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A NONLINEAR PROBLEM FOR AGE-STRUCTURED POPULATION DYNAMICS WITH SPATIAL DIFFUSION

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Abstract. We consider a nonlinear model for age-dependent population dynamics subject to a density dependent factor which regulates the selection of newborn at age zero. The initial-boundary value problem is studied using a vanishing viscosity method (in the age direction) together with the fixed point theory. Existence and uniqueness are obtained, and also the positivity of the solution to the problem.

1. Introduction

The paper addresses the problem of population dynamics with a boundary condition (birth condition) where the rate of newborns is a nonlinear function Φ of the usual integral expression $\int_0^\omega \beta(a)u(t,x,a)\,da$. Into this model, diffusion in a bounded space domain Ω of $\mathbb{R}^N$ is incorporated. For this system, a parabolic regularization (with respect to the age variable) is studied and results are obtained for the original problem.

The problems of the dynamic of population proposed by biologists have interested mathematicians years ago. The evolution of population in a region depends on many factors, but the direct causes determining the changes in population patern are births, deaths and migration. We are concerned by biology population subject to only birth and death parameters.

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Several authors studied these problems. For details on the general theory, see [1]. In [4], the problem considering the Lotka–Von Foerster model ([13]) was treated. We mention the work of Huyer ([7]) for size-structured population and [8] for the age-structured population. In [7], the density depends on the size of individuals. Note also the work of Chan and Guo ([3]) for age-size dependent population.

The dynamic considered in this paper is age-structured. It is described by a scalar function $u = u(t,x,a)$ representing the density of population having at time $t > 0$ the age a, and located at the geographic position x. We also incorporate a spatial diffusion term in the model. Notice the work of Gurtin [5] who was the first introducing spatial diffusion in age-structured population models. In these models, the selecting function Φ of new born (see (1) here after) is taken constant. Huyer [8] (see also Hernandez ([6]) and the references therein) studied the linear case with spatial diffusion and $\Phi = 1$. Unfortunately, when Φ is non-constant (which is generally the case in practice because of the birth process) only few papers treat this interesting case, well adapted to real populations. The problem was studied by Busenberg–Iannelli ([2]) for $\Phi > 1$, Φ increasing, without spatial diffusion, and by Ndiaye in his thesis ([12]) for a nonlinear Φ and for the spatial diffusion case. He used the method of separating variables as a first step, and the semigroup method in a second approach as used in almost of the papers of these authors. See [9] for a complete bibliography and the results obtained for population dynamics problems with spatial diffusion. We present here a different approach.

In [12] the author studied the following problem:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial u}{\partial a} + \mu u = 0 & \text{in }]0,T[\, \times \Omega \times \,]0,\omega[\,, \\
\dfrac{\partial u}{\partial \nu}(t,x,a) = 0 & \text{for } (t,x,a) \in \,]0,T[\, \times \partial\Omega \times \,]0,\omega[\,, \\
u(t,x,0) = F(\alpha_u) & \text{on }]0,T[\, \times \Omega \times \{0\}, \\
u(0,x,a) = u_0(x,a) & \text{on } \{0\} \times \Omega \times \,]0,\omega[\,,
\end{cases} \tag{1}
$$

where $\alpha_u = \int_0^\omega \beta(a) u(t,x,a)\, da$, and $F(t) = t\Phi(t)$ for all $t \in \mathbb{R}$, with the following hypothesis

Hypothesis.

- (H$_1$) $\mu \geq 0$, is continuous on $[0,\omega[$ and $\int_0^\omega \mu(a)\, da = \infty$. The last expression means that the probability of population to survive tends to zero when $a \to w$,
- (H$_2$) $\beta \geq 0$, is continuous on $[0,\omega]$, and $\beta \neq 0$,
- (H$_3$) Φ is a continuous and bounded function on $\mathbb{R}_+$,
- (H$_4$) $u_0 \geq 0$, is piecewise continuous on $[0,\omega]$,

(H$_5$) there is at least one $a_1 \leq \omega_1$ such that $u_0(a)$ is positif strictly, with $\omega_1 = \sup\{a \in [0,\omega] : \beta(a) \neq 0\}$, ω being the maximum age to attain,

(H$_6$) F is piecewise continuously differentiable, bounded on $\mathbb{R}_+$.

The functions μ and β denote the mortality rate and the fertility rate, respectively. The function Φ (which can be nonlinear) is the selecting function at age zero. Existence results have been obtained in [12], but not regularity and positivity of the solution which is essential to verify. To deal with this problem, we propose to construct an approximate solution $u_\varepsilon = u_\varepsilon(t,x,a)$, ($\varepsilon$ being a small viscous parameter), using a regularization via the age variable. We investigate then the parabolic approach proving that it is well adapted to describe problems of the population dynamics, with a nonlinear boundary condition as a natural condition of recruitment. Note that we need to satisfy only some of the above hypothesis (H$_1$)–(H$_5$).

The paper is organised as follows: Section 2 is devoted to present the method, and in Section 3 we prove that the approximate problem is well posed in some Hilbert space $\mathcal{V}$. We present then our main result using the Leray–Schauder fixed point theorem. Convergence to the unique solution of the problem (1) when $\varepsilon \to 0$ is proved in Section 4. At last, in Section 5, we turn to prove the positivity of the solution to the problem.

2. Parabolic regularization

The parabolic regularization needs the introduction of a sequence u_ε with the following problem

$$(2)\quad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \displaystyle\sum_{i=1}^{N} \frac{\partial^2 u_\varepsilon}{\partial x_i^2} - \varepsilon \frac{\partial^2 u_\varepsilon}{\partial a^2} + \frac{\partial u_\varepsilon}{\partial a} + \mu u_\varepsilon = 0 & \text{in }]0,T[\, \times \Omega \times\,]0,\omega[\,, \\ \dfrac{\partial u_\varepsilon}{\partial \nu}(t,x,a) = 0 & \text{on }]0,T[\, \times \partial\Omega \times\,]0,\omega[\,, \\ u_\varepsilon(t,x,0) - \varepsilon \dfrac{\partial u_\varepsilon}{\partial a}(t,x,0) = F(\alpha_{u_\varepsilon}) & \text{for } a = 0, \\ u_\varepsilon(t,x,\omega) = 0 & \text{for } a = \omega, \\ u_\varepsilon(0,x,a) = u_0(x,a) & \text{on } \{0\} \times \Omega \times\,]0,\omega[\,. \end{cases}$$

Remark 1. The condition $u_\varepsilon(t,x,w) = 0$ is natural. It expresses the mortality of population after the maximum attainable age w.

Notation. We set

- $\Theta = \Omega \times]0,\omega[$,
- $$a^\varepsilon(u,v) = \int_\Theta \nabla_x u \nabla_x v \, da\, dx + \varepsilon \int_\Theta \frac{\partial u}{\partial a}\frac{\partial v}{\partial a}\, da\, dx + \int_\Theta \frac{\partial u}{\partial a}\, v\, da dx + \int_\Theta \mu u v\, da\, dx + \int_\Omega u(\,\cdot\,,x,0) v(\,\cdot\,,x,0)\, dx,$$

- $$\mathcal{V} = \Big\{\varphi \in L^2(\Theta) : \nabla_x\varphi \in (L^2(\Theta))^N,\ \frac{\partial\varphi}{\partial a} \in L^2(\Theta),\ \sqrt{\mu}\varphi \in L^2(\Theta),\ \varphi|_{\Omega\times\{\omega\}} = 0,\ \varphi(\,\cdot\,,0) \in L^2(\Omega)\Big\},$$
- $$\mathcal{W}(0,T) = \Big\{\varphi \in L^2(0,T;\mathcal{V}),\ \frac{\partial\varphi}{\partial t} \in L^2(0,T;L^2(\Theta))\Big\}.$$

For $v \in \mathcal{V}$, we define the norm

$$\|v\|^2_{\mathcal{V}} = \|\nabla_x v\|^2_{L^2(\Theta)} + \Big\|\frac{\partial v}{\partial a}\Big\|^2_{L^2(\Theta)} + \|\sqrt{\mu}\,v\|^2_{L^2(\Theta)} + \|v(t,\,\cdot\,,0)\|^2_{L^2(\Omega)}. \tag{3}$$

It is easily to see that $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ and $\mathcal{W}$ (with its usual norm) are Hilbert spaces. We define also a function h such that:

$$\begin{gathered} h :]0,T[\,\times\,\Omega\times\mathcal{V} \to \mathbb{R} \\ (t,x,v) \mapsto h(t,x,v) = \begin{cases} F(\alpha_v) & \text{if } v > 0, \\ 0 & \text{if } v \le 0, \end{cases} \end{gathered} \tag{4}$$

and we state the following which is obvious:

Proposition 2.1. *The problem* (2) *is equivalent with the variational problem*

$$\begin{cases} \Big(\dfrac{\partial}{\partial t}u_\varepsilon, \varphi\Big) + a^\varepsilon(u_\varepsilon,\varphi) = \displaystyle\int_\Omega h(t,x,u_\varepsilon)\,\varphi(t,x,0)\,dx, \\ u_\varepsilon(0,x,a) = u_0(x,a). \end{cases} \tag{5}$$

3. Existence of approximate solutions

In this section we claim that for every $\varepsilon > 0$, the problem (2) admits at least one solution. For this, we apply the method of the fixed point of Leray–Schauder:

Theorem 3.1 (Leray–Schauder). *Suppose that*

(i) *the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is compact, $\mathcal{X}$ being a Banach space;*
(ii) (*a priori estimate*) *there exists an $r > 0$ such that if $v = \sigma\mathcal{T}(v)$ with $0 < \sigma < 1$ then $\|v\|_{\mathcal{X}} \le r$.*

Then the equation $v = \mathcal{T}(v)$ is solvable.

For a proof of the Theorem 3.1, see [14]. Now we prove the following:

Lemma 3.1. *The bilinear form $a^\varepsilon(\,\cdot\,,\,\cdot\,)$ is continuous and coercitive on $\mathcal{V}\times\mathcal{V}$.*

Proof. Let $t > 0$, then for every $u \in \mathcal{V}$, $v \in \mathcal{V}$, and for every $\varepsilon > 0$, $\varepsilon \ll 1$ we have

$$\begin{aligned} |a^\varepsilon(u,v)| \le \|\nabla_x u\|_{L^2}\|\nabla_x v\|_{L^2} + \varepsilon\Big\|\frac{\partial u}{\partial a}\Big\|_{L^2}\Big\|\frac{\partial v}{\partial a}\Big\|_{L^2} + \|\sqrt{\mu}u\|_{L^2}\|\sqrt{\mu}v\|_{L^2} \\ + \Big\|\frac{\partial u}{\partial a}\Big\|_{L^2}\|v\|_{L^2} + \|u(t,\,\cdot\,,0)\|_{L^2(\Omega)}\|v(t,\,\cdot\,,0)\|_{L^2(\Omega)} \le 5\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}, \end{aligned}$$

using the Cauchy–Schwartz inequality. Moreover, noting that

$$\begin{aligned}\int_\Omega\int_0^\omega \frac{\partial u}{\partial a}\cdot u\,da\,dx &= \int_\Omega\bigg(\int_0^\omega \frac{1}{2}\frac{\partial}{\partial a}(u^2)da\bigg)\,dx\\ &= \frac{1}{2}\int_\Omega u^2(t,x,\omega)\,dx - \frac{1}{2}\int_\Omega u^2(t,x,0)\,dx,\end{aligned}$$

for every $u \in \mathcal{V}$, we obtain

$$a^\varepsilon(u,u) = \|\nabla_x u\|_{L^2}^2 + \varepsilon\bigg\|\frac{\partial u}{\partial a}\bigg\|_{L^2}^2 + \|\sqrt{\mu}u\|_{L^2}^2 + \frac{1}{2}\|u(t,\,\cdot\,,0)\|_{L^2(\Omega)}^2. \tag{6}$$

Hence we have

$$a^\varepsilon(u,u) \geq \min(1/2,\varepsilon)\|u\|_{\mathcal{V}}^2. \tag{7}$$

□

Lemma 3.2. *Let $t > 0$. Then, under the hypothesis* (H_2) *and* (H_3)*, for every $v \in L^2(0,T;\mathcal{V})$ we have*

$$\int_\Omega |h(t,x,v)\varphi(t,x,0)|\,dx \leq \omega^2|\beta|_\infty|\Phi|_{L^\infty(\mathbb{R})}\|v\|_{L^2(\Theta)}\|\varphi\|_{\mathcal{V}}, \tag{8}$$

where $|\beta|_\infty = \sup_{0\leq a\leq\omega}|\beta(a)|$. In particular, the application

$$\varphi \mapsto \int_\Omega h(t,x,v)\varphi(t,x,0)\,dx,$$

is linear continuous on $\mathcal{V}$.

Proof. Since β is a bounded function on $[0,\omega]$ (hypothesis (H_2)), we have

$$\begin{aligned}&|h(t,x,v)\varphi(t,x,0)|\\ &\qquad\leq |\beta|_\infty\bigg(\int_0^\omega |v(t,x,a)|da\bigg)\bigg|\Phi\bigg(\int_0^\omega \beta(a)v(t,x,a)\,da\bigg)\bigg||\varphi(t,x,0)|.\end{aligned}$$

The hypothesis (H_3) yields $|\Phi(\int_0^\omega \beta(a)v(t,x,a))| \leq |\Phi|_{L^\infty(\mathbb{R}_+)}$. Hence

$$\begin{aligned}\int_\Omega |h(t,x,v)\varphi(t,x,0)|\,dx &\leq |\beta|_\infty\cdot|\Phi|_{L^\infty(\mathbb{R}_+)}\int_\Omega|\varphi(t,x,0)|\bigg(\int_0^\omega|v(t,x,a)|\,da\bigg)\,dx\\ &\leq \omega^2|\beta|_{L^\infty}|\Phi|_{L^\infty(\mathbb{R}_+)}\|v\|_{L^2(\Theta)}\|\varphi\|_{L^2(\Omega)}.\end{aligned}$$

We conclude, by using the definition of $\|\cdot\|_{\mathcal{V}}$. This gives (8) and ends the proof.□

Proposition 3.1. *For every $v \in \mathcal{W}(0,T)$ and $u_0 \in L^2(\Theta)$, there exists a unique solution $u_\varepsilon(v) = v_\varepsilon$ of the problem*

$$\begin{cases}\bigg(\dfrac{\partial}{\partial t}v_\varepsilon,\varphi\bigg) + a^\varepsilon(v_\varepsilon,\varphi) = \displaystyle\int_\Omega h(t,x,v)\varphi(t,x,0)\,dx,\\ \qquad\qquad v_\varepsilon(0,x,a) = u_0(x,a),\end{cases}$$

in $\mathcal{W}(0,T)$.

Proof. With Lemmas 3.1 and 3.2 we verify easily the conditions of the theorem of J. L. Lions [10], which allows to conclude. □

Fixed point. The Proposition 3.1 allows to construct an operator $\mathcal{T}$:

$$\begin{aligned} \mathcal{T} : L^2(0,T;L^2(\Theta)) &\to L^2(0,T;L^2(\Theta)), \\ v &\mapsto \mathcal{T}(v) = v_\varepsilon, \end{aligned} \tag{9}$$

where v_ε is given by (9).

REMARK 2. Now we need the hypothesis (H_6). This hypothesis is generally satisfied; It is the case in most of the known applications in the dynamic of population, as for the Beverton–Holt model, Chapmann model, or the "depensatory" model (see [12]).

Note that (H_6) implies in particular that $h \in L^\infty(]0,T[\times\Omega\times\mathcal{V})$. Denote its norm by $\|h\|_\infty < \infty$.

PROPOSITION 3.2. *Let $u_0 \in L^2(\theta)$. Then, under the hypothesis* (H_6)*, the operator $\mathcal{T}$ defined by* (9) *verifies the assertions* (i) *and* (ii) *of the Fixed Point Theorem* 3.1.

PROOF. From Theorem 3.1, the operator $\mathcal{T}$ admits a fixed point $u_\varepsilon = \mathcal{T}u_\varepsilon$, solution of the problem (5).

(i) Put $\varphi = v_\varepsilon$ in the equation (5), and integrate over $t \in]0,T[$. We have

$$\begin{aligned} \frac{1}{2}\|v_\varepsilon(T,\,\cdot\,,\,\cdot\,)\|^2_{L^2(\Theta)} &+ \int_0^T a^\varepsilon(v_\varepsilon, v_\varepsilon)\,dt \\ &= \int_0^T\int_\Omega h(t,x,v)v_\varepsilon(t,x,0)\,dx\,dt + \frac{1}{2}\|u_0\|^2_{L^2(\Theta)}. \end{aligned}$$

Then, for every $\varepsilon > 0$,

$$\begin{aligned} \min(1/2,\varepsilon)\|v_\varepsilon\|^2_{L^2(0,T;\mathcal{V})} &\le \|h\|_\infty \int_0^T\int_\Omega |v_\varepsilon(t,\,\cdot\,,0)|\,dt\,dx + \frac{1}{2}\|u_0\|^2_{L^2(\Theta)} \\ &\le \sqrt{T\mathrm{mes}(\Omega)}\|h\|_\infty\|v_\varepsilon(\,\cdot\,,\,\cdot\,,0)\|_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\|u_0\|^2_{L^2(\Theta)}, \end{aligned}$$

using the Cauchy–Schwartz formula and Remark 2. From the definition of the $\mathcal{V}$ space (3) we have the bound

$$\min(1/2,\varepsilon)\|v_\varepsilon\|^2_{L^2(0,T;\mathcal{V})} \le \sqrt{T\mathrm{mes}(\Omega)}\|h\|_\infty\|v_\varepsilon\|_{L^2(0,T;\mathcal{V})} + \frac{1}{2}\|u_0\|^2_{L^2(\Theta)}.$$

Hence, for every fixed $\varepsilon > 0$, the sequence (v_ε) is bounded in $L^2(0,T;\mathcal{V})$: $\|v_\varepsilon\|_{L^2(0,T;\mathcal{V})} \le C(\varepsilon)$. From the compactness of the injection of H^1 into L^2, and since $\mathcal{V} \subset H^1$ (with continuous injection), we deduce that $\mathcal{T}$ is compact. This ends the proof of the first point.

(ii) We claim that the set

$$\begin{aligned} \mathcal{A} &= \{v \in L^2(0,T;\mathcal{V})/\sigma\mathcal{T}(v) = v,\ \sigma \in]0,1[\} \\ &= \{v \in L^2(0,T;\mathcal{V})/v_\varepsilon = v/\sigma,\ \sigma \in]0,1[\} \end{aligned}$$

is bounded for every fixed $\sigma > 0$.

Indeed, let $v \in \mathcal{A}$. Changing v_ε to $v_\varepsilon = v/\sigma$ in (9) and taking $\varphi = v$, we obtain:

$$\begin{aligned}(11)\quad \frac{1}{2}\|v(T,\cdot,\cdot)\|^2_{L^2(\Theta)} &+ \int_0^T a^\varepsilon(v,v)\,dt \\ &= \int_0^T \int_\Omega \sigma h(t,x,v)v(t,x,0)\,dt\,dx + \frac{1}{2}\|\sigma u_0\|^2_{L^2(\Theta)}.\end{aligned}$$

With Remark 2 and the Cauchy–Schwartz inequality, we can write in the right hand side

$$\begin{aligned}(12)\quad \int_0^T\int_\Omega \sigma h(t,x,v)\,v(t,x,0)\,dx\,dt &\le \|h\|_\infty \int_0^T\int_\Omega |v(t,x,0)|\,dx\,dt \\ &\le \sqrt{T\mathrm{mes}(\Omega)}\|h\|_\infty |v(\cdot,\cdot,0)|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Then, thanks to (7), we have

$$\begin{aligned}(13)\quad \min(1/2,\varepsilon)\|v\|^2_{L^2(0,T;\mathcal{V})} &\le \sqrt{T\mathrm{mes}(\Omega)}\|h\|_\infty \|v(\cdot,\cdot,0)\|_{L^2(0,T;L^2(\Omega))} \\ &\quad + \frac{1}{2}\|u_0\|^2_{L^2(\Theta)} \\ &\le \sqrt{T\mathrm{mes}(\Omega)}\|h\|_\infty \|v\|_{L^2(0,T;\mathcal{V})} + \frac{1}{2}\|u_0\|^2_{L^2(\Theta)}.\end{aligned}$$

Hence, there exists a positif number $K = K(\varepsilon)$ such that for every $v \in \mathcal{A}$, $\|v\|_{L^2(0,T;\mathcal{V})} \le K$. Consequently, the norm $\|v\|_{L^2(0,T;L^2(\Theta))}$ is also bounded (by the continuity of the injection of $\mathcal{V}$ into the $L^2(\Theta)$ space). This completes the proof of Proposition 3.2. $\square$

4. Passage to the limit

We need the following theorem

Theorem 4.1. *Let $\varepsilon > 0$, $t \in]0,T[$, and $u_0 \in L^2(\Theta)$. Then, under the hypothesis* (H_2), (H_3), *and* (H_6), *there exists a positive constant C* (*independant from ε*) *such that*

$$\begin{aligned} |u_\varepsilon(T,\cdot,\cdot)|_{L^2(\Theta)} &\le C, \qquad |\nabla_x u_\varepsilon|_{L^2(0,T;L^2(\Theta))} \le C, \\ \sqrt{\varepsilon}\left|\frac{\partial u_\varepsilon}{\partial a}\right|_{L^2(0,T;L^2(\Theta))} &\le C, \qquad |\sqrt{\mu}u_\varepsilon|_{L^2(0,T;L^2(\Theta))} \le C, \\ |u_\varepsilon(\cdot,\cdot,0)|_{L^2(0,T;L^2(\Omega))} &\le C.\end{aligned}$$

Proof. Let us design by u_ε solution to the problem (5), and put $\varphi = u_\varepsilon$. After substitution of $a^\varepsilon(u_\varepsilon, u_\varepsilon)$ by the norm in (6), we obtain

$$\begin{aligned}(14)\quad & \frac{1}{2}|u_\varepsilon(T,\,\cdot\,,\,\cdot\,)|^2_{L^2(\Theta)} - \frac{1}{2}|u_\varepsilon(0,\,\cdot\,,\,\cdot\,)|^2_{L^2(\Theta)} + |\nabla_x u_\varepsilon|^2_{L^2(0,T;L^2(\Theta))} \\ &\qquad + \varepsilon\left|\frac{\partial u_\varepsilon}{\partial a}\right|^2_{L^2(0,T;L^2(\Theta))} + |\sqrt{\mu}u_\varepsilon|^2_{L^2(0,T;L^2(\Theta))} \\ &\qquad + \frac{1}{2}|u_\varepsilon(\,\cdot\,,\,\cdot\,,0)|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T\int_\Omega h(t,x,u_\varepsilon)\,u_\varepsilon(t,x,0)\,dx\,dt.\end{aligned}$$

Using the same techniques as above, we have

$$\begin{aligned}(15)\quad & \frac{1}{2}|u_\varepsilon(T,x,a)|^2_{L^2(\Theta)} + |\nabla_x u_\varepsilon|^2_{L^2(0,T;L^2(\Theta))} + \varepsilon\left|\frac{\partial u_\varepsilon}{\partial a}\right|^2_{L^2(0,T;L^2(\Theta))} \\ &\qquad + |\sqrt{\mu}u_\varepsilon|^2_{L^2(0,T;L^2(\Theta))} + \frac{1}{2}|u_\varepsilon(\,\cdot\,,\,\cdot\,,0)|^2_{L^2(0,T;L^2(\Omega))} \\ &\qquad \le \sqrt{T\mathrm{mes}(\Omega)}\|h\|_\infty |u_\varepsilon(\,\cdot\,,\,\cdot\,,0)\Big|_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}|u_0|^2_{L^2(\Theta)}.\end{aligned}$$

Assertions of the theorem are then easily checked noting that

$$\begin{aligned}&\frac{1}{2}|u_\varepsilon(\,\cdot\,,\,\cdot\,,0)|^2_{L^2(0,T;L^2(\Omega))} \\ &\qquad \le \sqrt{T\mathrm{mes}(\Omega)}\|h\|_\infty |u_\varepsilon(\,\cdot\,,\,\cdot\,,0)|_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}|u_0|^2_{L^2(\Theta)}\end{aligned}$$

gives $|u_\varepsilon(\,\cdot\,,\,\cdot\,,0)|_{L^2(0,T;L^2(\Omega))} \le C$. □

Now, we turn to prove the passage to the limit when ε tend to zero. We prepare by the two following lemmas.

Lemma 4.1. *If $u_\varepsilon \rightharpoonup u$ weak in $L^2(0,T;L^2(\Theta))$, then $\beta u_\varepsilon \rightharpoonup \beta u$ weakly in $L^2(0,T;L^2(\Theta))$.*

Proof. Indeed, since $\beta \in L^\infty(]0,\omega[)$, $\beta\psi \in L^2(0,T;L^2(\Theta))$ for every $\psi \in L^2(\Theta)$. Now, it is easily seen that

$$\int_\Theta \beta(a)u_\varepsilon\psi\,dx\,da = \int_\Theta u_\varepsilon(\beta\psi)\,dx\,da \to \int_\Theta u(\beta\psi)\,dx\,da,$$

which allows to conclude. □

Lemma 4.2. *Under the hypothesis of the Theorem 4.1, the assertions*

(i) *the strong convergence $u_\varepsilon \to u$ in $L^2(0,T;L^2(\Theta))$,*
(ii) *the weak convergence $h(u_\varepsilon) \rightharpoonup h(u)$ in $L^2(0,T;L^2(\Omega))$,*

are true.

Proof. (i) From the above Theorem 4.1, we have

$$\|\nabla_x u_\varepsilon\|_{L^2(]0,T[\times]0,\omega[;L^2(\Omega))} = \|\nabla_x u_\varepsilon\|_{L^2(0,T;L^2(\Theta))} \le C,$$

which gives by the Poincaré criterion a norm on $L^2(]0,T[\times]0,\omega[\,;H^1(\Omega))$ as we have

$$\|\nabla_x u_\varepsilon\|_{L^2(]0,T[\times]0,\omega[;L^2(\Omega))} = |u_\varepsilon\|_{L^2(]0,T[\times]0,\omega[;H^1(\Omega))} \leq C.$$

There is a subsequence (still noted u_ε) such that $u_\varepsilon \rightharpoonup u$ weakly in $L^2(]0,T[\times]0,\omega[\,;H^1(\Omega))$. Hence u_ε is strongly convergent to u in $L^2(]0,T[\times]0,\omega[\,;L^2(\Omega)) \equiv L^2(0,T;L^2(\Theta))$.

(ii) Let us prove the convergence

$$\int_\Omega [h(u_\varepsilon) - h(u)]\varphi(t,x,0)\,dx \to 0 \quad \text{when } \varepsilon \to 0,$$

where $\varphi(t,x,0)$ is any test function in $L^2(]0,T[\times\{0\};L^2(\Omega))$.

Introducing the function u we have

$$\begin{aligned}\int_\Omega &[h(u_\varepsilon) - h(u)]\varphi(t,x,0)\,dx\\
&= \int_\Omega \bigg[\int_0^\omega \beta(a)u_\varepsilon\,da\,\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\bigg)\\
&\quad - \int_0^\omega \beta(a)u\,da\,\Phi\bigg(\int_0^\omega \beta(a)u\bigg)\bigg]\varphi(t,x,0)\,dx\\
&= \iint \beta(a)(u_\varepsilon - u)\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\bigg)\varphi(t,x,0)\\
&\quad + \iint \beta(a)u\bigg\{\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\bigg) - \Phi\bigg(\int_0^\omega \beta(a)u\bigg)\bigg\}\varphi(t,x,0).\end{aligned}$$

Consider now the quantities A and B:

$$\begin{aligned}A &= \int_\Omega\int_0^\omega \beta(a)u(t,x,a)\bigg[\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\bigg) - \Phi\bigg(\int_0^\omega \beta(a)u\bigg)\bigg]\varphi(t,x,0)\,da\,dx,\\
B &= \int_\Omega\int_0^\omega \beta(a)(u_\varepsilon - u)\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\bigg)\varphi(t,x,0)\,da\,dx.\end{aligned}$$

We shall show that the terms A and B tend to zero.

$$\begin{aligned}A = &\int_\Omega\int_0^\omega \beta(a)u(t,x,a)\varphi(t,x,0)\Phi\bigg(\int_0^\omega u_\varepsilon\beta(a)\bigg)\,dx\,da\\
&- \int_\Omega\int_0^\omega \beta(a)u(t,x,a)\varphi(t,x,0)\Phi\bigg(\int_0^\omega u\beta(a)\bigg)\,dx\,da.\end{aligned}$$

To the term $\int_\Omega\int_0^\omega \beta(a)\,u(t,x,a)\Phi(\int_0^\omega u_\varepsilon\beta(a))\varphi(t,x,0)\,dx\,da$ we apply the Lebesgue's theorem. We use here the assertion (i). As

$$u_\varepsilon(t,x,a) \to u(t,x,a) \quad \text{strongly in } L^2(0,T;L^2(\Theta))$$

(one can extract a subsequence if necessary), we have

$$\int_0^\omega \beta(a)u_\varepsilon\,da \to \int_0^\omega \beta(a)u\,da$$

almost everywhere in $\Omega \times]0,\omega[$. Now since Φ is continuous, we have

$$\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\,da\bigg) \to \Phi\bigg(\int_0^\omega \beta(a)\,uda\bigg).$$

From another side $g(t,\,\cdot\,,\,\cdot\,) : (x,a) \mapsto g(t,x,a) = \beta(a)u(t,x,a)\varphi(t,x,0)$ is a bounded function in $\Omega \times]0,\omega[$, hence

$$g(t,x,a)\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\bigg)\,da \to g(t,x,a)\Phi\bigg(\int_0^\omega \beta(a)u\bigg)\,da \tag{17}$$

almost everywhere. Moreover,

$$\bigg|g(t,x,a)\Phi\bigg(\int_0^\omega \beta(a)u_\varepsilon\bigg)\bigg| \le \beta(a)u(t,x,a)\varphi(t,x,0)|\Phi|_{L^\infty(\mathbb{R})}, \tag{18}$$

which is an L^1-function. Lebesgue's assertions are satisfied by (17) and (18).

The term

$$B = \int_\Omega\int_0^\omega \beta(a)(u_\varepsilon - u)\Phi\bigg(\int \beta(a)u_\varepsilon\bigg)\varphi(t,x,0)\,da\,dx$$

tends clearly to zero (use (16) and Lemma 4.1). □

We can now state the theorem:

Theorem 4.2. *Let u_ε be a positive solution of* (2). *Then under the hypothesis of Theorem* 4.1*, the function $u = \lim_{\varepsilon\to 0} u_\varepsilon$ is the unique solution to the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^N \frac{\partial^2 u}{\partial {x_i}^2} + \frac{\partial u}{\partial a} + \mu u = 0 & \textit{in }]0,T[\, \times \Omega \times]0,\omega[\,, \\ \dfrac{\partial u}{\partial \nu}(t,x,a) = 0 & \textit{on }]0,T[\, \times \partial\Omega \times]0,\omega[\,, \\ u(t,x,0) = F(\alpha_u) & \textit{on }]0,T[\, \times \Omega \times \{0\}, \\ u(0,x,a) = u_0(x,a) & \textit{on } \{0\} \times \Omega \times]0,\omega[\,. \end{cases}$$

Proof. *Existence.* Let $\varepsilon \to 0$. We check easily the result, applying the estimations of the Theorem 4.1, and the Lemmas 4.1 and 4.2. We let the proof to the reader.

Uniqueness. Let u_1 and u_2 be two solutions of the initial problem. The function $\overline{u} = u_1 - u_2$ verifies

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} - \displaystyle\sum_{i=1}^N \frac{\partial^2 \overline{u}}{\partial {x_i}^2} + \frac{\partial \overline{u}}{\partial a} + \mu \overline{u} = 0 & \text{in }]0,T[\, \times \Omega \times]0,\omega[\,, \\ \dfrac{\partial \overline{u}}{\partial \nu}(t,x,a) = 0 & \text{on }]0,T[\, \times \partial\Omega \times]0,\omega[\,, \\ \overline{u}(t,x,0) = F(\alpha_{u_1}) - F(\alpha_{u_2}) & \text{on }]0,T[\, \times \Omega \times \{0\}, \\ \overline{u}(0,x,a) \equiv 0 & \text{on } \{0\} \times \Omega \times]0,\omega[\,. \end{cases}$$

We scalarize over Θ the first equation by $\overline{u}$. It follows

$$\frac{1}{2}\int_\Theta \frac{\partial}{\partial s}(\overline{u}^2)\,dx\,da + \int_\Theta |\nabla \overline{u}|^2\,dx\,da + \int_\Theta \mu\overline{u}^2\,dx\,da = \frac{1}{2}\int_\Omega \overline{u}^2(s,x,0)\,dx.$$

Integrating over $s \in [0,t]$ for $0 \le t \le T$, we obtain:

$$\begin{aligned}\frac{1}{2}\|\overline{u}(t,\,\cdot\,,\,\cdot\,)\|^2_{L^2(\Theta)} + \|\nabla\overline{u}\|^2_{L^2(0,t;L^2(\Theta))} + \|\sqrt{\mu}\overline{u}\|^2_{L^2(0,t;L^2(\Theta))}&\\ = \frac{1}{2}\int_0^t\int_\Omega (F(\alpha_{u_1}) - F(\alpha_{u_2}))^2.&\end{aligned}$$

Hence,

$$\tag{19} \begin{aligned}\frac{1}{2}\|\overline{u}(t,\,\cdot\,,\,\cdot\,)\|^2_{L^2(\Theta)} \le \frac{1}{2}\int_0^t\int_\Omega \biggl(F\biggl(\int_0^\omega \beta(a)u_1(s,x,a)\,da\biggr)&\\ - F\biggl(\int_0^\omega \beta(a)u_2(s,x,a)\,da\biggr)\biggr)^2.&\end{aligned}$$

From the hypothesis (H_6), F is Lipschitz on every bounded and closed set on $\mathbb{R}_+$. So there exists $C>0$ such that

$$\tag{20} \begin{aligned}&\biggl|F\biggl(\int_0^\omega \beta(a)u_1\,da\biggr) - F\biggl(\int_0^\omega \beta(a)u_2\,da\biggr)\biggr|\\ &\qquad\le C\biggl|\int_0^\omega \beta(a)u_1\,da - \int_0^\omega \beta(a)u_2\,da\biggr|,\end{aligned}$$

and using the Cauchy–Schwartz inequality, we obtain

$$\biggl|F\biggl(\int_0^\omega \beta(a)u_1\,da\biggr) - F\biggl(\int_0^\omega \beta(a)u_2\,da\biggr)\biggr| \le C|\beta|_\infty\sqrt{\omega}\int_0^\omega |\overline{u}(s,x,a)|^2\,da.$$

Henceforth,

$$\begin{aligned}\|\overline{u}(t,\,\cdot\,,\,\cdot\,)\|^2_{L^2(\Theta)} &\le C^2|\beta|^2_\infty\omega\int_0^t\biggl(\int_\Omega\int_0^\omega |\overline{u}(s,x,a)|^2\,da\,dx\biggr)\,ds\\ &\le C^2|\beta|^2_\infty\omega\int_0^t \|\overline{u}(s,\,\cdot\,,\,\cdot\,)^2\|^2_{L^2(\Theta)}\,ds.\end{aligned}$$

By the Gronwall lemma we deduce that $\|\overline{u}(t,\,\cdot\,,\,\cdot\,)\|_{L^2(\Theta)} \le 0$. It follows that $u_1 = u_2$ a.e. $\square$

5. Positivity

In this section we prove positivity for the density function u. We begin by the following remark.

Remark 3. It is easily seen that if v is in $\mathcal{V}$, then $v^+ = \sup(0,v)$ and $v^- = \sup(-v,0)$ (such that $v = v^+ - v^-$) are also in $\mathcal{V}$.

Proposition 5.1. *Let u be the solution of* (1). *Then for $u_0 \geq 0$, we have $u \geq 0$ almost everywhere on* $]0,T[\times\Omega\times]0,\omega[$.

Proof. We write $u_\varepsilon = u_\varepsilon^+ - u_\varepsilon^-$. With Remark 3 take $\varphi = u_\varepsilon^- \in \mathcal{V}$ in (5). This gives

$$-\frac{1}{2}|u_\varepsilon(T,\cdot,\cdot)^-|^2_{L^2(\Theta)} + \int_0^T a^\varepsilon(u_\varepsilon^+, u_\varepsilon^-) - \int_0^T a^\varepsilon(u_\varepsilon^-, u_\varepsilon^-) = \int_0^T\int_\Omega h(t,x,u_\varepsilon)\,u_\varepsilon^-\,dx\,dt - \frac{1}{2}|u_\varepsilon^-(0)|^2_{L^2(\Theta)}.$$

Since $u_\varepsilon(0) = u_0 \geq 0$, then $u_\varepsilon^-(0) = 0$. Hence

$$\begin{aligned} 0 &= \int_0^T a^\varepsilon(u_\varepsilon^-, u_\varepsilon^-) + \frac{1}{2}|u(T,\cdot,\cdot)^-|^2_{L^2(\Theta)} + \int_0^T\int_\Omega h(t,x,u_\varepsilon)\,u_\varepsilon^-\,dx\,dt \\ &\geq \min(1/2,\varepsilon)|u_\varepsilon^-|^2_{L^2(0,T;L^2(\mathcal{V})} + \frac{1}{2}|u(T,\cdot,\cdot)^-|^2_{L^2(\Theta)} \\ &\quad + \int_0^T\int_\Omega h(t,x,u_\varepsilon)\,u_\varepsilon^-\,dx\,dt. \end{aligned}$$

Now, since $h \geq 0$, the terms in the right hand side are equal to zero, which implies in particular that $u_\varepsilon^- = 0$ a.e. Passing to the limit on ε we deduce the positivity of the density function u solution to the initial problem. $\square$

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