**T**opological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 18, 2001, 1–16

# ON THE EXISTENCE OF THREE SOLUTIONS FOR JUMPING PROBLEMS INVOLVING QUASILINEAR OPERATORS

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Abstract. A jumping problem for quasilinear elliptic equations is considered. A local saddle argument in the framework of nonsmooth critical point theory is applied.

## Introduction

In this paper, we study the number of solutions of a quasilinear elliptic problem of the form

$$\text{(QP)}\qquad \begin{cases} -\displaystyle\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u \\ \qquad\qquad\qquad\qquad\qquad\qquad = g(x,u) + \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a_{ij}(x,s) = a_{ji}(x,s)$, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $\omega \in H^{-1}(\Omega)$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{s \to -\infty} \frac{g(x,s)}{s} = \alpha, \quad \lim_{s \to \infty} \frac{g(x,s)}{s} = \beta.$$

2000 *Mathematics Subject Classification.* 35J20, 58E05, 49J52.

*Key words and phrases.* Jumping problem, quasilinear elliptic equations, nonsmooth critical point theory, weak slope.

Setting $A_{ij}(x) = \lim_{|s|\to\infty} a_{ij}(x,s)$, let us denote with $\lambda_k$ the eigenvalues of the operator $-\sum_{i,j=1}^n D_j(A_{ij}D_i u)$ with homogeneous Dirichlet condition, repeated according to multiplicity.

In the semilinear case:

$$\text{(SP)} \qquad \begin{cases} -\Delta u = g(x,u) + \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

the number of the solutions of (SP), depending on the relation of $\alpha$ and $\beta$ with respect to the eigenvalues $\lambda_k$ of the operator $-\Delta$, has been widely investigated (see e.g. [17], [25], [21], [13] and references therein), starting from the pioneering paper [1]. The methods used are, often, a combination of topological and variational techniques.

In this paper, we suppose $\beta < \lambda_1$, $\alpha > \lambda_2$, $\omega_0 \in H^{-1}(\Omega)$ and we study (QP) when $\omega = t\varphi_1 + \omega_0$, where $\varphi_1$ is a positive eigenfunction corresponding to the first eigenvalue. We prove that (QP) has at least three solutions for $t$ large enough. Let us remark that $\alpha$ can be allowed to be one of the eigenvalues $\lambda_k$.

The case $\beta < \lambda_1 < \alpha$ has been already considered in [6], [7], where it is shown that (QP) has at least two solutions for $t$ large enough and no solutions for $t$ small enough.

As we pointed out in [5]–[8], in the case of quasilinear equations the first difficulty is that classical critical point theory fails. In fact, let us consider the associated functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx - \int_\Omega G(x,u)\, dx - \langle \omega, u \rangle,$$

where $G(x,s) = \int_0^s g(x,t)\, dt$. Under reasonable assumptions on $a_{ij}$ and $g$, it is possible to prove that $f$ is continuous, but we cannot expect $f$ to be of class $C^1$ or locally Lipschitz continuous.

On the other hand,

$$\left\{ u \mapsto -\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u - g(x,u) \right\}$$

is not well defined as an operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ and the classic topological methods, applied so far in the literature, cannot be directly adapted to this setting.

As in previous papers concerning quasilinear equations (see, e.g. [4]–[10], [22]), we will use variational methods based on the nonsmooth critical point theory of [11], [12] to find critical points of an associated functional which are also weak solutions of (QP).

Let us mention that similar abstract techniques have been developed also in [18], [19], while different techniques have been applied to quasilinear equations in [3], [26].

Let us emphasize that, in the semilinear case, an even stronger result holds for $\beta < \lambda_1$ and $\alpha > \lambda_2$, namely the existence of four solutions, as it was proved in [17] and [25], combining variational methods with degree or Morse theory. The same result can be obtained also by the introduction of suitable natural constraints, following the technique used in [16] for variational inequalities.

It seems to be hard to adapt such approaches to the quasilinear case, because of the lack of regularity we have already remarked. Thus, the problem of the existence of at least four solutions seems far from being solved in the quasilinear case.

Our approach, which is purely based on min-max theorems, is more similar to the techniques developed in [23], where a different proof of the existence of at least three solutions was given in the semilinear case.

Let us point out that also in [15] the nonsmooth critical point theory of [11], [12] is applied to obtain the same kind of result for the variational inequality associated with the constraint $u \geq \vartheta$, $\vartheta \in H_0^1(\Omega)$, $\vartheta^- \in L^\infty(\Omega)$. However such setting does not cover the case of equations. More precisely, in the proof of the min-max inequalities the presence of constraint provides some simplifications because in the asymptotic problem the constraint becomes $u \geq 0$ and this excludes all eigenfunctions $\varphi_k$ of the asymptotic linear problem with $k \geq 2$. As a consequence, in [15] it is used the classic Rabinowitz saddle theorem, whereas in this paper we have to apply a more refined local saddle argument.

After giving in Section 2 a brief exposition of nonsmooth critical point theory as developed in [11], [12], in Section 3, by means of some min-max inequalities, we prove the existence of a saddle point for the energy functional $f$. In Section 4 by studying the critical levels of $f$, we show that this solution cannot coincide with the other ones already found in [6], [7].

## 1. The main result

Let $\Omega$ be a connected bounded open subset of $\mathbb{R}^n$ $(n \geq 3)$. Let $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ $(1 \leq i,\ j \leq n)$ be such that

$$\begin{cases} \text{for all } s \in \mathbb{R} & a_{ij}(x,s) \text{ is measurable with respect to } x, \\ \text{for a.e. } x \in \Omega & a_{ij}(x,s) \text{ is of class } C^1 \text{ with respect to } s. \end{cases}$$

Let us make the following assumptions.
For a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, $1 \leq i, j \leq n$,

$$a_{ij}(x,s) = a_{ji}(x,s). \tag{a.1}$$

There exists $C > 0$ such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, $1 \le i,\ j \le n$,

$$\text{(a.2)} \qquad |a_{ij}(x,s)| \le C, \qquad |D_s a_{ij}(x,s)| \le C.$$

There exists $\nu > 0$ such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^n$,

$$\text{(a.3)} \qquad \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \ge \nu|\xi|^2.$$

There exists $R > 0$ such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^n$,

$$\text{(a.4)} \qquad |s| \ge R \Rightarrow \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \ge 0.$$

There exists a uniformly Lipschitz continuous bounded function $\theta : \mathbb{R} \to [0,\infty[$ such that for a.e. $x \in \Omega$, $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^n$

$$\text{(a.5)} \qquad \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \le s\theta'(s) \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j.$$

For a.e. $x \in \Omega$, $1 \le i,\ j \le n$,

$$\text{(a.6)} \qquad \lim_{s\to-\infty} a_{ij}(x,s) = \lim_{s\to\infty} a_{ij}(x,s).$$

Let us observe that by (a.4) such limits exist.

Now, let us consider a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$:

$$\text{(g.1)} \qquad |g(x,s)| \le a(x) + b(x)|s|$$

with $a \in L^{2n/(n+2)}(\Omega)$ and $b \in L^{n/2}(\Omega)$.

Moreover, assume there exist $\alpha, \beta \in \mathbb{R}$ such that for a.e. $x \in \Omega$:

$$\text{(g.2)} \qquad \lim_{s\to-\infty} \frac{g(x,s)}{s} = \alpha, \qquad \lim_{s\to\infty} \frac{g(x,s)}{s} = \beta.$$

Finally, setting

$$A_{ij}(x) = \lim_{s\to\pm\infty} a_{ij}(x,s),$$

let us denote with $\lambda_k$ the eigenvalues of the operator $-\sum D_j(A_{ij}D_i u)$ with homogeneous Dirichlet condition, repeated according to multiplicity. Let $\varphi_1$ be a nonnegative eigenfunction corresponding to $\lambda_1$.

It is known (see [14]) that $\varphi_1 \in H_0^1(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$ and $\varphi_1(x) > 0$ for every $x \in \Omega$.

Now, we can state the main result of the paper.

Theorem 1.1. *Let $a_{ij}$ and $g$ satisfy hypotheses* (a.1)–(a.6), (g.1)–(g.2) *and let $\omega \in H^{-1}(\Omega)$. Assume that $\beta < \lambda_1$ and $\alpha > \lambda_2$. Then there exists $t_0 \in \mathbb{R}^+$ such that for every $t > t_0$ the equation*

$$(1.1.1) \qquad -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u = g(x,u) + t\varphi_1 + \omega$$

*has at least three weak solutions in $H_0^1(\Omega)$. Moreover, if $\omega \in W^{-1,p}(\Omega)$ for some $p > n$ and $a, b \in L^r(\Omega)$ with $r > n/2$, such solutions belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$.*

Let us recall that for weak solutions belonging to $H_0^1(\Omega) \cap L^\infty(\Omega)$, further regularity results can be found in [20].

## 2. Functionals of the calculus of variations

In this section, we recall some results of the nonsmooth critical point theory developed in [11] and [12].

Let $X$ denote a metric space endowed with the metric $d$. Let us set $B_\rho(u) = \{v \in X : d(u,v) \leq \rho\}$ and $S_\rho(u) = \{v \in X : d(u,v) = \rho\}$.

Definition 2.1. Let $f : X \to \mathbb{R}$ be a continuous function and let $u \in X$. We denote by $|df|(u)$ the supremum of the $\sigma$'s in $[0, \infty[$ such that there exist $\delta > 0$ and a continuous map $H : B_\delta(u) \times [0,\delta] \to X$ such that

$$\begin{aligned} d(H(v,t),v) &\leq t && \text{for all } v \in B_\delta(u) \text{ and all } t \in [0,\delta], \\ f(H(v,t)) &\leq f(v) - \sigma t && \text{for all } v \in B_\delta(u) \text{ and all } t \in [0,\delta]. \end{aligned}$$

The extended real number $|df|(u)$ is called the weak slope of $f$ at $u$.

Based on weak slope we introduce the following fundamental notions.

Definition 2.2. Let $f : X \to \mathbb{R}$ be a continuous function. A point $u \in X$ is said to be (*lower*) *critical* for $f$, if $|df|(u) = 0$. A real number $c$ is said to be a (*lower*) *critical* value for $f$, if there exists $u \in X$ such that $|df|(u) = 0$ and $f(u) = c$.

Definition 2.3. Let $f : X \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. We say that $f$ satisfies $(\mathrm{PS})_c$, i.e. the Palais–Smale condition at level $c$, if from every sequence $(u_h)$ in $X$ with $|df|(u_h) \to 0$ and $f(u_h) \to c$ as $h \to \infty$ it is possible to extract a subsequence $(u_{h_k})$ converging in $X$.

The next results are extensions of two classical theorems to a continuous functional. (cf. [2], [24], [27], [21]).

Theorem 2.4 (cf. e.g. [6, Theorem 1.3]). *Let $X$ be complete and $f : X \to \mathbb{R}$ a continuous functional. Let $v_0, v_1 \in X$. Suppose that there exists $r > 0$ such that $d(v_1, v_0) > r$ and*

$$\inf\{f(u) : u \in X,\ d(u, v_0) = r\} > \max\{f(v_0), f(v_1)\}.$$

*Set*

$$\Gamma = \{\gamma : [0,1] \to X \text{ continuous with } \gamma(0) = v_0,\ \gamma(1) = v_1\},$$

$$c_1 = \inf_{B_r(v_0)} f \quad \text{and} \quad c_2 = \inf_{\gamma\in\Gamma} \max_{[0,1]}(f \circ \gamma).$$

*Assume that $c_1 > -\infty$, $\Gamma \neq \emptyset$ and that $f$ satisfies the Palais–Smale condition at the two levels $c_1$ and $c_2$. Then $c_1 < c_2$ and there exist a critical point $u_1$ of $f$ with $d(u_1, v_0) < r$ and $f(u_1) = c_1$ and a second critical point $u_2$ with $f(u_2) = c_2$.*

Theorem 2.5. *Let $X$ be a Banach space and $X_1$ and $X_2$ two closed subspaces of $X$ such that $X = X_1 \oplus X_2$ and $\dim X_1 < \infty$. Let $f : X \to \mathbb{R}$ be a continuous function and let us suppose that there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that*

$$\sup_{B_{\rho_1}(0)\cap X_1} f < \inf_{S_{\rho_2}(0)\cap X_2} f, \qquad \sup_{S_{\rho_1}(0)\cap X_1} f < \inf_{B_{\rho_2}(0)\cap X_2} f.$$

*Moreover, let us suppose that $f$ satisfies the $(\mathrm{PS})_c$ condition for every $c \in \mathbb{R}$. Then there exists at least a critical point $u_0$ for $f$ such that*

$$\inf_{B_{\rho_2}(0)\cap X_2} f \le f(u_0) \le \sup_{B_{\rho_1}(0)\cap X_1} f.$$

Proof. If $f \in C^1(E)$, the result can be found in [21, Theorem 2.3]. On the other hand, the Noncritical Interval Theorem has been extended to the continuous case in [11, Theorem 2.15]. Then the argument of [21, Theorem 2.3] can be easily adapted to our situation. □

Now, let $\Omega$, $a_{i,j}$ and $g$ as in the previous section. Let $\omega$ belong to $H^{-1}(\Omega)$.

Let us define $f : H^1_0(\Omega) \to \mathbb{R}$ by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u\, dx - \int_\Omega G(x,u)\, dx - \langle \omega, u\rangle, \tag{2.1}$$

where $G(x,s) = \int_0^s g(x,t)\, dt$.

The associated Euler equation is formally given by the quasilinear problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u \\ \qquad\qquad\qquad = g(x,u) + \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Definition 2.6. We say that $u$ is a *weak solution* of (2.2), if $u \in H_0^1(\Omega)$ and

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u = g(x,u) + \omega$$

in $D'(\Omega)$.

In order to apply variational methods, let us introduce a natural adaptation of Palais–Smale condition.

Definition 2.7. Let $c \in \mathbb{R}$. A sequence $(u_h)$ in $H_0^1(\Omega)$ is said to be a *concrete Palais–Smale sequence* at level $c$ ($(\mathrm{CPS})_c$-sequence, for short) for $f$, if $\lim_h f(u_h) = c$,

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u_h)D_i u_h) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u_h) D_i u_h D_j u_h - g(x,u_h) \in H^{-1}(\Omega)$$

eventually as $h \to \infty$ and

$$\Bigl(-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u_h)D_i u_h) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u_h) D_i u_h D_j u_h - g(x,u_h) - \omega\Bigr) \to 0$$

strongly in $H^{-1}(\Omega)$.

We say that $f$ satisfies the concrete Palais–Smale condition at level $c$ ($(\mathrm{CPS})_c$ for short), if every $(\mathrm{CPS})_c$-sequence for $f$ admits a strongly convergent subsequence in $H_0^1(\Omega)$.

Theorem 2.8 (cf. [8, Corollary 2.1.4]). *Let $u \in H_0^1(\Omega)$, $c \in \mathbb{R}$ and let $(u_h)$ be a sequence in $H_0^1(\Omega)$. Then the following facts hold*

(a) *if $u$ is a (lower) critical point of $f$, then $u$ is a weak solution of* (2.2),
(b) *if $(u_h)$ is a $(\mathrm{PS})_c$-sequence for $f$, then $(u_h)$ is a $(\mathrm{CPS})_c$-sequence for $f$,*
(c) *if $f$ satisfies $(\mathrm{CPS})_c$, then $f$ satisfies $(\mathrm{PS})_c$.*

## 3. Saddle point

In this section we prepare the proof of our main result.

Let us set $g_0(x,s) = g(x,s) - \beta s^+ + \alpha s^-$ and $G_0(x,s) = \int_0^s g_0(x,t)\,dt$. Of course, $g_0$ is a Carathéodory function satisfying

$$\lim_{|s|\to\infty} \frac{g_0(x,s)}{s} = 0 \quad \text{a.e. in } \Omega,$$
$$|g_0(x,s)| \le a(x) + \widetilde{b}(x)|s| \quad \text{with } \widetilde{b} \in L^{n/2}(\Omega).$$

Let us consider the energy functional $\widetilde{f}_t : H^1_0(\Omega) \to \mathbb{R}$, associated with (1.1.1),

$$\begin{aligned}\widetilde{f}_t(u) = {} & \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju\,dx - \frac{1}{2}\beta\int_\Omega (u^+)^2\,dx - \frac{1}{2}\alpha\int_\Omega (u^-)^2\,dx \\ & - \int_\Omega G_0(x,tu)\,dx - t\int_\Omega \varphi_1 u\,dx - \langle \omega, u\rangle,\end{aligned}$$

for $t > 0$ and define $f_t : H^1_0(\Omega) \to \mathbb{R}$ by $f_t(u) = t^{-2}\widetilde{f}_t(tu)$, namely

$$\begin{aligned}f_t(u) = {} & \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,tu)D_iuD_ju\,dx - \frac{1}{2}\beta\int_\Omega (u^+)^2\,dx - \frac{1}{2}\alpha\int_\Omega (u^-)^2\,dx \\ & - \frac{1}{t^2}\int_\Omega G_0(x,tu)\,dx - \int_\Omega \varphi_1 u\,dx - \frac{1}{t}\langle \omega, u\rangle.\end{aligned}$$

It is easy to verify that

$$|df_t|(u) = \frac{1}{t}|d\widetilde{f}_t|(tu). \tag{3.1}$$

We still define $f_\infty, \widehat{f}_\infty : H^1_0(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}f_\infty(u) = {} & \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}(x)D_iuD_ju\,dx \\ & - \frac{1}{2}\beta\int_\Omega (u^+)^2\,dx - \frac{1}{2}\alpha\int_\Omega (u^-)^2\,dx - \int_\Omega \varphi_1 u\,dx, \\ \widehat{f}_\infty(u) = {} & \frac{1}{2}\int_\Omega \sum_{i,j=1}^n A_{ij}(x)D_iuD_ju\,dx - \frac{1}{2}\alpha\int_\Omega u^2\,dx - \int_\Omega \varphi_1 u\,dx.\end{aligned}$$

THEOREM 3.1. *For every real number $c$ the functional $f_t$ satisfies* $(\mathrm{PS})_c$.

PROOF. It follows from [6, Theorem 3.4], [7, Theorem 4.5 and Lemma 3.14] and Theorem 2.8. □

THEOREM 3.2.

(i) *If $(t_h)$ is a sequence in $]0,\infty[$ with $t_h \to \infty$ and $(u_h)$ a sequence strongly convergent to $u$ in $H^1_0(\Omega)$, then $\lim_h f_{t_h}(u_h) = f_\infty(u)$.*

(ii) *If $(t_h)$ is a sequence in $]0,\infty[$ with $t_h \to \infty$ and $(u_h)$ a sequence weakly convergent to $u$ in $H^1_0(\Omega)$ such that $\limsup_h f_{t_h}(u_h) \le f_\infty(u)$, then $(u_h)$ strongly converges to $u$ in $H^1_0(\Omega)$.*

PROOF. (i) It is easy to prove.

(ii) Let us observe that by hypothesis

$$\begin{aligned}
&\limsup_h \frac{1}{2}\int_\Omega \sum_{ij} a_{ij}(x,t_h u_h) D_i u_h D_j u_h \, dx \\
&\quad = \limsup_h \left( \frac{1}{2}\int_\Omega \sum_{ij} a_{ij}(x,t_h u_h) D_i u_h D_j u_h \, dx - \frac{\beta}{2}\int_\Omega (u_h^+)^2 \, dx \right. \\
&\qquad \left. - \frac{\alpha}{2}\int_\Omega (u_h^-)^2 \, dx - \frac{1}{t^2}\int_\Omega G_0(x,t_h u_h)\, dx - \int_\Omega \varphi_1 u_h \, dx - \frac{1}{t_h}\langle \omega, u_h\rangle \right) \\
&\qquad + \frac{\beta}{2}\int_\Omega (u^+)^2 \, dx + \frac{\alpha}{2}\int_\Omega (u^-)^2 \, dx + \int_\Omega \varphi_1 u \, dx \\
&\quad \le \frac{1}{2}\int_\Omega \sum_{ij} A_{ij} D_i u D_j u \, dx.
\end{aligned}$$

Then, as in the proof of Lemma (3.2) in [6], let us observe that

$$\begin{aligned}
&\limsup_h \int_\Omega \sum_{ij} a_{ij}(x,t_h u_h) D_i(u_h-u) D_j(u_h-u)\, dx \\
&\quad = \limsup_h \int_\Omega \sum_{ij} a_{ij}(x,t_h u_h) D_i u_h D_j u_h \, dx \\
&\qquad - \int_\Omega \sum_{ij} A_{ij} D_i u D_j u \, dx \le 0.
\end{aligned} \tag{3.2.2}$$

By (3.2.2) and (a.3) we conclude that

$$\begin{aligned}
&\nu \limsup_h \|Du_h - Du\|_{L^2}^2 \\
&\qquad \le \limsup_h \int_\Omega \sum_{ij} a_{ij}(x,t_h u_h) D_i(u_h-u) D_j(u_h-u)\, dx \le 0.
\end{aligned}$$

Then $u_h$ converges strongly to $u$ in $H_0^1(\Omega)$. □

Corollary 3.3. *Let $K \subset H_0^1(\Omega)$ be a compact set. Then for each $\varepsilon > 0$ there exists $\bar t > 0$ such that, for all $t \ge \bar t$*

$$\max_K f_t \le \max_K f_\infty + \varepsilon.$$

Proof. If the assertion were false, then we could consider $\varepsilon > 0$, a sequence $(t_h) \subset \mathbb{R}$ tending to $\infty$ and a sequence $(u_h) \in K$ such that for every $h$

$$f_{t_h}(u_h) > \max_K f_\infty + \varepsilon. \tag{3.3.1}$$

Up to a subsequence, $u_h$ converges strongly to some $u \in K$ and, by Theorem 3.2, $\lim_h f_{t_h}(u_h) = f_\infty(u)$. Then passing to the limit in (3.3.1) we get

$$f_\infty(u) \ge \max_K f_\infty + \varepsilon$$

which is absurd. □

Corollary 3.4. *Let $C \subset H_0^1(\Omega)$ be a closed and bounded set. Then for each $\varepsilon > 0$ there exist $\overline{t} > 0$ and $\delta > 0$ such that, for all $t \geq \overline{t}$*

$$\inf_C f_t \geq \min\left\{\inf_C f_\infty - \varepsilon,\ \inf_{\overline{C}^w} f_\infty + \delta\right\},$$

*where $\overline{C}^w$ is the weak closure of $C$.*

Proof. If the assertion were false, then we could consider $\varepsilon > 0$, a sequence $(t_h) \subset \mathbb{R}$ tending to $\infty$ and a sequence $(u_h) \in C$ such that for every $h$

$$f_{t_h}(u_h) < \inf_C f_\infty - \varepsilon, \tag{3.4.1}$$

$$f_{t_h}(u_h) < \inf_{\overline{C}^w} f_\infty + \frac{1}{h}. \tag{3.4.2}$$

Up to a subsequence, $u_h$ weakly converges to some $u$ in $H_0^1(\Omega)$ and there exists

$$l = \lim_h f_{t_h}(u_h).$$

Let us suppose, as a first case, that $l \leq f_\infty(u)$. Then, by Theorem 3.2(ii), $u_h \to u$ strongly in $H_0^1(\Omega)$, and $u \in C$ since $C$ is closed. Thus, by Theorem 3.2(i), $\lim_h f_{t_h}(u_h) = f_\infty(u)$ and by (3.4.1), $f_\infty(u) \leq \inf_C f_\infty - \varepsilon$, that is absurd.

Now, let us consider the case $l > f_\infty(u)$. Since $u \in \overline{C}^w$, by (3.4.2)

$$\inf_{\overline{C}^w} f_\infty \geq l > f_\infty(u) \geq \inf_{\overline{C}^w} f_\infty$$

which is absurd. □

Now, let $\lambda_k < \alpha \leq \lambda_{k+1}$ with $k \geq 1$. Let us denote by $\widetilde{E}_-$ the subspace spanned by the eigenvectors associated to the first $k$ eigenvalues $(\lambda_1, \ldots, \lambda_k)$ and $E_+$ the closed subspace of $H_0^1(\Omega)$ spanned by the eigenvectors associated to the eigenvalues $(\lambda_{k+1}, \ldots)$. Let also $\varphi_k$ be an eigenfunction associated with $\lambda_k$. Recall that we have chosen $\varphi_1 \geq 0$.

Theorem 3.5. *There exists a subspace $E_- \subset \mathbb{R}\varphi_1 + C_0^\infty(\Omega)$ with $\dim E_- = \dim \widetilde{E}_-$ and $H_0^1(\Omega) = E_- \oplus E_+$ such that*

- (a) *for each $\rho > 0$ one has: $\sup_{S_\rho^-} \widehat{f}_\infty < \widehat{f}_\infty(-\varphi_1/(\alpha - \lambda_1))$,*
- (b) *there exists $\rho > 0$ such that $\widehat{f}_\infty(-\varphi_1/(\alpha - \lambda_1)) < \inf_{S_\rho^+} f_\infty$, where $S_\rho^\pm = -\varphi_1/(\alpha - \lambda_1) + (E_\pm \cap S_\rho(0))$.*

Proof. It is easy to verify that $-\varphi_1/(\alpha - \lambda_1)$ is a critical point for $\widehat{f}_\infty$ and

$$\widehat{f}''_\infty(u)(v)^2 = \int_\Omega \sum_{i,j} A_{ij} D_i v D_j v\, dx - \alpha \int_\Omega v^2\, dx \quad \text{for all } u, v \in H_0^1(\Omega).$$

Then, from the definition of $\widetilde{E}_-$, for all $\rho > 0$ we get

$$\sup_{\widetilde{E}_- \cap S_\rho(-\varphi_1/(\alpha - \lambda_1))} \widehat{f}_\infty < \widehat{f}_\infty\left(-\frac{\varphi_1}{\alpha - \lambda_1}\right). \tag{3.5.1}$$

Now, we can take $\psi_2, \dots, \psi_k \in C_0^\infty(\Omega)$ and consider $E_- = \operatorname{span}\{\varphi_1, \psi_2, \dots, \psi_k\}$. If $\psi_2 \dots \psi_k$ are sufficiently close in the $H_0^1$-norm to $\varphi_2, \dots, \varphi_k$, respectively, it is readily seen that the above inequality is true also for $\widetilde{E}_-$ replaced by $E_-$.

To prove assertion (b), denote by $Y$ the eigenspace associated to $\lambda_{k+1}$. Since $k+1 \geq 2$, there exists $\rho > 0$ such that

$$\text{for all } u \in -\frac{\varphi_1}{\alpha - \lambda_1} + Y : \quad u \leq 0 \text{ a.e. } \Rightarrow \left\| u + \frac{\varphi_1}{\alpha - \lambda_1} \right\| < \rho.$$

By contradiction, let $(u_h)$ be a sequence in $S_\rho^+$ with

$$\lim_h f_\infty(u_h) \leq \widehat{f}_\infty\left( - \frac{\varphi_1}{\alpha - \lambda_1}\right).$$

Up to a subsequence, $u_h$ is weakly convergent in $H_0^1(\Omega)$ to some $u \in -\varphi_1/(\alpha - \lambda_1) + E_+$. It follows

$$\lim_h f_\infty(u_h) \leq \widehat{f}_\infty\left( - \frac{\varphi_1}{\alpha - \lambda_1}\right) \leq \widehat{f}_\infty(u) \leq f_\infty(u),$$

so that $u_h$ is strongly convergent in $H_0^1(\Omega)$ to $u$, $f_\infty(u) = \widehat{f}_\infty(u) = \widehat{f}_\infty(-\varphi_1/(\alpha - \lambda_1))$ and $\|u + \varphi_1/(\alpha - \lambda_1)\| = \rho$. Therefore, $u + \varphi_1/(\alpha - \lambda_1) \in Y$ and

$$\frac{1}{2}(\alpha - \beta) \int_\Omega (u^+)^2 \, dx = f_\infty(u) - \widehat{f}_\infty(u) = 0,$$

namely $u \leq 0$ a.e. in $\Omega$. This is impossible by the choice of $\rho$ and (b) follows. □

Lemma 3.6. *Let $E_-$ be as in Theorem* 3.5*. Then there exists $\rho > 0$ such that for every $u \in E_- \cap B_\rho(-\varphi_1/(\alpha - \lambda_1))$ one has $u(x) \leq 0$ in $\Omega$.*

Proof. It is sufficient to recall that $\inf_K \varphi_1 > 0$ for every compact subset $K$ of $\Omega$. □

Now, let us formulate the main result of the section.

Theorem 3.7. *Let $a_{ij}$ and $g$ satisfy hypotheses* (a.1)–(a.6), (g.1), (g.2) *and let $\omega \in H^{-1}(\Omega)$. Assume that $\beta < \lambda_1 < \alpha$. Then for every $\varepsilon > 0$ there exists $\overline{t} \in \mathbb{R}^+$ such that for every $t > \overline{t}$ the functional $f_t$ has a critical point $\overline{u}_t$ with*

$$\left| f_t(\overline{u}_t) - f_\infty\left( - \frac{\varphi_1}{\alpha - \lambda_1}\right) \right| < \varepsilon.$$

Proof. Let $k \geq 1$ be such that $\lambda_k < \alpha \leq \lambda_{k+1}$, $E_-$ as in Theorem 3.5, $\rho_+ > 0$ as in (b) of Theorem 3.5 and $\rho_- > 0$ as in Lemma 3.6.

Set $B^\pm = -\varphi_1/(\alpha - \lambda_1) + (E_\pm \cap B_{\rho^\pm}(0))$ and $S^\pm = -\varphi_1/(\alpha - \lambda_1) + (E_\pm \cap S_{\rho_\pm}(0))$. Let us observe that $f_\infty(u) = \widehat{f}_\infty(u)$ for every $u \in B^-$ while in general $f_\infty(u) \geq \widehat{f}_\infty(u)$ for every $u \in H_0^1(\Omega)$. It is easy to prove that

$$\sup_{B^-} f_\infty = f_\infty\left( \frac{-\varphi_1}{\alpha - \lambda_1}\right) \tag{3.7.1}$$

and

$$f_\infty\left(\frac{-\varphi_1}{\alpha-\lambda_1}\right) = \inf_{B^+} \widehat{f}_\infty \le \inf_{B^+} f_\infty. \tag{3.7.2}$$

Moreover, by Theorem 3.5, it follows

$$\inf_{S^+} f_\infty > f_\infty\left(\frac{-\varphi_1}{\alpha-\lambda_1}\right). \tag{3.7.3}$$

Let us take

$$\varepsilon' = \frac{1}{2}\left[\inf_{S^+} f_\infty - f_\infty\left(\frac{-\varphi_1}{\alpha-\lambda_1}\right)\right]. \tag{3.7.4}$$

Applying Corollary 3.4 with $C = S^+$ and $\varepsilon'$ as in (3.7.4), we have that there exist $t_1 > 0$ and $\delta > 0$ such that, for all $t \ge t_1$

$$\inf_{S^+} f_t \ge \min\left\{\inf_{S^+} f_\infty - \varepsilon', \inf_{B^+} f_\infty + \delta\right\}. \tag{3.7.5}$$

Thus, there exists $\delta' \in (0, \varepsilon)$ such that, for all $t \ge t_1$

$$\inf_{S^+} f_t \ge f_\infty\left(\frac{-\varphi_1}{\alpha-\lambda_1}\right) + 2\delta'. \tag{3.7.6}$$

Now, by (3.7.1) and applying Corollary 3.3 with $K = B^-$, we have that there exists $t_2 > 0$ such that, for all $t \ge t_2$

$$\max_{B^-} f_t \le f_\infty\left(\frac{-\varphi_1}{\alpha-\lambda_1}\right) + \delta'. \tag{3.7.7}$$

By (3.7.6) and (3.7.7) we have that there exists $t_3 > 0$ such that, for all $t \ge t_3$

$$\max_{B^-} f_t < \min\left\{\inf_{S^+} f_t, f_\infty\left(\frac{-\varphi_1}{\alpha-\lambda_1}\right) + \varepsilon\right\}. \tag{3.7.8}$$

Now, with an analogous argument it can be proved that there exists $t_4 > 0$ such that, for all $t \ge t_4$

$$\max\left\{\max_{S^-} f_t, f_\infty\left(\frac{-\varphi_1}{\alpha-\lambda_1}\right) - \varepsilon\right\} < \inf_{B^+} f_t. \tag{3.7.9}$$

Let $\overline{t} = \max\{t_3, t_4\}$. By (3.7.8), (3.7.9) and Theorem 3.1, it is enough to apply Theorem 2.5 to have that for all $t \ge \overline{t}$ the functional $f_t$ has a critical point $\overline{u}_t$ such that

$$\left| f_t(\overline{u}_t) - f_\infty\left(-\frac{\varphi_1}{\alpha-\lambda_1}\right)\right| < \varepsilon. \qquad \square$$

## 4. Proof of the main result

Lemma 4.1. *Let $a_{ij}$ and $g$ satisfy hypotheses* (a.1–a.6), (g.1), (g.2) *and let $\omega \in H^{-1}(\Omega)$. Assume that $\beta < \lambda_1$ and $\alpha > \lambda_2$. Then there exists a continuous curve $\gamma : [0, \infty[ \longrightarrow H_0^1(\Omega)$ such that*

$$\gamma(0) = \frac{\varphi_1}{\lambda_1 - \beta}, \qquad \lim_{s \to \infty} f_\infty(\gamma(s)) = -\infty,$$
$$\sup_{s \geq 0} f_\infty(\gamma(s)) < f_\infty\bigg(-\frac{\varphi_1}{\alpha - \lambda_1}\bigg).$$

Proof. Let $k \geq 2$ be such that $\lambda_k < \alpha \leq \lambda_{k+1}$, $\psi_2$ as in the proof of Theorem 3.5, and $\rho$ as in Lemma 3.6. In the subspace spanned by $\{\varphi_1, \psi_2\}$, let us consider a curve $\gamma$ consisting of the union of $\gamma_1$, $\gamma_2$, $\gamma_3$ where $\gamma_1$ is given by the points on $\varphi_1$-axis between $-\varphi_1/(\alpha - \lambda_1) + \rho\varphi_1/\|\varphi_1\|$ and $\varphi_1/(\lambda_1 - \beta)$ with $\gamma_1(0) = \varphi_1/(\lambda_1 - \beta)$; $\gamma_2$ is the upper semicircle of radius $\rho$ and center $-\varphi_1/(\alpha - \lambda_1)$; $\gamma_3$ is given by the points $\tau\varphi_1$ with $\tau < -1/(\alpha - \lambda_1) - \rho/\|\varphi_1\|$.

By definition of $f_\infty$ and Theorem 3.5, $\gamma$ has the required properties. □

Now, let $\gamma$ be as in the previous lemma, let $\varepsilon > 0$ be such that

$$\sup_{s \geq 0} f_\infty(\gamma(s)) < f_\infty\bigg(-\frac{\varphi_1}{\alpha - \lambda_1}\bigg) - \varepsilon,$$

and let $\overline{t} \in \mathbb{R}$ as in Theorem 3.7.

Theorem 4.2. *Let $a_{ij}$ and $g$ satisfy hypotheses* (a.1)–(a.6), (g.1), (g.2) *and let $\omega \in H^{-1}(\Omega)$. Assume that $\beta < \lambda_1$ and $\alpha > \lambda_2$. Then there exists $t_0 \geq \overline{t}$ such that, for every $t > t_0$, the functional $f_t$ has two critical points $\underline{u}_t$ and $\widehat{u}_t$ with*

$$f_t(\underline{u}_t) < f_t(\widehat{u}_t) < f_t(\overline{u}_t)$$

*where $\overline{u}_t$ is the critical point found in Theorem* 3.7.

Proof. First of all, let us point out that from the definition of $f_\infty$ and hypothesis on $\alpha$ and $\beta$, it can be easily seen that there exists $r > 0$ such that

$$\inf_{S_r(\varphi_1/(\lambda_1 - \beta))} f_\infty > f_\infty\bigg(\frac{\varphi_1}{\lambda_1 - \beta}\bigg)$$

and

$$\min_{B_r(\varphi_1/(\lambda_1 - \beta))} f_\infty = f_\infty\bigg(\frac{\varphi_1}{\lambda_1 - \beta}\bigg).$$

Moreover, there exists $s$ large enough such that

$$f_\infty(\gamma(s)) \leq f_\infty\bigg(\frac{\varphi_1}{\lambda_1 - \beta}\bigg) \quad \text{and} \quad \bigg\|\gamma(s) - \frac{\varphi_1}{\lambda_1 - \beta}\bigg\| > r.$$

Now, let us apply Corollary 3.4 with $C = S_r(\varphi_1/(\lambda_1 - \beta))$ and

$$\varepsilon' = \frac{1}{2}\bigg[\inf_{S_r(\varphi_1/(\lambda_1-\beta))} f_\infty - f_\infty\bigg(\frac{\varphi_1}{\lambda_1 - \beta}\bigg)\bigg].$$

Then there exist $t_1 > 0$ and $\delta > 0$ such that for all $t \geq t_1$,

$$\inf_{S_r(\varphi_1/(\lambda_1-\beta))} f_t \geq \min\bigg\{\inf_{S_r(\varphi_1/(\lambda_1-\beta))} f_\infty - \varepsilon', \inf_{B_r(\varphi_1/(\lambda_1-\beta))} f_\infty + \delta\bigg\}.$$

In particular, there exists $\delta' > 0$ such that, for all $t \geq t_1$,

$$\inf_{S_r(\varphi_1/(\lambda_1-\beta))} f_t \geq f_\infty\bigg(\frac{\varphi_1}{\lambda_1 - \beta}\bigg) + 2\delta'. \tag{4.2.1}$$

Now, by applying Corollary 3.3 with $K = \{\varphi_1/(\lambda_1 - \beta), \gamma(s)\}$ we have that there exists $t_2 > 0$ such that, for all $t \geq t_2$,

$$\max_{\{\varphi_1/(\lambda_1-\beta),\gamma(s)\}} f_t \leq \max_{\{\varphi_1/(\lambda_1-\beta),\gamma(s)\}} f_\infty + \delta' = f_\infty\bigg(\frac{\varphi_1}{\lambda_1 - \beta}\bigg) + \delta'. \tag{4.2.2}$$

By (4.2.1) and (4.2.2), we have that there exists $t_3 > 0$ such that, for all $t \geq t_3$,

$$\inf_{S_r(\varphi_1/(\lambda_1-\beta))} f_t > \max_{\{\varphi_1/(\lambda_1-\beta),\gamma(s)\}} f_t. \tag{4.2.3}$$

Applying Corollary 3.3 with $K = \gamma([0, s])$, we have that there exists $t_0 > t_3$ such that, for all $t \geq t_0$,

$$\max_{\gamma([0,s])} f_t < f_\infty\bigg(-\frac{\varphi_1}{\alpha - \lambda_1}\bigg) - \varepsilon. \tag{4.2.4}$$

Then, by Theorem 3.1 and (4.2.3), we may apply Theorem 2.4. So, for all $t \geq t_0$ the functional $f_t$ has two distinct critical points $\underline{u}_t$ and $\widehat{u}_t$ with

$$f_t(\underline{u}_t) < f_t(\widehat{u}_t) < f_\infty\bigg(-\frac{\varphi_1}{\alpha - \lambda_1}\bigg) - \varepsilon < f_t(\overline{u}_t) \qquad \square$$

PROOF OF THEOREM 1.1. By Theorems 3.7 and 4.2, we deduce that the functional $f_t$ has at least three distinct critical points and then, by (3.1) and Theorem 2.8, that the equation (1.1.1) has at least three distinct weak solutions. For the $L^\infty$-regularity, we refer the reader to [7]. $\square$

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*Manuscript received February 16, 2001*

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