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WEAK COMPACTNESS OF SOLUTION SETS TO STOCHASTIC DIFFERENTIAL INCLUSIONS WITH CONVEX RIGHT-HAND SIDES

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Abstract. Necessary and sufficient conditions for the existence of weak solutions to stochastic differential inclusions with convex right-hand sides are given. The main results of the paper deal with the weak compactness with respect to the convergence in distribution of solution sets to such inclusions.

1. Introduction

The first papers concerning stochastic differential inclusions are due to F. Hiai [3] and M. Kisielewicz [7], where independently, stochastic differential inclusions have been defined as relations of the form

$$x_t - x_s \in \mathrm{cl}_{L^2}\bigg(\int_s^t F(\tau, x_\tau)\,d\tau + \int_s^t G(\tau, x_\tau)\,dB_\tau\bigg) \tag{1}$$

that have to be satisfied by L^2-continuous $\mathcal{F}_t$-nonanticipative stochastic process $(x_t)_{0\le t\le T}$ for every $0 \le s < t \le T$, i.e. by $\mathcal{F}_t$-nonanticipative square integrable process $(x_t)_{0\le t\le T}$ that is continuous with respect to the norm topology of the space $L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$. Such inclusions were considered on a given complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ satisfying the usual hypotheses, i.e. with the filtration $(\mathcal{F}_t)_{0\le t\le T}$ such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$ and

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$\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$. Apart from the set-valued mappings $F : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ and $G : [0,T]\times\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ or with values at $\mathrm{Cl}(H)$, where H is a Hilbert space, some $\mathcal{F}_t$-Brownian motions $(B_t)_{0\le t\le T}$ and $(W_t)_{0\le t\le T}$, with values at $\mathbb{R}$ or H, also have been given. As usualy $\mathrm{Cl}(X)$ denotes the space of all nonempty closed subsets of a metric space (X,ρ). Similarly as in the theory of stochastic differential equations, the process $(x_t)_{0\le t\le T}$ mentioned above, is said to be a strong solution to (1). Such solutions have been considered by J. P. Aubin and G. Da Prato [1], G. Da Prato and Frankowska [2], J. Motyl [9], [10], [11] and others. For the existence of strong solutions some Lipschitz type conditions for $F(t,\cdot)$ and $G(t,\cdot)$ have to be satisfied. Such assumptions are rather too strong for the practical applications. Therefore, we are interested in the weaker notation of solutions that are not restrictive in the existence theory and are extremaly useful and fruitful in both theory and applications. Such type solutions are known as weak ones ([4], [5]), and are understood in fact as systems including a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ satisfying the usual hypotheses, $\mathcal{F}_t$-Brownian motion $(B_t)_{0\le t\le T}$ and an L^2-continuous $\mathcal{F}_t$-nonanticipative process $(x_t)_{0\le t\le T}$ satisfying the relation (1), when F and G are given. In what follows we shall identify such system with a process $(x_t)_{0\le t\le T}$ depending on $(B_t)_{0\le t\le T}$ and denote it simply by $(x_t(B))_{0\le t\le T}$ or $(x_t)_{0\le t\le T}$ if dependence on $(B_t)_{0\le t\le T}$ is not important. We will say that $(x_t(B))_{0\le t\le T}$ is a weak solution to (1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. If $(x_t(B))_{0\le t\le T}$ is such that $P(x_0^{-1}(B)) = \mu$, where μ is a given probability measure on the Borel σ-algebra $\beta(\mathbb{R}^n)$ on $\mathbb{R}^n$, then we say that $(x_t(B))_{0\le t\le T}$ is a weak solution to (1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ with an initial distribution μ on $\beta(\mathbb{R}^n)$.

It was proved in [6] that for the existence of a weak solution to (1) with an initial distribution μ it is enough to assume that F and G are Borel measurable, bounded, convex valued and such that $F(t,\cdot)$ and $G(t,\cdot)$ are lower semicontinuous for fixed $t \in [0,T]$. Stochastic differential inclusions, considered in [6] and [7] are defined for one-dimensional Brownian motions. In the present paper we extend the notation to the general case with m-dimensional Brownian motion $(B_t)_{0\le t\le T}$. Therefore G has to take its values from the space $\mathrm{Cl}(\mathbb{R}^{n\times m})$, where $\mathbb{R}^{n\times m}$ denotes a space of all $n \times m$-type matrices. We shall consider $\mathbb{R}^{n\times m}$ as a normed space with the metric $\|\cdot\|$ defined by

$$(*) \qquad \|g\| = \Big(\sum_{i=1}^{n}\sum_{j=1}^{m} g_{ij}^2\Big)^{1/2} \quad \text{for } g = (g_{ij})_{n\times m}.$$

Throughout the paper we assume that F and G are convex-valued or that G is such that the set $\{g\cdot g^T : g \in G(t,x)\}$ is convex for $(t,x) \in [0,T]\times\mathbb{R}^n$, where g^T denotes the transposition of g.

Similarly as in [4, Definition II.7.1] we will say that a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \widetilde{\mathrm{P}})$ is an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ if and only if there exists an $(\widetilde{\mathcal{F}}, \mathcal{F})$-measurable mapping $\pi : \widetilde{\Omega} \to \Omega$ such that

(i) $\pi^{-1}(\mathcal{F}_t) \subset \widetilde{\mathcal{F}}_t$ for $t \in [0,T]$,
(ii) $P = \widetilde{\mathrm{P}} \circ \pi^{-1}$,
(iii) for every $x \in L^\infty(\Omega, \mathcal{F}, P)$ one has $\widetilde{E}(\widetilde{x}|\widetilde{\mathcal{F}}_t)(\widetilde{\omega}) = E(x|\mathcal{F}_t)(\pi(\widetilde{\omega}))$ for $\widetilde{\omega} \in \widetilde{\Omega}$, where $\widetilde{x}(\widetilde{\omega}) = x(\pi(\widetilde{\omega}))$.

2. Existence of weak solutions

Let $F : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ and $G : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ be Borel measurable and bounded, i.e. such that there exists $M > 0$ such that $\max\{|F(t,x)|, |G(t,x)|\} \le M$ for $(t,x) \in [0,T] \times \mathbb{R}^n$, where for a given subset A of a normed space $(X, |\cdot|)$ we define $|A| = \sup\{|a| : a \in A\}$. Assume that $(B_t)_{0\le t\le T}$ is an m-dimensional $\mathcal{F}_t$-Brownian motion and $x = (x_t)_{0\le t\le T}$ is an n-dimensional L^2-continuous $\mathcal{F}_t$-nonanticipative process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ satisfying the usual hypothesis. We can define stochastic set-valued integrals for mappings $(F \circ x)_t(\omega) = F(t, x_t(\omega))$ and $(G \circ x)_t(\omega) = G(t, x_t(\omega))$ setting

$$\int_s^t (F\circ x)_\tau \, d\tau = \left\{ \int_0^T \mathbb{1}_{[s,t]}(\tau) \cdot f_\tau \, d\tau : f \in S(F \circ x) \right\}, \tag{2}$$

$$\int_s^t (G\circ x)_\tau \, dB\tau = \left\{ \int_0^T \mathbb{1}_{[s,t]} \cdot g_\tau \, dB_\tau : g \in S(G \circ x) \right\}, \tag{3}$$

where $S(F \circ x)$ and $S(G \circ x)$ denote families of all $\mathcal{F}_t$-nonanticipative selectors for $F \circ x$ and $G \circ x$, respectively. These integrals (see [7]) are defined to be subsets of the space $L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$ and are denoted simply by $\int_s^t F(\tau, x_\tau)\, d\tau$ and $\int_s^t G(\tau, x_\tau)\, dB_\tau$, respectively. If F and G are assumed to be convex-valued then the last integrals are closed subsets of $L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$ [7]. Therefore, in what follows we shall consider stochastic differential inclusions written in the form:

$$x_t - x_s \in \int_s^t F(\tau, x_\tau)\, d\tau + \int_s^t G(\tau, x_\tau)\, dB_\tau \tag{4}$$

for $0 \le s < t \le T$. Similarly as in [7, Theorem 4] the following lemma can be proved.

Lemma 1. *Let $F : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ and $G : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ be measurable, bounded and convex valued. Assume $(x_t)_{0\le t\le T}$ and $(y_t)_{0\le t\le T}$ are n-dimensional L^2-continuous $\mathcal{F}_t$-nonanticipative processes and $(B_t)_{0\le t\le T}$ is an m-dimensional $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. Then*

$$y_t - y_s \in \int_s^t F(\tau, x_\tau)\, d\tau + \int_s^t G(\tau, x_\tau)\, dB_\tau$$

for every $0 \le s < t \le T$ *if and only if there are* $f \in S(F \circ x)$ *and* $g \in S(G \circ x)$ *such that*

$$y_t = y_0 + \int_0^t f_\tau \, d\tau + \int_0^t g_\tau \, dB_\tau \quad \textit{with } (\mathrm{P.1})$$

for $t \in [0,T]$.

It is clear now that if F and G are convex valued we can define a weak solution to (4) as a system consisting of a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, a continuous $\mathcal{F}_t$-adapted process $(x_t)_{0\le t\le T}$ and an $\mathcal{F}_t$-Brownian motion $(B_t)_{0\le t\le T}$ satisfying (4) for every $0 \le s < t \le T$. We shall still denote such systems by $(x_t(B))_{0\le t\le T}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$.

Denote by $C_b^2(\mathbb{R}^n)$ the space of all continuous bounded functions $h : \mathbb{R}^n \to \mathbb{R}$, having continuous and bounded derivatives h'_{x_i} and $h''_{x_i x_j}$ for $i, j = 1, \ldots, n$. For any given F, G and a continuous n-dimensional $\mathcal{F}_t$-adapted process $x = (x_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ we define a family

$$\mathcal{A}^x_{FG} = \{\mathcal{A}^x_{fg} : (f,g) \in S(F \circ x) \times S(G \circ x)\}$$

of linear operators on $C_b^2(\mathbb{R}^n)$ with values in the space of all $\mathcal{F}_t$-nonanticipative square integrable real valued processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, of the form:

$$(\mathcal{A}^x_{fg} h)_t = \sum_{i=1}^n h'_{x_i}(x_t) f_t^i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n h''_{x_i x_j}(x_t) \sigma_t^{ij} \tag{5}$$

a.e. on Ω and $t \in [0,T]$ for every $h \in C_b^2(\mathbb{R}^n)$, where $f = (f^i)_{1\times n}$, $g = (g^{ij})_{n\times m}$ and $\sigma = g \cdot g^T$. We will say that $\mathcal{A}^x_{fg} \in \mathcal{A}^x_{FG}$ generates on $C_b^2(\mathbb{R}^n)$ a family of continuous square integrable local $\mathcal{F}_t$-martingales, if for every $h \in C_b^2(\mathbb{R}^n)$ a process $[(\varphi_h^x)_t]_{0\le t\le T}$ defined by

$$(\varphi_h^x)_t = h(x_t) - h(x_0) - \int_0^t (\mathcal{A}^x_{fg} h)_\tau \, d\tau \quad \text{with (P.1)} \tag{6}$$

for $t \in [0,T]$ is a continuous square integrable local $\mathcal{F}_t$-martingales on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. By $\mathcal{M}^x_{FG}(C_b^2(\mathbb{R}^n))$ we denote the family of all $\mathcal{A}^x_{fg} \in \mathcal{A}^x_{FG}$ that generates on $C_b^2(\mathbb{R}^n)$ a family of continuous square integrable local $\mathcal{F}_t$-martingales.

Theorem 2. *Let* $F : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ *and* $G : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ *be Borel measurable, bounded and convex valued and let* μ *be a probability measure on* $\beta(\mathbb{R}^n)$. *Stochastic differential inclusion* (4) *has at least one weak solution with an initial distribution* μ *if and only if there are a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ *and an* n*-dimensional continuous* $\mathcal{F}_t$*-adapted process* $x = (x_t)_{0\le t\le T}$ *such that* $Px_0^{-1} = \mu$ *and* $\mathcal{M}^x_{FG}(C_b^2(\mathbb{R}^n)) \ne \emptyset$.

Proof. ($\Rightarrow$) Let $(x_t(B))_{0\le t\le T}$ be a weak solution to (4) on the space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P)$ with an initial distribution μ. For every $0\le s<t\le T$ we have

$$x_t - x_s \in \int_s^t F(\tau,x_\tau)\,d\tau + \int_s^t G(\tau,x_\tau)\,dB_\tau.$$

Therefore, by virtue of Lemma 1, there are $f\in S(F\circ x)$ and $g\in S(G\circ x)$ such that

$$x_t = x_0 + \int_0^t f_\tau\,d\tau + \int_0^t g_\tau\,dB_\tau \quad \text{with (P.1)}$$

for $t\in[0,T]$, and equivalently in the differential form $dx_t = f_t d\tau + g_t dB_t$ for $t\in[0,T]$. Hence, by the Itô formula, for every $h\in C_b^2(\mathbb{R}^n)$, we obtain

$$(7)\quad h(x_t)-h(x_0)-\int_0^t(\mathcal{A}_{fg}^x h)_\tau\,d\tau = \sum_{i=1}^n\sum_{j=1}^m\int_0^t h'_{x_i}(x_\tau)\cdot g_\tau^{ij}dB_\tau^j \quad \text{with (P.1)}$$

for $t\in[0,T]$, where $B_t=(B_t^1,\dots,B_t^m)^T$ and $g_t=(g_t^{ij})_{n\times m}$. By the definition of $(\varphi_h^x)_t$, the above equality (7) can be written in the form

$$(\varphi_h^x)_t = \sum_{i=1}^n\sum_{j=1}^m\int_0^t h'_{x_i}(x_\tau)\cdot g_\tau^{ij}dB_\tau^j \quad \text{with (P.1)}$$

for $t\in[0,T]$. Hence, by the properties of the Itô integrals, it follows that $\mathcal{A}_{fg}^x\in\mathcal{M}_{FG}^x(C_b^2(\mathbb{R}^n))$ and $\mathcal{M}_{FG}^x(C_b^2(\mathbb{R}^n))\ne\emptyset$.

($\Leftarrow$) Assume there are $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P)$ and an n-dimensional continuous $\mathcal{F}_t$-adapted process $x=(x_t)_{0\le t\le T}$ such that $Px_0^{-1}=\mu$ and $\mathcal{M}_{FG}^x(C_b^2(\mathbb{R}^n))\ne\emptyset$. Let $(f,g)\in S(F\circ x)\times S(G\circ x)$ be such that $\mathcal{A}_{fg}^x\in\mathcal{M}_{FG}^x(C_b^2(\mathbb{R}^n))$. Define a sequence $(\tau_l)_{l=1}^\infty$ of stopping times $\tau_l=\inf\{t\in[0,T]: x_t\notin K_l\}$, where $K_l=\{x\in\mathbb{R}^n: |x|\le l\}$ for $l=1,2,\dots$ Select now, for every $i=1,\dots,n$, $h_i\in C_b^2(\mathbb{R}^n)$ such that $h_i(x)=x_i$ for $x\in K_l$, where $x=(x^1,\dots,x^n)$. For such $h_i\in C_b^2(\mathbb{R}^n)$ we have

$$\int_0^{t\wedge\tau_l}(\mathcal{A}_{fg}^x h_i)_\tau\,d\tau = \int_0^{t\wedge\tau_l} f_\tau^i\,d\tau \quad \text{with (P.1)}$$

and therefore,

$$(\varphi_{h_i}^x)_{t\wedge\tau_l} = x_{t\wedge\tau_l}^i - x_0^i - \int_0^{t\wedge\tau_l} f_\tau^i\,d\tau \quad \text{with (P.1)}$$

for $i=1,\dots,n$, $l=1,2,\dots$ and $t\in[0,T]$. But $\mathcal{A}_{fg}^x\in\mathcal{M}_{FG}^x(C_b^2(\mathbb{R}^n))$. Then $[(\varphi_{h_i}^x)_{t\wedge\tau_l}]_{0\le t\le T}$ is for every $i=1,\dots,n$ and $l=1,2,\dots$ a continuous square integrable local $\mathcal{F}_t$-martingale on $(\Omega,\mathcal{F},P)$, which implies that also $[(\varphi_{h_i}^x)_t]_{0\le t\le T}$ is a continuous square integrable local $\mathcal{F}_t$-martingale on $(\Omega,\mathcal{F},P)$ for every $i=1,\dots,n$. Denote it by $(M_t^i)_{0\le t\le T}$, i.e. let $M_t^i=(\varphi_{h_i}^x)_t$ for $i=1,\dots,n$ and $t\in[0,T]$.

Similarly, taking $h_{ij} \in C_b^2(\mathbb{R}^n)$ such that $h_{ij}(x) = x^i x^j$ for $x \in K_l$ for $i,j = 1,\dots,n$, we obtain a family $(\mathcal{M}_t^{ij})_{0\le t\le T}$; $i,j = 1,\dots,n$ of continuous square integrable local $\mathcal{F}_t$-martingals on $(\Omega,\mathcal{F},P)$ such that

$$M_t^{ij} = x_t^i x_t^j - x_0^i x_0^j - \int_0^t [x_\tau^i f_\tau^j + x_\tau^j f_\tau^i + \sigma_\tau^{ij}]\,d\tau \quad \text{with (P.1)}$$

for $i,j = 1,\dots,n$ and $t \in [0,T]$. Now, similarly as in [3, Proposition 5.4.6], we conclude that

$$\langle M^i, M^j\rangle_t = \int_0^t \sigma_\tau^{ij}\,d\tau \quad \text{with (P.1)}$$

for $i,j = 1,\dots,n$ and $t \in [0,T]$. Therefore, by [2, Theorem II.7.1'], for every $j = 1,\dots,m$, there exists an $\widetilde{\mathcal{F}}_t$-Brownian motion $(\widetilde{B}_t^j)_{0\le t\le T}$ on an extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{0\le t\le T},\widetilde{\mathrm{P}})$ of $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P)$ such that

$$M_t^i = \sum_{j=1}^m \int_0^t \widetilde{g}_\tau^{ij}\,d\widetilde{B}_\tau \quad \text{with } (\widetilde{\mathrm{P}}.1)$$

for $i = 1,\dots,n$ and $t \in [0,T]$. Then

$$\widetilde{x}_t^i = \widetilde{x}_0^i + \int_0^t \widetilde{f}_\tau^i\,d\tau + \sum_{j=1}^m \int_0^t \widetilde{g}_\tau^{ij}\,d\widetilde{B}_\tau^j \quad \text{with } (\widetilde{\mathrm{P}}.1)$$

for $i = 1,\dots,n$ and $t \in [0,T]$, or equivalently

$$\widetilde{x}_t = \widetilde{x}_0 + \int_0^t \widetilde{f}_\tau\,d\tau + \int_0^t \widetilde{g}_\tau d\widetilde{B}_\tau \quad \text{with } (\widetilde{\mathrm{P}}.1)$$

for $t \in [0,T]$, where $\widetilde{x}_t(\widetilde{\omega}) = x_t(\pi(\widetilde{\omega}))$, $\widetilde{f}_t(\widetilde{\omega}) = f_t(\pi(\widetilde{\omega}))$ and $\widetilde{g}_t(\widetilde{\omega}) = g_t(\pi(\widetilde{\omega}))$ for $\widetilde{\omega} \in \widetilde{\Omega}$. Hence, for $0 \le s < t \le T$,

$$\widetilde{x}_t - \widetilde{x}_s \in \int_s^t F(\tau,\widetilde{x}_\tau) + \int_s^t G(\tau,\widetilde{x}_\tau)\,d\widetilde{B}_\tau$$

i.e. $(\widetilde{x}_t(\widetilde{B}))_{0\le t\le T}$ is a weak solution to (4) on $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{0\le t\le T},\widetilde{\mathrm{P}})$. □

3. Diagonally convex subsets of $\mathbb{R}^{n\times m}$ and diagonally convex valued multifunctions

A set $\mathcal{G} \subset \mathbb{R}^{n\times m}$ is said to be *diagonally convex* if a set $D(\mathcal{G}) = \{u\cdot u^T : u \in \mathcal{G}\}$ is a convex subset of $\mathbb{R}^{n\times n}$. It is easy to see that for every $\mathcal{G} \in \mathrm{Cl}(\mathbb{R}^{n\times m})$ we also have $D(\mathcal{G}) \in \mathrm{Cl}(\mathbb{R}^{n\times n})$, because $D(\mathcal{G}) = l(\mathcal{G})$, where $l(u) = u\cdot u^T$ for $u \in \mathbb{R}^{n\times m}$. It is also clear, that for every bounded set $\mathcal{G} \subset \mathbb{R}^{n\times m}, D(\mathcal{G})$ is bounded. It is natural to ask when the convexity of $\mathcal{G} \subset \mathbb{R}^{n\times m}$ implies the convexity of $D(\mathcal{G})$. We show that this is true for sets $\mathcal{G} \subset \mathbb{R}^{1\times m}$. It will be follow from the below general result.

Proposition 1. *If $\mathcal{G} \subset \mathbb{R}^{n\times m}$ is convex then for every $u, v \in \mathcal{G}$ and $\lambda \in [0,1]$ there exists $x_\lambda \in \mathcal{G}$ such that $\lambda\|u\|^2 + (1-\lambda)\|v\|^2 = \|x_\lambda\|^2$, where $\|\cdot\|$ is the norm in $\mathbb{R}^{n\times m}$ defined above.*

Proof. Let $u, v \in \mathcal{G}$ be given. If $\|u\| = \|v\|$ then for every $\lambda \in [0,1]$ we can take $x_\lambda = u$. Suppose $0 < \|u\| < \|v\|$ and let $z_\lambda = \lambda\|u\|^2 + (1-\lambda)\|v\|^2$ for fixed $\lambda \in [0,1]$. Let $r_\lambda = \sqrt{z_\lambda}$. It is easy to see that $\|u\|^2 < z_\lambda < \|v\|^2$. Then $B_{\|u\|} \subset B_{r_\lambda} \subset B_{\|v\|}$, where B_r denotes the closed ball in $\mathbb{R}^{n\times m}$ centered at the origin with a radius $r > 0$. Furthermore, we have $u \in \partial B_{\|u\|}$ and $v \in \partial B_{\|v\|}$, where $\partial B_r = \{z \in \mathbb{R}^{n\times m} : \|z\| = r\}$. Denoting $l_{uv} = \{\lambda u + (1-\lambda)v : 0 \le \lambda \le 1\}$ we obtain $l_{uv} \cap \partial B_{r_\lambda} \neq \emptyset$ for every $\lambda \in [0,1]$. By the convexity of $\mathcal{G}$ we have $l_{uv} \cap \partial B_{r_\lambda} \subset \mathcal{G}$ for every $\lambda \in [0,1]$. Therefore, for every $\lambda \in [0,1]$ there is $x_\lambda \in \mathcal{G}$ such that $\|x_\lambda\| = r_\lambda$. If $0 < \|v\| < \|u\|$ we can also select $x_\lambda \in \mathcal{G}$ to every $\lambda \in [0,1]$ such that $\|x_\lambda\| = r_\lambda$. Suppose now that $\|u\| = 0$ and $\|v\| > 0$. Taking $z_\lambda = (1-\lambda)\|v\|^2$ and $r_\lambda = \sqrt{1-\lambda}\|v\|$ we obtain that $B_{r_\lambda} \subset B_{\|v\|}$ and that $l_{ov} \cap \partial B_{r_\lambda} \neq \emptyset$ and $l_{ov} \cap \partial B_{r_\lambda} \subset \mathcal{G}$. Hence it follows the existence of $x_\lambda \in \mathcal{G}$ such that $\|x_\lambda\|^2 = (1-\lambda)\|v\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2$ with $\|u\| = 0$ for $\lambda \in [0,1]$. $\square$

Now, from Proposition 1, we immediately obtain:

Proposition 2. *If $\mathcal{G} \subset \mathbb{R}^{1\times m}$ is convex then it is also diagonally convex.*

Proof. Let us observe that for every $u \in \mathbb{R}^{1\times m}$ we have $u \cdot u^T = \|u\|^2$. Therefore for every $z_1, z_2 \in D(\mathcal{G})$ and $\lambda \in [0,1]$ there are $u, v \in \mathcal{G}$ such that $\lambda z_1 + (1-\lambda)z_2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2$. By Proposition 1, for every $\lambda \in [0,1]$ there exists $x_\lambda \in \mathcal{G}$ such that $\|x_\lambda\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2 = \lambda z_1 + (1-\lambda)z_2$ and $\|x_\lambda\|^2 = x_\lambda \cdot x_\lambda^T \in D(\mathcal{G})$. $\square$

In what follows we shall deal with set valued mappings $G : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ that are assumed to be diagonally convex valued. It is clear that all regularity properties of G can be extended for $D(G) : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times n})$ defined by $D(G)(t,x) = D(G(t,x))$, because $D(G(t,x)) = l(G(t,x))$ and $l : \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times n}$ defined by $l(u) = u \cdot u^T$ for $u \in \mathbb{R}^{n\times m}$ is continuous. Hence in particular, it follows that for a given n-dimensional continuous $\mathcal{F}_t$-adapted process $x = (x_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ one has $S(D(G\circ x)) = D(S(G\circ x))$. Hence, in particular it follows that for every $\sigma \in S(D(G\circ x))$ there is $g \in S(G\circ x)$ such that $\sigma = g\cdot g^T$. More precisely we can state this result as the following Proposition.

Proposition 3. *Assume $G : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ is measurable and bounded. Then for every n-dimensional continuous $\mathcal{F}_t$-adapted process $x = (x_t)_{0\le t\le T}$ and $n\times n$-dimensional $\mathcal{F}_t$-nonanticipative process $\sigma_t = (\sigma_t^{ij})_{n\times n}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ such that $\sigma_t \in D(G(t,x_t))$ on $[0,T]\times\Omega$ there is $g \in S(G\circ x)$ such that $\sigma = g\cdot g^T$.*

PROOF. The result follows immediately from [8, Theorem II.3.12] applied to the function $l(u) = u \cdot u^T$ for $u \in \mathbb{R}^{n\times m}$ and set-valued mapping $\Gamma_t(\omega) = G(t, x_t(\omega))$ that is Σ-measurable on $[0,T]\times\Omega$ with $\Sigma = \{Z \in \beta_T \otimes \mathcal{F} : Z_t \in \mathcal{F}_t$ for each $t \in [0,T]\}$, where Z_t denotes the t-section of the set Z. It is easy to see that Σ-measurability of Γ is equivalent with its $\mathcal{F}_t$-nonanticipativity. Now, by ([8], Theorem II.3.12) $\sigma_t(\omega) \in l(\Gamma_t(\omega))$ for $(t,\omega) \in [0,T]\times\Omega$ implies the existence of $g \in S(G\circ x)$ such that $\sigma_t(\omega) = l(g_t(\omega))$ on $[0,T]\times\Omega$. □

4. Selection properties of some set-valued mappings

A continuous n-dimensional stochastic process $x = (x_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ can be equivalently defined as $(\mathcal{F}, \beta(C_T))$-measurable random function $x : \Omega \to C_T$, where $C_T = C([0,T], \mathbb{R}^n)$ and $\beta(C_T)$ denotes the Borel σ-algebra on C_T. Such defined continuous process determines on $\beta(C_T)$ its distribution, denoted by Px^{-1} and understood as a probability measure on $\beta(C_T)$ of the form $(Px^{-1})(A) = P(x^{-1}(A))$ for every $A \in \beta(C_T)$, where $x^{-1}(A) = \{\omega \in \Omega : x(\omega) \in A\}$. It admits the definition of convergence of sequences of continuous processes in distribution, calling also a weak convergence. Recall, a sequence $(x^r)_{r=1}^\infty$ of continuous processes defined on a sequence $\{(\Omega^r, \mathcal{F}^r, P^r)\}_{r=1}^\infty$ of probability spaces is said to be convergent in distribution to a continuous process x on $(\Omega, \mathcal{F}, P)$ if the sequence $\{P(x^r)^{-1}\}_{r=1}^\infty$ converges weakly to Px^{-1} as $r \to \infty$. It is well known (see [5]) that it is equivalent to $\lim_{r\to\infty} E_r f(x^r) = Ef(x)$ for every continuous bounded function $f : C_T \to \mathbb{R}$, where E_r and E denote expectations with respect to P^r and P, respectively. In particular, for continuous processes x and $\widetilde{x}$ on $(\Omega, \mathcal{F}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathrm{P}})$, respectively such that $Px^{-1} = \widetilde{\mathrm{P}}\widetilde{x}^{-1}$ we have $Ef(x) = \widetilde{E}f(\widetilde{x})$ for every continuous bounded function $f : C_T \to \mathbb{R}$. Denote $\mathcal{C}_n = C_b(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathcal{C}_{n\times n} = C_b(\mathbb{R}^n, \mathbb{R}^{n\times n})$ and define on $\mathcal{C}_n \times \mathbb{R}^n \times \mathbb{R}^n$ and $\mathcal{C}_{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$ set-valued mappings Φ and Ψ by settings

$$\Phi(\varphi, u)(z) = \sum_{i=1}^n \varphi_i(z)\cdot u_i, \tag{8}$$

$$\Psi(\psi, u)(z) = \sum_{i=1}^n \sum_{j=1}^n \psi_{ij}(z)\cdot v^{ij}, \tag{9}$$

for $\varphi \in \mathcal{C}_n$, $\psi \in \mathcal{C}_{n\times n}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^{n\times n}$ and $z \in \mathbb{R}^n$, where $\varphi = (\varphi_1, \dots, \varphi_n)$, $\psi = (\psi_{ij})_{n\times n}$, $u = (u^1, \dots, u^n)$ and $v = (v^{ij})_{n\times n}$. In what follows we shall restrict functional parameters φ and ψ to the set $K_k = \{z \in \mathbb{R}^n : |z| \le k\}$ for fixed $k = 1, 2, \dots$ and consider Φ and Ψ on the restricted spaces $\mathcal{C}_n^k = C_b(K_k, \mathbb{R}^n)$ and $\mathcal{C}_{n\times n}^k = C_b(K_k, \mathbb{R}^{n\times n})$ instead of $\mathcal{C}_n$ and $\mathcal{C}_{n\times n}$, respectively. We shall also consider Φ and Ψ with the restricted domain to the sets $\{\varphi(h) : h \in C_b^2(\mathbb{R}^n)\} \times \mathbb{R}^n \times \mathbb{R}^n$ and $\{\psi(h) : h \in C_b^2(\mathbb{R}^n)\} \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$, where $\varphi(h) = (h'_{x_1}, \dots, h'_{x_n})$

and $\psi(h) = (h''_{x_i x_j})_{n\times n}$ for every $h \in C_b^2(\mathbb{R}^n)$. Immediately from the above definitions we obtain:

Lemma 3. *Let $F : [0,T]\times\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ and $G : [0,T]\times\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ be measurable and bounded.*

(i) *If F and G are convex and diagonally convex valued, respectively then $\Phi(\varphi, F(t,z))(z)$ and $\Psi(\psi, D(G(t,z)))(z)$ are bounded closed and convex subsets of $\mathbb{R}$ for fixed $\varphi \in \mathcal{C}_n$, $\psi \in \mathcal{C}_{n\times n}$, $z \in \mathbb{R}^n$ and $t \in [0,T]$.*

(ii) *$\Phi(\varphi, F(\,\cdot\,,\,\cdot\,))(\,\cdot\,)$ and $(\psi, D(G(\,\cdot\,,\,\cdot\,)))(\,\cdot\,)$ are measurable on $[0,T]\times\mathbb{R}^n$ for fixed $\varphi \in \mathcal{C}_n$ and $\psi \in \mathcal{C}_{n\times n}$.*

(iii) *$\Phi(\,\cdot\,, F(t,z))(z)$ and $\Psi(\,\cdot\,, D(G(t,z)))(z)$ are continuous for fixed $(t,z) \in [0,T]\times\mathbb{R}^n$.*

(iv) *If $F(t,\,\cdot\,)$ and $G(t,\,\cdot\,)$ are continuous then $\Phi(\varphi, F(t,\,\cdot\,))(\,\cdot\,)$ and $\Psi(\psi, D(G(t,\,\cdot\,)))(\,\cdot\,)$ are continuous for fixed $\varphi \in \mathcal{C}_n$, $\psi \in \mathcal{C}_{n\times n}$ and $t \in [0,T]$.*

Lemma 4. *Assume $F : [0,T]\times\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ and $G : [0,T]\times\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ are measurable and bounded and are such that $F(t,\,\cdot\,)$ and $G(t,\,\cdot\,)$ are continuous for fixed $t \in [0,T]$. Let x and $\widetilde{x}$ be n-dimensional continuous processes on $(\Omega, \mathcal{F}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathrm{P}})$, respectively, such that $Px^{-1} = \widetilde{\mathrm{P}}\widetilde{x}^{-1}$. Then for every $l \in \mathcal{C}_1$, $\varphi \in \mathcal{C}_n$ and $\psi \in \mathcal{C}_{n\times n}$ one has*

$$E\bigg(l(x_s)\int_s^t \Phi(\varphi, F(\tau, x_\tau))(x_\tau)\,d\tau\bigg) = \widetilde{E}\bigg(l(\widetilde{x}_s)\int_s^t \Phi(\varphi, F(\tau, \widetilde{x}_\tau))(\widetilde{x}_\tau)\,d\tau\bigg)$$

and

$$\begin{aligned} E\bigg(l(x_s)\int_s^t \Psi(\psi, D(G(\tau, x_\tau)))(x_\tau)\,d\tau\bigg) \\ = \widetilde{E}\bigg(l(\widetilde{x}_s)\int_s^t \Psi(\psi, D(G(\tau, \widetilde{x}_\tau)))(\widetilde{x}_\tau)\,d\tau\bigg) \end{aligned}$$

for every $0 \le s < t \le T$.

Proof. Let $0 \le s < t \le T$, $\varphi \in \mathcal{C}_n, \psi \in \mathcal{C}_{n\times n}$ and $l \in \mathcal{C}_1$ be fixed. Define, for fixed $p \in \mathbb{R}$, mappings $U_p : C_T \to \mathbb{R}$ and $V_p : C_T \to \mathbb{R}$ by

$$\begin{aligned} U_p(x) &= \mathcal{J}\bigg(p, \int_s^t l(x_s)\Phi(\varphi, F(\tau, x_\tau))(x_\tau)\,d\tau\bigg), \\ V_p(x) &= \mathcal{J}\bigg(p, \int_s^t l(x_s)\Psi(\psi, D(G(\tau, x_\tau)))(x_\tau)\,d\tau\bigg), \end{aligned}$$

for $x \in C_T$, where $\mathcal{J}(\,\cdot\,, A)$ denotes the support function of a set $A \subset \mathbb{R}$. It can be verified (see [8]) that U_p and V_p are continuous and bounded on C_T for every $p \in \mathbb{R}$. Therefore, by the properties of x and $\widetilde{x}$, we get $EU_p(x) = \widetilde{E}U_p(\widetilde{x})$ and $EV_p(x) = \widetilde{E}V_p(\widetilde{x})$ for every $p \in \mathbb{R}$, where E and $\widetilde{E}$ denote the expectations with respect to P and $\widetilde{\mathrm{P}}$, respectively. By the properties of Aumann's integral (see [8]) we obtain

$$\begin{aligned}
EU_p(x) &= \mathcal{J}\bigg(p, E\int_s^t l(x_s)\Phi(\varphi, F(\tau, x_\tau))(x_\tau)\,d\tau\bigg),\\
\widetilde{E}U_p(\widetilde{x}) &= \mathcal{J}\bigg(p, \widetilde{E}\int_s^t l(\widetilde{x}_s)\Phi(\varphi, F(\tau, \widetilde{x}_\tau))(\widetilde{x}_\tau)\,d\tau\bigg),\\
EV_p(x) &= \mathcal{J}\bigg(p, E\int_s^t l(x_s)\Psi(\psi, D(G(\tau, x_\tau)))(x_\tau)\,d\tau\bigg),\\
\widetilde{E}V_p(\widetilde{x}) &= \mathcal{J}\bigg(p, \widetilde{E}\int_s^t l(\widetilde{x}_s)\Psi(\psi, D(G(\tau, \widetilde{x}_\tau)))(\widetilde{x}_\tau)\,d\tau\bigg),
\end{aligned}$$

for every $p \in \mathbb{R}$. Hence the convexity of Aumann's integrals implies the result.$\square$

Let us extend now the definition of the operator $\mathcal{A}^x_{fg}$ on the space $\mathcal{C}_n \times \mathcal{C}_{n\times n}$ by taking

$$\mathcal{A}^x_{fg}(\varphi,\psi)_t = \sum_{i=1}^n \varphi_i(x_t)\cdot f^i_t + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \psi_{ij}(x_t)\sigma^{ij}_t$$

for $t \in [0,T]$, $\varphi = (\varphi_i)_{1\times n} \in \mathcal{C}_n$ and $\psi = (\psi_{ij})_{n\times n} \in \mathcal{C}_{n\times n}$, where for a given continuous process $x = (x_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, P)$ we have $f \in S(F\circ x)$, $g \in S(G\circ x)$ and $\sigma = g\cdot g^T$. It is clear that for every $h \in C^2_b(\mathbb{R}^n)$ we have $(\mathcal{A}^x_{fg}h)_t = \mathcal{A}^x_{fg}(\varphi(h), \psi(h))_t$ for $t \in [0,T]$.

Lemma 5. *Assume F and G satisfy the assumptions of Lemma 4 and are convex and diagonally convex valued, respectively, and let $(x_t(B))_{0\le t\le T}$ be a weak solution to (4) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. Assume $\widetilde{x} = (\widetilde{x}_t)_{0\le t\le T}$ is a continuous n-dimensional $\widetilde{\mathcal{F}}_t$-adapted process on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \widetilde{\mathrm{P}})$ such that $Px^{-1} = \widetilde{\mathrm{P}}\widetilde{x}^{-1}$. Then for every $l \in \mathcal{C}_1$, $\varphi \in \mathcal{C}_n$ and $\psi \in \mathcal{C}_{n\times n}$ there are $\widetilde{\mathcal{F}}_t$-nonanticipative processes $(\widetilde{\alpha}_t(l,\varphi))_{0\le t\le T}$ and $(\widetilde{\beta}_t(l,\psi))_{0\le t\le T}$ such that*

(i) $\widetilde{\alpha}_t(l,\varphi) \in \Phi(\varphi, F(t, \widetilde{x}_t))(\widetilde{x}_t)$ *with* $(\widetilde{\mathrm{P}}.1)$,

(ii) $\widetilde{\beta}_t(l,\psi) \in \Psi(\psi, D(G(t, \widetilde{x}_t)))(\widetilde{x}_t)$ *with* $(\widetilde{\mathrm{P}}.1)$,

(iii) $E\bigg(l(x_s)\displaystyle\int_s^t \mathcal{A}^x_{fg}(\varphi,\psi)_\tau\,d\tau\bigg) = \widetilde{E}\bigg(l(\widetilde{x}_s)\displaystyle\int_s^t \Big[\widetilde{\alpha}_\tau(l,\varphi) + \frac{1}{2}\widetilde{\beta}_\tau(l,\psi)\Big]\,d\tau\bigg)$,

for every $0 \le s < t \le T$, where $f \in S(F\circ x)$ and $g \in S(G\circ x)$ are such that $dx_t = f_t dt + g_t dB_t$.

Proof. Let $l \in \mathcal{C}_1$, $\varphi \in \mathcal{C}_n$ and $\psi \in \mathcal{C}_{n\times n}$ be fixed. Denote

$$\alpha_t = \sum_{i=1}^n \varphi_i(x_t) f^i_t \quad\text{and}\quad \beta_t = \sum_{i=1}^n\sum_{j=1}^n \psi_{ij}(x_t)\sigma^{ij}_t \quad \text{with (P.1)}$$

for $t \in [0,T]$. We have $\alpha_t \in \Phi(\varphi, F(t,x_t))(x_t)$ and $\beta_t \in \Psi(\psi, D(G(t,x_t)))(x_t)$ a.e. on $[0,T]\times\Omega$. Hence, and by Lemma 4, it follows that

$$E\bigg(l(x_s)\int_s^t \alpha_\tau\,d\tau\bigg) \in \widetilde{E}\bigg(l(\widetilde{x}_s)\int_s^t \Phi(\varphi, F(\tau,\widetilde{x}_\tau))(\widetilde{x}_\tau))\,d\tau\bigg),$$
$$E\bigg(l(x_s)\int_s^t \beta_\tau\,d\tau\bigg) \in \widetilde{E}\bigg(l(\widetilde{x}_s)\int_s^t \Psi(\psi, D(G(\tau,\widetilde{x}_\tau)))(\widetilde{x}_\tau))\,d\tau\bigg)$$

for every $0 \le s < t \le T$.

Let $L = \sup_{x\in\mathbb{R}^n}|l(x)| < \infty$ and $\widetilde{M}_t = \sup\{|a| : a \in \Phi(\varphi, F(t,\widetilde{x}_t))(\widetilde{x}_t)\}$.

By the definition of Φ and properties of F it follows that $(\widetilde{M}_t)_{0\le t\le T}$ is bounded on $[0,T]\times\widetilde{\Omega}$. Assume $L > 0$ and let $\varepsilon > 0$. Select $\delta > 0$ such that $\sup_{0\le t\le T} E\int_t^{t+\delta}\alpha_\tau\,d\tau < \varepsilon/4L$, $\sup_{0\le t\le T}\widetilde{E}\int_t^{t+\delta}\widetilde{M}_\tau\,d\tau < \varepsilon/4L$, $\sup_{0\le t\le T}|x_t - x_{t+\delta}| \le \varepsilon$ and $\sup_{0\le t\le T}|\widetilde{x}_t - \widetilde{x}_{t+\delta}| \le \varepsilon$ with (P.1) and ($\widetilde{\mathrm{P}}$.1), respectively. Let $\tau_0 = 0$ and $\tau_k = k\delta$ for $k = 1,\dots,N$, where N is such that $(N-1)\delta < T \le N\delta$. By the definitions of set-valued integrals for every $k = 1,\dots,N$ there is $(\widetilde{\alpha}_t^k)_{0\le t\le T} \in S(\Phi(\varphi, F(\,\cdot\,,\widetilde{x})(\widetilde{x}))$ such that $E[\int_{\tau_{k-1}}^{\tau_k} l(x_{\tau_{k-1}})\alpha_\tau\,d\tau] = \widetilde{E}[\int_{\tau_{k-1}}^{\tau_k} l(\widetilde{x}_{\tau_{k-1}})\widetilde{\alpha}_\tau^k\,d\tau]$. Define $x^\varepsilon = 1_{[0,\tau_1)}x_0 + 1_{[\tau_1,\tau_2)}x_{\tau_1} + \ldots + 1_{[\tau_{N-1},T]}x_{\tau_{N-1}}$, $\widetilde{x}^\varepsilon = 1_{[0,\tau_1)}\widetilde{x}_0 + 1_{[\tau_1,\tau_2)}\widetilde{x}_{\tau_1} + \ldots + 1_{[\tau_{N-1},T]}\widetilde{x}_{\tau_{N-1}}$ and $\widetilde{\alpha}^\varepsilon = 1_{\{0\}}\widetilde{\alpha}_0^1 + 1_{(\tau,\tau_1]}\widetilde{\alpha}^1 + \ldots + 1_{(\tau_{N-1},T]}\widetilde{\alpha}^N$. For every $t \in [0,T]$ there is $k \in \{1,\dots,N\}$ such that $t \in [\tau_{k-1},\tau_k)$. Therefore, $|x_t - x_t^\varepsilon| = |x_t - x_{\tau_{k-1}}| \le \varepsilon$ and $|\widetilde{x}_t - \widetilde{x}_t^\varepsilon| = |\widetilde{x}_t - \widetilde{x}_{\tau_{k-1}}| \le \varepsilon$, which implies that $\sup_{0\le t\le T}|x_t - x_t^\varepsilon| \le \varepsilon$ and $\sup_{0\le t\le T}|\widetilde{x}_t - \widetilde{x}_t^\varepsilon| \le \varepsilon$ with (P.1) and ($\widetilde{\mathrm{P}}$.1), respectively. By the definition of $\widetilde{\alpha}^\varepsilon$ we get $\widetilde{\alpha}^\varepsilon \in S(\Phi(\varphi, F(\,\cdot\,,\widetilde{x}))(\widetilde{x}))$ because $S(\Phi(\varphi, F(\,\cdot\,,\widetilde{x}))(\widetilde{x}))$ is a decomposable set (see [8] p. 50). For every fixed $0 \le s < t \le T$ there are positive integers $1 \le r < l \le N$ such that $s \in (\tau_{r-1},\tau_r]$ and $t \in (\tau_l,\tau_{l-1}]$ or $s,t \in (\tau_{r-1},\tau_r]$ or $s,t \in (\tau_{l-1},\tau_l]$. In the last two cases we get

$$\bigg|E\bigg(\int_s^t l(x_s^\varepsilon)\alpha_\tau d\tau\bigg) - \widetilde{E}\bigg(\int_s^t l(\widetilde{x}_s^\varepsilon)\widetilde{\alpha}_\tau^\varepsilon\bigg)\bigg|$$
$$\le LE\bigg(\int_s^t|\alpha_\tau|\,d\tau\bigg) + L\widetilde{E}\bigg(\int_s^t \widetilde{M}_\tau\,d\tau\bigg) \le \varepsilon.$$

If $s \in (\tau_{r-1},\tau_r]$ and $t \in (\tau_{l-1},\tau_l]$ we obtain

$$\begin{aligned}
&\bigg|E\bigg(\int_s^t l(x_s^\varepsilon)\alpha_\tau d\tau\bigg) - \widetilde{E}\bigg(\int_s^t l(\widetilde{x}_s^\varepsilon)\widetilde{\alpha}_\tau^\varepsilon d\tau\bigg)\bigg| \\
&\quad\le \bigg|E\bigg(\int_s^{\tau_r} l(x_s^\varepsilon)\alpha_\tau\,d\tau\bigg) - \widetilde{E}\bigg(\int_s^{\tau_r} l(\widetilde{x}_s^\varepsilon)\widetilde{\alpha}_\tau^\varepsilon\,d\tau\bigg)\bigg| \\
&\qquad+ \sum_{i=r+1}^{l-1}\bigg|E\bigg(\int_{\tau_{i-1}}^{\tau_i} l(x_{\tau_{i-1}})\alpha_\tau\,d\tau\bigg) - \widetilde{E}\bigg(\int_{\tau_{i-1}}^{\tau_i} l(\widetilde{x}_{\tau_{i-1}})\widetilde{\alpha}_\tau^i\,d\tau\bigg)\bigg| \\
&\qquad+ \bigg|E\bigg(\int_{\tau_{l-1}}^t l(x_{\tau_{l-1}})\alpha_\tau\,d\tau\bigg) - \widetilde{E}\bigg(\int_{\tau_{l-1}}^t l(\widetilde{x}_{\tau_{l-1}})\widetilde{\alpha}_\tau^l\,d\tau\bigg)\bigg|
\end{aligned}$$

$$\leq LE\biggl(\int_s^{\tau_r}|\alpha_\tau|\,d\tau\biggr)+L\widetilde{E}\biggl(\int_s^{\tau r}\widetilde{M}_\tau\,d\tau\biggr)+LE\biggl(\int_{\tau_{l-1}}^t|\alpha_\tau|\,d\tau\biggr)$$
$$+L\widetilde{E}\biggl(\int_{\tau_{l-1}}^t\widetilde{M}_\tau\,d\tau\biggr)\leq\varepsilon.$$

By the weak compactness of $S(\Phi(\varphi,F(\,\cdot\,,\widetilde{x}))(\widetilde{x}))$ (see [7], [8]) in the space of all $\widetilde{\sum}$-measurable ($\mathcal{F}_t$-nonanticipative) bounded functions, for every sequence $(\varepsilon_n)_{n=1}^\infty$ with $\varepsilon_n\to0$ we can select its subsequence, say $(\varepsilon_{n_k})_{k=1}^\infty$ such that $(\widetilde{\alpha}^{\varepsilon_{n_k}})_{k=1}^\infty$ converges weakly to some $\widetilde{\alpha}\in S(\Phi(\varphi,F(\,\cdot\,,\widetilde{x})))(\widetilde{x}))$ as $k\to\infty$. Furthermore, we have $\sup_{0\leq t\leq T}|x_t-x_t^{\varepsilon_{n_k}}|\to0$ and $\sup_{0\leq t\leq T}|\widetilde{x}_t-\widetilde{x}_t^{\varepsilon_{n_k}}|\to0$ as $k\to\infty$ with (P.1) and ($\widetilde{\text{P}}$.1), respectively. Therefore, we finally get

$$E\biggl(\int_s^t l(x_s)\alpha_\tau\,d\tau\biggr)=\widetilde{E}\biggl(\int_s^t l(\widetilde{x}_s)\widetilde{\alpha}_\tau\,d\tau\biggr)$$

for $0\leq s<t\leq T$. Denoting $\widetilde{\alpha}(l,\varphi)=\widetilde{\alpha}$ we obtain condition (i). In a similar way we obtain the existence of $\widetilde{\beta}(l,\psi)=\widetilde{\beta}\in S(\Psi(\psi,D(G(\,\cdot\,,\widetilde{x})))(\widetilde{x})$ such that

$$E\biggl(\int_s^t l(x_s)\beta_\tau\,d\tau\biggr)=\widetilde{E}\biggl(\int_s^t l(\widetilde{x}_s)\widetilde{\beta}_\tau\,d\tau\biggr)$$

for every $0\leq s<t\leq T$. Thus (ii) is satisfied. By the definition of $\mathcal{A}_{fg}^x(\varphi,\psi)$ it follows that (i) and (ii), imply that also (iii) is satisfied. □

LEMMA 6. *Let assumptions of Lemma* 5 *be satisfied and let* $\tau_k=\inf\{t\in[0,T]:x_t\notin K_k\}$ *and* $\widetilde{\tau}_k=\inf\{t\in[0,T]:\widetilde{x}\notin K_k\}$, *where* $K_k=\{z\in\mathbb{R}^n:|z|\leq k\}$ *for* $k=1,2,\ldots$ *Then, for every* $l\in\mathcal{C}_1$, $\varphi\in\mathcal{C}_n$, $\psi\in\mathcal{C}_{n\times n}$ *and* $k=1,2,\ldots$ *there are* $\widetilde{\mathcal{F}}_t$*-nonanticipative processes* $(\widetilde{\alpha}_t^k(l,\varphi)_{0\leq t\leq T}$ *and* $(\widetilde{\beta}_t^k(l,\psi)_{0\leq t\leq T}$ *such that*

(i) $\widetilde{\alpha}_t^k(l,\varphi)\in\Phi(\varphi,F(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k}))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}))$ *with* ($\widetilde{\text{P}}$.1),

(ii) $\widetilde{\beta}_t^k(l,\psi)\in\Psi(\psi,D(G(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k})))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}))$ *with* ($\widetilde{\text{P}}$.1),

(iii) *for every* $0\leq s<t\leq T$

$$E(l(x_{s\wedge\tau_k})\int_{s\wedge\tau_k}^{t\wedge\tau_k}\mathcal{A}_{fg}^x(\varphi,\psi)\,d\tau)=\widetilde{E}(l(\widetilde{x}_{s\wedge\widetilde{\tau}_k})\int_{s\wedge\widetilde{\tau}_k}^{t\wedge\widetilde{\tau}_k}[\widetilde{\alpha}_\tau^k(l,\varphi)+\frac{1}{2}\widetilde{\beta}_\tau^k(l,\psi)]\,d\tau),$$

(iv) $\widetilde{\alpha}_t^k$ *and* $\widetilde{\beta}_t^k$ *are continuous on* $\mathcal{C}_1^k\times\mathcal{C}_n^k$ *and* $\mathcal{C}_1^k\times\mathcal{C}_{n\times n}^k$, *respectively for fixed* $t\in[0,T]$ *and* $k=1,2,\ldots$

PROOF. Let us observe that $\mathcal{C}_1^k,\mathcal{C}_n^k$ and $\mathcal{C}_{n\times n}^k$ are separable metric space for $k=1,2,\ldots$ Denote their countable dense subsets by $\mathcal{D}_1^k,\mathcal{D}_n^k$ and $\mathcal{D}_{n\times n}^k$, respectively and assume that $\mathcal{D}_1^k=\{l_1,l_2,\ldots\}$, $\mathcal{D}_n^k=\{\varphi_1,\varphi_2,\ldots\}$ and $\mathcal{D}_{n\times n}^k=\{\psi_1,\psi_2,\ldots\}$ Similarly as in Lemma 5 we can show that for every fixed $k=1,2,\ldots$ and $i=1,2,\ldots$ there are $\widetilde{\mathcal{F}}_t$-nonanticipative processes $(\widetilde{\alpha}_t^i)_{0\leq t\leq T}$ and $(\widetilde{\beta}_t^i)_{0\leq t\leq T}$ such that

(i') $\widetilde{\alpha}_t^i\in\Phi(\varphi_i,F(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k}))(\widetilde{x}_{t\wedge\widetilde{\tau}_k})$ with ($\widetilde{\text{P}}$.1), (ii') $\widetilde{\beta}_t^i \in \Psi(\psi_i, D(G(t, \widetilde{x}_{t\wedge\widetilde{\tau}_k})))(\widetilde{x}_{t\wedge\widetilde{\tau}_k})$ with $(\widetilde{\mathrm{P}}.1)$,

(iii') for every $0 \le s < t \le T$

$$E\biggl(l_i(x_{s\wedge\tau_k})\int_{s\wedge\tau_k}^{s\wedge\tau_k} \mathcal{A}_{fg}^x(\varphi_i,\psi_i)_\tau\, d\tau\biggr) = \widetilde{E}\biggl(l_i(\widetilde{x}_{s\wedge\widetilde{\tau}_k})\int_{s\wedge\widetilde{\tau}_k}^{s\wedge\widetilde{\tau}_k}\left[\widetilde{\alpha}_\tau^i + \frac{1}{2}\widetilde{\beta}_\tau^i\right] d\tau\biggr).$$

Define now multifunctions Φ_F^i and Ψ_G^i by setting

$$\Phi_F^i(t,\widetilde{\omega},l,\varphi) = \begin{cases} \Phi(\varphi, F(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k}))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}) & \text{for } (l,\varphi) \ne (l_i,\varphi_i), \\ \widetilde{\alpha}_t^i & \text{for } (l,\varphi) = (l_i,\varphi_i), \end{cases}$$

$$\Psi_G^i(t,\widetilde{\omega},l,\varphi) = \begin{cases} \Psi(\psi, D(G(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k})))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}) & \text{for } (l,\psi) \ne (l_i,\psi_i), \\ \widetilde{\beta}_t^i & \text{for } (l,\varphi) = (l_i,\varphi_i), \end{cases}$$

for $i = 1,2,\dots$ It is easy to see that Φ_G^i and Ψ_G^i are closed convex valued. Furthermore $\Phi_F^i(\,\cdot\,,\cdot\,,l,\varphi)$ and $\Psi_G^i(\,\cdot\,,\cdot\,,l,\psi)$ are $\widetilde{\sum}$-measuarble (i.e. $\widetilde{\mathcal{F}}_t$-non-anticipative) and $\Phi_F^i(t,\widetilde{\omega},\,\cdot\,,\cdot\,)$ and $\Psi_G^i(t,\widetilde{\omega},\,\cdot\,,\cdot\,)$ are continuous on $\mathcal{C}_1^k \times \mathcal{C}_n^k$ and $\mathcal{C}_1^k \times \mathcal{C}_{n\times n}^k$, respectively. Therefore, by [12, Theorem 2] for every $i = 1,2,\dots$ there are $\widetilde{\Sigma}\otimes\beta_n^1$ and $\widetilde{\Sigma}\otimes\beta_{n\times n}^1$-measurable, respectively mappings $\gamma^i : [0,T]\times\widetilde{\Omega}\times\mathcal{C}_1^k\times\mathcal{C}_n^k \to \mathbb{R}$ and $\lambda^i : [0,T]\times\widetilde{\Omega}\times\mathcal{C}_1^k\times\mathcal{C}_{n\times n}^k \to \mathbb{R}$, where β_n^1 and $\beta_{n\times n}^1$ denote the Borel σ-algebras on $\mathcal{C}_1^k\times\mathcal{C}_n^k$ and $\mathcal{C}_1^k\times\mathcal{C}_{n\times n}^k$, respectively, such that $\gamma^i(t,\widetilde{\omega},\,\cdot\,,\cdot\,)$ and $\lambda^i(t,\widetilde{\omega},\,\cdot\,,\cdot\,)$ are continuous on $\mathcal{C}_1^k\times\mathcal{C}_n^k$ and $\mathcal{C}_1^k\times\mathcal{C}_{n\times n}^k$, respectively, $\gamma^i(t,\omega,l,\varphi) \in \Phi(\varphi, F(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k}))(\widetilde{x}_{t\wedge\widetilde{\tau}_k})$, $\lambda^i(t,\omega,l,\psi) \in \Psi(\psi, D(G(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k})))(\widetilde{x}_{t\wedge\widetilde{\tau}_k})$, $\gamma^i(t,\omega,l_i,\varphi_i) = \widetilde{\alpha}_t^i(\omega)$ and $\lambda^i(t,\omega,l_i,\psi_i) = \widetilde{\beta}_t^i(\omega)$ for a.e. $(t,\omega)\in[0,T]\times\widetilde{\Omega}$ and $i = 1,2,\dots$

Let $(U_i^k)_{i=1}^\infty$ and $(V_i^k)_{i=1}^\infty$ be a countable open covering for $\mathcal{C}_1^k\times\mathcal{C}_n^k$ and $\mathcal{C}_1^k\times\mathcal{C}_{n\times n}^k$, respectively such that $(l_i,\varphi_i)\in U_i$ and $(l_i,\psi_i)\in V_i^k$ for $i=1,2,\dots$ Select continuous locally finite partitions of the unity $(p_i)_{i=1}^\infty$ and $(q_i)_{i=1}^\infty$ subordinate to $(U_i^k)_{i=1}^\infty$ and $(V_i^k)_{i=1}^\infty$, respectively. Define now $\widetilde{\alpha}_t^k(l,\varphi)$ and $\widetilde{\beta}_t^k(l,\varphi)$ by

$$\widetilde{\alpha}_t^k(l,\varphi)(\omega) = \sum_{i=1}^\infty p_i(l,\varphi)\cdot\gamma^i(t,\omega,l,\varphi),$$
$$\widetilde{\beta}_t^k(l,\psi)(\omega) = \sum_{i=1}^\infty q_i(l,\psi)\cdot\lambda^i(t,\omega,l,\psi),$$

for $l\in\mathcal{C}_1^k$, $\varphi\in\mathcal{C}_n^k$, $\psi\in\mathcal{C}_{n\times n}^k$ and $(t,\omega)\in[0,T]\times\widetilde{\Omega}$. It is clear that

$$\widetilde{\alpha}_t^k(l,\varphi) \in \Phi(\varphi, F(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k}))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}) \quad \text{with } (\widetilde{\mathrm{P}}.1),$$
$$\widetilde{\beta}_t^k(l,\psi) \in \Psi(\psi, D(G(t,\widetilde{x}_{t\wedge\widetilde{\tau}_k})))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}) \quad \text{with } (\widetilde{\mathrm{P}}.1),$$

for a.e. $t\in[0,T]$, because $(p_i)_{i=1}^\infty$ and $(q_i)_{i=1}^\infty$ are locally finite and multifunctions $\Phi(\varphi,F(t,z))(z)$ and $\Psi(\psi,D(G(t,z)))(z)$ are convex valued. Immediatelly from the above definitions, it follows that $\widetilde{\alpha}_t^k$ and $\widetilde{\beta}_t^k$ are continuous on $\mathcal{C}_1^k \times \mathcal{C}_n^k$ and $\mathcal{C}_1^k \times \mathcal{C}_{n\times n}^k$ with ($\widetilde{\mathrm{P}}$.1) for fixed $t \in [0,T]$. Finally, by the above definitions we get

$$\begin{aligned}\Lambda_{st}(l,\varphi,\psi) &= E\bigg(l(x_{s\wedge\tau_k})\int_{s\wedge\tau_k}^{t\wedge\tau_k} \mathcal{A}_{fg}^x(\varphi,\psi)_\tau\,d\tau\bigg)\\ &\quad - \widetilde{E}\bigg(l(\widetilde{x}_{t\wedge\widetilde{\tau}_k})\int_{s\wedge\tau_k}^{t\wedge\widetilde{\tau}_k}\Big[\widetilde{\alpha}_\tau^k(l,\varphi)+\frac{1}{2}\widetilde{\beta}_\tau^k(l,\psi)\Big]\,d\tau\bigg)\\ &= \sum_{i=1}^\infty\sum_{j=1}^\infty p_i(l,\varphi)q_j(l,\psi)\bigg[E\bigg(l(x_{s\wedge\tau_k})\int_{s\wedge\tau_k}^{t\wedge\tau_k}\mathcal{A}_{fg}^x(\varphi,\psi)_\tau\,d\tau\bigg)\\ &\quad - \widetilde{E}\bigg(l(\widetilde{x}_{t\wedge\widetilde{\tau}_k})\int_{s\wedge\widetilde{\tau}_k}^{t\wedge\widetilde{\tau}_k}\Big[\gamma^i(\tau,\cdot\,,l,\varphi)+\frac{1}{2}\lambda^j(\tau,\cdot\,,l,\psi)\Big]\,d\tau\bigg)\bigg]\end{aligned}$$

for $0 \le s < t \le T$, $l \in \mathcal{C}_1^k$, $\varphi \in \mathcal{C}_n^k$ and $\psi \in \mathcal{C}_{n\times n}^k$. Hence, by the properties of γ^i and λ^i and (iii)' it follows that $\Lambda_{st}(l_i,\varphi_i,\psi_i) = 0$ for every $i = 1,2,\ldots$ and $0 \le s < t \le T$. But Λ_{st} is continuous on $\mathcal{C}_1^k \times \mathcal{C}_n^k \times \mathcal{C}_{n\times n}^k$ and is equal to zero on $\mathcal{D}_1^k \times \mathcal{D}_n^k \times \mathcal{D}_{n\times n}^k$. Then by the density of the last set we finally have $\Lambda_{st}(l,\varphi,\psi) = 0$ for $l \in \mathcal{C}_1^k$, $\varphi \in \mathcal{C}_n^k$ and $\psi \in \mathcal{C}_{n\times n}^k$. □

LEMMA 7. *Let assumptions and notations of Lemma* 6 *be true. Then for every* $k = 1,2,\ldots$ *there are* $\widetilde{f}^k \in S(F\circ\widetilde{x}^k)$ *and* $\widetilde{g}^k \in S(G\circ\widetilde{x}^k)$ *such that for* $k = 1,2,\ldots$ *we have*

$$E\bigg(l(x_{s\wedge\tau_k})\int_{s\wedge\tau_k}^{t\wedge\tau_k}\mathcal{A}_{fg}^x(\varphi,\psi)_\tau\,d\tau\bigg) = \widetilde{E}\bigg(l(\widetilde{x}_{s\wedge\widetilde{\tau}_k})\int_{s\wedge\widetilde{\tau}_k}^{t\wedge\widetilde{\tau}_k}\mathcal{A}_{\widetilde{f}^k\widetilde{g}^k}^{\widetilde{x}}(\varphi,\psi)(\widetilde{x}_\tau)\,d\tau\bigg) \tag{10}$$

for every $0 \le s < t \le T$, $l \in \mathcal{C}_1^k$, $\varphi \in \mathcal{C}_n^k$ *and* $\psi \in \mathcal{C}_{n\times n}^k$, *where* $\widetilde{x}^k = (\widetilde{x}_{t\wedge\widetilde{\tau}_k})_{0\le t\le T}$.

PROOF. Let $(\widetilde{\alpha}_t^k(l,\varphi))_{0\le t\le T}$ and $(\widetilde{\beta}_t^k(l,\psi))_{0\le t\le T}$ be such as in Lemma 6 and let us define multifunctions K and Q by setting

$$K_t(\widetilde{\omega}) = F(t,\widetilde{x}_t)\cap\{u\in\mathbb{R}^n : \sup_{(l,\varphi)\in\mathcal{C}_1^k\times\mathcal{C}_n^k}\operatorname{dist}(\widetilde{\alpha}_t^k(l,\varphi)(\widetilde{\omega}),\Phi(\varphi,u)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega})))\}$$

and

$$Q_t(\widetilde{\omega}) = D(G(t,\widetilde{x}_t))\cap\{v\in\mathbb{R}^{n\times n} : \sup_{(l,\psi)\in\mathcal{C}_1^k\times\mathcal{C}_{n\times n}^k}\operatorname{dist}(\widetilde{\beta}_t^k(l,\psi)(\widetilde{\omega}),\Psi(\psi,v)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega})))\}$$

for $t \in [0,T]$, $\widetilde{\omega} \in \widetilde{\Omega}$. By the continuity of $\operatorname{dist}(\widetilde{\alpha}_t^k(\cdot\,,\cdot)(\widetilde{\omega}),\Phi(\cdot\,,u)\ (\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega})))$ and $\operatorname{dist}(\widetilde{\beta}_t^k(\cdot\,,\cdot)(\widetilde{\omega}),\Psi(\cdot\,,v)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega})))$ for fixed $(t,\widetilde{\omega}) \in [0,T]\times\widetilde{\Omega}$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n\times n}$ and the separability of metric spaces $\mathcal{C}_1^k \times \mathcal{C}_n^k$ and $\mathcal{C}_1^k \times \mathcal{C}_{n\times n}^k$ we have

$$K_t(\widetilde{\omega}) = F(t,\widetilde{x}_t)\cap\{u\in\mathbb{R}^n : \sup_{(l,\varphi)\in\mathcal{D}_1^k\times\mathcal{D}_n^k}\operatorname{dist}(\widetilde{\alpha}_t^k(l,\varphi)(\widetilde{\omega}),\Phi(\varphi,u)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega})))\}$$

and

$$Q_t(\widetilde{\omega}) = D(G(t,\widetilde{x}_t)) \cap \{v \in \mathbb{R}^{n\times n} : \sup_{(l,\psi)\in\mathcal{D}_1^k\times\mathcal{D}_n^k} \operatorname{dist}(\widetilde{\beta}_t^k(l,\psi)(\widetilde{\omega}), \Psi(\psi,v)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega})))\}$$

for $(t,\widetilde{\omega}) \in [0,T]\times\widetilde{\Omega}$. By the continuity of the functions mentioned above, it follows that mappings

$$[0,T]\times\widetilde{\Omega} \ni (t,\widetilde{\omega}) \to \operatorname{dist}(\widetilde{\alpha}_t^k(l,\varphi)(\widetilde{\omega}), \Phi(\varphi,u)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega}))) \in \mathbb{R},$$
$$[0,T]\times\widetilde{\Omega} \ni (t,\widetilde{\omega}) \to \operatorname{dist}(\widetilde{\beta}_t^k(l,\psi)(\widetilde{\omega}), \Psi(\psi,v)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega}))) \in \mathbb{R},$$

are $\widetilde{\sum}$-measurable, i.e. $\widetilde{\mathcal{F}}_t$-nonanticipative for fixed $l \in \mathcal{D}_1^k$, $\varphi \in \mathcal{D}_n^k$, $\psi \in \mathcal{D}_{n\times n}^k$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n\times n}$. Then, by the countability of $\mathcal{D}_1^k \times \mathcal{D}_n^k$ and $\mathcal{D}_1^k \times \mathcal{D}_{n\times n}^k$, also mappings

$$[0,T]\times\widetilde{\Omega} \ni (t,\widetilde{\omega}) \to \sup_{(l,\varphi)\in\mathcal{D}_1^k\times\mathcal{D}_n^k} \operatorname{dist}(\widetilde{\alpha}_t^k(l,\varphi)(\widetilde{\omega}), \Phi(\varphi,u)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega}))) \in \mathbb{R},$$
$$[0,T]\times\widetilde{\Omega} \ni (t,\widetilde{\omega}) \to \sup_{(l,\psi)\in\mathcal{D}_1^k\times\mathcal{D}_n^k} \operatorname{dist}(\widetilde{\beta}_t^k(l,\psi)(\widetilde{\omega}), \Psi(\psi,v)(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega}))) \in \mathbb{R},$$

are $\widetilde{\sum}$-measuarble for fixed $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^{n\times n}$. Hence, similarly as in the proof of [8, Theorem II.3.12], it follows that $(K_t)_{0\le t\le T}$ and $(Q_t)_{0\le t\le T}$ are $\widetilde{\mathcal{F}}_t$-nonanticipative. Therefore, by virtue of Kuratowski and Ryll–Nardzewski measurable selection theorem, there are $\widetilde{\mathcal{F}}_t$-nonanticipative selectors $\widetilde{f}^k = (\widetilde{f}_t^k)_{0\le t\le T}$ and $\widetilde{\sigma}^k = (\widetilde{\sigma}_t^k)_{0\le t\le T}$ for $(K_t)_{0\le t\le T}$ and $(Q_t)_{0\le t\le T}$, respectively. By the definitions of $K_t(\widetilde{\omega})$ and $Q_t(\widetilde{\omega})$ it follows that $\widetilde{f}^k \in S(F\circ\widetilde{x})$ and $\widetilde{\sigma}^k \in S(D(G\circ\widetilde{x}))$. Furthermore, we have

$$\sup_{(l,\varphi)\in\mathcal{C}_1^k\times\mathcal{C}_n^k} \operatorname{dist}(\widetilde{\alpha}_t^k(l,\varphi)(\widetilde{\omega}), \Phi(\varphi,\widetilde{f}_t^k(\widetilde{\omega}))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega}))) = 0,$$
$$\sup_{(l,\psi)\in\mathcal{C}_1^k\times\mathcal{C}_{n\times n}^k} \operatorname{dist}(\widetilde{\beta}_t^k(l,\psi)(\widetilde{\omega}), \Psi(\psi,\widetilde{\sigma}_t^k(\widetilde{\omega}))(\widetilde{x}_{t\wedge\widetilde{\tau}_k}(\widetilde{\omega}))) = 0,$$

a.e. on $[0,T]\times\widetilde{\Omega}$. By virtue of Proposition 4, $\widetilde{\sigma}^k \in S(D(G\circ\widetilde{x}))$ implies the existence of $\widetilde{g}^k \in S(G\circ\widetilde{x})$ such that $\widetilde{\sigma} = \widetilde{g}^k\cdot(\widetilde{g}^k)^T$. Hence the properties of $(\widetilde{\alpha}_t^k(l,\varphi))_{0\le t\le T}$ and $(\widetilde{\beta}_t^k(l,\psi))_{0\le t\le T}$, and the definition of $\mathcal{A}_{fg}^x(\varphi,\psi)_t$, imply that (10) is satisfied. $\square$

LEMMA 8. *Let $F : [0,T]\times\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ and $G : [0,T]\times\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n\times m})$ be bounded measurable and convex and diagonally convex valued, respectively and such that $F(t,\cdot)$ and $G(t,\cdot)$ are continuous for fixed $t \in [0,T]$. Let $(x_t(B))_{0\le t\le T}$ be a weak solution to* (4) *on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P)$. Assume $\widetilde{x} = (\widetilde{x}_t)_{0\le t\le T}$ is an*

n-dimensional continuous $\widetilde{\mathcal{F}}_t$-adapted process on $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{0\le t\le T},\widetilde{\mathrm{P}})$ such that $Px^{-1}=\widetilde{\mathrm{P}}\widetilde{x}^{-1}$. Then there are $\widetilde{f}\in S(F\circ\widetilde{x})$ and $\widetilde{g}\in S(G\circ\widetilde{x})$ such that

$$E\biggl(l(x_s)\int_s^t \mathcal{A}^x_{fg}(\varphi,\psi)_\tau\,d\tau\biggr)=\widetilde{E}\biggl(l(\widetilde{x}_s)\int_s^t \mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}}(\varphi,\psi)_\tau\,d\tau\biggr)\tag{11}$$

for every $0\le s<t\le T$, $l\in\mathcal{C}_1$, $\varphi\in\mathcal{C}_n$, $\psi\in\mathcal{C}_n$ and $\psi\in\mathcal{C}_{n\times n}$.

Proof. Let $(\tau_k)_{k=1}^\infty$ and $(\widetilde{\tau}_k)_{k=1}^\infty$ be such as in Lemma 6. We have $0<\tau_1<\tau_2,\dots,\ 0<\widetilde{\tau}_1<\widetilde{\tau}_2,\dots,\ \lim_{k\to\infty}\tau_k=T$ and $\lim_{k\to\infty}\widetilde{\tau}_k=T$ with (P.1) and ($\widetilde{\mathrm{P}}$.1), respectively. Denote by l_k, φ_k and ψ_k the restrictions of $l\in\mathcal{C}_1$, $\varphi\in\mathcal{C}_n$, and $\psi\in\mathcal{C}_{n\times n}$ to the set K_k, for $k=1,2,\dots$, respectively. We have $l_k(x_{s\wedge\tau_k})=l(x_{s\wedge\tau_k})$, $l_k(\widetilde{x}_{s\wedge\widetilde{\tau}_k})=l(\widetilde{x}_{s\wedge\widetilde{\tau}_k})$, $\int_{s\wedge\tau_k}^{t\wedge\tau_k}\mathcal{A}^x_{fg}(\varphi_k,\psi_k)_\tau\,d\tau=\int_{s\wedge\tau_k}^{t\wedge\tau_k}\mathcal{A}^x_{fg}(\varphi,\psi)_\tau\,d\tau$ and $\int_{s\wedge\widetilde{\tau}_k}^{t\wedge\widetilde{\tau}_k}\mathcal{A}^{\widetilde{x}}_{\widetilde{f}^k\widetilde{g}^k}(\varphi_k,\psi_k)_\tau\,d\tau=\int_{s\wedge\widetilde{\tau}_k}^{t\wedge\widetilde{\tau}_k}\mathcal{A}^{\widetilde{x}}_{\widetilde{f}^k\widetilde{g}^k}(\varphi,\psi)_\tau\,d\tau$ with (P.1) and ($\widetilde{\mathrm{P}}$.1), respectively, where $\widetilde{f}^k\in S(F\circ\widetilde{x})$ and $\widetilde{g}^k\in S(G\circ\widetilde{x})$ are such that (10) is satisfied. Put now $\widetilde{f}=1_{\{0\}}\widetilde{f}^1_0+1_{(0,\widetilde{\tau}_1]}\widetilde{f}^1+1_{(\widetilde{\tau}_1,\widetilde{\tau}_2]}\widetilde{f}^2+\dots$ and $\widetilde{g}=1_{\{0\}}\widetilde{g}^1_0+1_{(0,\widetilde{\tau}_1]}\widetilde{g}^1+1_{(\widetilde{\tau}_1,\widetilde{\tau}_2]}\widetilde{g}^2+\dots$ Let us observe that by the decomposability of $S(F\circ\widetilde{x})$ and $S(G\circ\widetilde{x})$ we have $\widetilde{f}\in S(F\circ\widetilde{x})$ and $\widetilde{g}\in S(G\circ\widetilde{x})$. Furthermore

$$E\biggl(l(x_{s\wedge\tau_k})\int_{s\wedge\tau_k}^{t\wedge\tau_k}\mathcal{A}^x_{fg}(\varphi,\psi)_\tau\,d\tau\biggr)=\widetilde{E}\biggl(l(\widetilde{x}_{s\wedge\widetilde{\tau}_k})\int_{s\wedge\widetilde{\tau}_k}^{t\wedge\widetilde{\tau}_k}\mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}}(\varphi,\psi)_\tau\,d\tau\biggr)$$

for $0\le s<t\le T$, $l\in\mathcal{C}_1$, $\varphi\in\mathcal{C}_n$, and $\psi\in\mathcal{C}_{n\times n}$. Hence, in the limit as $k\to\infty$, we obtain

$$E\biggl(l(x_s)\int_s^t \mathcal{A}^x_{fg}(\varphi,\psi)_\tau\,d\tau\biggr)=\widetilde{E}\biggl(l(\widetilde{x}_s)\int_s^t \mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}}(\varphi,\psi)_\tau\,d\tau\biggr)$$

for $0\le s<t\le T$, $l\in\mathcal{C}_1$, $\varphi\in\mathcal{C}_n$, and $\psi\in\mathcal{C}_{n\times n}$. $\square$

Lemma 9. *Let assumptions of Lemma 8 be satisfied. Then there are $\widetilde{f}\in S(F\circ\widetilde{x})$ and $\widetilde{g}\in S(G\circ\widetilde{x})$ such that*

$$E\biggl(l(x_s)\int_s^t (\mathcal{A}^x_{fg}h)_\tau\,d\tau\biggr)=\widetilde{E}\biggl(l(\widetilde{x}_s)\int_s^t (\mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}}h)_\tau\,d\tau\biggr)\tag{12}$$

for every $0\le s<t\le T$, $l\in\mathcal{C}_1$ and $h\in C^2_b(\mathbb{R}^n)$.

Proof. The proof follows immediately from Lemma 8. Indeed, by Lemma 8 there are $\widetilde{f}\in S(F\circ\widetilde{x})$ and $\widetilde{g}\in S(G\circ\widetilde{x})$ such that (11) is satisfied for $0\le s<t\le T$ and every $l\in\mathcal{C}_1$, $\varphi\in\mathcal{C}_n$, and $\psi\in\mathcal{C}_{n\times n}$. Then (11) is in particular, satisfied for $0\le s<t\le T$, $\varphi(h)\in\mathcal{C}_n$ and $\psi(h)\in\mathcal{C}_{n\times n}$ for every $l\in\mathcal{C}_1$ and $h\in C^2_b(\mathbb{R}^n)$. But for every $h\in C^2_h(\mathbb{R}^n)$ one has

$$\begin{aligned}\mathcal{A}^x_f(\varphi(h),\psi(h))_t&=(\mathcal{A}^x_{fg}h)_t\quad\text{with (P.1)},\\ \mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}}(\varphi(h),\psi(h))_t&=(\mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}}h)_t\quad\text{with }(\widetilde{\mathrm{P}}.1),\end{aligned}$$

for $t\in[0,T]$. Thus (12) is satisfied. $\square$

Lemma 10. *Assume that cponditions of Lemma 8 are satisfied and let G be convex valued. Then there are $\widetilde{f} \in S(F \circ \widetilde{x})$ and $\widetilde{g} \in S(G \circ \widetilde{x})$ such that $\mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}} \in \mathcal{M}^{\widetilde{x}}_{FG}(C^2_b(\mathbb{R}^n))$.*

Proof. By virtue of Theorem 2, there are $f \in S(F \circ x)$ and $g \in S(G \circ x)$ such that $\mathcal{A}^x_{fg} \in \mathcal{M}^x_{FG}(C^2_b(\mathbb{R}^n))$. Then, for every $h \in C^2_b(\mathbb{R}^n)$ a process $[(\varphi^x_h)_t]_{0\le t\le T}$, with $(\varphi^x_h)_t$ defined by (6), is a continuous square integrable local $\mathcal{F}_t$-martingale on $(\Omega, \mathcal{F}, P)$. Therefore, there exists a sequence $(r_k)^\infty_{k=1}$ of $\mathcal{F}_t$-stopping times on $(\Omega, \mathcal{F}, P)$ such that $r_{k-1} \le r_k$ for $k = 1, 2, \dots$, with $r_0 = 0$, $\lim_{k\to\infty} r_k = +\infty$ with (P.1) and such that $[(\varphi^x_h)_{t\wedge r_k}]_{0\le t\le T}$ are, for every $k = 1, 2, \dots$, continuous square integrable $\mathcal{F}_t$-martingales on $(\Omega, \mathcal{F}, P)$. Hence, in particular it follows that for every $0 \le s < t \le T$ one has $E[(\varphi^x_h)_{t\wedge r_k}|\mathcal{F}_s] = (\varphi^x_h)_{s\wedge r_k}$ with (P.1). Thus, for every $0 \le s < t \le T$ and $h \in C^2_b(\mathbb{R}^n)$ we have $E\{[(\varphi^x_h)_{t\wedge r_k}) - (\varphi^x_h)_{s\wedge r_k}]|\mathcal{F}_s\} = 0$ with (P.1). By the continuity of $l \in \mathcal{C}_1$ and $\mathcal{F}_s$-measurability of x_s, a random variable $l(x_s)$ is also $\mathcal{F}_s$-measurable. Therefore $E\{(l(x_s)[(\varphi^x_h)_{t\wedge r_k}) - (\varphi^x_h)_{s\wedge r_k}]|\mathcal{F}_s\} = 0$ with $(P.1)$ for every $0 \le s < t \le T$, $l \in \mathcal{C}_1$ and $h \in C^2_b(\mathbb{R}^n)$, which in particular implies that $E(l(x_s)[(\varphi^x_h)_{t\wedge r_k}) - (\varphi^x_h)_{s\wedge r_k}]) = 0$ for every $0 \le s < t \le T$, $l \in \mathcal{C}_1$ and $h \in C^2_b(\mathbb{R}^n)$. Hence, in the limit as $k \to \infty$, we obtain $E(l(x_s)[(\varphi^x_h)_t - (\varphi^x_h)_s]) = 0$, that can be written in the form

$$E(l(x_s)[(h(x_t) - h(x_s)]) = E\left(l(x_s)\int_s^t (\mathcal{A}^x_{fg}h)_\tau \, d\tau\right) \tag{13}$$

for every $0 \le s < t \le T$, $l \in \mathcal{C}_1$ and $h \in C^2_b(\mathbb{R}^n)$. By Lemma 9, there are $\widetilde{f} \in S(F \circ \widetilde{x})$ and $\widetilde{g} \in S(G \circ \widetilde{x})$ such that (12) is satisfied. By the continuity of $l \in \mathcal{C}_1$ and $h \in C^2_b(\mathbb{R}^n)$ and the equality $Px^{-1} = \widetilde{\mathrm{P}}\widetilde{x}^{-1}$ it follows that $E(l(x_s)[h(x_t) - h(x_s)]) = \widetilde{E}(l(\widetilde{x}_s)[h(\widetilde{x}_t) - h(\widetilde{x}_s)])$ for every $0 \le s < t \le T$. Hence, from (12) and (13) one obtains

$$\widetilde{E}(l(\widetilde{x}_s)[h(\widetilde{x}_t) - h(\widetilde{x}_s)]) = \widetilde{E}\left(l(x_s)\int_s^t (\mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}}h)_\tau \, d\tau\right) \tag{14}$$

for $0 \le s < t \le T$, $l \in \mathcal{C}_1$ and $h \in C^2_b(\mathbb{R}^n)$, i.e. $\widetilde{E}\{l(\widetilde{x}_s)[(\varphi^{\widetilde{x}}_h)_t - (\varphi^{\widetilde{x}}_h)_s]\} = 0$ for $0 \le s < t \le T$, $l \in \mathcal{C}_1$ and $h \in C^2_b(\mathbb{R}^n)$. Hence, in particular, $\widetilde{E}(l(\widetilde{x}_s) \cdot \widetilde{E}\{[(\varphi^{\widetilde{x}}_h)_t - (\varphi^{\widetilde{x}}_h)_s]|\widetilde{\mathcal{F}}_s\}) = 0$ for $0 \le s < t \le T$ every $l \in \mathcal{C}_1$ and $h \in C^2_b(\mathbb{R}^n)$. Taking in particular, $l(\widetilde{x}_s) = \widetilde{E}\{[(\varphi^{\widetilde{x}}_h)_t - (\varphi^{\widetilde{x}}_h)_s]|\widetilde{\mathcal{F}}_s\}$ with $(\widetilde{\mathrm{P}}.1)$ we get $\widetilde{E}(\widetilde{E}\{[(\varphi^{\widetilde{x}}_h)_t - (\varphi^{\widetilde{x}}_h)_s]|\widetilde{\mathcal{F}}_s\})^2 = 0$ for $0 \le s < t \le T$ and $h \in C^2_b(\mathbb{R}^n)$. Therefore, $\widetilde{E}\{[(\varphi^{\widetilde{x}}_h)_t - (\varphi^{\widetilde{x}}_h)_s]|\widetilde{\mathcal{F}}_s\} = 0$ with $(\widetilde{\mathrm{P}}.1)$ for $0 \le s < t \le T$ and $h \in C^2_b(\mathbb{R}^n)$. Then $\widetilde{E}[(\varphi^{\widetilde{x}}_h)_t|\widetilde{\mathcal{F}}_s] = (\varphi^{\widetilde{x}}_h)_s$ with $(\widetilde{\mathrm{P}}.1)$ for every $0 \le s < t \le T$ and $h \in C^2_b(\mathbb{R}^n)$. Therefore, $\mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}} \in \mathcal{M}^{\widetilde{x}}_{FG}(C^2_b(\mathbb{R}^n))$. $\square$

Lemma 11. *Assume F and G satisfy the assumptions of Lemma 8 and G is convex valued. Let $(x^r_t(B^r))_{0\le t\le T}$ be weak solutions to (4) on the space*

$(\Omega^r, \mathcal{F}^r, (\mathcal{F}_t^r)_{0\le t\le T}, P^r)$ for $r = 1,2,\ldots$ Let $\widetilde{x}^r = (\widetilde{x}_t^r)_{0\le t\le T}$ and $\widetilde{x} = (\widetilde{x}_t)_{0\le t\le T}$ be for $r = 1,2,\ldots$ continuous n-dimensional $\widetilde{\mathcal{F}}_t$-adapted processes on the space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \widetilde{\mathrm{P}})$ such that $\widetilde{\mathrm{P}}(\widetilde{x}^r)^{-1} = P^r[x^r(B^r)]^{-1}$ for $r = 1,2,\ldots$ and $\lim_{r\to\infty}\sup_{0\le t\le T}|\widetilde{x}_t^r - \widetilde{x}_t| = 0$ with $(\widetilde{\mathrm{P}}.1)$. Then there are $\widetilde{f} \in S(F\circ\widetilde{x})$ and $\widetilde{g} \in S(G\circ\widetilde{x})$ such that $\mathcal{A}_{\widetilde{f}\widetilde{g}}^{\widetilde{x}} \in \mathcal{M}_{FG}^{\widetilde{x}}(C_b^2(\mathbb{R}^n))$.

PROOF. By virtue of Lemma 10, for every $r = 1,2,\ldots$ there are $\widetilde{f}^r \in S(F\circ\widetilde{x})$ and $\widetilde{g}^r \in S(G\circ\widetilde{x})$ such that $\mathcal{A}_{\widetilde{f}^r\widetilde{g}^r}^{\widetilde{x}^r} \in \mathcal{M}_{FG}^{\widetilde{x}^r}(C_b^2(\mathbb{R}^n))$. Similarly as in the proof of Lemma 10, it follows

$$\widetilde{E}(l(\widetilde{x}_s^r)[h(\widetilde{x}_t^r) - h(\widetilde{x}_s^r)]) = \widetilde{E}\bigg(l(\widetilde{x}_s^r)\int_s^t (\mathcal{A}_{\widetilde{f}^r\widetilde{g}^r}^{\widetilde{x}^r}h)_\tau\, d\tau\bigg) \tag{15}$$

for every $r = 1,2,\ldots,$ $0 \le s < t \le T$, and $h \in C_b^2(\mathbb{R}^n)$.

By the continuity of $l \in \mathcal{C}_1$ and $h \in C_b^2(\mathbb{R}^n)$ we get $\lim_{r\to\infty}\widetilde{E}(l(\widetilde{x}_s^r)[h(\widetilde{x}_t^r) - h(\widetilde{x}_s^r)]) = \widetilde{E}(l(\widetilde{x}_s)[h(\widetilde{x}_t) - h(\widetilde{x}_s)])$, for every $0 \le s < t \le T$, $l \in \mathcal{C}_1$ and $h \in C_b^2(\mathbb{R}^n)$. By the boundedeness of F we obtain that a sequence $(\widetilde{f}_t^r)_{0\le t\le T}$, $r = 1,2,\ldots$ is uniformly integrable ([8]) and therefore it is weakly compact. Then there exists an $\widetilde{\mathcal{F}}_t$-nonanticipative process $(\widetilde{f}_t)_{0\le t\le T}$ such that for every $A \in \beta_T\otimes\widetilde{\mathcal{F}}$ one has $\int_A \widetilde{f}_\tau^r\, d\tau d\widetilde{\mathrm{P}} \to \int_A \widetilde{f}_\tau\, d\tau d\widetilde{\mathrm{P}}$ as $r\to\infty$. Therefore, for every $\varepsilon > 0$ and $A \in \beta_T\otimes\widetilde{\mathcal{F}}$ we also get

$$\begin{aligned}
&\mathrm{dist}\bigg(\int_A \widetilde{f}_\tau\, d\tau\, d\widetilde{\mathrm{P}}, \int_A F(\tau,\widetilde{x}_\tau)\, d\tau\, d\widetilde{\mathrm{P}}\bigg)\\
&\quad\le \bigg|\int_A \widetilde{f}_\tau\, d\tau\, d\widetilde{\mathrm{P}} - \int_A \widetilde{f}_\tau^r\, d\tau\, d\widetilde{\mathrm{P}}\bigg| + \mathrm{dist}\bigg(\int_A \widetilde{f}_\tau^r\, d\tau\, d\widetilde{\mathrm{P}}, \int_A F(\tau,\widetilde{x}_\tau^r)\, d\tau\, d\widetilde{\mathrm{P}}\bigg)\\
&\qquad + H\bigg(\int_A F(\tau,\widetilde{x}_\tau^r)\, d\tau\, d\widetilde{\mathrm{P}}, \int_A F(\tau,\widetilde{x}_\tau^r)\, d\tau\, d\widetilde{\mathrm{P}}\bigg) \le \varepsilon
\end{aligned}$$

for sufficiently large $r = 1,2,\ldots,$ because $\widetilde{f}_\tau^r \in F(\tau,\widetilde{x}_\tau^r)$ a.e. on $[0,T]\times\widetilde{\Omega}$, $\int_A \widetilde{f}_\tau^r\, d\tau d\widetilde{\mathrm{P}} \to \int_A \widetilde{f}_\tau\, d\tau d\widetilde{\mathrm{P}}$ and $H(\int_A F(\tau,\widetilde{x}_\tau^r)\, d\tau d\widetilde{\mathrm{P}}, \int_A F(\tau,\widetilde{x}_\tau^r)\, d\tau d\widetilde{\mathrm{P}}) \to 0$ as $r\to\infty$, where H is a generalized Haussdorff metric on $\mathrm{Cl}(\mathbb{R}^n)$. Hence it follows that $\widetilde{f} \in S(F\circ\widetilde{x})$. On the other hand, by the linearity of the mapping $\Phi(\varphi,\cdot)$, defined by (8) we obtain

$$\lim_{r\to\infty}\widetilde{E}\bigg(l(\widetilde{x}_s^r)\int_s^t \Phi(\varphi(h),\widetilde{f}_\tau^r)(\widetilde{x}_\tau^r)\, d\tau\bigg) = \widetilde{E}\bigg(l(\widetilde{x}_s)\int_s^t \Phi(\varphi(h),\widetilde{f})(\widetilde{x}_\tau)\, d\tau\bigg)$$

for every $0 \le s < t \le T$, $l \in \mathcal{C}_1$ and $h \in C_b^2(\mathbb{R}^n)$, because $l \in \mathcal{C}_1$, $h \in C_b^2(\mathbb{R}^n)$ and $\lim_{r\to\infty}\sup_{0\le t\le T}|\widetilde{x}_t^r - \widetilde{x}_t| = 0$ with $(\widetilde{\mathrm{P}}.1)$.

Similarly, by the boundedeness of $D(G(t,z))$, for $(t,z) \in [0,T]\times\mathbb{R}^n$, a sequence $(\widetilde{\sigma}_t^r)_{0\le t\le T}$, defined for each $r = 1,2,\ldots$ by setting $\widetilde{\sigma}_t^r = \widetilde{g}_t^r\cdot(\widetilde{g}_t^r)^T$ is also weakly compact, because $\widetilde{\sigma}_t^r \in D(t,\widetilde{x}_t^r)$ for a.e. $(t,\widetilde{\omega}) \in [0,T]\times\widetilde{\Omega}$ and each

$r = 1, 2, \ldots$ Then there exists an $n \times n$-dimensional $\widetilde{\mathcal{F}}_t$-nonanticipative process $\widetilde{\sigma} = (\widetilde{\sigma}_t)_{0 \le t \le T}$ such that $\sigma \in S(D(G \circ \widetilde{x})) = D(S(G \circ \widetilde{x}))$ and such that

$$\lim_{r \to \infty} \widetilde{E}\left(l(x_s^r) \int_s^t \Psi(\psi(h), \widetilde{\sigma}_\tau^r)(\widetilde{x}_\tau^r)\, d\tau \right) = \widetilde{E}\left(l(x_s) \int_s^t \Psi(\psi(h), \widetilde{\sigma}_\tau)(\widetilde{x}_\tau)\, d\tau \right)$$

for every $0 \le s < t \le T$, $l \in \mathcal{C}^1$ and $h \in C_b^2(\mathbb{R}^n)$, which similarly as in the proof of Lemma 10, implies that there is $\widetilde{g} \in S(G \circ \widetilde{x})$ such that $\widetilde{\sigma} = \widetilde{g} \cdot \widetilde{g}^T$ and $\mathcal{A}_{\widetilde{f}\widetilde{g}}^{\widetilde{x}} \in \mathcal{M}_{FG}^{\widetilde{x}}(C_b^2(\mathbb{R}^n))$. □

5. Weak compactness of solution sets

For given $F : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$, $G : [0,T] \times \mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^{n \times m})$ and a probability measure μ on $\beta(\mathbb{R}^n)$ we denote by $\mathcal{X}_\mu(F, G)$ a set of all weak solutions to (4) with an initial distribution μ. A sequence $(x^r(B^r))_{r=1}^\infty$ of $\mathcal{X}_\mu(F, G)$ is said to be convergent in distribution if there is a probability measure $\mathcal{P}$ on $\beta(C_T)$, such that $P^r(x^r(B^r))^{-1} \Rightarrow \mathcal{P}$ as $r \to \infty$, where $\Rightarrow$ denotes the weak convergence of probability measures. We prove now the main results of the paper.

Theorem 12. *Assume that F and G satisfy the assumptions of Lemma 8 and G is convex valued. Then, for every probability measure μ on $\beta(\mathbb{R}^n)$, the set $\mathcal{X}_\mu(F, G)$ is nonempty and sequentially weakly closed with respect to the convergence in distribution.*

Proof. By virtue of ([12], Theorem 2) there are measurable mappings $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ such that $f(t, \cdot)$ and $g(t, \cdot)$ are continuous for fixed $t \in [0,T]$ and such that $f(t,x) \in F(t,x)$ and $g(t,x) \in G(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}^n$. Now, immediately from [4, Theorem IV.2.2] it follows that a stochastic differential equation $dx_t = f(t, x_t)dt + g(t, x_t)\, dB_t$ has at least one weak solution $(x_t(B))_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ with an initial distribution μ. By the definitions of mappings f and g it easy follows that $(x_t(B))_{0 \le t \le T} \in \mathcal{X}_\mu(F, G)$. Thus $\mathcal{X}_\mu(F, G) \neq \emptyset$.

Assume $\{(x_t^r(B^r))_{0 \le t \le T}\}_{r=1}^\infty$ is a sequence in $\mathcal{X}_\mu(F, G)$ convergent in distributions. Then there is a probability measure $\mathcal{P}$ on $\beta(C_T)$ such that $P^r[(x^r(B^r)]^{-1} \Rightarrow \mathcal{P}$ as $r \to \infty$, where $(\Omega^r, \mathcal{F}^r, (\mathcal{F}_t^r)_{0 \le t \le T}, P^r)$ is such that $x^r(B_r)$ satisfies (4) on this space. We have also $P[x_0^r(B^r)]^{-1} = \mu$ for every $r = 1, 2, \ldots$

By virtue of [4, Theorem I.2.7] there is $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0 \le t \le T}, \widetilde{\mathrm{P}})$ and random functions $\widetilde{x}^r : \widetilde{\Omega} \to C_T$ and $\widetilde{x} : \widetilde{\Omega} \to C_T$; each $r = 1, 2, \ldots$ such that $P^r[x^r(B^r)]^{-1} = \widetilde{\mathrm{P}}(\widetilde{x}^r)^{-1}$ for $r = 1, 2, \ldots$, $\widetilde{\mathrm{P}}\widetilde{x}^{-1} = \mathcal{P}$ and $\lim_{r \to \infty} \sup_{0 \le t \le T} |\widetilde{x}_t^r - \widetilde{x}_t| = 0$ with $(\widetilde{\mathrm{P}}.1)$. Then, by Lemma 11 there are $\widetilde{f} \in S(F \circ \widetilde{x})$ and $\widetilde{g} \in S(G \circ \widetilde{x})$ such that $\mathcal{A}_{\widetilde{f}\widetilde{g}}^{\widetilde{x}} \in \mathcal{A}_{FG}^{\widetilde{x}}(C_b^2(\mathbb{R}^n))$. Therefore, by Theorem 2, there exists $\widehat{\mathcal{F}}_t$-Brownian motion $(\widehat{B}_t)_{0 \le t \le T}$ on an extension $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{0 \le t \le T}, \widehat{P})$ of $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0 \le t \le T}, \widetilde{\mathrm{P}})$ and such that $(\widehat{x}_t(\widehat{B}))_{0 \le t \le T}$ is a weak solution to (4) on $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{0 \le t \le T}, \widehat{P})$, where $\widehat{x}_t(\widehat{B})(\widehat{\omega}) = \widetilde{x}_t(\pi(\widehat{\omega}))$ for every $\widehat{\omega} \in \widehat{\Omega}$. It is easy to see that $\widehat{P}[\widehat{x}_0(\widehat{B})] = \mu$. On the other hand we have $P^r[x^r(B^r)]^{-1} = \widetilde{\mathrm{P}}(\widetilde{x}^r)^{-1}$ and $\widetilde{x}^r_t \to \widetilde{x}_t$ uniformly with respect to $t \in [0,T]$ with $(\widetilde{\mathrm{P}}.1)$. Then $\widetilde{\mathrm{P}}(\widetilde{x}^r)^{-1} \Rightarrow \widetilde{\mathrm{P}}\widetilde{x}^{-1}$ as $r \to \infty$ and $\widetilde{\mathrm{P}}(\widetilde{x}^r)^{-1} = P^r[x^r(B^r)]^{-1}$; each $r = 1, 2, \ldots$ Therefore, $P^r[x^r(B^r)]^{-1} \Rightarrow \widetilde{\mathrm{P}}\widetilde{x}^{-1}$ as $r \to \infty$. By the definition of the extension $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{0\le t\le T}, \widehat{P})$ of $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \widetilde{\mathrm{P}})$ it follows that $\widehat{P}[\widehat{x}(\widehat{B})]^{-1}(A) = \widehat{P}[\widehat{x}(\widehat{B})^{-1}(A)] = \widehat{P}[(\pi^{-1} \circ \widetilde{x}^{-1})(A)] = (\widehat{P} \circ \pi^{-1})(\widetilde{x}^{-1}(A)) = \widetilde{\mathrm{P}}(\widetilde{x}^{-1}(A)) = (\widetilde{\mathrm{P}}\widetilde{x}^{-1}(A)$ for every $A \in \beta(C_T)$. Therefore, $\widehat{P}[\widehat{x}(\widehat{B})]^{-1} = \widetilde{\mathrm{P}}\widetilde{x}^{-1}$ and $P^r[x^r(B^r)]^{-1} \Rightarrow \widehat{P}[\widehat{x}(\widehat{B})]^{-1}$ as $r \to \infty$, which completes the proof. □

Remark 1. For the nonemptiness of $\mathcal{X}_\mu(F,G)$ it is enough only to assume that F and G are measurable bounded closed and convex valued and such that $F(t,\cdot)$ and $G(t,\cdot)$ are lower semicontinuous for fixed $t \in [0,T]$.

Denote now by $\mathcal{X}_\mu(F,G,\Omega)$ set of all weak solutions to (4) on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ with an initial distribution μ. We have of course $\mathcal{X}_\mu(F,G,\Omega) \subset \mathcal{X}_\mu(F,G)$.

Theorem 13. *If F and G satisfy the assumptions of Theorem 12 then for every filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ and every probability measure μ on $\beta(\mathbb{R}^n)$ the set $\mathcal{X}_\mu(F,G,\Omega)$ is nonempty and relatively sequentially weakly compact with respect to the convergence in distribution.*

Proof. Similarly as in the proof of Theorem 12 we obtain $\mathcal{X}_\mu(F,G,\Omega) \neq \emptyset$ for fixed $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ and μ. Let $\{(x^r_t(B^r))_{0\le t\le T}\}_{r=1}^\infty$ be a sequence of weak solutions to (4) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ with an initial distribution μ. By virtue of Lemma 1 for every $r = 1, 2, \ldots$ there are $f^r \in S(F \circ x^r)$ and $g^r \in S(G \circ x^r)$ such that $dx^r_t = f^r_t\,dt + g^r_t\,dB^r_t$ for $t \in [0,1]$. Now, similarly as in [4, Theorem IV.2.2], for every $k = 1, 2, \ldots$, there is a number C_k such that

$$\sup_{r\ge1}\sup_{0\le t\le T} E\{|x^r_t|^{2k}\} \le C_k \quad \text{and} \quad \sup_{r\ge1}\sup_{0\le t\le T} E\{|x^r_t|^{2k}\} \le C_k|t-s|^k$$

for every $t, s \in [0,T]$. Therefore, by [4, Theorems I.4.2 and I.4.3] there exist an increasing sequence $(r_l)_{l=1}^\infty$ of positive integers, a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \widetilde{\mathrm{P}})$ and continuous n-dimensional processes $(\widetilde{x}^{n_l}_t)_{0\le t\le T}$ and $(\widetilde{x}_t)_{0\le t\le T}$, each $l = 1, 2, \ldots$ such that $P(x^{n_l})^{-1} = \widetilde{\mathrm{P}}(\widetilde{x}^{n_l})^{-1}$ for $l = 1, 2, \ldots$ and $\lim_{l\to\infty}\sup_{0\le t\le T}|\widetilde{x}^{n_l} - \widetilde{x}_t| = 0$ with $(\widetilde{\mathrm{P}}.1)$. Hence, by virtue of Lemma 11 there are $\widetilde{f} \in S(F \circ \widetilde{x})$ and $\widetilde{g} \in S(G \circ \widetilde{x})$ such that $\mathcal{A}^{\widetilde{x}}_{\widetilde{f}\widetilde{g}} \in \mathcal{M}^{\widetilde{x}}_{FG}(C^2_b(\mathbb{R}^n))$, which by Theorem 2 implies the existence of $\widehat{\mathcal{F}}_t$-Brownian motion on an extension of $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{0\le t\le T}, \widehat{P})$ of $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \widetilde{\mathrm{P}})$ such that $(\widehat{x}_t(\widehat{B}))_{0\le t\le T}$ is a weak solution to (4) on $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{0\le t\le T}, \widehat{P})$, where $\widehat{x}_t(\widehat{B})(\widehat{\omega}) = \widetilde{x}_t(\pi(\widehat{\omega})$ for $\widehat{\omega} \in \widehat{\Omega}$. Similarly as in the proof of Theorem 12 we can show that $P[x^{n_l}(B^{n_l})]^{-1} \Rightarrow \widehat{P}[\widehat{x}(\widehat{B})]$ as $l \to \infty$. Then the set $\mathrm{cl}^d_w(\mathcal{X}_\mu(F,G,\Omega))$ is weakly compact with respect to the convergence in distribution, where cl_w^d denotes the weak closure with respect to the convergence in distribution. □

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