 0$) such that, up to a subsequence,*

$$\pi_{\varepsilon_n}(t_n, u_n) \to u_0 \quad \text{in } L^2(\Omega) \text{ as } n \to \infty.$$

In particular, choosing $\varepsilon_n = \varepsilon_0$ for all n, we obtain that the semiflow π_{ε_0} is asymptotically L^2-compact.

Proof. By Propositions 4.6 and 4.7, there exist two positive constants K_1 and K_2 such that

$$\|u_\varepsilon(t)\|^2_{H^1_\varepsilon} \leq K_1R^2 + K_2\|g\|^2_{L^2} \quad \text{for all } t \geq 0 \text{ and for all } n.$$

It follows that there exists u_0 in $H^1(\Omega)$ such that, up to a subsequence,

$$\pi_{\varepsilon_n}(t_n, u_n) \rightharpoonup u_0 \quad \text{in } H^1(\Omega) \text{ as } n \to \infty.$$

Moreover,

$$\begin{aligned} \|\nabla_y \pi_{\varepsilon_n}(t_n,u_n)\|^2_{L^2} &= 0 && \text{if } \varepsilon_n = 0, \\ \|\nabla_y \pi_{\varepsilon_n}(t_n,u_n)\|^2_{L^2} &\leq \frac{1}{\varepsilon_n^2}(K_1R^2 + K_2\|g\|^2_{L^2}) && \text{if } \varepsilon_n > 0, \end{aligned}$$

so $u_0 \in H^1_s(\Omega)$ if $\varepsilon_0 = 0$. In order to recover strong L^2-convergence, we just need to show that the set

$$\{\pi_{\varepsilon_n}(t_n,u_n) \mid n \in \mathbb{N}\}$$

is precompact in $L^2(\Omega)$. To this end, we apply Propositions 5.1 and 5.2. Let $\eta > 0$ and choose $k > \max\{k(\eta), k'(\eta)\}$ like in the proof of Proposition 5.1. Moreover, take $\overline{n}$ such that $t_n \geq T(R) + \lambda^{-1}\log(\eta^{-1})$ for all $n \geq \overline{n}$. Then, for $n \geq \overline{n}$,

$$
\begin{aligned}
(5.1)\qquad &\{\pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\}\\
&= \{\Theta_k \pi_{\varepsilon_n}(t_n, u_n) + (I - \Theta_k)\pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\}\\
&\subset \{\Theta_k \pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\} + \{(I - \Theta_k)\pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\}\\
&\subset B_\eta(0) + \{(I - \Theta_k)\pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\},
\end{aligned}
$$

where $B_\eta(0)$ is the ball of radius η centered at 0 in $L^2(\Omega)$. The set

$$\{(I - \Theta_k)\pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\}$$

consists of functions of $H^1(\Omega)$ which are equal to zero outside the ball $B_{\sqrt{2}k}(0)$ in $\mathbb{R}^{N+P}$. On the other hand, the $H^1(\Omega)$ norm of these functions is bounded by the constant $(K_1 R^2 + K_2\|g\|^2_{L^2})^{1/2}$. Then, by Rellich Theorem, we deduce that the set $\{(I - \Theta_k)\pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\}$ is precompact in $L^2(\Omega)$. Hence we can cover it by a finite number of balls of radius η in $L^2(\Omega)$. This observation, together with (5.1), implies that the set $\{\pi_{\varepsilon_n}(t_n, u_n) \mid n \in \mathbb{N}\}$ is totally bounded and hence precompact in $L^2(\Omega)$. □

Remark. In the case $\varepsilon_0 > 0$, Theorem 5.3 is due to B. Wang (see [18]). Indeed, he considers a single fixed concrete reaction-diffusion equation and his proof is based on energy estimates and weak continuity of solutions with respect to initial data. However, his technique seems not to apply to the singular problem we are dealing with. Our proof is simpler and the singular behaviour of the problem does not introduce any further difficulties. On the other hand, the advantage of Wang's technique is that it applies as well to different classes of problems, like Navier-Stokes equations and damped wave equations.

Now, thanks to Theorems 4.6 and 5.3, one could easily prove the existence of compact global $(L^2 - L^2)$-attractors for the semiflows π_ε, $\varepsilon > 0$ (see [18]). However, since our phase space is H^1, we are mostly interested in the existence of $(H^1 - H^1)$-attractors. To this end, we need to establish the asymptotic H^1-compactness of the semiflows π_ε. We argue like in [12]. In fact, Theorems 5.3 and 4.8 together imply the following stronger compactness result:

Theorem 5.4. *Under the same assumptions of Theorem* 5.3, *there exists* $u_0 \in H^1(\Omega)$ ($u_0 \in H^1_s(\Omega)$ *if* $\varepsilon_0 = 0$) *such that, up to a subsequence,*

$$\|\pi_{\varepsilon_n}(t_n, u_n) - u_0\|_{H^1_{\varepsilon_n}} \to 0 \qquad \text{as } n \to \infty.$$

In particular, choosing $\varepsilon_n = \varepsilon_0$ *for all* n, *we obtain that the semiflow* π_{ε_0} *is asymptotically* H^1*-compact.*

Proof. Let $\bar{t} > 0$, $s_n := t_n - \bar{t}$; then $s_n \to \infty$ and by Theorem 5.3 there exists $v_0 \in H^1_s(\Omega)$ such that, up to a subsequence,

$$\pi_{\varepsilon_n}(s_n, u_n) \to v_0 \quad \text{in } L^2(\Omega) \text{ as } n \to \infty.$$

Let $u_0 := \pi_0(\bar{t}, v_0)$. Then $u_0 \in H^1_s(\Omega)$ and, by Theorem 4.8,

$$\begin{aligned}\|\pi_{\varepsilon_n}(t_n, u_n) - u_0\|_{H^1_{\varepsilon_n}} &= \|\pi_{\varepsilon_n}(\bar{t} + s_n, u_n) - \pi_0(\bar{t}, v_0)\|_{H^1_{\varepsilon_n}} \\ &= \|\pi_{\varepsilon_n}(\bar{t}, \pi_{\varepsilon_n}(s_n, u_n)) - \pi_0(\bar{t}, v_0)\|_{H^1_{\varepsilon_n}} \to 0\end{aligned}$$

as $n \to \infty$. □

Now we are able to prove existence of the global attractors for the semiflows π_ε, $\varepsilon \geq 0$.

Theorem 5.5. *For every $\varepsilon \geq 0$, the semiflow π_ε has a global attractor $\mathcal{A}_\varepsilon$ in $H^1(\Omega)$ (in $H^1_s(\Omega)$ if $\varepsilon = 0$). The set $\mathcal{A}_\varepsilon$ is H^1-compact, connected, and consists of all the full bounded solutions of equation* (4.1) (*of equation* (4.2) *if* $\varepsilon = 0$).

Proof. By Proposition 4.6 (Proposition 4.7 if $\varepsilon = 0$), the global semiflow π_ε has an absorbing set in $H^1(\Omega)$. Moreover, by Theorem 5.4, the semiflow π_ε is asymptotically H^1-compact. The conclusion follows from the classical results of [6], [11], [16] and [1]. □

Finally, we can prove the upper-semicontinuity result announced in the Introduction:

Theorem 5.6. *For $\varepsilon \geq 0$, let $\mathcal{A}_\varepsilon$ be the attractor of the semiflow π_ε. Then for every $\delta > 0$ there exists $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon < \bar{\varepsilon}$*

$$\mathrm{d}_{H^1_\varepsilon}(\mathcal{A}_\varepsilon, \mathcal{A}_0) := \max_{u \in \mathcal{A}_\varepsilon} \mathrm{d}_{H^1_\varepsilon}(u, \mathcal{A}_0) < \delta.$$

Proof. If the theorem is not true, there exist $\bar{\delta} > 0$, a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers, $\varepsilon_n \to 0$, and a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in \mathcal{A}_{\varepsilon_n}$ for all $n \in \mathbb{N}$, such that

$$\mathrm{d}_{H^1_{\varepsilon_n}}(u_n, \mathcal{A}_0) \geq \bar{\delta} \quad \text{for all } n \in \mathbb{N}.$$

Since $u_n \in \mathcal{A}_{\varepsilon_n}$, for every $n \in \mathbb{N}$ there exists a full bounded solution $\sigma_n(t)$ of equation (4.1), with $\varepsilon = \varepsilon_n$, passing through u_n. This means that

(i) the function $t \mapsto \sigma_n(t)$ is a bounded solution of equation (4.1), with $\varepsilon = \varepsilon_n$,

(ii) $\sigma_n(t) = \pi_{\varepsilon_n}(t, u_n)$ for $t \geq 0$.

Moreover, by Proposition 4.6, there exists $K > 0$, independent of n, such that

(iii) $\|\sigma_n(t)\|_{H^1_{\varepsilon_n}} \leq K$ for $t \in \mathbb{R}$.

Let k be a positive integer and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $h_n \to \infty$; by (iii) and by Proposition 5.4, there exists $\overline{u}_k \in H^1_s(\Omega)$ such that, up to a subsequence,

$$\|\sigma_n(-k) - \overline{u}_k\|_{H^1_{\varepsilon_n}} = \|\pi_{\varepsilon_n}(h_n, \sigma_n(-k-h_n)) - \overline{u}_k\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

By a Cantor diagonal procedure, we can assume that

$$\|\sigma_n(-k) - \overline{u}_k\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty$$

for every positive integer k. By Theorem 4.8, for every $t > 0$,

$$\|\pi_{\varepsilon_n}(t, \sigma_n(-k)) - \pi_0(t, \overline{u}_k)\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty. \tag{5.2}$$

Choosing $t = k$ we get

$$\|u_n - \pi_0(k, \overline{u}_k)\|_{H^1_{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Notice that $\pi_0(k, \overline{u}_k)$ is independent of k, so we can define $u_0 := \pi_0(k, \overline{u}_k)$. The proof will be complete if we show that $u_0 \in \mathcal{A}_0$. To this end, let us define $\sigma_0(t) := \pi_0(t+k, \overline{u}_k)$, $t \geq -k$. By (5.2) and Theorem 4.8, for every $t > -k$

$$\|\sigma_n(t) - \pi_0(t+k, \overline{u}_k)\|_{H^1_{\varepsilon_n}} = \|\pi_{\varepsilon_n}(t+k, \sigma(-k)) - \pi_0(t+k, \overline{u}_k))\|_{H^1_{\varepsilon_n}} \to 0$$

as $n \to \infty$. It follows that $\pi_0(t+k, \overline{u}_k)$ is independent of k and therefore $\sigma_0(t)$ is unambiguously defined for every $t \in \mathbb{R}$. Moreover, $\sigma_0(t)$ is a solution of (4.2) and $\|\sigma_0(t)\|_{H^1_s} \leq K$ for every $t \in \mathbb{R}$. Thus there is a full bounded solution of equation (4.2) through u_0. This finally implies that $u_0 \in \mathcal{A}_0$. □

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