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VARIATIONAL AND BOUNDARY VALUE PROBLEMS WITH PERTURBATIONS

Stanisław Walczak

Abstract. In the paper an optimization problem with parameters is considered. Some sufficient conditions under which the solutions of the problem continuosly depend on parameters (in the weak or the strong topology of a Banach space) are proved. Moreover, some applications to the eigenvalue and boundary value problems for differential operators are given.

Introduction

Consider an optimization problem with parameters

$$\min_{x\in\mathbb{X}} F(x,u) \quad \text{subject to } x\in G(u). \tag{1.1}$$

Denote by $V=V(u)$ a set of solutions of problem (1.1). We shall prove some sufficient conditions under which problem (1.1) possesses at least one solution i.e. $V(u)$ is a nonempty set and the set-valued mapping $V=V(u)$ is continuous or semicontinuous. Throughout this paper, $\mathbb{X}$ is a reflexive Banach space, while the perturbation u belongs to some metric space. In particular cases when we consider boundary value and eigenvalue problems $\mathbb{X}$ is a Sobolev space.

The problem of the existence of a solution for (1.1) and its dependence on a variable parameter u was considered in many papers and monographs and is

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usually referred to as the sensitivity analysis of systems. One of the first results in this direction was published in [5]–[7].

In most of the papers devoted to the finite dimensional optimization problems with perturbations one considers the question of the continuous (or Lipschitz continuous) dependence and the directional differentiability of optimal solutions with respect to parameters (cf. [2], [8], [15], [18], [19]). Recently in the journal SIAM Review there appeared a paper *"Optimization problems with perturbations: A guided tour"* (cf. [3]). In this work, one can find wide information on recent results and the development of the stability and sensitivity analysis of optimization problems. In this analysis two assumptions are crucial. Namely:

(1.2) the linear independence of gradients of active constraints,

(1.3) the strong second-order sufficient optimality condition.

In our paper we give a direct method of the stability analysis for problem (1.1) which allows us to omit strong optimality condition (1.3) and, in many cases, condition (1.2). We prove some sufficient conditions under which the set of optimal solutions of problem (1.1) is a continuous or semicontinuous function of parameters with respect to the weak or strong topology in $\mathbb{X}$ and the metric topology in the set of parameters. In the concluding part of the paper we consider some boundary value and eigenvalue problems for differential operators with variable parameters defined in the Sobolev spaces.

The question of the existence of a solution for boundary value problem was considered in several monographs and papers, (cf. [14], [13], [20] and references therein). The literature devoted to the question of the continuous dependence on parameters of the solutions of nonlinear boundary value problems is not extensive. Some papers based on direct methods and deal with the scalar equations only were published on seventh's years (cf. [12], [17]). Multi-dimensional systems with variable parameters and boundary data of the Dirichlet type were investigating in [4], [10], [21], [22]. In this papers based on variational methods some sufficient conditions under which solutions of the Dirichlet problem continuously depend on variable parameters and boundary data are proved.

2. Optimization problems with perturbations

Let $\mathbb{X}$ be a reflexive Banach space and $\mathbb{U}$ a metric space with metric $\rho = \rho_u(u_1, u_2)$. The space $\mathbb{U}$ will be referred to as the set of parameters, $\mathbb{X}$ – the space of states or arguments. On $\mathbb{X}$ and $\mathbb{U}$ there are defined two functions:

$$F : \mathbb{X} \times \mathbb{U} \to \mathbb{R} \quad \text{and} \quad G : \mathbb{U} \to 2^{\mathbb{X}}.$$

For a fixed $u \in \mathbb{U}$, consider an optimization problem

$$\min F(x, u) \text{ subject to } x \in G(u). \tag{2.1}$$

We shall impose the following conditions on F and G:

(2.2) $F(x,\cdot)$ and $F(\cdot,u)$ are continuous in the metric and the norm topology, respectively for any $x\in\mathbb{X}$ and $u\in\mathbb{U}$,

(2.3) $F(\cdot,u)$ is lower semicontinuous in the weak topology of $\mathbb{X}$ for any $u\in\mathbb{U}$,

(2.4) $G(u)\subset\mathbb{X}$ is a nonempty and weakly closed set for any $u\in\mathbb{U}$,

(2.5) there exist $\overline{x}\in\mathbb{X}$, $\overline{r}>0$ and $\overline{s}>0$, such that $\mathcal{L}(\overline{s},u)\neq\phi$ and $\mathcal{L}(\overline{s},u)\subset B(\overline{x},\overline{r})$, for all $u\in\mathbb{U}$, where $\mathcal{L}(\overline{s},u)=\{x\in G(u):F(x,u)\le\overline{s}\}$ is the Lebesgue set for the functional $F(\cdot,u)$, $B(\overline{x},\overline{r})=\{v\in X:\|v-\overline{x}\|\le\overline{r}\}$,

(2.6) $F(\cdot,u)$ tends to $F(\cdot,u_0)$ uniformly on $B(\overline{x},\overline{r})$ if u tends to u_0 in $\mathbb{U}$,

(2.7) the set-valued mapping $G(u)$ is locally continuous i.e. $G(u)\cap B(\overline{x},\overline{r})\to G(u_0)\cap B(\overline{x},\overline{r})$ with respect to the Hausdorff distance d_H provided $u\to u_0$ in $\mathbb{U}$,where $\overline{x}$ and $\overline{r}$ are the same as in (2.5) and (2.6).

Remark 2.1. Let us recall that the Hausdroff distance is defined by the formula

$$d_H(A(u),A(u_0))=\max\{\sup_{x\in A(u)}\operatorname{dist}(x,A(u_0)),\sup_{x\in A(u_0)}\operatorname{dist}(x,A(u))\},\tag{2.8}$$

where $\operatorname{dist}(x,A(u))=\inf_{v\in A(u)}\|x-v\|$.

Let $\{u_k\}\subset\mathbb{U}$ be a fixed sequence and

(2.9) $V_k=V(u_k)=\{v\in G(u_k)\subset\mathbb{X}:F(v,u_k)=\min F(x,u_k),x\in G(u_k)\}$ for $k=0,1,2,\dots$

Remark 2.2. The set $\widetilde{V}$ of all cluster points of sequences $\{x_k\}$, $x_k\in V_k$ is denoted by $\widetilde{V}=\operatorname{Lim\,sup}V_k$ and referred to as the upper limit in the sense of Poinleve–Kuratowski cf. [16]. In the case when cluster points are understood in the sense of the weak topology or the strong topology of $\mathbb{X}$, we shall write $(w)\operatorname{Lim\,sup}$ or $(s)\operatorname{Lim\,sup}$, respectively.

Now we prove

Theorem 2.1. *If the functional F and the multifunction G satisfy assumptions* (2.2)–(2.7) *and u_k tends to u_0 in $\mathbb{U}$, then*

1. *the set $V(u_k)$ is not empty for any $u_k\in\mathbb{U}$,*
2. *$V(u_k)\subset B(\overline{x},\overline{r})$ for any $u_k\in\mathbb{U}$ and $\overline{x},\overline{r}$ as in assumption* (2.5), *$k=0,1,\dots,$*
3. *$(w)\operatorname{Lim\,sup}V(u_k)\neq\phi$,*
4. *$(w)\operatorname{Lim\,sup}V(u_k)\subset V(u_0)$.*

Moreover, if $F(\cdot,u)$ is strictly convex and $G(u)$ is convex for $u\in\mathbb{U}$, then $V(u_k)$ is a singleton x_k, and $x_k\to x_0$ weakly in $\mathbb{X}$.

Conditions (3) *and* (4) *mean that the set-valued mapping $V:\mathbb{U}\to 2^{\mathbb{X}}$ is upper semicontinuous with respect to the metric topology in the set of parameters and the weak topology in $\mathbb{X}$.*

PROOF. By assumptions (2.3)–(2.5), the functional $F(\,\cdot\,,u)$ is weakly lower semicontinuous, while the set $\mathcal{L}(\overline{s},u)$ is nonempty and weakly compact. Thus $V(u_k)\neq\phi$ and $V(u_k)\subset B(\overline{x},\overline{r})$ for any $u_k\in\mathbb{U}$, i.e. we have proved assertions (1) and (2). Denote by μ_k the optimal value for $u=u_k$ i.e. $\mu_k=\min\{F(x,u_k):x\in G(u_k)\}=\min\{F(x,u_k):x\in\mathcal{L}(\overline{s},u_k)\subset B(\overline{x},\overline{r})\}$, $k=0,1,\dots$ Since $V(u_k)\neq\phi$, there exists $x_k\in V(u_k)$ such that $\mu_k=F(x_k,u_k)$ for $k=0,1\dots$ We have

$$\begin{aligned}\mu_k-\mu_0&=F(x_k,u_k)-F(x_0,u_0)\\&=[F(x_k,u_k)-F(x_k,u_0)]+[F(x,u_0)-F(y_0^k,u_0)]\\&\quad+[F(y_0^k,u_0)-F(x_0,u_k)]+[F(x_0,u_k)-F(x_0,u_0)],\end{aligned}$$

where $y_0^k\in G(u_0)\cap B(\overline{x},\overline{r})$ and is arbitrarily close to x_k. Such a point y_0^k does exist by (2.7). More precisely, for any $\varepsilon>0$ there exists $K>0$ such that, for all $x_k\in G(u_k)$ with $k>K$, there exists $y_0^k\in G(u_0)\cap B(\overline{x},\overline{r})$ such that $\|x_k-y_0^k\|<\varepsilon$. It is easy to notice that the terms in the first and third brackets tend to null by (2.6), while the second and fourth one by (2.2). Thus we have proved that

$$\mu_k\to\mu_0\quad\text{as } k\to\infty. \tag{2.10}$$

Let $\{x_k\}$ be any sequence of minimizers i.e. $x_k\in V(u_k)$, $k=1,2,\dots$. We have just noticed that $V(u_k)\subset B(\overline{x},\overline{r}))$. This implies that $\{x_k\}$ is weakly compact and without lost of generality, we can assume that x_k tends to some $\widetilde{x}\in\mathbb{X}$ in the weak topology of the space $\mathbb{X}$. By (2.7) we have $\widetilde{x}\in G(u_0)$. This means that $(w)\mathrm{Lim\,sup}\,V_k=\widetilde{V}\neq\phi$ and condition (3) is fulfilled. Suppose that $\widetilde{x}$ does not belong to $V(u_0)$. The set $V(u_0)$ is not empty, thus there exists $x_0\in V(u_0)$ and $F(\widetilde{x},u_0)-F(x_0,u_0)=c>0$.

We have

$$\begin{aligned}\mu_k-\mu_0&=F(x_k,u_k)-F(x_0,u_0)\\&=[F(x_k,u_k)-F(x_k,u_0)]+[F(x_k,u_0)-F(\widetilde{x},u_0)]+c.\end{aligned}$$

It is easy to see that $\lim(\mu_k-\mu_0)=0$ by (2.10), $\lim[F(x_k,u_k)-F(x_k,u_0)]=0$ by (2.6) and $\liminf[F(x_k,u_0)-F(\widetilde{x},u_0)]\geq0$ by (2.3). We have thus got a contradiction. This means that $\widetilde{x}\in V(u_0)$ and $(w)\mathrm{Lim\,sup}\,V_k\subset V_0$. In this way we have completed the proof of Theorem 2.1. □

In the next theorem we shall consider a special case of problem (2.1), namely,

$$\min F(x,u)\text{ subject to } g_i(x,u)\leq0,\ i=1,\dots,p,\text{ where } g_i:\mathbb{X}\times\mathbb{U}\to\mathbb{R}. \tag{2.11}$$

If we put

$$\overline{G}(u)=\{x\in X:g_i(x,u)\leq0,\ i=1,\dots,p\} \tag{2.12}$$

then (2.11) reduces to problem (2.1) with $G = \overline{G}$. In this case we assume that

(2.13) $g_i(x,u)$ tends to $g_i(x,u_0)$ uniformly on $B(\overline{x},\overline{r})$ as $u \to u_0$ in $\mathbb{U}$, where $B(\overline{x},\overline{r})$ is the ball defined in (2.5) with $G = \overline{G}$. Moreover, $g_i(\,\cdot\,,u)$ are weakly lower semicontinuous on $B(\overline{x},\overline{r})$ and for any $\widetilde{u} \in \mathbb{U}$ there exists $\widetilde{x} \in \mathbb{X}$, such that $g_i(\widetilde{x},\widetilde{u}) \le 0$ for $i = 1,\dots,p$.

(2.14) The functional $F(\,\cdot\,,\,\cdot\,)$ and the multifunction $\overline{G}(\,\cdot\,)$ satisfy assumption (2.2), (2.3), (2.5) and (2.6) with $G = \overline{G}$. (Let us notice that (2.4) immediately follows from (2.13)).

(2.15) For $\alpha > 0$ and $(x_0,u_0) \in B^0(\overline{x},\overline{r}) \times \mathbb{U}$ such that $g_i(x_0,u_0) \le 0$, $i = 1,\dots,p$ there exists $\widetilde{x} \in B^0(\overline{x},\overline{r})$ such that $\|\widetilde{x}-x_0\| < \alpha$ and $g_i(\widetilde{x},u_0) \le 0$, for $i = 1,\dots,p$, where $B^0 = \operatorname{Int} B$ (the local Slater condition). In the case when $g_i(\,\cdot\,,\,\cdot\,)$ are convex it is enough to assume that, for any $u \in \mathbb{U}$, there exists $\widetilde{x} \in B^0(\overline{x},\overline{r})$ such that $g_i(\widetilde{x},u_0) \le 0$, for $i = 1,\dots,p$, (the local Slatler condition). (It is easy to show that the global Slater condition implies the local one.) For a given $u_k \in \mathbb{U}$, denote by $\overline{V}_k = \overline{V}(u_k)$ the set of solutions of problem (2.11) described by formula (2.9) with $G = \overline{G}$, i.e.

$$(2.16)\quad \overline{V}(u_k) == \{v \in \mathbb{X} : F(v,u_k) = \min F(x,u_k) \text{ subject to } g_i(x,u_k) \le 0,\ i = 1,\dots,p\}.$$

Theorem 2.2. *If the functionals F and g_i $i = 1,\dots,p$ satisfy assumptions* (2.13)–(2.15) *and $u_k \to u_0$ in $\mathbb{U}$, then the sets $\overline{V}_k = \overline{V}(u_k)$* (*cf.* (2.16)) *of solutions of problem* (2.11) *satisfy assertions* (1)–(4) *of Theorem* 2.1 *with $V = \overline{V}$.*

Proof. Let $u_k \to u_0$ in $\mathbb{U}$ and $x_0 \in \overline{G}(u_0) \cap B^0(\overline{x},\overline{r})$ i.e. $g_i(x_0,u_0) \le 0$ for $i = 1,\dots,p$, and $x_0 \in B^0(\overline{x},\overline{r})$. By the local Slater condition (2.15) for any $\alpha > 0$ there exists $\widetilde{x} \in B^0(\overline{x},\overline{r})$ with $\|x_0 - \widetilde{x}\| < \alpha$ and $g_i(\widetilde{x},u_0) < -a < 0$ for some $a > 0$ and $i = 1,\dots,p$. Assumption (2.13) implies that there exists $K > 0$ which depends only on α, such that $g_i(\widetilde{x},u_k) < -a/2 < 0$ for $k > K$ and $i = 1,\dots,p$. This means that $\widetilde{x} \in \overline{G}(u_k) \cap B(\overline{x},\overline{r})$ and $\operatorname{dist}(x_0,\overline{G}(u_k) \cap B(\overline{x},\overline{r})) < \alpha$ for $k > K$ and for any $x_0 \in \overline{G}(u_0) \cap B(\overline{x},\overline{r})$. Similarly we can show that $\operatorname{dist}(x_k,\overline{G}(u_0) \cap B(\overline{x},\overline{r})) < \alpha$ for $k > K$ and for each $x_k \in \overline{G}(u_k) \cap B(\overline{x},\overline{r})$. Passing with α to null, we see that $\overline{G}(u_k) \cap B(\overline{x},\overline{r})$ tends to $\overline{G}(u_0) \cap B(\overline{x},\overline{r})$ with respect to the Hausdorff distance (2.8). In this way, we have shown that the functional F and the multifunction $\overline{G}$ satisfy assumptions (2.2)–(2.7) with $G = \overline{G}$. Applying Theorem 2.1, we get the assertions of this one. □

Next, let us consider a finite-dimensional optimization problem where, besides the inequality constraints, there can also appear an equality one i.e.

(2.17) $\min F(x,u)$ subject to $g_i(x,u) = 0$ for $i = 1,\dots,q$ and $g_i(x,u) \le 0$ for $i = q+1,\dots,p$. By $I^a(x,u)$ we shall denote the set of active indices at the point (x,u), i.e. $k \in I^a(x,u)$ if and only if $1 \le k \le p$ and $g_k(x,u) = 0$.

Denote by

$$G^0(u) = \{x \in X : g_i(x,u) = 0 \text{ for } i = 1,\dots,q \text{ and } g_i(x,u) \le 0 \text{ for } i = q+1,\dots,p\} \tag{2.18}$$

and

$$V_k^0 = V^0(u_k) = \{v \in X : F(v,u_k) = \min_{x \in G^0(u_k)} F(x,u_k)\}. \tag{2.19}$$

We assume that $G^0(u)$ is a nonempty set for $u \in \mathbb{U}$.

The following theorem holds:

THEOREM 2.3. *If $\mathbb{X} = R^n$, U is an open subset of a Banach space,*

(1) *the functionals F and g_i, $i = 1,\dots,p$, satisfy conditions* (2.13), (2.3), (2.5) *and* (2.6),

(2) *the functionals g_i are C^1 class on some open overset of $B(\overline{x},\overline{r}) \times \mathbb{U}$, $i = 1,\dots,p$,*

(3) *the gradients $\nabla_x g_i(x,u) = \partial g_i(x,u)/\partial x$ are linearly independent for $i \in I^a(x,u)$ and $(x,u) \in B(\overline{x},\overline{r}) \times \mathbb{U}$,*

(4) *the sequence of parameters $u_k \to u_0$ in $\mathbb{U}$,*

then the sequence $V_k^0 = V^0(u_k)$ (cf. (2.19) *of optimal solutions of problem* (2.17) *satisfies conditions* (1)–(4) *of Theorem* 2.1 *with $V_k = V_k^0$. (With respect to the finite dimension of $\mathbb{X}$, the weak and the strong topologies coinside in this case.)*

PROOF. Similarly as in the previous theorem it is enough to show that $G^0(u_k)$ tends to $G^0(u_0)$ locally with respect to the Hausdorff distance. Let $x_0 \in G^0(u_0) \cap B(\overline{x},\overline{r})$. This means that $g_i(x_0,u_0) = 0$ for $i \in I^a(x_0,u_0)$ and $g_i(x_0,u_0) < 0$ for $i \notin I^a(x_0,u_0)$. Since $g_i(x,u_k) \rightrightarrows g_i(x,u_0)$ on $B(\overline{x},\overline{r})$ (cf. (1) and (2,13)), one can find a neighbourhood $N(x_0,\alpha) = \{x : \|x - x_0\| < \alpha\}$, $\alpha > 0$, such that $g_i(x,u_k) < 0$ for $i \notin I^a(x_0,u_0)$ and $k > K(\alpha)$. Consider a system of equations

$$g_i(x,u) = 0, \quad i \in I^a(x_0,u_0). \tag{2.20}$$

Taking into account assumptions (2) and (3), it is easy to see that system (2.20) satisfies the conditions of the Graves implicit function theorem (cf. [9]). Thus for any $\alpha > 0$ and u_k, $k > K(\alpha)$, there exists at least one x_k, $\|x_k - x_0\| < \alpha$, such that $g_i(x_k,u_k) = 0$ for $i \in I^a(x_0,u_0)$. This means that $x_k \in G^0(u_k)$ and $\|x_k - x_0\| < \alpha$. Since our space $\mathbb{X}$ is finite-dimensional, the $K(\alpha)$ can be chosen independently of $x_0 \in G^0(u_0)$. In this way we have proved that $\operatorname{dist}(x_0, G^0(u_k) \cap B(\overline{x},\overline{r})) < \alpha$ for all $x_0 \in G^0(u_0)$.

Similarly one can show that $\operatorname{dist}(x_k, G^0(u_0) \cap B(\overline{x}, \overline{r})) < \alpha$ for $k > K(\alpha)$. Passing with $\alpha \to 0$ we see that $G^0(u_k) \cap B(\overline{x}, \overline{r}) \to G^0(u_0) \cap B(\overline{x}, \overline{r})$ with respect to the Hausdorff distance. Thus, applying Theorem 2.1, we have completed our proof. □

In Theorems 2.1 and 2.2 we have proved that the sequence of optimal solutions $V(u_k)$ tends to $V(u_0)$ weakly in $\mathbb{X}$, provided that u_k tends to u_0 in the metric space of parameters. In Theorem 2.3, with respect to the finite dimension of $\mathbb{X}$, the weak convergence implies the strong one.

For a special class of functionals defined on a Hilbert space, we are able to show that the convergence of u_k to u_0 in $\mathbb{U}$ implies the strong convergence of optimal solutions. We shall consider functionals of the form

$$F(x,u) = \frac{1}{2}\|x\|^2 + f(x,u) \tag{2.21}$$

and an optimization problem

(2.22) $\min F(x,u)$ subject to $g_i(x,u) \leq 0$, $i = 1, \dots, p$, where $x \in \mathbb{X}$, $u \in \mathbb{U}$, $\mathbb{X}$ is a Hilbert space and F is given by (2.21).

Denote by

$$G^s(u) = \{x \in \mathbb{X} : g_i(x,u) \leq 0,\ i = 1, \dots, p\} \tag{2.23}$$

and

$$V_k^s = V^s(u_k) = \{v \in \mathbb{X} : F(v, u_k) = \min_{x \in G^s(u_k)} F(x, u_k)\}, \tag{2.24}$$

where F is given by (2.21) and G^s by (2.23).

In the next theorem we impose the following conditions on f and g_i:

(2.25) The functions $\partial f/\partial x$, $\partial g_i/\partial x$, $i = 1, \dots, p$, are continuous with respect to the weak topology of $\mathbb{X}$ and the metric topology in $\mathbb{U}$.

(2.26) The functional F given by (2.21) and the sets $G^s(u)$ (cf. (2.23)) satisfy conditions (2.2), (2.3), (2.5) and (2.6) with $G = G^s$.

(2.27) The functionals g_i satisfy (2.13) and the global Slater condition (cf. (2.15)).

Let $\{u_k\} \subset U$ be a sequence and $u_k \to u_0 \in U$. We shall assume that

(2.28) the gradients $\partial g_i(x_0, u_0)/\partial x$ are linearly independent for active constraints where $x_0 \in V^s(u_0)$.

Theorem 2.4. *If the above assumptions* (2.25)–(2.27) *are satisfied and* $u_k \to u_0$ *in* $\mathbb{U}$*, then the sets of optimal solutions* V_k^s*,* $k = 0, 1, \dots$ *(cf.* (2.24)*), satisfy the conditions*

(1) *the sets* $V_k^s = V^s(u_k)$ *are not empty and commonly bounded, i.e. there exists a ball* $B(\overline{x}, \overline{r})$ *such that* $V_k^s \subset B(\overline{x}, \overline{r})$ *for* $k = 0, 1, \dots$

(2) $(s)\mathrm{Lim\,sup}\, V^s(u_k)$ *is a nonempty set and* $(s)\mathrm{Lim\,sup}\, V^s(u_k) \subset V^s(u_0)$ *where* $(s)\mathrm{Lim\,sup}$ *denotes the upper limit with respect to the strong topology of* $\mathbb{X}$.

Moreover, if $F(\,\cdot\,, u)$ *is a strictly convex functional then* $V^s(u_k)$ *is a singleton* $\{x_k\}$ *for* $k = 0, 1, 2, \ldots,$ *and* x_k *tends to* x_0 *in the strong topology of* $\mathbb{X}$.

Proof. It is easy to check that all the assumptions of Theorem 2.2 are satisfied. Thus $(w)\mathrm{Lim\,sup}\, V^s(u_k) \subset V^s(u_0)$, i.e. the set-valued mapping $\mathbb{U} \ni u \to V^s(u) \subset 2^{\mathbb{X}}$ is upper semicontinuous with respect to the weak topology in $\mathbb{X}$. We shall show that, in our case, the weak convergence of x_k to x_0 implies the strong one. Let $\{x_k\}$, $x_k \in V^s(u_k)$, be any sequence weakly converging to $x_0 \in V^s(u_0)$. It is easy to check that, in our case, all the assertions of the Kuhn–Tucker theorem are fulfilled (cf. [11]). Thus

$$L'_x(x_k, u_k, \lambda^k) = x_k + f'_x(x_k, u_k) + \sum_{i=1}^{p} \lambda_i^k g'_i(x_k, u_k) = 0 \tag{2.29}$$

where $\lambda_i^k \ge 0$, $\lambda_i^k g_i(x_k, u_k) = 0$ for $k = 1, \ldots, p$, and $L(x, u, \lambda) = \|x\|^2/2 + f(x, u) + \sum_{i=1}^{p} \lambda_i g_i(x, u)$.

Denote by $I^a(x_0, u_0) = I_0^a$ the set of active indices at the point (x_0, u_0). For $k = 0$, equality (2.29) takes the form

$$L'_x(x_0, u_0, \lambda^0) = x_0 + f'_x(x_0, u_0) + \sum_{i \in I_0^a}^{p} \lambda_i^0 g'_i(x_0, u_0) = 0. \tag{2.30}$$

By (2.25), for sufficiently large $k (k > K)$, we have

$$L'_x(x_k, u_k, \lambda^k) = x_k + f'_x(x_k, u_k) + \sum_{i \in I_0^a}^{p} \lambda_i^k g'_i(x_k, u_k) = 0, \tag{2.31}$$

because $g_i(x_k, u_k) < 0$ for $k > K$ and $i \notin I_0^a$. Taking into account (2.25) and (2.28), we see that the gradients $g'_i(x_k, u_k)$, $i \in I_0^a$, $k > K$, are linearly independent and equalities (2.30) and (2.31) imply that $\lambda_i^k \to \lambda_i^0$ for $i \in I_0^a$ as $k \to \infty$. It is easy to check, that for $k > K$,

$$\begin{aligned} &\langle L'_x(x_k, u_k, \lambda^k) - L'_x(x_0, u_0, \lambda^0), x_k - x_0 \rangle \\ &\quad = \|x_k - x_0\|^2 + \langle f'_x(x_k, u_k) - f'_x(x_0, u_0), x_k - x_0 \rangle \\ &\quad\quad + \sum_{i \in I_0^a} \langle \lambda_i^k g'_i(x_k, u_k) - \lambda_i^0 g'_i(x_0, u_0), x_k - x_0 \rangle = 0. \end{aligned} \tag{2.32}$$

We know that $u_k \to u_0$ in $\mathbb{U}$, $x_k \to x_0$ weakly in $\mathbb{X}$ and $\lambda_i^k \to \lambda_i^0$ in $\mathbb{R}$. Thus assumption (2.25) and equality (2.32) imply that $\|x_k - x_0\| \to 0$. □

Remark 2.3. Condition (2) of Theorem 2.4 means that the set-valued mapping $\mathbb{U} \ni u \to V^s(u) \subset \mathbb{X}$ is upper semicontinuous with respect to the metric topology in $\mathbb{U}$ and the norm topology in $\mathbb{X}$ provided that the assertions of Theorem 2.4 are satisfied.

Example 2.1. Let $\mathbb{X} = L^2([0,1],\mathbb{R})$ and $\mathbb{U} = \{u : [0,1] \to \mathbb{R} : u(\,\cdot\,)$ is measurable and $|u(t)| \le 1/2\}$ with $\rho_u(u_1, u_2) = (\int_0^1 |u_1(t) - u_2(t)|^2\, dt))^{1/2}$. Define

$$F(x,u) = \frac{1}{2}\int_0^1 x^2(t)\,dt + \frac{1}{4}\int_0^1 \bigg(u(t)\int_0^t x(\tau)\,d\tau\bigg)^2 dt + \int_0^1 (u(t)+1)x(t)\,dt$$

and

$$g_1(x,u) = \int_0^1 \bigg(a\bigg(\int_0^t x(\tau)\,d\tau\bigg)^3 + u(t)x(t) - \frac{1}{4}\bigg)\,dt$$

for $(x,u) \in \mathbb{X}\times\mathbb{U}$, $a \in [0,1]$ and consider an optimization problem with perturbation

$$\min F(x,u) \text{ subject to } g_1(x,u) \le 0. \tag{2.33}$$

It is easy to check that for $a \in [0,1]$ problem (2.33) satisfies the assertions of Theorem 2.2 and the functional $F(\,\cdot\,,u)$ is strictly convex. Thus, for any $u \in U$, there exists exactly one solution x_u of problem (2.33) and x_u tends to x_v weakly in L^2 as u tends to v in $\mathbb{U}$. In the case of $a = 0$, the functionals F and g_1 satisfy the assumptions of Theorem 2.4. In this case x_u tends to x_v in the norm of L^2 provided $u \to v$ in $\mathbb{U}$. For $u = 0$ and $a = 0$, problem (2.33) possesses a unique solution $x_0 = -1$. Thus x_u tends to -1 in L^2 if u tends to 0 in $\mathbb{U}$ and $a = 0$.

3. Applications to the eigenvalue problem and to the boundary value problem

Denote by $\mathbb{X} = H_0^1(\Omega,\mathbb{R}^N)$ the Sobolev space of functions defined on $\Omega \subset \mathbb{R}^n$, $n \ge 1$, where Ω is a bounded domain with a Lipschitzian boundary, $N \ge 1$ (if $n = 1$ we put $\Omega = [a,b]$). Let the metric space of parameters be defined by the formula $\mathbb{U} = \{u \in L^\infty(\Omega,\mathbb{R}^m) : u(t) \in M\}$ with $\rho(u_1,u_2) = \operatorname{vraisup}|u_1(t) - u_2(t)|$ where M is a bounded subset of $\mathbb{R}^m$, $t = (t^1,\ldots,t^n)$. Consider an optimization problem

$$\min_{x\in\mathbb{X}} F(x,u) \text{ subject to } \int_\Omega |x(t)|^2 dt - 1 \le 0 \tag{3.1}$$

where

$$F(x,u) = \int_\Omega \frac{1}{2}|\nabla x(t)|^2 dt + \int_\Omega \varphi(t,x(t),u(t))\,dt,$$

$$
\begin{aligned}
\varphi &: \Omega \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R},\\
x &= (x^1, \dots , x^N),\\
\nabla x &= (\nabla x^1, \dots , \nabla x^N),\\
\nabla x^i &= \left(\frac{\partial}{\partial t^1} x^i, \dots , \frac{\partial}{\partial t^n} x^i\right).
\end{aligned}
\tag{3.2}
$$

We shall assume that

(3.3) the functions φ and φ_x are measurable with respect to t and continuous with respect to (x, u),

(3.4) if $n > 1$, we assume that, there exists $C > 0$ such that

$$
|\varphi(t, x, u)| \le C(1 + |x|^s), \qquad |\varphi_x(t, x, u)| \le C(1 + |x|^{s-1})
$$

for $t \in \Omega$, $u \in M$ and $x \in \mathbb{R}^N$ with $s \in [1, 2n/(n - 2))$ if $n \ge 3$ and $s > 1$, if $n = 2$, if $n = 1$ we assume that, for any bounded set $A \subset \mathbb{R}^N$, there exists an integrable function $h : \Omega \to \mathbb{R}^+$ such that

$$
|\varphi(t, x, u)| \le h(t) \quad \text{and} \quad |\varphi_x(t, x, u)| \le h(t)
$$

for all $x \in A$, $u \in M$ and $t \in [a, b]$ a.e.

(3.5) there exists some constants a, b, c such that

$$
\varphi(t, x, u) \ge a|x|^2 + b|x| + c,
$$

(3.6) for $\alpha > 0$, $\varphi(t, \alpha x, u) = \alpha^2 \varphi(t, x, u)$ for $x \in \mathbb{X}, t \in \Omega$ a.e. and any $u \in M$, i.e. the function φ is a homogeneous function of order two with respect to x,

(3.7) if $n > 1$,we assume that,for any $\widetilde{u} \in M$ and $\delta > 0$, there exists $L > 0$, such that

$$
\begin{aligned}
|\varphi(t, x, u^1) - \varphi(t, x, u^2)| &\le L(1 + |x|^2)|u^1 - u^2|,\\
|\varphi_x(t, x, u^1) - \varphi_x(t, x, u^2)| &\le L(1 + |x|)|u^1 - u^2|,
\end{aligned}
$$

for $t \in \Omega$ a.e. $x \in \mathbb{R}^N$ and for any u^1, u^2 with $|u^1 - \widetilde{u}| < \delta$, and $|u^2 - \widetilde{u}| < \delta$, if $n = 1$ it is enough to assume that for any bounded set $A \subset \mathbb{R}^N$, any $\widetilde{u} \in M$ and $\delta > 0$, there exists $h(\,\cdot\,) \in L^1([a, b], \mathbb{R}^+)$ such that

$$
\begin{aligned}
|\varphi(t, x, u^1) - \varphi(t, x, u^2)| &\le h(t)|u^1 - u^2|,\\
|\varphi_x(t, x, u^1) - \varphi_x(t, x, u^2)| &\le h(t)|u^1 - u^2|,
\end{aligned}
$$

for $t \in [a, b]$ a.e. $u^1, u^2 \in M$ such that $|u^1 - \widetilde{u}| < \delta$ and $|u^2 - \widetilde{u}| < \delta$.

Remark 3.1. Let us recall that, in the space H^1_0, the norm is defined by the formula

$$
\|x\|^2 = \int_\Omega |\nabla x(t)|^2 \, dt
$$

and the Poincare inequality has the form

$$\int_{\Omega} |x(t)|^2 dt \le d^2 \int_{\Omega} |\nabla x(t)|^2 \, dt,$$

where d is the diameter of Ω (cf. [20, Theorem A8]). In the one-dimensional case usually $\Omega = [0, \pi]$ and, in this case,

$$\int_0^{\pi} |x(t)|^2 \le \int_0^{\pi} |\dot{x}(t)|^2 \, dt.$$

This estimate is sharp. Let $P : \mathbb{X} \to \mathbb{Y}$ be an operator where $\mathbb{X}$ and $\mathbb{Y}$ are some Hilbert spaces. Suppose that there exist $\lambda \in \mathbb{R}$ and $x \in \mathbb{X}$ with $\|x\| = 1$, such that $P(x) = \lambda x$. In such a case, the number λ is called an eigenvalue of the operator P, while the vector x an eigenvector of P, corresponding to λ.

We shall prove.

Theorem 3.1. *Let u_k tend to u^0 in $\mathbb{U}$. If*

(a) *the function φ satisfies conditions* (3.3)–(3.7)*,*
(b) *for any $\overline{u} \in \mathbb{U}$, there exists $\overline{x} \in H_0^1$ such that $F(\overline{x}, \overline{u}) < 0$,*

then the set $V^e(u_k)$ of optimal solutions of problem (3.1) *with $u = u_k \in \mathbb{U}$ satisfies the conditions*

(1) *$V^e(u_k) \neq \phi$ and $V^e(u_k) \subset S$ where S is the unit sphere in $L^2(\Omega, \mathbb{R}^N)$,*
(2) *$(s)\mathrm{Limsup} V^e(u_k) \neq \phi$ and $(s)\mathrm{Limsup} V^e(u_k) \subset V^e(u_0)$ provided $u_k(\,\cdot\,) \to u_0(\,\cdot\,)$ in $L^\infty(\Omega, \mathbb{R}^N)$,*
(3) *for any u_k, there exist $\lambda_k < 0$ and $x_k \in V^e(u_k)$ such that $\Delta x_k(t) - \varphi_k(t, x_k(t), u_k(t)) = \lambda_k x_k(t)$, $k = 0, 1, \ldots$, and $\lambda_k \to \lambda_0$ if $u_k \to u_0$ in L^∞, i.e. x_k is an eigenvector and λ_k is an eigenvalue of the elliptic differential operator $\Delta x - \varphi_k(t, x, u_k)$ where $\Delta x = (\Delta x^1, \ldots, \Delta x^N)$ and $\Delta x^i = \sum_{s=1}^n \partial^2 x^i / \partial (t^s)^2$, $i = 1, \ldots, N$.*

Proof. For an arbitrary ball $B(0, r) \subset H_0^1(\Omega, \mathbb{R}^N)$, any $x \in B(0, r)$ and $\varepsilon > 0$, we have

$$\begin{aligned} |F(x, u_k) - F(x, u_0)| &= \left| \int_{\Omega} (\varphi(t, x, u_k) - \varphi(t, x, u_0)) \, dt \right| \\ &\le \int_{\Omega} L(1 + |x(t)|^2) |u_k(t) - u_0(t)| \, dt. \end{aligned}$$

By the Poincare inequality (cf. Remark 3.1), we get

$$|F(x, u_k) - F(x, u_0)| \le L(1 + r^2)\varepsilon$$

for k sufficiently large. Since $\varepsilon > 0$ is an arbitrarily small number, the last inequality means that $F(x, u_k)$ tends to $F(x, u_0)$ uniformly on $B(0, r)$ for any $r > 0$. By (3.5),

$$F(x, u) \geq \frac{1}{2}\|x\|^2 + C \quad \text{for all } u \in \mathbb{U} \text{ and } x \in G(u), \tag{3.8}$$

where

$$G(u) = G = \Big\{x : \int_\Omega |x(t)|^2\,dt \leq 1\Big\}. \tag{3.9}$$

Let us put (by (3.4))

$$\overline{s} = \sup_{u\in\mathbb{U}} F(0, u) = \sup \int_\Omega \varphi(t, 0, u(t))\,dt < \infty. \tag{3.10}$$

Inequalities (3.8) and (3.10) imply that the sets $\mathcal{L}(\overline{s}, u)$ and $G(u) = G$ satisfy the conditions: $\mathcal{L}(\overline{s}, u) \cap G \neq \phi$ and $\mathcal{L}(\overline{s}, u) \subset B(0, \overline{r})$ for some $\overline{r} > 0$ where G is given by (3.9) and $\mathcal{L}(\overline{s}, u) = \{x \in G : F(x, u) \leq \overline{s}\}$. It is well known that the space H_0^1 is compactly embedded into L^s if $n > 1$ and into C^0 if $n = 1$ (cf. [1]). Basing ourselves on this fact and the Krasnosiel'skiĭ theorem on the continuity of the Nemyckiĭ operator we can easily show that all the remaining assumptions of Theorem 2.4 are fulfilled (cf. Theorem 1.4 [14] and Theorem C1 [20]), (conditions (2.27) and (2.28) are trivial in this case). We have thus proved assertion (2) of our theorem. Moreover, (3.6) and assumption (b) imply that the functional F given by (3.2) attains its minimum on the boundary of the set G (cf. (3.9)) for any fixed $u \in \mathbb{U}$. Thus

$$V_k^e = V^e(u_k) \subset S = \Big\{x \in H_0^1 : \int_\Omega |x(t)|^2\,dt = 1\Big\}$$

for all $u_k \in \mathbb{U}$. Let h be an arbitrary test function, i.e. $h \in C_0^\infty(\Omega, \mathbb{R}^N)$. Consider a scalar function

$$\psi(\tau) = \frac{1}{\|x_k(\tau)\|^2} F(x_k(\tau), u_k) = F\bigg(\frac{x_k(\tau)}{\|x_k(\tau)\|}, u_k\bigg), \tag{3.11}$$

where $\tau \in \mathbb{R}$, $x_k(\tau) = x_k + \tau h$, $x_k \in V_k^e \subset S$. It is easy to see that $\psi(\,\cdot\,)$ is a C^1-class function and $\psi(0) = \min \psi(\tau)$. Thus $\psi'(0) = 0$ and by a direct calculation, using formula (3.11), we obtain

$$2F(x_k, u_k)\langle x_k, h\rangle = F_x(x_k, u_k)h. \tag{3.12}$$

Let us put $\lambda_k = 2\mu_k$ where $\mu_k = F(x_k, u_k) = \min F(x, u_k) < 0$ (by (b)). We have proved (cf. (2.10)) that $\mu_k \to \mu_0$. Since equality (3.11) holds for any $h \in C_0^\infty$, therefore x_k is a generalized solution of the equation

$$\Delta x(t) - \varphi_x(t, x(t), u_k(t)) = \lambda_k x(t),$$

i.e. λ_k is an eigenvalue and x_k is an eigenvector of the nonlinear elliptic differential operator $\Delta x - \varphi_x(t,x,u_k)$, where $\Delta x = (\Delta x^1,\dots,\Delta x^N)$ and $\Delta x^i(t) = \sum_{s=1}^n \partial^2 x^i/\partial(t^s)^2(t)$, $i=1,\dots,N$. Thus, our proof is concluded. □

Next, let us consider a Dirichlet boundary value problem with parameter

$$\Delta x(t) = \varphi_x(t,x(t),u(t)), x(t) = 0 \quad \text{for } t \in \partial\Omega. \tag{3.13}$$

It is easy to see that (3.13) is the Euler–Lagrange system for the functional of action

$$F(x,u) = \int_\Omega \left[\frac{1}{2}|\nabla x(t)|^2 + \varphi(t,x,(t),u(t))\right] dt \tag{3.14}$$

where $x(\,\cdot\,) \in H_0^1(\Omega,\mathbb{R}^N)$, $u(\,\cdot\,) \in \mathbb{U}$ and the sets Ω and $\mathbb{U}$ are described at the beginning of this section. The following theorem on the continuous dependence on parameters of the solutions of boundary value problem (3.13) holds.

Theorem 3.2. *Let u_k tend to u_0 in $\mathbb{U}$. If the function φ satisfies conditions* (3.3), (3.4), (3.7) *and condition* (3.5) *with $a > -1/2d^2$, then the set $V^b(u_k)$ of solutions of boundary value problem* (3.13) *satisfies the conditions*

(1) *$V^b(u_k) \neq \phi$ and there exists a ball $B(0,\overline{r}) \subset H_0^1$ such that $V^b(u_k) \subset B(0,\overline{r})$ for $k = 0,1,\dots,$*
(2) *$(s)\mathrm{Limsup} V^b(u_k) \neq \theta$ and $(s)\mathrm{Limsup} V^b(u_k) \subset V^b(u_0)$.*

If functional (3.14) *is strictly convex, then, for any $u_k \in \mathbb{U}$, problem* (3.13) *possesses a unique solution $x_k \in H_0^1$ and x_k tends to x_0 in H_0^1 provided u_k tends to u_0 in $\mathbb{U}$. In the case when $n = 1$ and $\Omega = [0,\pi]$, it is enough to assume that $a > -1/2$.*

Proof. We shall apply Theorem 2.4 with $G^s(u) = \mathbb{X}$. Identically as in the previous theorem, one can show that $F(x,u_k)$ tends to $F(x,u_0)$ uniformly on any ball $B(0,r)$ provided that u_k tends to u_0 in $\mathbb{U}$. From assumption (3.5) with $a > -1/2d^2$ and the Poincare inequality we obtain

$$\begin{aligned} F(x,u) &= \frac{1}{2}\|x\|^2 + \int_\Omega \varphi(t,x(t),u(t))\,dt \\ &\geq \frac{1}{2}\|x\|^2 + \int_\Omega [a|x(t)|^2 + b|x(t)|^2 + c]\,dt \\ &\geq \left(\frac{1}{2} + ad^2\right)\|x\|^2 + \alpha \end{aligned} \tag{3.15}$$

for some $\alpha \in \mathbb{R}$, all $u \in \mathbb{U}$ and $x \in H_0^1$.

On the other hand, by (3.4),

$$F(0,u) \leq C \tag{3.16}$$

for all $u \in \mathbb{U}$, where C is some constant. Since $1/2 + ad^2 > 0$, inequalities (3.15) and (3.16) imply that there exist $\overline{r} > 0$ and $\overline{s} = C$, such that $\mathcal{L}(\overline{s}, u) \subset B(0, \overline{r})$ for all $u \in \mathbb{U}$ where $\mathcal{L}(\overline{s}, u) = \{x \in X : F(x, u) \leq \overline{s}\}$. Thus all the assumptions of Theorem 2.4 are satisfied and applying this one, we get the assertions of Theorem 3.2. $\square$

Example 3.1. Let $\mathbb{X} = H_0^1([0,\pi], \mathbb{R}^2)$ and $M = [0, l]$ with $l > 0$. Consider a functional

$$F(x,u) = \int_0^\pi \left[\frac{1}{2}|\dot{x}(t)|^2 + a(t)x^1(t)\sqrt{(x^1(t))^2 + (x^2(t))^2 u(t)}\right] dt, \tag{3.17}$$

where $x(\,\cdot\,) \in H_0^1$, $u(\,\cdot\,) \in L^\infty([0,\pi],[0,l])$, $a(\,\cdot\,) \in L^\infty([0,\pi],\mathbb{R})$, $\int_0^\pi a(t)\,dt \neq 0$.

Let us notice that, for any $\overline{u}$, there exists $\overline{x} = \pm(1,0)$ such that $F(\overline{x}, \overline{u}) < 0$. It is easy to check that functional (3.17) satisfies all the remaining assumptions of Theorem 3.2. Thus, for any admissible parameter u_k, there exist an eigenvalue λ_k and at least one eigenvector x_k, $\|x_k\| = 1$ such that

$$\begin{aligned} \ddot{x}^1(t) - a(t)\sqrt{(x^1)^2 + (x^2)^2 u}(t) - a(t)\frac{(x^1)^2}{\sqrt{(x^1)^2 + (x^2)^2 u}}(t) &= \lambda x^1(t), \\ \ddot{x}^2(t) - a(t)\frac{x^1 x^2 u}{\sqrt{(x^1)^2 + (x^2)^2 u}}(t) &= \lambda x^2(t), \end{aligned}$$

for $t \in [0,\pi]$ a.e. $x = x_k = (x_k^1, x_k^2)$ and $\lambda = \lambda_k$, $k = 0, 1, \ldots$ Moreover, if $u_k \to u_0$ in L^∞, then λ_k tends to λ_0 and any cluster point of the sequence $\{x_k\}$ is an eigenvector corresponding to the eigenvalue λ_0.

Example 3.2. Let $\Omega = \{t \in \mathbb{R}^2 : |t| \leq 1\}$ and let $M \subset \mathbb{R}^2$ any bounded set. Consider an elliptic boundary value problem with control

$$\begin{cases} \Delta x^1(t) - 6h(t)(x^1(t))^5(x^2(t))^2 + \dfrac{1}{8}x^2(t) = u^1(t), \\ \Delta x^2(t) - 2h(t)(x^1(t))^6 x^2(t) + \dfrac{1}{8}x^1(t) = u^2(t), \end{cases} \tag{3.18}$$

subject to $x(t) = 0$ for $t \in S = \{t \in \mathbb{R}^2 : |t| = 1\}$, where $h(\,\cdot\,) \in L^p(\Omega, \mathbb{R}^+)$, $p > 1$, $u(t) \in M$ for $t \in \Omega$ a.e. It is easy to notice that the functional of action for system (3.18) is of the form

$$\begin{aligned} F(x,u) = \int_\Omega \bigg[\frac{1}{2}|\nabla x(t)|^2 &+ h(t)\cdot(x^1(t))^6\cdot(x^2(t))^2 \\ &- \frac{1}{8}x^1(t)\cdot x^2(t) + u^2(t)\cdot x(t)\bigg] dt, \end{aligned}$$

$x(\,\cdot\,) \in H_0^1(\Omega, \mathbb{R}^N)$, and satisfies all the conditions of Theorem 3.2. Moreover, this functional is strictly convex because $h(t) \geq 0$ for $t \in \Omega$ a.e. Thus Theorem 3.2 implies that, for any parameter u_k, $k = 0, 1, 2 \ldots$, there exists a unique solution

$x_k \in H_0^1$ and x_k tends to x_0 in H_0^1 provided $u_k \to u_0$ in $L^\infty(\Omega, M)$. If $u_0 = 0$ system (3.17) possesses only a trivial solution $x_0 = 0$. In this case x_k tends to 0 in H_0^1 provided that u_k tends to null in L^∞. The strong convergence of x_k to x_0 in H_0^1 means that ∇x_k tends to ∇x_0 and x_k tends to x_0 in $L^2(\Omega, \mathbb{R}^2)$. Since $H_0^1(\Omega, \mathbb{R}^N)$ with $\Omega \subset \mathbb{R}^2$ is continuously embedded into $L^s(\Omega, \mathbb{R}^N)$ for any $s > 1$, therefore $x_k \to x_0$ in L^s with $s > 1$.

REFERENCES

[1] R. A. ADAMS, *Sobolev spaces*, Academic Press, N.Y., 1975.

[2] J. F. BONNANS, *A semi-strong sufficiency condition for optimality in non convex programming and its connection to the perturbation problem*, J. Optim. Theory Appl. **60** (1989), 7–18.

[3] J. F. BONNANS AND A. SHAPIRO, *Optimization problems with perturbation: a quided tour*, SIAM Rev. **40** (1998), 228–264.

[4] D. BORS AND S. WALCZAK, *Dirichlet problems with variable boundary data*, Lecture Notes in Nonlinear Analysis **23** (1998), 57–71.

[5] J. M. DANSKIN, *The Theory of Max-Min and Its Applications to Weapons Allocation Problems*, Springer–Verlag, New York, 1967.

[6] V. F. DEM'YANOV AND V.N. MALOZEMOV, *Introduction to Minimax*, Wiley, New York, 1974.

[7] A. V. FIACCO AND G. P. MCCORMICK, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, Wiley, New York, 1968.

[8] J. GAUVIN AND R. JANIN, *Directional behaviour of optimal solutions in nonlinear mathematical programming*, Math. Oper. Res. **13** (1988), 629–649.

[9] L. M. GRAVES, *Some mapping theorems*, Duke Math. J. **17** (1950), 111–114.

[10] D. IDCZAK, *Stability in semilinear problems*, J. Differential Equations **162** (2000), 64–90.

[11] A. D. IOFFE AND V. M. TIKHOMIROV, *Theory of Extremal Problems*, North-Holland, Amsterdam, 1979.

[12] G. KLASSEN, *Dependence of solution on boundary conditions for second order ordinary differential equations*, J. Differential Equations **7** (1970), 24–33.

[13] J. MAWHIN, *Problemes de Dirichlet Variationnels Non-Lineaires*, Les Presses de L'Universite de Montreal Canada, 1987; WNT, Warszawa, 1999. (Polish)

[14] J. MAWHIN AND M. WILLEM, *Critical Point Theory and Hamiltonian Systems*, Springer–Verlag, New York, 1989.

[15] S. M. ROBINSON, *Strongly regular generalized equations*, Math. Oper. Res. **5** (1980), 43–62.

[16] R. T. ROCKAFELLAR AND R. J.-B. WETS, *Variational Analysis*, Springer–Verlag, Berlin, 1998.

[17] S. SĘDZIWY, *Dependence of solution on boundary data for a system of two ordinary differential equations*, J. Differential Equations **9** (1971), 381–389.

[18] A. SHAPIRO, *Second-order sensitivity analysis and asymptotic theory of parametrized nonlinear programs*, Math. Programing **33** (1985), 280–299.

[19] ______, *First and second order analysis of nonlinear semidefinite programs*, Math. Programming (B) **77** (1997), 301–320.

[20] M. STRUWE, *Variational Methods*, Springer–Verlag, Berlin, 1990.

[21] S. Walczak, *On the continuous dependence on parameters of solution of the Dirichlet problem, Part I. Coercive case, Part II. The case of saddle points*, Bulletin de la Clase des Sciences de l'Academie Royal de Belgique, Tome VI **7** (1995), 247–261, 263–273.

[22] ———, *Continuous dependence on parameters and boundary data for nonlinear PDE. Coercive case*, Differential Integral Equations **11** (1998), 35–46.

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Stanisław Walczak
Faculty of Mathematics
University of Łódź
Stefana Banacha 22
90-238 Łódź, POLAND

E-mail address: stawal@math.uni.lodz.pl