Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 18, 2001, 17–39

MULTIPLE POSITIVE SYMMETRIC SOLUTIONS OF A SINGULARLY PERTURBED ELLIPTIC EQUATION

Mónica Clapp — Gustavo Izquierdo

Abstract. This paper is concerned with the multiplicity of positive solutions of the Dirichlet problem

$$-\varepsilon^2\Delta u + u = K(x)|u|^{p-2}u \quad \text{in } \Omega,$$

where Ω is a smooth domain in $\mathbb{R}^N$ which is either bounded or has bounded complement (including the case $\Omega = \mathbb{R}^N$), $N \geq 3$, K is continuous and p is subcritical. It is known that critical points of K give rise to multibump solutions of this type of problems. It is also known that, in general, the presence of symmetries has the effect of producing many additional solutions. So, we consider domains Ω which are invariant under the action of a group G of orthogonal transformations of $\mathbb{R}^N$, we assume that K is G-invariant, and study the combined effect of symmetries and the nonautonomous term K on the number of positive solutions of this problem. We obtain multiplicity results which extend previous results of Benci and Cerami (1994), Cingolani and Lazzo (1997) and Qiao and Wang (1999).

2000 *Mathematics Subject Classification.* 35J60, 35J20.

Key words and phrases. Singularly perturbed elliptic problems, symmetric solutions, nonlinear PDE's.

Research supported by CONACyT under grant 28031-E.

1. Introduction and statement of results

We consider the singularly perturbed problem

$$(\mathrm{P}_{\varepsilon,K}) \qquad \begin{cases} -\varepsilon^2\Delta u + u = K(x) \mid u \mid^{p-2} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth domain in $\mathbb{R}^N$ which is either bounded or has a bounded complement (Ω might be all of $\mathbb{R}^N$), $N \geq 3$, $2 < p < 2^* := 2N/(N-2)$, and K is a Hölder continuous function on $\overline{\Omega}$ which satisfies

$$\inf_{\Omega} K > 0 \quad \text{and} \quad \lim_{|x|\to\infty} K(x) = K_\infty < \infty.$$

Singularly perturbed elliptic equations have attracted much attention in recent years and various interesting existence and multiplicity results have been obtained, see for example [19] and the references therein.

Here we are interested in studying the effect of the topology of certain subsets of the domain related to this problem on the number of solutions of it. In the autonomous case $K \equiv 1$ Benci and Cerami have shown that, for bounded domains, there is an influence of the domain topology on the number of single-bump solutions of this problem for ε small enough ([4], [5]). Results of this kind for more general domains are also known, see for example [6], [7]. In the nonautonomous case ground state solutions concentrate at maxima of K as $\varepsilon \to 0$ ([25], [28], [29]), and this set of maxima has an effect on the number of single-bump solutions of this problem ([8], [9], [24]). Similar results for the Neumann problem are also known ([1], [24], [30]).

It has been shown that critical points of K give rise to multibump solutions for this type of problems, see for example [17], [19], [13]. On the other hand, it is well known that the presence of symmetries has usually the effect of producing additional solutions. Here we shall study the combined effect of both of these factors. We consider domains Ω which are invariant under the action of a group G of orthogonal transformations of $\mathbb{R}^N$ (i.e. $gx \in \Omega$ for all $g \in G$, $x \in \Omega$). We assume that the function K is G-invariant (i.e. $K(gx) = K(x)$ for all $g \in G$, $x \in \Omega$) and look for solutions u which are also G-invariant, that is, we consider the problem

$$(\mathrm{P}^G_{\varepsilon,K}) \qquad \begin{cases} -\varepsilon^2\Delta u + u = K(x) \mid u \mid^{p-2} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(gx) = u(x) & \text{for all } g \in G,\ x \in \Omega, \end{cases}$$

where G is a closed subgroup of $O(N)$, Ω is a G-invariant smooth domain in $\mathbb{R}^N$ which is either bounded or has a bounded complement, $N \geq 3$, $2 < p < 2^*$ and

K is a G-invariant Hölder continuous function on $\overline{\Omega}$ which satisfies

$$\inf_{\Omega} K > 0 \quad \text{and} \quad \lim_{|x|\to\infty} K(x) = K_\infty < \infty.$$

We shall show that the G-invariant ground-state solutions of this problem tend to concentrate near G-orbits of the set

$$M = M(G) = \left\{ y \in \overline{\Omega} \;\middle|\; \frac{\#Gy}{K(y)^{2/(p-2)}} = \min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}} \right\}$$

and that the orbit space of M has an effect on the number of solutions of this problem for ε small enough. More precisely, for $\rho > 0$, let

$$\begin{aligned} M_\rho^- &= M(G)_\rho^- = \{y \in M \mid \operatorname{dist}(y, \partial\Omega) \geq \rho\}, \\ M_\rho^+ &= M(G)_\rho^+ = \{y \in \mathbb{R}^N \mid \operatorname{dist}(y, M) \leq \rho\}. \end{aligned}$$

We denote by $X/G = \{Gx \mid x \in X\}$ the G-*orbit space* of X. Its elements are the G-*orbits* $Gx := \{gx \mid g \in G\}$ of X and it has the quotient space topology. We write $\#Gx$ for the cardinality of Gx. Let $\mu_{1,\mathbb{R}^N}$ be the energy of the ground state solution of the problem

$$(\mathrm{P}_\infty) \qquad \begin{cases} -\Delta u + u = | u |^{p-2} u & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

We shall prove the following results.

THEOREM 1.1. *Assume that Ω is bounded and that it contains a finite G-orbit. Then, for every $\rho > 0$ and every $\gamma_1 < (\min_{x\in\overline{\Omega}} \#Gx/(K(x)^{2/(p-2)})\mu_{1,\mathbb{R}^N} < \gamma_2$, there exists an $\overline{\varepsilon} > 0$ such that, for every $0 < \varepsilon < \overline{\varepsilon}$, problem $(\mathrm{P}^G_{\varepsilon,K})$ has at least*

$$\operatorname{cat}_{M_\rho^+/G}(M_\rho^-/G)$$

solutions u which satisfy

$$\frac{2p}{p-2}\gamma_1 < \varepsilon^{-N} \int_\Omega (\varepsilon^2 |\nabla u|^2 + |u|^2)\, dx < \frac{2p}{p-2}\gamma_2.$$

THEOREM 1.2. *Assume Ω has bounded (possibly empty) complement and that it contains a finite G-orbit. Assume further that*

$$(\mathrm{C}_G) \qquad \min x \in \overline{\Omega} \frac{\#Gx}{K(x)^{2/(p-2)}} < \min x \in \overline{\Omega} \frac{\#Gx}{K_\infty^{2/(p-2)}}.$$

Then, for every $\rho > 0$ and every $\gamma_1 < (\min x \in \overline{\Omega}\#Gx/K(x)^{2/(p-2)})\mu_{1,\mathbb{R}^N} < \gamma_2$ there exists an $\overline{\varepsilon} > 0$ such that, for every $0 < \varepsilon < \overline{\varepsilon}$, problem $(\mathrm{P}^G_{\varepsilon,K})$ has at least

$$\operatorname{cat}_{M_\rho^+/G}(M_\rho^-/G)$$

solutions u which satisfy

$$\frac{2p}{p-2}\gamma_1 < \varepsilon^{-N}\int_\Omega (\varepsilon^2|\nabla u|^2 + |u|^2)\,dx < \frac{2p}{p-2}\gamma_2.$$

Here $\mathrm{cat}_Z(Y)$ denotes the Lusternik–Schnirelmann category of Y in Z, that is, the smallest number of open subsets of Z which are contractible in Z and cover Y.

If G is the trivial group Theorem 1.1 is due to Benci and Cerami [5] for $K \equiv 1$, and to Qiang and Wang [24] for arbitrary K. For $\Omega = \mathbb{R}^N$ with no group action, Theorem 1.2 is due to Cingolani and Lazzo [8], [9]. The technics used there, however, cannot be adapted to our case. In all of these papers the result is obtained by showing that, for ε small enough, the "barycenter" of low energy functions lies near enough Ω or, respectively, near enough the set of minima of the potential. But the "barycenter map" on symmetric functions is trivial, so it is of no use to obtain symmetric results. On the other hand, as in the non-symmetric case, there is a concentration behavior of G-invariant ground-state solutions as $\varepsilon \to 0$. Our results are based on a careful study of this concentration phenomenon. We show that, as $\varepsilon \to 0$, low energy G-invariant functions concentrate near G-orbits of M in an adequate way, that is, low energy G-invariant functions tend to look as a sum of highly concentrated ground state solutions of the limiting problem (P_∞) centered at each point of a G-orbit of M (see Theorem 4.3 below). Moreover, we show that, for ε small enough, there is a unique such sum which minimizes the distance to a low energy function (see Proposition 5.5 below). This way we produce a "baryorbit map" which will yield the above results.

We would like to point out that, if the action of G is not free, the points of M are not necessarily local maxima of K. Notice also that, if Ω and K are G-invariant, Theorems 1.1 and 1.2 provide in fact additional (multibump) solutions to those obtained in [5], [8], [24], namely, the following holds.

Corollary 1.3. *Assume that Ω is a bounded G-invariant domain, K is strictly positive and G-invariant, and that there is a finite sequence of subgroups $\{\mathrm{id}\} = G_0 \subset \ldots \subset G_m = G$ of G such that*

$$\min_{x\in\overline{\Omega}} \frac{\#G_i x}{K(x)^{2/(p-2)}} < \min_{x\in\overline{\Omega}} \frac{\#G_{i+1} x}{K(x)^{2/(p-2)}} < \infty$$

for $i = 0, \ldots, m-1$. Then, given $\rho > 0$, there exists an $\overline{\varepsilon} > 0$ such that, for every $0 < \varepsilon < \overline{\varepsilon}$, problem $(\mathrm{P}_{\varepsilon,K})$ has at least

$$\mathrm{cat}_{M(G_i)^+_\rho/G_i}(M(G_i)^-_\rho/G_i)$$

solutions which are G_i-invariant but not G_{i+1}-invariant. In particular, $(\mathrm{P}_{\varepsilon,K})$ *has at least*

$$\sum_{i=0}^{m} \operatorname{cat}_{M(G_i)^+_\rho/G_i}(M(G_i)^-_\rho/G_i)$$

solutions.

Corollary 1.4. *Assume that Ω has bounded* (*possibly empty*) *complement and that*

$$\inf_\Omega K > 0 \quad \textit{and} \quad \lim_{|x|\to\infty} K(x) = K_\infty < \max_{x\in\overline{\Omega}} K(x).$$

Assume further that Ω and K are G-invariant, and there is a finite sequence of subgroups $\{\mathrm{id}\} = G_0 \subset \ldots \subset G_m = G$ of G such that

$$\min_{x\in\overline{\Omega}} \frac{\#G_i x}{K(x)^{2/(p-2)}} < \min_{x\in\overline{\Omega}} \frac{\#G_{i+1} x}{K(x)^{2/(p-2)}} < \infty$$

for $i = 0,\ldots,m-1$. Then, given $\rho > 0$, there exists an $\overline{\varepsilon} > 0$ such that, for every $0 < \varepsilon < \overline{\varepsilon}$, problem $(\mathrm{P}_{\varepsilon,K})$ *has at least*

$$\operatorname{cat}_{M(G_i)^+_\rho/G_i}(M(G_i)^-_\rho/G_i)$$

solutiona which are G_i-invariant but not G_{i+1}-invariant. In particular, $(\mathrm{P}_{\varepsilon,K})$ *has at least*

$$\sum_{i=0}^{m} \operatorname{cat}_{M(G_i)^+_\rho/G_i}(M(G_i)^-_\rho/G_i)$$

solutions.

This paper is organized as follows: in Section 2 we describe the variational setting for problem $(\mathrm{P}^G_{\varepsilon,K})$ and in Section 3 we derive some useful properties of the ground state of this problem. In Section 3 we make a careful analysis of the concentration behavior of G-invariant "Palais–Smale sequences" of the energy as $\varepsilon \to 0$. Finally, Section 4 is devoted to the proof of the above results.

2. The variational setting

We shall use the following notation:

$$\begin{aligned}
\langle u, v\rangle_\varepsilon &:= \int_\Omega (\varepsilon\nabla u \cdot \varepsilon\nabla v + uv)\,dx,\\
\|u\|_\varepsilon &:= \bigg(\int_\Omega (\varepsilon^2|\nabla u|^2 + |u|^2)\,dx\bigg)^{1/2},\\
|u|_{p,K} &:= \bigg(\int_\Omega K(x)|u|^p\,dx\bigg)^{1/p}.
\end{aligned}$$

The action of G on Ω induces a G-action on $H^1_0(\Omega)$ given by

$$(gu)(x) := u(g^{-1}x) \quad \text{for } g\in G,\ u\in H^1_0(\Omega),\ x\in\Omega.$$

This is an orthogonal action on $H_0^1(\Omega)$ for any of the scalar products $\langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon$, $\varepsilon > 0$, and it preserves the norm $|\cdot|_{p,K}$, that is,

$$\langle gu, gv\rangle_\varepsilon = \langle u, v\rangle_\varepsilon \quad \text{and} \quad |gu|_{p,K} = |u|_{p,K}$$

for all $u, v \in H_0^1(\Omega)$, $g \in G$. Therefore, the functional $E_{\varepsilon,K} : H_0^1(\Omega) \to \mathbb{R}$,

$$E_{\varepsilon,K}(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + |u|^2)\,dx - \frac{1}{p}\int_\Omega K(x)|u|^p\,dx = \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{p}|u|_{p,K}^p$$

is G-invariant and, by the Principle of Symmetric Criticality [23], the positive critical points of its restriction

$$E_{\varepsilon,K}^G : H_0^1(\Omega)^G \to \mathbb{R}$$

to the fixed point space $H_0^1(\Omega)^G = \{u \in H_0^1(\Omega) \mid gu = u \text{ for all } g \in G\}$ are precisely the solutions of $(\mathrm{P}_{\varepsilon,K}^G)$.

The non-trivial critical points of $E_{\varepsilon,K}^G$ lie on the *Nehari manifold*

$$\begin{aligned}\mathcal{N}_{\varepsilon,K}^G &= \{u \in H_0^1(\Omega)^G \setminus \{0\} \mid DE_{\varepsilon,K}(u)u = 0\} \\ &= \{u \in H_0^1(\Omega)^G \mid \|u\|_\varepsilon^2 = |u|_{p,K}^p,\ u \neq 0\},\end{aligned}$$

which is a $C^{1,1}$-manifold, radially dipheomorphic to each of the unit spheres

$$\Sigma_\varepsilon^G = \{u \in H_0^1(\Omega)^G \mid \|u\|_\varepsilon = 1\}, \quad \Sigma_{p,K}^G = \{u \in H_0^1(\Omega)^G \mid |u|_{p,K} = 1\}.$$

The diffeomorphisms are given by

$$\Sigma_\varepsilon^G \to \mathcal{N}_{\varepsilon,K}^G, \quad u \mapsto |u|_{p,K}^{p/(2-p)}u, \quad \Sigma_{p,K}^G \to \mathcal{N}_{\varepsilon,K}^G, \quad u \mapsto \|u\|_\varepsilon^{2/(p-2)}u. \tag{2.1}$$

For $u \in \mathcal{N}_{\varepsilon,K}^G$ the functional $E_{\varepsilon,K}^G$ is simply

$$E_{\varepsilon,K}^G(u) = \frac{p-2}{2p}\|u\|_\varepsilon^2 = \frac{p-2}{2p}|u|_{p,K}^p,$$

and the positive critical points of this functional on $\mathcal{N}_{\varepsilon,K}^G$ are precisely the non-trivial G-invariant solutions of $(\mathrm{P}_{\varepsilon,K}^G)$, (see [5], [31]).

Let

$$\mu_{\varepsilon,K}^G := \inf\{E_{\varepsilon,K}^G(u) \mid u \in \mathcal{N}_{\varepsilon,K}^G\} > 0.$$

Then, following Benci and Cerami [4] (cf. [10]), one can easily show that

Proposition 2.1. *If $u \in \mathcal{N}_{\varepsilon,K}^G$ is a critical point of $E_{\varepsilon,K}^G$ such that $E_{\varepsilon,K}^G(u) < 2\mu_{\varepsilon,K}^G$ then either $u > 0$ or $u < 0$ in Ω. Hence $|u|$ is a solution of $(\mathrm{P}_{\varepsilon,K}^G)$.*

3. Properties of $\mu^G_{\varepsilon,K}$

From the diffeomorphisms (2.1) defined above it follows that

$$\mu^G_{\varepsilon,K} = \frac{p-2}{2p} \inf_{u\in\Sigma^G_\varepsilon} |u|^{2p/(2-p)}_{p,K} = \frac{p-2}{2p} \inf_{u\in\Sigma^G_{p,K}} \|u\|^{2p/(p-2)}_\varepsilon.$$

With this remark the following properties are easily verified.

Proposition 3.1.

(a) *If K is a constant function then*

$$\mu^G_{\varepsilon,K} = K^{2/(2-p)}\mu^G_{\varepsilon,1}.$$

(b) *If $0 < K_1 \le K_2$ then $\mu^G_{\varepsilon,K_1} \ge \mu^G_{\varepsilon,K_2}$. In particular, if Ω_0 is a G-invariant subdomain of Ω and $K_0 := K \mid \overline{\Omega}_0$ is the restriction of K to $\overline{\Omega}_0$, then $\mu^G_{\varepsilon,K_0} \ge \mu^G_{\varepsilon,K}$.*

(c) *If $0 < \varepsilon_1 \le \varepsilon_2$ then $\mu^G_{\varepsilon_1,K} \le \mu^G_{\varepsilon_2,K}$.*

(d) *If Γ is a closed subgroup of G, then $\mu^\Gamma_{\varepsilon,K} \le \mu^G_{\varepsilon,K}$.*

If $K \equiv 1$ on Ω we write $\mu^G_{\varepsilon,\Omega}$ for $\mu^G_{\varepsilon,1}$, and if $G = \{\mathrm{Id}\}$ is the trivial group we denote $\mu^G_{\varepsilon,K}$ simply by $\mu_{\varepsilon,K}$. Let $B(x,\rho)$ the open ball of radius ρ centered at x in $\mathbb{R}^N$. It is easy to see that

$$\varepsilon^{-N}\mu_{\varepsilon,B(0,\rho)} = \mu_{1,B(0,\varepsilon^{-1}\rho)} \quad \text{and} \quad \varepsilon^{-N}\mu_{\varepsilon,\mathbb{R}^N} = \mu_{1,\mathbb{R}^N}.$$

Using the exponential decay of the ground state solution of the limiting problem in $\mathbb{R}^N$ Benci and Cerami have shown [4] that

$$\lim_{\varepsilon\to 0} \varepsilon^{-N}\mu_{\varepsilon,B(0,\rho)} = \lim_{r\to\infty} \mu_{1,B(0,r)} = \mu_{1,\mathbb{R}^N}. \tag{3.1}$$

Let $\#Gx$ be the cardinality of the G-orbit $Gx = \{gx \mid g \in G\}$ of x.

Proposition 3.2. *The following inequalities hold:*

(a) $(\sup_{x\in\overline{\Omega}} K(x))^{2/(2-p)}\mu_{1,\mathbb{R}^N} \le \varepsilon^{-N}\mu^G_{\varepsilon,K}$,

(b) $\limsup_{\varepsilon\to 0} \varepsilon^{-N}\mu^G_{\varepsilon,K} \le (\inf_{x\in\overline{\Omega}} \#Gx/K(x)^{2/(p-2)})\mu_{1,\mathbb{R}^N}$.

Proof. (a) Let $\overline{K} \equiv \sup_{x\in\overline{\Omega}} K(x)$. By Proposition 3.1,

$$\begin{aligned}\overline{K}^{2/(2-p)}\mu_{1,\mathbb{R}^N} &= \overline{K}^{2/(2-p)}\varepsilon^{-N}\mu_{\varepsilon,\mathbb{R}^N} \le \overline{K}^{2/(2-p)}\varepsilon^{-N}\mu_{\varepsilon,\Omega} \\ &= \varepsilon^{-N}\mu_{\varepsilon,\overline{K}} \le \varepsilon^{-N}\mu^G_{\varepsilon,\overline{K}} \le \varepsilon^{-N}\mu^G_{\varepsilon,K}.\end{aligned}$$

(b) If all orbits Gx are infinite there is nothing to prove. So let $x \in \Omega$ be such that $\#Gx < \infty$. For every $\rho > 0$ small enough so that $B(x,\rho) \subset \Omega$ and $B(gx,\rho) \cap B(x,\rho) = \emptyset$ if $gx \ne x$, let

$$U_\rho := \bigcup_{g\in G} B(gx,\rho).$$

Let $K_\rho := K \mid \overline{U}_\rho$ be the restriction of K to $\overline{U}_\rho$, and let $\underline{K}_\rho := \inf_{y \in B(x,\rho)} K(y)$. Then, by Proposition 3.1,

$$\begin{aligned}\varepsilon^{-N}\mu^G_{\varepsilon,K} \leq \varepsilon^{-N}\mu^G_{\varepsilon,K_\rho} \leq \varepsilon^{-N}\mu^G_{\varepsilon,\underline{K}_\rho} &= \frac{1}{(\underline{K}_\rho)^{2/(p-2)}}\varepsilon^{-N}\mu^G_{\varepsilon,U_\rho} \\ &= \frac{\#Gx}{(\underline{K}_\rho)^{2/(p-2)}}\varepsilon^{-N}\mu_{\varepsilon,B(0,\rho)}\end{aligned}$$

and, by (3.1) above,

$$\limsup_{\varepsilon\to 0}\varepsilon^{-N}\mu^G_{\varepsilon,K} \leq \frac{\#Gx}{(\underline{K}_\rho)^{2/(p-2)}}\mu_{1,\mathbb{R}^N}.$$

Letting $\rho \to 0$ we get

$$\limsup_{\varepsilon\to 0}\varepsilon^{-N}\mu^G_{\varepsilon,K} \leq \frac{\#Gx}{K(x)^{2/(p-2)}}\mu_{1,\mathbb{R}^N}$$

for all $x \in \Omega$ and, by [10, Lemma 8], also for all $x \in \overline{\Omega}$. □

In particular, if K attains its maximum at some fixed point of the action of G on $\overline{\Omega}$, then

$$\lim_{\varepsilon\to 0}\varepsilon^{-N}\mu^G_{\varepsilon,K} = (\max_{x\in\overline{\Omega}} K(x))^{2/(2-p)}\mu_{1,\mathbb{R}^N}.$$

We shall show that in fact, under an appropriate compactness condition (Condition 4.2 below),

$$\lim_{\varepsilon\to 0}\varepsilon^{-N}\mu^G_{\varepsilon,K} = \left(\min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}}\right)\mu_{1,\mathbb{R}^N}.$$

4. G-invariant minimizing sequences

Let $0 < \mu^G_K := \liminf_{\varepsilon\to 0}\varepsilon^{-N}\mu^G_{\varepsilon,K} \leq \infty$.

Definition 4.1. A G-PS-*sequence for* $E^G_{*,K}$ is sequence (ε_n, u_n) such that

(i) $\varepsilon_n > 0$, $\varepsilon_n \to 0$,
(ii) $\varepsilon_n^{-N}E^G_{\varepsilon_n,K}(u_n) \to \gamma \in \mathbb{R}$,
(iii) $\varepsilon_n^{-N/2}\|\nabla E^G_{\varepsilon_n,K}(u_n)\|_{\varepsilon_n} \to 0$.

If $\gamma = \mu^G_K < \infty$ then (ε_n, u_n) will be called a *minimizing* G-PS-*sequence for* $E^G_{*,K}$.

$\nabla E^G_{\varepsilon_n,K}$ denotes the gradient of $E^G_{\varepsilon_n,K}$ in the Hilbert space $(H^1_0(\Omega), \langle\,\cdot\,,\,\cdot\,\rangle_\varepsilon)$.

We wish to describe minimizing G-PS-sequences. We shall need the following compactness condition.

CONDITION 4.2. *There exists a $\beta > 0$ such that the set*

$$\left\{ y \in \overline{\Omega} \;\middle|\; \frac{\#Gy}{K(y)^{2/(p-2)}} \leq \left(\inf_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}} \right) + \beta \right\}$$

is compact.

Observe that Condition 4.2 always holds if Ω is bounded. If Ω has bounded complement Condition 4.2 is equivalent to

$$(\mathrm{C}_G) \qquad \min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}} < \min_{x\in\overline{\Omega}} \frac{\#Gx}{K_\infty^{2/(p-2)}},$$

which is the one given in Theorem 1.2.

Without loss of generality we assume that K is defined on all of $\mathbb{R}^N$.

Let $\omega = \pm|\omega|$ denote either the positive or the negative ground state solution of the limiting problem

$$(\mathrm{P}_\infty) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

which is radially symmetric with respect to the origin.

There is a strong analogy between the behaviour of G-PS-sequences for $E^G_{*,K}$ and those occuring in problems where invariance under dilations or under tranlations produces a lack of compactness [20], [26], [3], [27]. The following theorem gives a precise description of minimizing G-PS-sequences for $E^G_{*,K}$.

We write $G_x := \{g \in G \mid gx = x\}$ for the G-isotropy subgroup of x.

THEOREM 4.3. *Assume that Condition 4.2 holds. Then, for every minimizing G-PS-sequence (ε_n, u_n) for $E^G_{*,K}$, there exist a subsequence, also denoted by (ε_n, u_n), a closed subgroup Γ of finite index in G, and a sequence (y_n) in Ω such that*

(a) $G_{y_n} = \Gamma$,
(b) $y_n \to y \in \overline{\Omega}$ *with* $G_y = \Gamma$ *and*

$$\frac{\#Gy}{K(y)^{2/(p-2)}} = \min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}},$$

(c) $\varepsilon_n^{-N/2} \| u_n - \sum_{[g]\in G/\Gamma} K(y)^{1/2-p} \omega(\varepsilon_n^{-1}(\,\cdot\, - gy_n) \|_{\varepsilon_n} \to 0$,
(d) $\mu^G_K = \lim_{\varepsilon\to 0} \varepsilon^{-N} \mu^G_{\varepsilon,K} = (\min_{x\in\overline{\Omega}} \#Gx/K(x)^{2/(p-2)})\mu_{1,\mathbb{R}^N}$.

PROOF. Let $\widetilde{u}_n \in H^1(\mathbb{R}^N)$ and $\widetilde{K}_n : \mathbb{R}^N \to \mathbb{R}$ be

$$\widetilde{u}_n(z) := u_n(\varepsilon_n z), \quad \widetilde{K}_n(z) := K(\varepsilon_n z).$$

Then,

$$\varepsilon_n^{-N}\|u_n\|_{\varepsilon_n}^2 = \|\widetilde{u}_n\|^2 := \int(|\nabla\widetilde{u}_n|^2 + \widetilde{u}_n^2),$$
$$\varepsilon_n^{-N}|u_n|_{p,K}^p = \|\widetilde{u}_n\|_{p,\widetilde{K}_n} := \int \widetilde{K}_n|\widetilde{u}_n|^p.$$

In the usual way [27], [31] one can show that the sequence $\varepsilon_n^{-N}\|u_n\|_{\varepsilon_n}^2 = \|\widetilde{u}_n\|^2$ is bounded and, therefore, that

$$|\widetilde{u}_n|_{p,\widetilde{K}_n}^p = \varepsilon_n^{-N}|u_n|_{p,K}^p \to \frac{2p}{p-2}\gamma > 0.$$

We apply the Concentration–Compactness Principle [20]. By Lemma 1.21 in [31] vanishing does not occur. Therefore, there exists an $0 < \alpha \leq 1$ and a subsequence $(\widetilde{u}_n)$ such that, for every $\delta > 0$, there exist $R > 0$, a sequence (z_n) in $\mathbb{R}^N$ and a sequence $R_n \to \infty$ satisfying, for all n large enough.

$$\begin{gathered}\left|\frac{2p}{p-2}\gamma\alpha - \int_{B(z_n,R)} \widetilde{K}_n\|\widetilde{u}_n\|^p\right| < \delta,\\ \left|\frac{2p}{p-2}\gamma(1-\alpha) - \int_{\mathbb{R}^N\setminus B(z_n,R_n)} \widetilde{K}_n|\widetilde{u}_n|^p\right| < \delta.\end{gathered} \tag{4.1}$$

We denote now $\mathbb{R}^N = V$. For a subgroup H of G, we denote by

$$V^H = \{z \in V \mid gz = z \text{ for all } g \in H\}$$

the H-fixed point set of V and by z^H the orthogonal projection of $z \in V$ onto V^H. We need the following.

Lemma 4.4. *If for some closed subgroup H of G*

$$\operatorname{dist}(z_n, V^H) \to \infty, \tag{4.2}$$

then there exists a proper closed subgroup K of H such that $|H/K| < \infty$ and a subsequence, denoted again by (z_n), such that

(a) $H_{z_n^K} = K$ *and*
(b) *for every $r > 0$ there is an $n(r) \in \mathbb{N}$ such that*

$$B(gz_n, r) \cap B(g'z_n, r) = \emptyset \quad \textit{for all } [g] \neq [g'] \in H/K \textit{ and } n \geq n(r).$$

Proof. Let $z_n^\perp$ be the orthogonal projection of z_n onto the orthogonal complement $(V^H)^\perp$ of V^H in V. Then, up to a subsequence, $z_n^\perp \neq 0$ and

$$z_n' = \frac{z_n^\perp}{|z_n^\perp|} \to z' \in (V^H)^\perp.$$

Since H acts on $(V^H)^\perp$ without non-trivial fixed points, the isotropy subgroup $K = H_{z'}$ is a proper subgroup of H. For every set of classes $[g_1], \dots, [g_m] \in H/K$ there is a $\rho > 0$ such that

$$B(g_i z', \rho) \cap B(g_j z', \rho) = \emptyset \quad \text{for } i \neq j.$$

So, since $|z_n^\perp| = \operatorname{dist}(z_n, V^H) \to \infty$,

$$B(g_i z_n^\perp, r) \cap B(g_j z_n^\perp, r) = \emptyset \quad \text{for } i \neq j \text{ and } n \text{ large},$$

and therefore

$$B(g_i z_n, r) \cap B(g_j z_n, r) = \emptyset \quad \text{for } i \neq j \text{ and } n \text{ large}.$$

Since $\widetilde{u}_n$ is G-invariant,

$$m \int_{B(z_n,r)} \widetilde{K}_n |\widetilde{u}_n|^p \leq \int \widetilde{K}_n |\widetilde{u}_n|^p = \frac{2p}{p-2}\gamma + o(1).$$

Hence, $|H/K| < \infty$. Finally, since $(z_n')^K \to (z')^K = z'$, the isotropy subgroup $H_{(z_n^\perp)^K} = H_{(z_n')^K} \subset K = H_{z'}$. and therefore $H_{z_n^K} = K$. $\square$

We go on with the proof of Theorem 4.3. Starting with $H = G$ we apply Lemma 4.4 inductively as many times as (4.2) is satisfied (maybe none) until we arrive at a closed subgroup Γ of finite index in G and a subsequence (z_n) such that

$$\begin{gathered} \operatorname{dist}(z_n, V^\Gamma) < C < \infty, \\ G_{z_n^\Gamma} = \Gamma, \\ B(g z_n, r) \cap B(g' z_n, r) = \emptyset \quad \text{for all } [g] \neq [g'] \in G/\Gamma,\ n \geq n(r). \end{gathered} \tag{4.3}$$

(Observe that, if (4.2) does not hold for $H = G$, then $\Gamma := G$ satisfies these three conditions). Let $\zeta_n := z_n^\Gamma$ be the orthogonal projection of z_n onto V^Γ and let

$$\overline{u}_n = \widetilde{u}_n(\,\cdot\, + \zeta_n), \quad \overline{K}_n = \widetilde{K}_n(\,\cdot\, + \zeta_n).$$

Since $\|\overline{u}_n\| = \|\widetilde{u}_n\|$ is bounded, a subsequence

$$\begin{aligned} \overline{u}_n &\rightharpoonup \overline{u} \quad && \text{weakly in } H^1(\mathbb{R}^N), \\ \overline{u}_n &\to \overline{u} && \text{a.e. on } \mathbb{R}^N, \\ \overline{u}_n &\to \overline{u} && \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N). \end{aligned}$$

It follows from (4.1) and (4.3) that, for $r \geq R$ and n large enough,

$$\begin{aligned} \frac{2p}{p-2}\gamma\alpha - \delta &\leq \int_{B(z_n,R)} \widetilde{K}_n |\widetilde{u}_n|^p \leq \int_{B(\zeta_n, C+r)} \widetilde{K}_n |\widetilde{u}_n|^p \\ &\leq \int_{B(z_n, 2C+r)} \widetilde{K}_n |\widetilde{u}_n|^p \leq \int_{B(z_n, R_n)} \widetilde{K}_n |\widetilde{u}_n|^p \leq \frac{2p}{p-2}\gamma\alpha + 2\delta. \end{aligned}$$

Since $\overline{u}_n \to \overline{u}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\int_{B(\zeta_n,C+r)} \widetilde{K}_n|\widetilde{u}_n|^p = \int_{B(0,C+r)} \overline{K}_n|\overline{u}_n|^p \to \lim_{n\to\infty} K(\varepsilon_n\zeta_n) \int_{B(0,C+r)} |\overline{u}|^p$$

as $n \to \infty$. Letting first $n \to \infty$ and then $r \to \infty$ we obtain

$$\frac{2p}{p-2}\gamma\alpha - \delta \le \lim_{n\to\infty} K(\varepsilon_n\zeta_n) \int |\overline{u}|^p \le \frac{2p}{p-2}\gamma\alpha + 2\delta$$

for all $\delta > 0$ and, therefore,

$$\lim_{n\to\infty} K(\varepsilon_n\zeta_n) \int |\overline{u}|^p = \frac{2p}{p-2}\gamma\alpha. \tag{4.4}$$

In particular, $\overline{u} \neq 0$. Furthermore, since $\widetilde{u}_n$ is G-invariant, it follows from (4.3) that

$$\begin{aligned} |G/\Gamma| \int_{B(0,C+r)} \overline{K}_n|\overline{u}_n|^p &\le |G/\Gamma| \int_{B(z_n,2C+r)} \widetilde{K}_n|\widetilde{u}_n|^p \\ &\le \int \widetilde{K}_n|\widetilde{u}_n|^p = \frac{2p}{p-2}\gamma + o(1) \end{aligned}$$

for n large enough. So letting first $n \to \infty$ and then $r \to \infty$ we obtain

$$\alpha \le |G/\Gamma|^{-1}. \tag{4.5}$$

Moreover, for $\delta < 2p/(p-2)\gamma\alpha$ and all n large enough,

$$0 < \int_{B(\zeta_n,C+R)} \widetilde{K}_n|\widetilde{u}_n|^p = \varepsilon_n^{-N} \int_{B(\varepsilon_n\zeta_n,\varepsilon_n(C+R))} K|u_n|^p.$$

Hence $\mathrm{dist}(\varepsilon_n\zeta_n, \Omega) < \varepsilon_n(C+R)$. Let

$$y_n := \varepsilon_n\zeta_n, \quad \widehat{K} := \lim_{n\to\infty} K(y_n).$$

Then $G_{y_n} = \Gamma$. If

$$\liminf_{n\to\infty} \varepsilon_n^{-1}\mathrm{dist}(y_n, \partial\Omega) < \infty$$

we may assume that

$$\lim_{n\to\infty} \varepsilon_n^{-1}\mathrm{dist}(y_n, \partial\Omega) = d.$$

It is then easy to verify that, up to a rotation, the sets $\Omega_n := \{z \in \mathbb{R}^N \mid \varepsilon_n z + y_n \in \Omega\}$ satisfy

$$\bigcap_{k=1}^{\infty} \bigg(\bigcup_{n=k}^{\infty} \Omega_n \bigg) = \mathbb{H}^N,$$

where $\mathbb{H}^N = \{(z_1, \ldots, z_N) \in \mathbb{R}^N : z_N > -d\}$. Hence, $\overline{u}$ is a solution of the limiting problem

$$\begin{cases} -\Delta u + u = \widehat{K}|u|^{p-2}u & \text{in } \mathbb{H}^N, \\ u \in H^1_0(\mathbb{H}^N), \end{cases}$$

in the half-space $\mathbb{H}^N$ and, by [16], $\overline{u}$ must be zero. This is a contradiction. Therefore,

$$\lim_{n\to\infty} \varepsilon_n^{-1}\mathrm{dist}(y_n, \partial\Omega) = \infty$$

and $\overline{u}$ is a solution of the limiting problem

$$(\mathrm{P}_{\infty,\widehat{K}}) \qquad \begin{cases} -\Delta u + u = \widehat{K}|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

If $\gamma = \mu_K^G$, Proposition 3.3 and Equations (4.4) and (4.5) above imply

$$\frac{\#Gy_n}{\widehat{K}^{2/(p-2)}}\mu_{1,\mathbb{R}^N} \le |G/\Gamma|\frac{p-2}{2p}\widehat{K}\int |\overline{u}|^p \le \mu_K^G \le \Big(\min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}}\Big)\mu_{1,\mathbb{R}^N}.$$

Condition 4.2 implies that (y_n) is bounded. Therefore a subsequence $y_n \to y \in \overline{\Omega}$, and the inequalities above imply $G_y = \Gamma$, $K(y) = \widehat{K}$,

$$\frac{|G/\Gamma|}{\widehat{K}^{2/(p-2)}} = \frac{\#Gy}{K(y)^{2/(p-2)}} = \min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}}, \tag{4.6}$$

$\overline{u}$ is a ground state solution of $(\mathrm{P}_{\infty,\widehat{K}})$, that is

$$\overline{u} = K(y)^{1/2-p}\omega, \tag{4.7}$$

and

$$\mu_K^G = \Big(\min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}}\Big)\mu_{1,\mathbb{R}^N}. \tag{4.8}$$

We now prove (c). Since $\varepsilon_n^{-1}|gy_n - y_n| \to \infty$ if $g \notin \Gamma$, it follows that $\overline{u}(\,\cdot\, + \varepsilon_n^{-1}(gy_n - y_n)) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. Therefore, for every subset S of G/Γ which does not contain the identity class,

$$\begin{aligned} \varepsilon_n^{-N}\Big\|u_n - \sum_{[g]\in S}\overline{u}(\varepsilon_n^{-1}(\,\cdot\, - gy_n))\Big\|_{\varepsilon_n}^2 &= \Big\|\overline{u}_n - \sum_{[g]\in S}\overline{u}(\,\cdot\, + \varepsilon_n^{-1}(gy_n - y_n))\Big\|^2 \\ &= \Big\|\overline{u}_n - \overline{u} - \sum_{[g]\in S}\overline{u}(\,\cdot\, + \varepsilon_n^{-1}(gy_n - y_n))\Big\|^2 + \|\overline{u}\|^2 + o(1) \\ &= \varepsilon_n^{-N}\Big\|u_n - \overline{u}(\varepsilon_n^{-1}(\,\cdot\, - y_n)) - \sum_{[g]\in S}\overline{u}(\varepsilon_n^{-1}(\,\cdot\, - gy_n))\Big\|_{e_n}^2 \\ &\quad + \frac{2p}{p-2}\widehat{K}^{2/(2-p)}\mu_{1,\mathbb{R}^N} + o(1). \end{aligned}$$

Choose $[g_0] \in S$ and let $S' = S \setminus \{[g_0]\}$. Then, since $\| \cdot \|_\varepsilon$ is G-invariant and $\overline{u}$ is radially symmetric,

$$\Big\| u_n - \sum_{[g]\in S} \overline{u}(\varepsilon_n^{-1}(\,\cdot\, - gy_n))\Big\|_{\varepsilon_n}^2 = \Big\| g_0 u_n - \overline{u}(\varepsilon_n^{-1}(\,\cdot\, - g_0 y_n)) - \sum_{[g]\in S'} \overline{u}(\varepsilon_n^{-1}(\,\cdot\, - g_0 g y_n))\Big\|_{\varepsilon_n}^2.$$

Since u_n is G-invariant, we may proceed inductively to obtain

$$\varepsilon_n^{-N}\|u_n\|_{\varepsilon_n}^2 = \varepsilon_n^{-N}\Big\| u_n - \sum_{[g]\in G/\Gamma} \overline{u}(\varepsilon_n^{-1}(\,\cdot\, - gy_n))\Big\|_{\varepsilon_n}^2 + |G/\Gamma|\frac{2p}{p-2}\widehat{K}^{2/(2-p)}\mu_{1,\mathbb{R}^N} + o(1),$$

and (c) follows from and equations (4.6)–(4.8). □

An immediate consequence is the following.

Corollary 4.5. *Assume Condition* 4.2 *holds. Then* $\mu_K^G = \infty$ *if and only if every* G*-orbit of* Ω *is infinite.*

For the critical exponent problem a result similar to Theorem 4.3 was proved in [10], however without the condition that the isotropy subgroups of the y_n's coincide with the one of its limit point y. This fact shall be rather useful for proving Theorems 1.1 and 1.2.

5. Proof of Theorems 1 and 2

The Nehari-manifold $\mathcal{N}_{\varepsilon,K}^G$ is symmetric with respect to the origin and the functional $E_{\varepsilon,K}^G : \mathcal{N}_{\varepsilon,K}^G \to \mathbb{R}$ is even. So critical points appear in pairs $\{u, -u\}$. According to Proposition 2.1, the number of critical pairs with

$$E_{\varepsilon,K}^G(u) = E_{\varepsilon,K}^G(-u) < 2\mu_{\varepsilon,K}^G$$

is a lower bound for the number solutions of $(P_{\varepsilon,K}^G)$.

It is well known that, if the functional $E_{\varepsilon,K}^G : \mathcal{N}_{\varepsilon,K}^G \to \mathbb{R}$ satisfies the *Palais–Smale condition* in the interval $[\mu_{\varepsilon,K}^G, \nu]$, that is, if

(PS) Every sequence $(u_n) \in \mathcal{N}_{\varepsilon,K}^G$ such that

$$E_{\varepsilon,K}^G(u_n) \to c \in [\mu_{\varepsilon,K}^G, \nu] \quad \text{and} \quad \|\nabla E_{\varepsilon,K}^G(u_n)\|_\varepsilon \to 0$$

has a convergent subsequence,

then the number of critical antipodal pairs of $E_{\varepsilon,K}^G : \mathcal{N}_{\varepsilon,K}^G \to \mathbb{R}$ with values $\le \nu$ is at least

$$\{\mathbb{Z}/2\}\text{-cat}(E_{\varepsilon,K}^G)^{\le \nu},$$

where $(E^G_{\varepsilon,K})^{\leq\nu} := \{u \in \mathcal{N}^G_{\varepsilon,K} : E^G_{\varepsilon,K}(u) \leq \nu\}$ and $\{\mathbb{Z}/2\}$-cat(X) denotes the *equivariant* $\{\mathbb{Z}/2\}$*-category of* X that is, the smallest number of open subsets which cover X each of which can be deformed into a pair $\{x,-x\}$ in X through a an odd deformation (see for example [2], [12]).

If Y is a subspace of Z we denote by $\operatorname{cat}_Z(Y)$ the *Lusternik–Schnirelmann category of* Y in Z [22], [18], that is, the smallest number of open subsets of Z, each of them contractible in Z, which cover Y, and we write $\operatorname{cat}(Z) := \operatorname{cat}_Z(Z)$.

We shall need the following easy lemma:

Lemma 5.1. *Let X be a fixed point free $\mathbb{Z}/2$-space and assume there is a space Z, a subspace $Y \subset Z$ and maps*

$$Y \xrightarrow{\iota} X \xrightarrow{\beta} Z$$

such that $\beta(x) = \beta(-x)$ for every $x \in X$, and $\beta\circ\iota(y) = y$ for all $y \in Y$. Then,

$$\operatorname{cat}_Z(Y) \leq \{\mathbb{Z}/2\}\text{-}\operatorname{cat}(X).$$

Proof. Since $\mathbb{Z}/2$ acts freely on X, $\{\mathbb{Z}/2\}$-cat$(X) = \operatorname{cat}(\widehat{X})$ where $\widehat{X}$ is the quotient space of X obtained by identifying each x with $-x$. Then β induces a map $\widehat{\beta} : \widehat{X} \to Z$. If $\widehat{\iota} : Y \to \widehat{X}$ denotes the composition of ι with the quotient map $X \to \widehat{X}$, then $\widehat{\beta}\circ\widehat{\iota}(y) = y$ for all $y \in Y$ and the result follows from [11, 1.3(3)]. □

Let

$$M := \left\{ y \in \overline{\Omega} \;\middle|\; \frac{\#Gy}{K(y)^{2/(p-2)}} = \min_{x\in\overline{\Omega}} \left(\frac{\#Gx}{K(x)^{2/(p-2)}} \right) \right\}$$

and, for $\rho > 0$, let

$$\begin{aligned} M^-_\rho &:= \{y \in M \mid \operatorname{dist}(y,\partial\Omega) \geq \rho\},\\ M^+_\rho &:= \{y \in \mathbb{R}^N \mid \operatorname{dist}(y,M) \leq \rho\}. \end{aligned}$$

We shall show that, given $\rho > 0$ and $\gamma_2 > \mu^G_K$, there exist $\overline{\varepsilon} > 0$ and $\mu^G_K < \gamma < \min\{\gamma_2, 2\mu^G_K\}$ with the following property: For every $0 < \varepsilon < \overline{\varepsilon}$, the inequalities $\mu^G_{\varepsilon,K} \leq \varepsilon^N\gamma < 2\mu^G_{\varepsilon,K}$ hold, and there exist two continuous functions

$$M^-_\rho \xrightarrow{\iota_{\rho,\varepsilon}} (E^G_{\varepsilon,K})^{\leq\varepsilon^N\gamma} \xrightarrow{\beta_{\rho,\varepsilon}} M^+_\rho/G$$

such that $\iota_{\rho,\varepsilon}$ is G-invariant, $\beta_{\rho,\varepsilon}(u) = \beta_{\rho,\varepsilon}(-u)$ and $\beta_{\rho,\varepsilon}(\iota_{\rho,\varepsilon}(y)) = Gy$. Lemma 5.1 then yields Theorems 1.1 and 1.2 provided the Palais–Smale condition holds.

We shall assume throughout that Ω contains a finite G-orbit and that Condition 4.2 is satisfied. This implies, in particular, that M is compact and that every G-orbit in M is finite. Now, isotropy subgroups satisfy that $G_{gx} = gG_xg^{-1}$, therefore the set of isotropy subgroups of a G-space consists of complete conjugacy classes [14]. Let $\{(\Gamma_1),\ldots,(\Gamma_m)\}$ be the set of conjugacy classes of those subgroups of G which occur as isotropy subgroups in M. Fix a subgroup Γ_i in each conjugacy class (Γ_i) and let

$$M_i := \{y \in M \mid G_y = \Gamma_i\}.$$

Then $M = GM_1 \cup \ldots \cup GM_m$ where $GM_i := \{gy \mid g \in G,\ y \in M_i\}$ and, since

$$\frac{|G/\Gamma_i|}{K(y)^{2/(p-2)}} = \min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}}$$

for each $y \in GM_i$, K is constant on GM_i. We denote by K_i the value of K on GM_i.

It follows easily from the definition of M that each M_i is compact. So we may fix a $\overline{\rho} > 0$ such that

$$\begin{aligned} |y - gy| &> 2\overline{\rho} \quad \text{if } gy \neq y \in M, \\ \operatorname{dist}(GM_i, GM_j) &> 2\overline{\rho} \quad \text{if } i \neq j. \end{aligned} \tag{5.1}$$

Proposition 5.3. *Given $\rho > 0$ and $\mu_K^G < \gamma < 2\mu_K^G$ there exists $\varepsilon_1 = \varepsilon_1(\rho, \gamma) > 0$ such that, for every $0 < \varepsilon < \varepsilon_1$,*

(a) *$\varepsilon^N \gamma < 2\mu_{\varepsilon,K}^G$ and*

(b) *there exits a G-invariant map $\iota_{\rho,\varepsilon} : M_\rho^- \to (E_{\varepsilon,K}^G)^{\leq \varepsilon^N\gamma}$.*

Proof. Let $\omega_{\varepsilon^{-1}\rho}$ be the positive ground state solution of the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } B(0, \varepsilon^{-1}\rho), \\ u = 0 & \text{on } \partial B(0, \varepsilon^{-1}\rho), \end{cases}$$

that is, $\|\omega_{\varepsilon^{-1}\rho}\|^2 = |\omega_{\varepsilon^{-1}\rho}|_p^p$ and $p - 2/2p\|\omega_{\varepsilon^{-1}\rho}\|^2 = \mu_{1,B(0,\varepsilon^{-1}\rho)}$. For $\rho \leq \overline{\rho}$, let $\iota_{\rho,\varepsilon} : M_\rho^- \to \mathcal{N}_{\varepsilon,K}^G$ be given by

$$\iota_{\rho,\varepsilon}(y) := \sum_{[g]\in G/G_y} \bigg(\frac{|\omega_{\varepsilon^{-1}\rho}(\varepsilon^{-1}(\,\cdot\, - gy))|_{p,K}^p}{\|\omega_{\varepsilon^{-1}\rho}(\varepsilon^{-1}(\,\cdot\, - gy))\|_\varepsilon^2} \bigg)^{1/2-p} \omega_{\varepsilon^{-1}\rho}(\varepsilon^{-1}(\,\cdot\, - gy)) \in \mathcal{N}_{\varepsilon,K}^G.$$

Then, using (5.2), we obtain

$$\begin{aligned} &\varepsilon^{-N} E_{\varepsilon,K}(\iota_{\rho,\varepsilon}(y)) \\ &= \sum_{[g]\in G/G_y} \bigg(\frac{|\omega_{\varepsilon^{-1}\rho}(\varepsilon^{-1}(\,\cdot\, - gy))|_{p,K}^p}{\|\omega_{\varepsilon^{-1}\rho}(\varepsilon^{-1}(\,\cdot\, - gy))\|_\varepsilon^2} \bigg)^{2/(2-p)} \frac{p-2}{2p} \varepsilon^{-N} \|\omega_{\varepsilon^{-1}\rho}(\varepsilon^{-1}(\,\cdot\, - gy))\|_\varepsilon^2 \\ &= \sum_{[g]\in G/G_y} \bigg(\frac{\int K(\varepsilon(\,\cdot\,) + gy)|\omega_{\varepsilon^{-1}\rho}|^p}{\|\omega_{\varepsilon^{-1}\rho}\|^2} \bigg)^{2/(2-p)} \frac{p-2}{2p} \|\omega_{\varepsilon^{-1}\rho}\|^2 \\ &= \frac{|G/G_y|}{K(y)^{2/(p-2)}} \mu_{1,\mathbb{R}^N} + o_\varepsilon(1) = \mu_K^G + o_\varepsilon(1), \end{aligned}$$

where $o_\varepsilon(1) \to 0$ uniformily in M. Since $\varepsilon^{-N}\mu^G_{\varepsilon,K} \to \mu^G_K$ as $\varepsilon \to 0$ there exists $\varepsilon_1 > 0$, depending on ρ, such that for all $0 < \varepsilon < \varepsilon_1$ and all $y \in M^-_\rho$,

$$\varepsilon^{-N} E_{\varepsilon,K}(\iota_{\rho,\varepsilon}(y)) < \gamma < 2\varepsilon^{-N}\mu^G_{\varepsilon,K}.$$

Finally, if $\rho \geq \overline{\rho}$, we define $\iota_{\rho,\varepsilon} = \iota_{\overline{\rho},\varepsilon}$. □

The outgoing map $\beta_{\rho,\varepsilon}$ requires some more work. For each $\rho \leq \overline{\rho}$ we consider the ρ-neighbourhood

$$M_i^\rho := \{x \in V_i \mid \operatorname{dist}(x, M_i) \leq \rho\}$$

of M_i in the Γ_i-fixed point space $V_i := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in \Gamma_i\}$. Notice that $GM_1^\rho \cup \ldots \cup GM_m^\rho \subset M^+_\rho$ and that $G_y = \Gamma_i$ for every $y \in M_i^\rho$ if $\rho \leq \overline{\rho}$. Let

$$\theta_{\varepsilon,y} := \sum_{[g]\in G/\Gamma_i} K_i^{1/2-p}\omega(\varepsilon^{-1}(\,\cdot\, - gy))$$

and, for $\varepsilon > 0$, $\rho \leq \overline{\rho}$, and $\delta > 0$, consider the sets

$$\begin{aligned}
\Theta_{\varepsilon,\rho} &:= \{\theta_{\varepsilon,y} \mid y \in M_1^\rho \cup \ldots \cup M_m^\rho\},\\
\Theta^\delta_{\varepsilon,\rho} &:= \{v \in H^1_0(\Omega)^G \mid \varepsilon^{-N/2}\|v - \theta_{\varepsilon,y}\|_\varepsilon < \delta \text{ for some } \theta_{e,y} \in \Theta_{\varepsilon,\rho}\}.
\end{aligned}$$

Then,

Proposition 5.4. *Given $\delta > 0$ and $\rho > 0$ there exist an $\varepsilon_2 = \varepsilon_2(\delta,\rho) > 0$ and a $\gamma = \gamma(\delta,\rho) > \mu^G_K$ such that, for every $0 < \varepsilon < \varepsilon_2$,*

$$(E^G_{\varepsilon,K})^{\leq \varepsilon^N\gamma} \subset \Theta^\delta_{\varepsilon,\rho}.$$

Proof. If this were not so, then for some $\delta > 0$ and some $\rho > 0$ there would exist a sequence of positive numbers $\varepsilon_n \to 0$ and a sequence (u_n) in $\mathcal{N}^G_{\varepsilon,K}$ such that $\varepsilon_n^{-N}E_{\varepsilon_n,K}(u_n) \leq \mu^G_K + 1/n$ and $u_n \notin \Theta^\delta_{\varepsilon_n,\rho}$. As in Ekeland's Variational Principle [15] we may assume that (ε_n, u_n) is a minimizing PS-sequence. This contradicts Theorem 4.3. □

We wish to show that

Proposition 5.5. *Given $0 < \rho < \overline{\rho}$ there exist a $\gamma = \gamma(\rho) > \mu^G_K$ and an $\varepsilon_3 = \varepsilon_3(\rho) > 0$ with the property that, for every $0 < \varepsilon < \varepsilon_3$ and every $u \in (E^G_{\varepsilon,K})^{\leq\varepsilon^N\gamma}$, there is exactly one G-orbit $Gy_{\varepsilon,u}$, such that $y_{\varepsilon,u} \in M_1^\rho \cup \ldots \cup M_m^\rho$ and*

$$\varepsilon^{-N/2}\|u - \theta_{\varepsilon,y_{\varepsilon,u}}\|_\varepsilon = \min_{\theta\in\Theta_{\varepsilon,\rho}} \varepsilon^{-N/2}\|u - \theta\|_\varepsilon. \tag{5.2}$$

We need the following lemma.

Lemma 5.6.

(a) $\varepsilon^{-N}\|\theta_{\varepsilon,y}\|_\varepsilon^2 = (2p/(p-2))\mu_K^G + o_\varepsilon(1)$ *where* $o_\varepsilon(1) \to 0$ *uniformly in* $M_1^{\overline{\rho}} \cup \ldots \cup M_m^{\overline{\rho}}$.

(b) *Given* $r > 0$ *there is an* $\varepsilon_4 = \varepsilon_4(r) > 0$ *such that, if* $0 < \varepsilon < \varepsilon_4$ *and* $y, y' \in M_1^{\overline{\rho}} \cup \ldots \cup M_m^{\overline{\rho}}$ *satisfy*

$$\varepsilon^{-N}\|\theta_{\varepsilon,y} - \theta_{\varepsilon,y'}\|_\varepsilon^2 < \frac{2p}{p-2}\mu_K^G,$$

then $\operatorname{dist}(Gy, Gy') < r$.

Proof. Since $|\omega|$ and $|\nabla\omega|$ decay exponentially as $|z| \to \infty$,

$$|\langle\omega, \omega(\,\cdot\, + z)\rangle| \le Me^{-a|z|}$$

where M and a are positive constants independent of z. Therefore, for $y \in M_i^{\overline{\rho}}$,

$$\begin{aligned}\varepsilon^{-N}\|\theta_{\varepsilon,y}\|_\varepsilon^2 &= \|\sum_{[g]\in G/\Gamma_i} K_i^{1/2-p}\omega(\,\cdot\, - \varepsilon^{-1}gy)\|^2 \\ &= |G/\Gamma_i|K_i^{2/(2-p)}\|\omega\|^2 + 2\sum_{[g]\neq[g']} K_i^{2/2-p}\langle\omega, \omega(\,\cdot\, - \varepsilon^{-1}(gy - g'y))\rangle \\ &= \frac{2p}{p-2}\mu_K^G + o_\varepsilon(1)\end{aligned}$$

with $|o_\varepsilon(1)| \le M'e^{-2a\varepsilon^{-1}\overline{\rho}}$. This proves the first assertion.

Now, if $y \in M_i^{\overline{\rho}}$, $y' \in M_j^{\overline{\rho}}$ and $\varepsilon^{-N}\|\theta_{\varepsilon,y} - \theta_{\varepsilon,y'}\|_\varepsilon^2 < 2p/(p-2)\mu_K^G$, then

$$\begin{aligned}\frac{2p}{p-2}\mu_K^G + o_e(1) &\le \varepsilon^{-N}(\|\theta_{\varepsilon,y}\|_\varepsilon^2 + \|\theta_{\varepsilon,y'}\|_\varepsilon^2) - \frac{2p}{p-2}\mu_K^G \\ &< 2\varepsilon^{-N}\langle\theta_{\varepsilon,y}, \theta_{\varepsilon,y'}\rangle_\varepsilon \\ &= 2\sum_{[g]\in G/\Gamma_i}\sum_{[g']\in G/\Gamma_j}(K_iK_j)^{1/2-p}\langle\omega, \omega(\,\cdot\, - \varepsilon^{-1}(gy - g'y'))\rangle \\ &\le M''e^{-a\varepsilon^{-1}\operatorname{dist}(Gy,Gy')}\end{aligned}$$

This implies that $\operatorname{dist}(Gy, Gy') \le C\varepsilon$ and (b) follows. □

Proof of Proposition 5.5. Choose

$$0 < \delta < \min\left\{\sqrt{\frac{p}{p-2}\mu_K^G}, \frac{1}{2}\left\|\frac{\partial\omega}{\partial z_1}\right\|^2\left(N\max_{i,j,k} K_i^{1/2-p}\left\|\frac{\partial^2\omega}{\partial z_j\partial z_k}\right\|\right)^{-1}\right\} \tag{5.3}$$

and let $\varepsilon_2 = \varepsilon_2(\delta, \rho/3) > 0$ and a $\gamma' = \gamma(\delta, \rho/3) > \mu_K^G$ be as in Proposition 5.4. Let $\varepsilon_4 = \varepsilon_4(\rho/3)$ be as in Lemma 5.6. Then, for every $0 < \varepsilon < \min\{\varepsilon_2, \varepsilon_4\}$ and $u \in (E_{\varepsilon,K}^G)^{\le\varepsilon^N\gamma'}$ there is a $y \in M_i^{\rho/3}$ such that $\varepsilon^{-N/2}\|u - \theta_{\varepsilon,y}\|_\varepsilon < \delta$. Thus, if $\overline{y} \in M_j^\rho$ satisfies (5.2), then

$$\varepsilon^{-N/2}\|u - \theta_{\varepsilon,\overline{y}}\|_\varepsilon = \min_{\theta\in\Theta_{\varepsilon,\rho}} \varepsilon^{-N/2}\|u - \theta\|_\varepsilon < \delta \tag{5.4}$$

and, therefore,

$$\varepsilon^{-N}\|\theta_{\varepsilon,y}-\theta_{\varepsilon,\overline{y}}\|_{\varepsilon}^2<\frac{2p}{p-2}\mu_K^G.$$

By Lemma 5.6, $\operatorname{dist}(Gy,G\overline{y})<\rho/3$. Hence, (5.1) implies that $\overline{y}\in M_i^{2\rho/3}$. So our problem reduces to showing that any two minima of the function

$$f_{\varepsilon,u}(y)=\varepsilon^{-N}\|u-\theta_{\varepsilon,y}\|_{\varepsilon}^2=\|\widetilde{u}-\widetilde{\theta}_{\varepsilon,y}\|^2$$

defined on M_i^ρ lie on the same G-orbit. Here we write $\widetilde{v}(z):=v(\varepsilon z)$. The function $f_{\varepsilon,u}$ is twice differentiable and, for every $y\in M_i^\rho$ and $h\in V_i$,

$$\begin{aligned}D^2f_{\varepsilon,u}(y)(h,h)=2\varepsilon^{-2}\Bigg[\Bigg\|&\sum_{[g]\in G/\Gamma_i}K_i^{1/2-p}D\omega(\,\cdot\,-\varepsilon^{-1}gy)gh\Bigg\|^2\\&-\Bigg\langle\widetilde{u}-\widetilde{\theta}_{\varepsilon,y},\sum_{[g]\in G/\Gamma_i}K_i^{1/2-p}D^2\omega(\,\cdot\,-\varepsilon^{-1}gy)(gh,gh)\Bigg\rangle\Bigg]\end{aligned}$$

By (5.2), $|y-gy|\geq 2(\overline{\rho}-\rho)$ if $gy\neq y$, $y\in M_i^\rho$, so following the argument in the proof of Lemma 5.6 we obtain

$$\Bigg\|\sum_{[g]\in G/\Gamma_i}K_i^{1/2-p}D\omega(\,\cdot\,-\varepsilon^{-1}g\overline{y})gh\Bigg\|^2=|h|^2\left(\frac{|G/\Gamma_i|}{K_i^{2/(p-2)}}\left\|\frac{\partial\omega}{\partial z_1}\right\|^2+o_e(1)\right)$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$ independently of $y\in M_i^\rho$. If $\overline{y}$ is a minimum of $f_{\varepsilon,u}$ it follows from (5.2) and (5.3) that there exist $0<\varepsilon_3'\leq\min\{\varepsilon_2,\varepsilon_4\}$ and positive constant C independent of u,ε, $\overline{y}$ and h such that, if $0<\varepsilon<\varepsilon_3'$, then

$$\begin{aligned}&D^2f_{\varepsilon,u}(\overline{y})(h,h)\geq 2\frac{|G/\Gamma_i|}{K_i^{2/(p-2)}}\varepsilon^{-2}|h|^2\\&\quad\left(\left\|\frac{\partial\omega}{\partial z_1}\right\|^2+o_e(1)-NK_i^{1/2-p}\max_{j,k}\left\|\frac{\partial^2\omega}{\partial z_j\partial z_k}\right\|\|\widetilde{u}-\widetilde{\theta}_{\varepsilon,\overline{y}}\|\right)\geq C\varepsilon^{-2}|h|^2.\end{aligned}$$

It follows that there is an $R>0$, independent of $0<\varepsilon<\varepsilon_3'$ and $u\in(E_{\varepsilon,K}^G)^{\leq\varepsilon^N\gamma'}$, such that any two minima $\overline{y}_1\neq\overline{y}_2$ of $f_{\varepsilon,u}$ in M_i^ρ satisfy

$$|\overline{y}_1-\overline{y}_2|\geq\varepsilon R. \tag{5.5}$$

We now argue by contradiction: Assume there are sequences $\gamma_n\to\mu_K^G$, $\varepsilon_n\to 0$ and $u_n\in(E_{\varepsilon_n,K}^G)^{\leq\varepsilon_n^N\gamma_n}$ such that f_{ε_n,u_n} has at least two minima $\overline{y}_{n,1},\overline{y}_{n,2}$ in M_i^ρ with $G\overline{y}_{n,1}\neq G\overline{y}_{n,2}$. Then Theorem 4.3 asserts that $\varepsilon_n^{-N}\|u_n-\theta_{\varepsilon_n,\overline{y}_{n,i}}\|_{\varepsilon_n}^2\to 0$ for $i=1,2$ and therefore

$$\begin{aligned}&K_i^{2/(2-p)}\Bigg\|\sum_{[g]\in G/\Gamma_i}(\omega(\,\cdot\,-\varepsilon_n^{-1}g\overline{y}_{n,1})-\omega(\,\cdot\,-\varepsilon_n^{-1}g\overline{y}_{n,2}))\Bigg\|^2\\&\qquad\qquad=\varepsilon_n^{-N}\|\theta_{\varepsilon_n,\overline{y}_{n,1}}-\theta_{\varepsilon_n,\overline{y}_{n,2}}\|_{\varepsilon_n}^2\to 0\end{aligned}$$

as $n \to \infty$. But (5.5) says that $\mathrm{dist}(G\varepsilon_n^{-1}\overline{y}_{n,1}, G\varepsilon_n^{-1}\overline{y}_{n,2}) \geq R$ for all n sufficiently large. This is a contradiction. $\square$

Corollary 5.7. *Given $0 < \rho < \overline{\rho}$ and $\gamma_2 > \mu_K^G$ there exist a $\mu_K^G < \gamma < \min\{2\mu_K^G, \gamma_2\}$ and an $\widetilde{\varepsilon} > 0$ with the property that, for every $0 < \varepsilon < \widetilde{\varepsilon}$,*

(a) $\varepsilon^N \gamma < 2\mu^G_{\varepsilon,K}$,

(b) *there exist two continuous functions*

$$\iota_{\rho,\varepsilon} : M_\rho^- \to (E^G_{\varepsilon,K})^{\leq \varepsilon^N\gamma} \quad \text{and} \quad \beta_{\rho,\varepsilon} : (E^G_{\varepsilon,K})^{\leq \varepsilon^N\gamma} \to M_\rho^+/G$$

such that $\iota_{\rho,\varepsilon}$ is G-invariant, $\beta_{\rho,\varepsilon}(u) = \beta_{\rho,\varepsilon}(-u)$ and $\beta_{\rho,\varepsilon}(\iota_{\rho,\varepsilon}(y)) = Gy$.

Proof. Let $\gamma(\rho) > \mu_K^G$ and $\varepsilon_3 = \varepsilon_3(\rho) > 0$ be as in Proposition 5.5, let $\gamma := \min\{2\mu_K^G, \gamma_2, \gamma(\rho)\}$, let $\varepsilon_1 = \varepsilon_1(\rho,\gamma) > 0$ be as in Proposition 5.3 and let $\widetilde{\varepsilon} := \min\{\varepsilon_1, \varepsilon_2\}$. Then, for every $0 < \varepsilon < \widetilde{\varepsilon}$, (a) holds. We take $\iota_{\rho,\varepsilon} : M_\rho^- \to (E^G_{\varepsilon,K})^{\leq \varepsilon^N\gamma}$ as in Proposition 5.3 and define

$$\beta_{\rho,\varepsilon} : (E^G_{\varepsilon,K})^{\leq \varepsilon^N\gamma} \to M_\rho^+/G, \quad \beta_{\rho,\varepsilon}(u) := Gy_{\varepsilon,u},$$

with $y_{\varepsilon,u}$ as in Proposition 5.5. It is easy to check that $\iota_{\rho,\varepsilon}$ and $\beta_{\rho,\varepsilon}$ has the desired properties. $\square$

Proof of Theorems 1.1 and 1.2. We may assume that $\rho < \overline{\rho}$. Let $\mu_K^G < \gamma < \min\{2\mu_K^G, \gamma_2\}$ and $\widetilde{\varepsilon} > 0$ be as in Corollary 5.7 above. According to the discussion at the beginning of this section, Lemma 5.1 and Corollary 5.7 imply that, if $0 < \varepsilon < \widetilde{\varepsilon}$, then $E^G_{\varepsilon,K} : \mathcal{N}^G_{\varepsilon,K} \to \mathbb{R}$ has at least

$$\mathrm{cat}_{M_\rho^+/G}(M_\rho^-/G)$$

critical points u with $E^G_{\varepsilon,K}(u) < 2\mu^G_{\varepsilon,K}$ and such that

$$\frac{2p}{p-2}\varepsilon^{-N}\mu^G_{\varepsilon,K} \leq \varepsilon^{-N}\int_\Omega(\varepsilon^2|\nabla u|^2 + |u|^2) = \frac{2p}{p-2}\varepsilon^{-N}E^G_{\varepsilon,K}(u) < \frac{2p}{p-2}\gamma_2$$

provided it satisfies the Palais–Smale condition (PS) in $[\mu^G_{\varepsilon,K}, \varepsilon^N\gamma]$. Moreover, Theorem 4.3(d) says there is an $0 < \overline{\varepsilon} < \widetilde{\varepsilon}$ such that $\varepsilon^{-N}\mu^G_{\varepsilon,K} > \gamma_1$ for all $0 < \varepsilon < \overline{\varepsilon}$. So in order to complete the proof of Theorems 1.1 and 1.2 all we need to show is that $E^G_{\varepsilon,K}$ satisfies (PS) in $[\mu^G_{\varepsilon,K}, \varepsilon^N\gamma]$. This is true and well known if Ω is bounded, so Theorem 1.1 follows immediately.

If Ω has bounded complement then Condition 4.2 is equivalent to

$$\min_{x\in\overline{\Omega}} \frac{\#Gx}{K(x)^{2/(p-2)}} < \min_{x\in\overline{\Omega}} \frac{\#Gx}{K_\infty^{2/(p-2)}}.$$

We choose γ so that it also satisfies

$$\gamma < \mu_\infty^G := \bigg(\min_{x\in\overline{\Omega}} \frac{\#Gx}{K_\infty^{2/(p-2)}}\bigg)\mu_{1,\mathbb{R}^N}.$$

Let $(u_n) \in \mathcal{N}^G_{\varepsilon,K}$ be a Palais–Smale sequence for $E^G_{\varepsilon,K}$ such that $E^G_{\varepsilon,K}(u_n) \to c \in [\mu^G_{\varepsilon,K}, \varepsilon^N \gamma]$. Let $\widetilde{u}_n(z) := u_n(\varepsilon z)$ and $\widetilde{K}(z) := K(\varepsilon z)$. Then $(\widetilde{u}_n)$ is a G-invariant Palais–Smale sequence for

$$E^G_{1,\widetilde{K}}(v) := \frac{1}{2}\int(|\nabla v|^2 + |v|^2) - \frac{1}{p}\int \widetilde{K}(z)|v|^p, \quad v \in H^1_0(\varepsilon^{-1}\Omega)^G,$$

with $E^G_{1,\widetilde{K}}(\widetilde{u}_n) \to \varepsilon^{-N} c < \mu^G_\infty$, and $\varepsilon^{-1}\Omega := \{\varepsilon^{-1} y \mid y \in \Omega\}$ has bounded complement. By Benci and Cerami's Compactness Lemma [3] $(\widetilde{u}_n)$ has a convergent subsequence. So $E^G_{\varepsilon,K} : \mathcal{N}^G_{\varepsilon,K} \to \mathbb{R}$ satisfies the Palais–Smale condition (PS) in $[\mu^G_{\varepsilon,K}, \varepsilon^N\gamma]$ for every $0 < \varepsilon < \overline{\varepsilon}$ and Theorem 1.2 follows. □

The existence of a G-invariant ground state solution under the hypotheses of Theorem 1.2 was shown in [21]. We would like to point out that our Theorem 1.2 does not include the autonomous case because Condition 4.2 does not hold for constant K on unbounded domains. But using the "baryorbit map" $\beta_{\rho,\varepsilon}$ one should be able to obtain similar results to those of [7] for the G-equivariant case.

Proof of Corollaries 1.3 and 1.4. Assume first Ω is bounded. It follows from Theorems 1.1 and 4.3(d) that there is an $\overline{\varepsilon} > 0$ such that, for every $0 < \varepsilon < \overline{\varepsilon}$, problem $(\mathrm{P}_{\varepsilon,K})$ has at least

$$\operatorname{cat}_{M(G_i)^+_\rho/G_i}(M(G_i)^-_\rho/G_i)$$

G_i-invariant solutions u with

$$E_{\varepsilon,K}(u) = \frac{p-2}{2p}\int_\Omega(\varepsilon^2|\nabla u|^2 + |u|^2) < \mu^{G_{i+1}}_{\varepsilon,K}.$$

Therefore these solutions are not G_{i+1}-invariant. This proves Corollary 1.3. The proof of Corollary 1.4 is completely analogous. Just observe that, since $K_\infty < \max_{x\in\overline{\Omega}} K(x)$, Condition (C_{G_i}) in Theorem 1.2 holds for all $i = 0, \dots, m$. □

References

[1] N. Ackermann, *Multiple single-peaked solutions of a class of semilinear Neumann problems via the category of the domain boundary*, Calc. Var. Partial Differential Equations **7** (1998), 263–292.

[2] Th. Bartsch, *Topological methods for variational problems with symmetries*, Lecture Notes in Math **1560** (1993), Springer–Verlag.

[3] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains*, Arch. Rational Mech. Anal. **99** (1987), 283–300.

[4] ———, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **11** (1991), 79–93.

[5] ———, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Calc. Var. Partial Differential Equations **2** (1994), 29–48.

[6] G. Cerami, *The role of the domain shape on the existence and multiplicity of positive solutions of some elliptic nonlinear problems*, Variational Methods in Nonlinear Analysis

(A. Ambrosetti and K. C. Chang, eds.), Gordon and Breach Pubs., Switzerland, 1995, pp. 263–286.

[7] G. Cerami and D. Passaseo, *Existence and multiplicity results for semilinear elliptic Dirichlet problems in exterior domains*, Nonlinear Anal. **24** (1995), 1533–1547.

[8] S. Cingolani and M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equation*, Topol. Methods. Nonlinear Anal. **10** (1997), 397–408.

[9] ______, *Multiple positive solutions to nonlinear Schrödinger equations with compiting potential functions*, J. Differential Equations **160** (2000), 118–138.

[10] M. Clapp, *On the number of positive symmetric solutions of a non-autonomous semilinear elliptic problem*, Nonlinear Anal. **42** (2000), 405–422.

[11] M. Clapp and D. Puppe, *Invariants of the Lusternik–Schnirelmann type and the topology of critical sets*, Trans. Amer. Math. Soc. **298** (1986), 603–620.

[12] ______, *Critical point theory with symmetries*, J. Reine Angew. Math. **418** (1991), 1–29.

[13] M. Del Pino and P. Felmer, *Multi-peak bound states for a nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré **15** (1998), 127–149.

[14] T. tom Dieck, *Transformation groups*, de Gruyter Stud. Math., vol. 8, de Gruyter, Berlin–New York, 1987.

[15] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.

[16] M. J. Esteban and P. J. Lions, *Existence and non-existence results for semilinear elliptic problems in unbounded domains*, Proc. Roy. Soc. Edinburgh Sect A **93** (1982), 1–14.

[17] C. Gui, *Existence of multi-bump solutions for nonlinear Schrödinger equations via variational methods*, Comm. Part. Differential Equations **21** (1996), 787–820.

[18] I. M. James, *On category, in the sense of Lusternik–Schnirelmann*, Topology **17** (1978), 331–348.

[19] Y. Y. Li, *On a singularly perturbed ellipitic equation*, Adv. Differential Equations **2** (1997), 955–980.

[20] P. L. Lions, *The concentration-compacteness principle in the calculus of variations. The locally compact case*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145, 223–283.

[21] ______, *Symmetries and the concentration-compactness method*, Nonlinear Variational Problems, Pitman, London, 1985, pp. 47–56.

[22] L. Lusternik and L. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationels*, Hermann, Paris, 1934.

[23] R. S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), 19–30.

[24] N. Qiao and Z.-Q. Wang, *Multiplicity results for positive solutions to non-autonomous elliptic problems*, Electron. J. Differential Equations **28** (1999), 1–28.

[25] X. Ren, *Least-energy solutions to a non-autonomous semilinear problem with small diffusion coefficient*, Electron. J. Differential Equations **5** (1993), 1–21.

[26] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), 511–517.

[27] ______, *Variational Methods*, Springer–Verlag, Berlin, 1990.

[28] X. Wang, *On a concentration of positive bound states of nonlinear Schrödinger equations*, Comm. Math. Phys. **153** (1993), 223–243.

[29] X. Wang and B. Zeng, *On concentration of positive bound states of nonlinear Schrödinger equations with compiting potential functions*, SIAM J. Math. Anal. **28** (1997), 633–655.

[30] Z.-Q. WANG, *On the existence of multiple, single-peaked solutions for a semilinear Neumann problem*, Arch. Rational Mech. Anal. **120** (1992), 375–399.

[31] M. WILLEM, *Minimax Theorems*, PNLDE 24, Birkhäuser, Boston–Basel–Berlin, 1996.

Manuscript received January 25, 2001

MÓNICA CLAPP
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
04510 México, D. F. MEXICO

E-mail address: mclapp@math.unam.mx

GUSTAVO IZQUIERDO
Departamento de Matemáticas
Universidad Autónoma Metropolitana – Iztapalapa
Av. Michoacán y la Purísima, Col. Vicentina
09340 Iztapalapa, D. F. MEXICO

E-mail address: iubg@xanum.uam.mx