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EXISTENCE AND CONCENTRATION OF LOCAL MOUNTAIN PASSES FOR A NONLINEAR ELLIPTIC FIELD EQUATION IN THE SEMI-CLASSICAL LIMIT

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Abstract. In this paper we are concerned with the problem of finding solutions for the following nonlinear field equation

$$-\Delta u + V(hx)u - \Delta_p u + W'(u) = 0,$$

where $u : \mathbb{R}^N \to \mathbb{R}^{N+1}$, $N \geq 3$, $p > N$ and $h > 0$. We assume that the potential V is positive and W is an appropriate singular function. In particular we deal with the existence of solutions obtained as critical (not minimum) points for the associated energy functional when h is small enough. Such solutions will eventually exhibit some notable behaviour as $h \to 0^+$. The proof of our results is variational and consists in the introduction of a modified (penalized) energy functional for which mountain pass solutions are studied and soon after are proved to solve our equation for h sufficiently small. This idea is in the spirit of that used in [15], [16] and [17], where "local mountain passes" are found in certain nonlinear Schrödinger equations.

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1. Introduction

This paper has been motivated by several recent works concerning the existence and the concentration behaviour of bound states (i.e. solutions with finite energy) for the following nonlinear elliptic system:

$$-h^2\Delta v + V(x)v - h^p\Delta_p v + W'(v) = 0, \tag{1.1}$$

where

- $h > 0$,
- $v : \mathbb{R}^N \to \mathbb{R}^{N+1}$,
- $N \geq 3$, $p > N$,
- $V : \mathbb{R}^N \to \mathbb{R}$,
- $W : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^{N+1}$ an open set, denoting W' the gradient of W.

Here $\Delta v = (\Delta v_1, \dots, \Delta v_{N+1})$, being Δ the classical Laplacian operator, while $\Delta_p v$ denotes the (N+1)-vector whose j-th component is given by

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v_j).$$

Making the change of variables $x \to hx$, (1.1) can be rewritten as

$$-\Delta u + V_h(x)u - \Delta_p u + W'(u) = 0 \tag{1.2}$$

where $V_h(x) = V(hx)$ and $u(x) = v(hx)$.

Equations like (1.1) or (1.2), but without the potential term $V(x)v$, have been introduced in a set of recent papers (see [6]–[11]). In such works the authors look for soliton-like solutions, i.e. solutions whose energy is finite and which preserve their shape after interactions; in this respect solitons have a particle like behaviour. The interest in studying solitons is due to different reasons: they occur in many physical phenomena and they might represent a model for elementary particles. We refer to [7], [9] and [11] for a more precise description of such developments.

The study of equation (1.1) has been carried on in [4] and [5] where the authors achieved the existence of "ground states", i.e. solutions with least energy: in [4] under the assumption $\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x)$ it was proved that if h is small enough (1.2) possesses at least a solution obtained as a minimum for the associated energy functional; furthermore this solution concentrates around an absolute minimum of V as $h \to 0^+$, in the sense that its shape is a sharp peak near that point, while it vanishes everywhere else. In [5] the authors removed any global assumption on V except for $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and constructed solutions with multiple peaks which concentrate at any prescribed finite set of local minimum points of V in the semi-classical limit (i.e. as $h \to 0^+$). Again such solutions are captured as minima for the energy functional and the technique is based on the analysis of the behaviour of sequences with bounded energy, in the spirit of the concentration-compactness principle ([21]). This paper intends to get existence results for equation (1.2) complementary to those of [4] and [5]; in particular it is a first attempt to deal with critical points, instead of minima: its goal is to show how variational methods based on variants of the Mountain Pass Theorem can be used in order to obtain a critical value for the associated functional characterized by a min-max argument. Furthermore the related solution exhibits some concentration behaviour as $h \to 0^+$.

The appearance of solutions exhibiting a "spike-layer" pattern as $h \to 0^+$ is a phenomenon which has been widely studied in relation with various elliptic equations. For instance, in recent years a large number of works has been devoted in finding single and multiple spike solutions for the Schrödinger equation:

$$ih\frac{\partial \psi}{\partial t} = -\frac{h^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi, \tag{1.3}$$

where $\gamma > 0$, $p > 1$ and $\psi : \mathbb{R}^N \to \mathbb{C}$.

Looking for standing waves of (1.3), i.e. solutions of the form $\psi(x,t) = \exp(-iEt/h)v(x)$, the equation for v becomes

$$-h^2\Delta v + V(x)v - |v|^{p-1}v = 0 \tag{1.4}$$

where we have assumed $\gamma = 2m = 1$ and the parameter E has been absorbed by V. The first result in this line, at our knowledge, is due to Floer and Weinstein ([18]). These authors considered the one-dimensional case and constructed for small $h > 0$ such a concentrating family via a Lyapunov–Schmidt reduction around any non-degenerate critical point of the potential V, under the condition that V is bounded and $p = 3$. In [22] and [23] Oh generalized this result to higher dimensions when $1 < p < (N+2)/(N-2)$ $(N \geq 3)$ and V exhibits "mild oscillations" at infinity. Variational methods based on variants of Mountain Pass Lemma are used in [25] to get existence results for (1.4) where V lies in some class of highly oscillatory V's which are not allowed in [22]–[23]. Under the condition $\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x)$ in [26] Wang established that these mountain-pass solutions concentrate at global minimum points of V as $h \to 0^+$; moreover, a point at which a sequence of solutions concentrate must be critical for V. This line of research has been extensively pursued in a set of papers by Del Pino and Felmer ([15]–[17]). In particular the work [16] seems to be the first attempt to use a localization approach in order to deal with the case of degenerate critical points; in [15] the same authors devised a penalization approach which permitted to find "local mountain-passes" around a local minimum of V with arbitrary degeneracy. Finally we also recall the nonlinear finite dimensional reduction used in [2] and the papers by Grossi ([19]) and Li ([20]).

In most of the above examples, the method employed, local in nature, seems to use in an essential way the splitting of the functional space into a direct sum of good invariant subspaces of the linearized operator; in such a linearization process the nondegeneracy of the concentration points plays a basic role, even though this assumption can be somewhat relaxed. However the finite dimensional reduction does not seem possible in the study of equation (1.1) because of the presence of the p-Laplacian operator. Instead the direct use of variational methods, relying on topological tools, permits to obtain good results under relatively minimal assumptions and this is exactly the direction we will follow. Indeed W is chosen to be a suitable singular function so that the presence of the term $W'(u)$ in (1.1) implies that the solutions have to be searched among the maps which take value in a certain open set $\Omega \subset \mathbb{R}^{N+1}$ (see hypotheses (a)–(g) below). So the nontrivial topological properties of Ω allows us to give a topological classification of such maps. This classification is carried out by means of a topological invariant, the topological charge, which is an integer number depending only on the behaviour of the function on a bounded set (see Definition 3.1).

Throughout this paper we always assume the following assumptions:

(a) $V \in C^1(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and $\lim_{|x|\to\infty} V(x) = \infty$,
(b) $W \in C^1(\Omega, \mathbb{R})$ where $\Omega = \mathbb{R}^{N+1} \setminus \{\overline{\xi}\}$ for some $\overline{\xi}$ with $|\overline{\xi}| = 1$,
(c) W is two times differentiable in 0,
(d) $W(\xi) \geq W(0) = 0$ for all $\xi \in \Omega$,
(e) there exist c, $r > 0$ such that

$$|\xi| < r \Rightarrow W(\overline{\xi} + \xi) > c|\xi|^{-q} \quad \text{where} \quad \frac{1}{q} = \frac{1}{N} - \frac{1}{p}, \quad N \geq 3,\ p > N,$$

(f) there exists $\overline{\varepsilon} \in (0,1)$ such that for every $\xi \in \Omega$ with $|\xi| \leq \overline{\varepsilon}$:

$$W(\xi) \leq \inf\{W(\eta) \mid \eta \in \Omega,\ |\eta| > \overline{\varepsilon}\},$$

(g) for every ξ, $\eta \in \Omega$:

$$|\xi| = |\eta| \leq \overline{\varepsilon} \Rightarrow W(\xi) = W(\eta),$$

furthermore the function $\varphi : [0, \overline{\varepsilon}] \to \mathbb{R}$ defined by

$$\varphi(s) = W(\xi), \quad \xi \in \Omega,\ |\xi| = s,$$

is nondecreasing.

Hypothesis (g) implies in particular

$$W'(\xi)\xi \geq 0 \quad \text{for all } \xi \in \Omega \text{ with } |\xi| \leq \overline{\varepsilon}. \tag{1.5}$$

For $N = 3$ and $p = 6$ (hence $q = 6$) a simple function W which satisfies assumptions (b)–(e) is the following

$$W(\xi) = \frac{|\xi|^2}{|\xi - \bar{\xi}|^6} + |\xi|^2.$$

Modifying it in a suitable way in a neighbourhood of the origin so as to satisfy (g) we obtain a W right for our purposes.

Under the regularity assumptions on V and W it is standard to check that the weak solutions of (1.2) correspond to the nontrivial critical points for the associated energy functional

$$E_h(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}(|\nabla u|^2 + V_h(x)|u|^2) + \frac{1}{p}|\nabla u|^p + W(u) \right) dx. \tag{1.6}$$

In what follows we state precisely our local assumptions on V: we suppose that there is a compact set A_0 and a positive number $\delta_0 > 0$ such that,

(I) $V_0 \equiv V(x)$ for every $x \in A_0$,

and, setting

$$A_0^\delta \equiv \{x \in \mathbb{R}^N \mid \operatorname{dist}(x, A_0) < \delta\},$$

(II) $V_0 < \inf_{x \in \partial A_0^\delta} V(x)$ for every $0 < \delta < \delta_0$.

In other words A_0 is one of the maximal connected set of local minima for the potential V associated to the value V_0. We additionally assume that V_0 is not the global minimum of V, i.e. there exists a point $x_1 \notin A_0^{\delta_0}$ verifying $V(x_1) < V_0$; hence we can choose $\delta_0 > 0$ sufficiently small so as

(III) $V_1 \equiv \sup_{x \in B_{\delta_0}(x_1)} V(x) < V_0$.

The main result of this paper is the following one:

Theorem. *Assume that hypotheses* (a)–(g) *and* (I)–(III) *hold. Then for* $h > 0$ *sufficiently small there exists a solution* v_h *of equation* (1.1). *Furthermore if we consider the sets*

$$S_h = \{x \in \mathbb{R}^N \mid W'(v_h(x))v_h(x) < 0\} \subset \mathbb{R}^N,$$

and put $S_\delta = \{x \in \mathbb{R}^N \mid \operatorname{dist}(x, S) < \delta\}$, *then the family* v_h *decays uniformly to zero for* x *outside every neighbourhood of* S.

Since we are in a case where the local reductions do not immediately apply, it is natural to ask if the penalization method developed in [15]–[17] can be adapted to our situation. This paper gives a positive answer to the above question. However our current framework is more delicate, since the functional (1.6) is well defined in an open subset of a Banach space, and this fact put an obstacle to a direct application of the Critical Points Theory for a general C^1-functional in a Banach space. Roughly speaking, the main argument consists of "stopping up" in some sense the singularity $\overline{\xi}$ by defining a suitable modification of the nonlinearity W; the functional J_h associated to the new nonlinear term turns out to be defined in a closed subspace of $W^{1,2}(\mathbb{R}^N,\mathbb{R}^{N+1})\cap W^{1,p}(\mathbb{R}^N,\mathbb{R}^{N+1})$; furthermore the coercivity of V becomes crucial in obtaining the Palais–Smale condition for this functional, while the structure assumptions (a)–(g) guarantee the validity of the results obtained in [5] for the same equation and permit us to construct the geometry of the Mountain Pass Theorem; hence this theorem applies providing the existence of critical values for J_h. Then, taking advantage of the min-max characterization of the "mountain pass" critical values, one finally shows that the solutions found are prevented from approaching the singularity and then they solves our original problem when h is sufficiently small. In this sense we call them "local mountain passes". Finally the associated family of solutions $\{v_h\}$ to equations (1.1) vanishes uniformly outside a bounded set in $\mathbb{R}^N$.

An improvement of the above stated results would be the exact localization of the concentration set of the family $\{v_h\}$ so as to fully recover the achievements of Del Pino–Felmer in [15], [16] and [17]. However in the attempt to extend their approach to our current framework, some technical difficulties arise and make a direct application of their methods disadvantageous. Probably the technique has to be changed.

This paper is organized in the following way. In Section 2 we introduce the abstract setting, i.e. the functional set in which the energy functional is defined. Section 3 is devoted to the definition of a topological device, the "topological charge", and briefly sketches the arguments which are straightforward transposition of [4] and [5]. In Section 4 several technical results are established such as various smoothness and qualitative properties of the energy functional E_h. Section 5 deals with the proof of some preliminary results and the explanation of some general facts which pave the way for the construction of a suitable modified functional in Section 6. After a detailed study of the properties of this penalized functional, among which the validity of the Palais-Smale condition, in the last section we define an appropriate min-max value which will yield a critical point of the original functional E_h for small h. Finally, at the end of Section 7, the study of the behaviour of the above found solutions in the semi-classical limit is developed and a concentration result is achieved.

Notations. We fix the following notations we will use from now on.

- $x\,y$ is the standard scalar product between $x,\, y\in\mathbb{R}^N$.
- $|x|$ is the Euclidean norm of $x\in\mathbb{R}^N$. Analogously $|M|$ is the Euclidean norm of a $m\times n$ real matrix M.
- $W^{1,2}(\mathbb{R}^N,\mathbb{R}^{N+1})$ and $W^{1,p}(\mathbb{R}^N,\mathbb{R}^{N+1})$ are the standard Sobolev spaces.
- For $u:\mathbb{R}^N\to\mathbb{R}^{N+1}$, ∇u is the $(N+1)\times N$ real matrix whose rows are given by the gradient of each component function u_j. Furthermore, for

$u, v : \mathbb{R}^N \to \mathbb{R}^{N+1}$, we put

$$|\nabla u| = \sqrt{\sum_{i,j} \left(\frac{\partial u^j}{\partial x_i} \right)^2}, \quad (\nabla u | \nabla v) = \sum_{i,j} \frac{\partial u^j}{\partial x_i} \frac{\partial v^j}{\partial x_i}.$$

- For any $U \subset \mathbb{R}^N$, $\operatorname{int}(U)$ is its internal part, $\overline{U}$ its closure and ∂U its boundary. Furthermore χ_U denote the characteristic function of U.
- By $\operatorname{meas}(U)$ we intend the Lebesgue measure of a set $U \subset \mathbb{R}^N$, while $\operatorname{dist}(x, U)$ is the Euclidean distance between a point $x \in \mathbb{R}^N$ and U, i.e. $\operatorname{dist}(x, U) = \inf_{y \in U} |x - y|$.
- If $x \in \mathbb{R}^N$ and $r > 0$, then $B_r(x)$ or $B(x, r)$ is the open ball with centre in x and radius r.
- For a Banach space H we denote its dual by H'.
- If H is a Banach space, by $\langle F, u \rangle$ we indicate the duality between $F \in H'$ and $u \in H$.

2. Functional setting

In order to obtain critical points for the functional E_h we choose a suitable Banach space: for every $h > 0$ let H_h denote the subspace of $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1})$ consisting of functions u such that

$$\|u\|_{H_h} \equiv \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V_h(x)|u|^2)\, dx \right)^{1/2} + \left(\int_{\mathbb{R}^N} |\nabla u|^p\, dx \right)^{1/p} < \infty. \tag{2.1}$$

The space H_h can also be defined as the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{R}^{N+1})$ with respect to the norm (2.1). The main properties of H_h are summarized in the following lemma.

Lemma 2.1. *For every $h > 0$ the following statements hold:*

(i) *H_h is continuously embedded in $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1})$ and $W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$.*

(ii) *There exist two constants C_0, $C_1 > 0$ such that, for every $u \in H_h$,*

$$\|u\|_{L^\infty} \le C_0 \|u\|_{H_h}$$

and

$$|u(x) - u(y)| \le C_1 |x - y|^{(p-N)/p} \|\nabla u\|_{L^p} \quad \text{for all } x, y \in \mathbb{R}^N. \tag{2.2}$$

(iii) *For every $u \in H_h$*

$$\lim_{|x| \to \infty} u(x) = 0. \tag{2.3}$$

(iv) *If $\{u_n\}$ converges weakly in H_h to some function u, then it converges uniformly on every compact set in $\mathbb{R}^N$.*

Proof. We confine ourselves to prove the continuous immersion $H_h \subset W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$, since the other assertions are direct consequences of the Sobolev embedding theorems. Taking into account of (2.1), it is sufficient to prove the continuous embedding $H_h \subset L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$. Let $\{\ell_k\}$ be the sequences of numbers in $(0, \infty]$ defined by recurrence as follows:

$$\begin{aligned} \ell_1 &= 2^* \equiv \frac{2N}{N-2}, \\ \ell_{k+1} &= \begin{cases} \ell_k^* \equiv \dfrac{N\ell_k}{N-\ell_k} & \text{if } \ell_k < N, \\ \infty & \text{if } \ell_k \geq N. \end{cases} \end{aligned}$$

It is immediate to prove that

$$\lim_{k\to\infty} \ell_k = \infty. \tag{2.4}$$

Notice that, by hypothesis $N > 2$, hence $2^* < \infty$; then distinguish two cases: either

(a_1) $\ell_1 \equiv 2^* \geq p$, or (b_1) $\ell_1 = 2^* < p$.

In the case (a_1) the Sobolev continuous inclusion $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1}) \subset L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$ permits us to conclude.

Let us consider case (b_1). We have $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1}) \subset L^{2^*}(\mathbb{R}^N, \mathbb{R}^{N+1})$ with a continuous immersion, and moreover, taking $\alpha = 2(p-2^*)/2^*(p-2) \in (0,1)$, since $2 < 2^* < p$,

$$\|\nabla u\|_{L^{2^*}} \leq \|\nabla u\|_{L^2}^{\alpha} \|\nabla u\|_{L^p}^{1-\alpha}$$

and then, from (2.1), $\|\nabla u\|_{L^{2^*}} \leq \|u\|_{H_h}$, which implies $H_h \subset W^{1,2^*}(\mathbb{R}^N, \mathbb{R}^{N+1})$ continuously. Again we have an alternative: either

(a_2) $\ell_2 \geq p$, or (b_2) $\ell_2 < p$.

If case (a_2) holds true, we get $H_h \subset W^{1,2^*}(\mathbb{R}^N, \mathbb{R}^{N+1}) \subset L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$ with continuous inclusions. In the case b_2) it is $H_h \subset L^{\ell_2}(\mathbb{R}^N, \mathbb{R}^{N+1})$ continuously and we repeat the same argument used for (b_1). This alternative precess terminates in a finite number of steps. Indeed, using (2.4), it makes sense to define $k_0 = \inf\{k \in \mathbb{N} \mid \ell_k \geq p\} \in (2, \infty)$. Then we deduce that the case (a_{k_0}) occurs and so we conclude. □

The fact that $\inf_{x\in\mathbb{R}^N} V(x) > 0$ assures that, for every $h > 0$, the space H_h is continuously embedded in $H_0 \equiv W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1}) \cap W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$ when endowed with the norm

$$\|u\|_{H_0} \equiv \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\, dx \right)^{1/2} + \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{1/p}.$$

It's immediate to get that last norm is invariant with respect to the group of rotations and translations in $\mathbb{R}^N$.

Notice that from (2.2) we derive the following property we are going to use several times in the proofs of our results: given $\{u_\alpha\} \subset H_0$ a family of functions verifying $\|\nabla u_\alpha\|_{L^p} \le M$ for some $M \ge 0$, then there results: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x, y \in \mathbb{R}^N : \quad |x - y| \le \delta \Rightarrow |u_\alpha(x) - u_\alpha(y)| \le \varepsilon \quad \forall \alpha.$$

We refer to the above property as to the "equi-uniform continuity" of the family $\{u_\alpha\}$. As an immediate consequence we get the following lemma.

Lemma 2.2. *Let $h > 0$ and $\{u_n\} \subset H_h$ a sequence verifying*

$$u_n \rightharpoonup u \quad \text{weakly in } H_h$$

for some $u \in H_h$. Then, up to a subsequence,

$$u_n \to u \quad \text{uniformly in } \mathbb{R}^N.$$

Proof. Fix $\gamma \in (0,1)$ arbitrarily and consider $R > 0$ such that, according to (2.3),

$$\forall x \in \mathbb{R}^N \setminus \overline{B}_R(0) : |u(x)| < \gamma/2.$$

The object is to prove that, up to a subsequence,

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^N \setminus \overline{B}_R(0) : \quad |u_n(x)| < \gamma/2. \tag{2.5}$$

Let $x_n \in \mathbb{R}^N$ be a point with $|u_n(x_n)| \ge \gamma/2$. First we'll prove that the sequence $\{x_n\}$ is bounded in $\mathbb{R}^N$. Indeed, arguing by contradiction, we assume that, up to a subsequence,

$$|x_n| \to \infty \quad \text{as } n \to \infty. \tag{2.6}$$

We claim that

$$\exists \overline{r} > 0 \quad \text{s.t. } B_{\overline{r}}(x_n) \subset \{x \in \mathbb{R}^N : |u_n(x)| > \gamma/4\}. \tag{2.7}$$

Indeed, since $\{u_n\}$ is bounded in H_h, by (2.2) there exists $\delta > 0$ such that

$$\forall n \in \mathbb{N},\ \forall x, y \in \mathbb{R}^N : |x - y| < \delta \ \Rightarrow\ |u_n(x) - u_n(y)| < \gamma/4. \tag{2.8}$$

Now, if (2.7) were false, there would exist $n \in \mathbb{N}$ and $\overline{x}_n \in \mathbb{R}^N$, with $|x_n - \overline{x}_n| < \delta$, such that

$$|u_n(\overline{x}_n)| \le \gamma/4.$$

Then we conclude

$$|u_n(x_n) - u_n(\overline{x}_n)| \ge |u_n(x_n)| - |u_n(\overline{x}_n)| \ge \gamma/2 - \gamma/4,$$

in contradiction with (2.8). Then we can write

$$\int_{\mathbb{R}^N} V_h(x)|u_n|^2\,dx \ge \int_{B_{\overline{r}}(x_n)} V_h(x)|u_n|^2\,dx \ge \frac{\gamma^2}{16}\int_{B_{\overline{r}}(x_n)} V_h(x)\,dx.$$

Because of the coercivity of V_h and (2.6) the integral on the right side diverges for large n, but this contradicts the fact that $\{u_n\}$ is bounded in H_h. Therefore the sequence $\{x_n\}$ turns out to be bounded in $\mathbb{R}^N$, up to subsequence we obtain

$$x_n \to x \in \mathbb{R}^N \quad \text{as } n \to \infty.$$

Now we have

$$|u_n(x_n) - u(x)| \le |u_n(x_n) - u_n(x)| + |u_n(x) - u(x)|.$$

For n large enough both terms on the right side are arbitrary small, the first because of the equi-uniform continuity of $\{u_n\}$ and the second because of (iv) of Lemma 2.1. This implies

$$u_n(x_n) \to u(x) \quad \text{as } n \to \infty$$

and then $|u(x)| \ge \gamma/2$. The choice of γ yields $|x| < R$, which implies $|x_n| < R$ for n sufficiently large. So (2.5) holds.

Now combine (2.5) with (iv) of Lemma 2.1: what we deduce is the existence of a subsequence $\{u_n^1\}$ such that

$$|u_n^1(x) - u(x)| < \gamma \quad \text{for all } x \in \mathbb{R}^N \text{ and all } n \in \mathbb{N}.$$

Repeating the same argument for $\gamma^2 > 0$, there exists a subsequence $\{u_n^2\}$ of $\{u_n^1\}$ verifying

$$|u_n^2(x) - u(x)| < \gamma^2 \quad \text{for all } x \in \mathbb{R}^N, \text{ and all } n \in \mathbb{N}$$

and so on for γ^3, γ^4 etc. Now we apply a diagonal method and consider the sequence

$$\widehat{u}_n = u_n^n \quad \text{for all } n \in \mathbb{N}.$$

Obviously $\{\widehat{u}_n\}$ is a subsequence of the original $\{u_n\}$. It remains to prove that $\{\widehat{u}_n\}$ converges uniformly to u in $\mathbb{R}^N$. Let us fix $\eta > 0$; there is $\alpha \in \mathbb{N}$ such that $\gamma^\alpha < \eta$. By definition

$$|\widehat{u}_\beta(x) - u(x)| < \eta \quad \text{for all } \beta > \alpha \text{ and all } x \in \mathbb{R}^N,$$

which is the thesis. □

Since for every $h \ge 0$ the functions in H_h are continuous, we can consider the set

$$\Lambda_h = \{u \in H_h \mid \text{for all } x \in \mathbb{R}^N : u(x) \ne \overline{\xi}\}.$$

By (ii) and (iii) of Lemma 2.1, it is easy to obtain that Λ_h is open in H_h. The boundary of Λ_h is given by

$$\partial\Lambda_h = \{u \in H_h \mid \text{exists } \overline{x} \in \mathbb{R}^N : u(\overline{x}) = \overline{\xi}\}.$$

Now we want to give a topological classification of the maps $u \in \Lambda_h$. More precisely, we introduce a topological invariant with suitable "localization" properties in the sense that, roughly speaking, it depends on the compact region where u is concentrated. This invariant consists of an integer number called "topological charge" and it will be defined by means of the topological degree.

3. Topological charge

In this section we take from [13] some crucial definitions and results. In the open set $\Omega = \mathbb{R}^{N+1} \setminus \{\overline{\xi}\}$ we consider the N-sphere centered at $\overline{\xi}$

$$\Sigma = \{\xi \in \mathbb{R}^{N+1} \mid |\xi - \overline{\xi}| = 1\}.$$

On Σ we take the north and the south pole, denoted by ξ_N and ξ_S, with respect to the axis joining the origin with $\overline{\xi}$, i.e. since $|\overline{\xi}| = 1$,

$$\xi_N = 2\overline{\xi}, \quad \xi_S = 0.$$

Then we consider the projection $P : \Omega \to \Sigma$ defined by

$$P(\xi) = \overline{\xi} + \frac{\xi - \overline{\xi}}{|\xi - \overline{\xi}|} \quad \text{for all } \xi \in \Omega.$$

Notice that, by definition, it follows:

$$P(\xi) = 2\overline{\xi} \Leftrightarrow \xi = (1 + |\xi - \overline{\xi}|)\overline{\xi},$$

which leads to

$$P\,(\xi) = 2\overline{\xi} \Rightarrow |\xi| > 1. \tag{3.1}$$

Using the above-mentioned notation we can give the following definition.

Definition 3.1. Given $u \in \Lambda_\hbar$ and $U \subset \mathbb{R}^N$ an open set such that $|u(x)| \le 1$ if $x \in \partial U$, then we define the (topological) charge of u in the set U as the following integer number

$$\operatorname{ch}(u, U) = \deg(P \circ u, U \cap K(u), 2\overline{\xi}),$$

where $K(u)$ is the open set

$$K(u) = \{x \in \mathbb{R}^N \mid |u(x)| > 1\}.$$

We recall the convention $\deg(P \circ u, \emptyset, 2\overline{\xi}) = 0$. Furthermore, for given $u \in \Lambda_\hbar$, we define the (topological) charge of u as the integer number

$$\operatorname{ch}(u) = \deg(P \circ u, K(u), 2\overline{\xi}).$$

Notice that the choice of the value 1 in Definition 3.1 depends on the norm of the singularity $\overline{\xi}$. In other words, if $|u(x)| \le 1$ for $x \in \partial U$, the topological charge of u in U is the Brower topological degree of $P \circ u$ in $K(u) \cap U$ with respect to the north pole of Σ, and it is well defined thanks to (2.3) and (3.1).

The following lemma shows how the topological charge has some invariance property.

LEMMA 3.1. *Let $u \in \Lambda_h$ and $U \subset \mathbb{R}^N$ an open bounded set with $|u(x)| \leq 1$ for $x \in \partial U$. Then there results*

$$\mathrm{ch}(u, U) = \deg(P \circ u, U, 2\overline{\xi}).$$

As a corollary, for every $u \in \Lambda_\hbar$ and for every $R > 0$ such that $K(u) \subset B_R(0)$,

$$\mathrm{ch}(u) = \deg(P \circ u, B_R(0), 2\overline{\xi}).$$

The proof is essentially the same as in Proposition 3.3 of [13] and is based on the excision property of the topological degree.

From well known properties of the topological degree we get other useful properties of the topological charge. For example notice that if $U \subset \mathbb{R}^N$ is open and such that $|u(x)| \leq 1$ for $x \in \partial U$ and if U consists of m connected components $U_1, \dots, U_m$, then by the additivity property of the degree we get

$$\mathrm{ch}(u, U) = \sum_{j=1}^{m} \mathrm{ch}(u, U_j). \tag{3.2}$$

Another consequence is that the topological charge is stable under uniform convergence in the sense specified by the following lemma.

LEMMA 3.2. *Let $\{u_n\} \subset \Lambda_\hbar$, $u \in \Lambda_\hbar$ and $U \subset \mathbb{R}^N$ an open set such that*

$$u_n \to u \text{ uniformly in } U,$$
$$|u_n(x)| \leq 1, \ |u(x)| \leq 1 \quad \text{for all } x \in \partial U \text{ and all } n \in \mathbb{N}.$$

Then, for n large enough,

$$\mathrm{ch}(u, U) = \mathrm{ch}(u_n, U).$$

As a consequence, for every $u \in \Lambda_\hbar$ there exists $\varrho = \varrho(u) > 0$ such that, for every $v \in \Lambda_\hbar$,

$$\|u - v\|_{L^\infty} \leq \varrho \Rightarrow \mathrm{ch}(u) = \mathrm{ch}(v).$$

The proof can be found in [4], Lemma 3.2 and Corollary 3.1.

Finally we set

$$\Lambda_h^* = \{u \in \Lambda_h \mid \mathrm{ch}(u) \neq 0\}.$$

By Theorem 3.3, Λ_h^* is open in H_h and, by (3.1), we deduce

$$\|u\|_{L^\infty} > 1 \quad \text{for all } u \in \Lambda_h^*. \tag{3.3}$$

In [5] the authors proved the existence of a minimum for the functional E_h defined by (1.6) in some open subset of Λ_h^* so as to obtain the existence of solutions for (1.2) among the fields u with nontrivial charge. These results will be recalled in Section 5 and will constitute the starting point for our search of mountain-pass solutions.

4. The energy functional

Now we are going to study the properties of the functional E_h; first of all we want to show that it is well defined in the space Λ_h, i.e. for every $u \in \Lambda_h$ we have

$$E_h(u) < \infty. \tag{4.1}$$

Indeed, by hypothesis (b)–(d), for every $\xi \in \Omega$ we can write

$$|W(\xi)| \leq |W''(0)| \cdot |\xi|^2 + |o(\xi)| \tag{4.2}$$

where $o(\xi)$ is a real function bounded on bounded sets and satisfying

$$\lim_{|\xi| \to 0} o(\xi)/|\xi|^2 = 0.$$

Here we have denoted by $W''(0)$ the Hessian matrix of the function W in 0. Then there exist ϱ, $\varepsilon > 0$ such that, for all $\xi \in \Omega$,

$$|\xi| \leq \varrho \Rightarrow |o(\xi)| \leq \varepsilon |\xi|^2$$

which implies

$$\int_{\mathbb{R}^N} W(u)\, dx \leq |W''(0)| \int_{\mathbb{R}^N} |u|^2\, dx + \varepsilon \int_{|u| \leq \varrho} |u|^2\, dx + \int_{|u| > \varrho} |o(u)|\, dx.$$

By (2.3) the set $\{|u| > \varrho\}$ has compact closure, then we obtain (4.1). Obviously E_h is bounded from below and is coercive in the H_h-norm:

$$\lim_{\|u\|_{H_h} \to \infty} E_h(u) = \infty. \tag{4.3}$$

Moreover, in [4] we have proved that the energy functional E_h belongs to the class $C^1(\Lambda_h, \mathbb{R})$ under the assumptions (a)–(d). An immediate corollary is that the critical points $u \in \Lambda_h$ for the functional E_h are weak solutions of equation (1.2).

For sake of brevity we call the internal energy the functional defined on H_0:

$$E_i(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\inf_{x \in \mathbb{R}^N} V(x))|u|^2)\, dx + \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p\, dx. \tag{4.4}$$

Next two propositions deal with some other properties of the functional E_h; we omit the proofs because they are the same as in [13], provided that we substitute "E_h" for "E" and "Λ_h" for "Λ". The first deals with the behaviour of E_h when u approaches the boundary of Λ_h.

Proposition 4.1. *Let* $\{u_n\} \subset \Lambda_h$ $(h \geq 0)$ *be bounded in the* H_h*-norm and weakly converging to* $u \in \partial\Lambda_h$*, then*

$$\int_{\mathbb{R}^N} W(u_n)\,dx \to \infty \quad \text{as } n \to \infty.$$

As a consequence, if $\{u_n\} \subset \Lambda_h$ *is weakly converging to* u *and such that* $E_h(u_n)$ *is bounded, then* $u \in \Lambda_h$*.*

(See [13, Lemma 3.7, p. 326]).

The second proposition states the weakly lower semi-continuity of the energy functional E_h.

Proposition 4.2. *For every* $u \in \Lambda_h$ *and for every sequence* $\{u_n\} \subset \Lambda_h$*, if* $\{u_n\}$ *weakly converges to* u*, then*

$$\liminf_{n\to\infty} E_h(u_n) \geq E_h(u).$$

(See [13, Proposition 3.10, p. 328])

The following is the key result of this section and it will be very useful for the construction of the penalized functional. For sake of simplicity we fix the following notation: for $u \in H_0$ and $a > 0$ we define

$$\Upsilon(u,a) \equiv \{x \in \mathbb{R}^N \mid |u(x)| \geq a\} \tag{4.5}$$

and

$$\Pi(u,a) \equiv \{x \in \mathbb{R}^N \mid |u(x) - \overline{\xi}| \geq a\}. \tag{4.6}$$

Lemma 4.1. *For every* α, $\beta > 0$ *there exists* $d > 0$ *such that for every* $u \in H_0$ *with* $E_i(u) \leq \alpha$ *:*

$$\int_{\Pi(u,d/2)} W(u)\,dx \leq \beta \Rightarrow \min_{x\in\mathbb{R}^N} |u(x) - \overline{\xi}| > d. \tag{4.7}$$

Proof. Arguing by contradiction, assume the existence of α, $\beta > 0$ and a sequence $\{u_n\} \subset H_0$ such that

$$E_i(u_n) \leq \alpha, \quad \int_{\Pi(u,1/2n)} W(u_n)\,dx \leq \beta \tag{4.8}$$

and

$$\min_{x\in\mathbb{R}^N} |u_n(x) - \overline{\xi}| \leq 1/n. \tag{4.9}$$

For every $n \in \mathbb{N}$, by (2.3), there exists $x_n \in \mathbb{R}^N$ such that

$$|u_n(x_n) - \overline{\xi}| = \min_{x\in\mathbb{R}^N} |u_n(x) - \overline{\xi}|. \tag{4.8}$$

Then we consider the sequence $w_n = u_n(\,\cdot\, + x_n)$. Since $E_i(w_n) = E_i(u_n) \le \alpha$, we deduce that $\{w_n\}$ is bounded in H_0; hence, up to a subsequence, it weakly converges to some $w \in H_0$. Without loss of generality we may assume $w_n \to w$ a.e. in $\mathbb{R}^N$ too. Now, from (4.8), (4.9) and the definition of w_n, we obtain:

$$w(0) = \lim_{n\to\infty} w_n(0) = \overline{\xi},$$

therefore $w \in \partial\Lambda_0$. According to (2.3), the set $\{x \in \mathbb{R}^N \mid w(x) = \overline{\xi}\}$ is compact and non empty, hence it makes sense to choose $\overline{x} \in \mathbb{R}^N$ with the following properties:

$$w(\overline{x}) = \overline{\xi} \quad \text{and} \quad \|\overline{x}\| = \max\{\|x\| \mid x \in \mathbb{R}^N \text{ s.t. } w(x) = \overline{\xi}\}. \tag{4.10}$$

Observe that since W is nonnegative, the desired conclusion will follow if we show the existence of $\rho > 0$ sufficiently small such that

$$\int_{B_\rho(\overline{x}+x_n)\cap\Pi(u_n,1/2n)} W(u_n)\,dx \equiv \int_{B_\rho(\overline{x})\cap\Pi(w_n,1/2n)} W(w_n)\,dx \to \infty$$

as $n \to \infty$, contrary to (4.7). In the first place we prove the following assertion: there exists $\rho > 0$ such that, for every $x \in B_\rho(\overline{x})$ and for n sufficiently large,

$$|w_n(x) - \overline{\xi}| < r, \tag{4.11}$$

where r has been defined in hypothesis (e). Indeed, since $\{w_n\}$ is bounded in H_0, in particular, $\{\nabla w_n\}$ is bounded in L^p. Using (2.2) we have, for every $x \in \mathbb{R}^N$,

$$|w_n(x) - w_n(\overline{x})| \le \overline{c}\,|x - \overline{x}|^{(p-N)/p}$$

for some constant $\overline{c} > 0$. Then, for all $x \in \mathbb{R}^N$, there results

$$|w_n(x) - \overline{\xi}| \le \overline{c}\,|x - \overline{x}|^{(p-N)/p} + |w_n(\overline{x}) - \overline{\xi}|, \tag{4.12}$$

by which, taking into account that $w_n(\overline{x}) \to \overline{\xi}$ as $n \to \infty$ and choosing ρ sufficiently small, we obtain (4.11).

Now, using (4.11) and hypothesis (e), for every $x \in B_\rho(\overline{x})$ we get

$$W(w_n(x)) \ge \frac{c}{|w_n(x) - \overline{\xi}|^q},$$

and (4.12) yields

$$\int_{B_\rho(\overline{x})\cap\Pi(w_n,1/2n)} W(w_n)\,dx \ge \int_{B_\rho(\overline{x})\cap\Pi(w_n,1/2n)} \frac{c}{\big(\overline{c}\,|x-\overline{x}|^{(p-N)/p} + |w_n(\overline{x}) - \overline{\xi}|\big)^q}\,dx.$$

Note that because of (4.10) for every $x \in \mathbb{R}^N$ with $\|x\| > \|\overline{x}\|$ it is $w(x) \ne \overline{\xi}$ and then straightforward calculations imply:

$$\liminf_{n\to\infty} \chi_{\Pi(w_n,1/2n)}(x) = 1.$$

Finally Fatou's lemma leads to

$$\liminf_{n\to\infty}\int_{B_\rho(\overline{x})\cap\Pi(w_n,1/2n)} W(w_n)\,dx \geq \int_{B_\rho(\overline{x})\setminus B_{\|\overline{x}\|}(0)} \frac{c}{\overline{c}^q\,|x-\overline{x}|^N}\,dx = \infty$$

and the theorem is proved. □

The development of our arguments starts essentially with the next section, where some crucial technical results are provided.

5. Preliminary results

This section is devoted to establish some preliminary results concerning the properties of the energy functional E_h.

Let us go back to Theorem 5.2 of [5] and try to apply it with respect to the open set $A_0^{\delta_0}$. First of all observe that the hypotheses (a)–(g) permit us to reconstruct exactly the same framework considered in [5] and, moreover, the geometric properties (I) and (II) of the set $A_0^{\delta_0}$ guarantee that all the assumptions used in the above quoted theorem are satisfied; for sake of completeness we summarize the crucial result.

For all $h>0$ consider the set

$$\widetilde{\Lambda}_h^* = \{u\in\Lambda_h^* \mid \text{for all } x\in\mathbb{R}^N\setminus A_0^{\delta_0}/h : |u(x)|<1\},$$

where, with obvious notation, $A_0^{\delta_0}/h = \{x/h \mid x\in A_0^{\delta_0}\}$. According to Lemma 3.2 and (ii) of Lemma 2.1 each $\widetilde{\Lambda}_h^*$ is open in Λ_h. Now we define

$$\widetilde{E}_h^* = \inf_{u\in\widetilde{\Lambda}_h^*} E_h(u).$$

Theorem 5.2 of [5] can be reformulated in the following way.

Theorem 5.1. *There exists $h_0>0$ such that the minimum $\widetilde{E}_h^*$ is attained by a function $u_h^0\in\widetilde{\Lambda}_h^*$ for every $h\in(0,h_0)$. Furthermore the family $\{u_h^0\}$ satisfies the following property:*

- *u_h^0 has at least one local maximum point $x_h^0\in(1/h)A_0^{\delta_0}$ with $|u_h^0(x_h^0)|>1$. Also, for every $\varepsilon>0$, there exists $R_\varepsilon>0$ such that, for h sufficiently small, it holds:*

$$|u_h^0(x)|\leq\varepsilon \quad \text{for all } x\in\mathbb{R}^N \text{ with } |x-x_h^0|\geq R_\varepsilon. \tag{5.1}$$

Finally, for every sequence $h_n\to 0^+$, there exists a subsequence, still denoted by h_n, such that

$$h_n x_{h_n}^0 \to x_0 \quad \text{as } n\to\infty,\ x_0\in A_0. \tag{5.2}$$

REMARK 5.1. If the set A_0 consists of a single point, then all the family $\{h\,x_h^0\}$ converges to that point as $h \to 0^+$. In general we have $V(h\,x_h^0) \to V_0$ as $h \to 0^+$; moreover, combining (5.1) with (5.2), we obtain

$$\lim_{h\to 0^+} \sup_{x\notin (1/h)A_0^{\delta_0}} |u_h^0(x)| = 0. \tag{5.3}$$

For the proof and more details of Lemma 5.1 we refer to [5, Section 5].

Furthermore Lemma 5.1 of [5] gives us another important information: the family $\{E_h(u_h^0)\}$ is bounded for h small enough, hence, provided we choose h_0 sufficiently small, it follows

$$\sup_{h\in(0,h_0)} E_h(u_h^0) < \infty. \tag{5.4}$$

As an immediate consequence we get the following simple result.

LEMMA 5.1. *There exists $r_0 > 0$ such that, for every $h \in (0, h_0)$,*

$$B_{r_0}(x_h^0) \subset \{x \in \mathbb{R}^N \mid |u_h^0(x)| \geq 1/2\}. \tag{5.5}$$

PROOF. By (5.4) it immediately follows that the family $\{\|\nabla u_h^0\|_p\}_{h\in(0,h_0)}$ is bounded, hence from (2.2) we deduce the equi-uniform continuity of $\{u_h^0\}$, then there exists $r_0 > 0$ such that

$$|x - y| \leq r_0 \Rightarrow |u_h^0(x) - u_h^0(y)| \leq 1/2 \tag{5.6}$$

for all $h \in (0, h_0)$ and all $x, y \in \mathbb{R}^N$. Now take $h \in (0, h_0)$ arbitrarily and $z \in \mathbb{R}^N$ with $|x_h^0 - z| \leq r_0$, then we get

$$|u_h^0(z)| \geq |u_h^0(x_h^0)| - |u_h^0(x_h^0) - u_h^0(z)| > 1 - 1/2 = 1/2.$$

Hence the desired conclusion follows. □

Now let us consider the open set

$$U_h^* = \{u \in \Lambda_h^* \mid |u(x)| < \overline{\varepsilon} \quad \text{for all } x \in \mathbb{R}^N \setminus (1/h)A_0^{\delta_0}\}, \tag{5.7}$$

where $\overline{\varepsilon}$ has been defined in hypotheses (f)–(g). The boundary of U_h^* is given by

$$\partial U_h^* = \{u \in \Lambda_h^* \mid |u(x)| \leq \overline{\varepsilon} \text{ for all } x \in \mathbb{R}^N \setminus (1/h)A_0^{\delta_0} \\ \text{and exists } \overline{x} \in \partial((1/h)A_0^{\delta_0}) \text{ s.t. } |u(\overline{x})| = \overline{\varepsilon}\}.$$

With the help of these notations we give the following lemma which will play a fundamental role for catching our solutions; it essentially states that if h is sufficiently small, then u_h^0 is a strict minimum point for E_h.

Lemma 5.2. *For h sufficiently small it is*

$$u_h^0 \in U_h^* \quad \text{and} \quad \inf_{u\in\partial U_h^*} E_h(u) > E_h(u_h^0).$$

Proof. Choose $h \in (0, h_0)$, small enough so that, according to (5.3), u_h^0 verifies:

$$|u_h^0(x)| < \bar{\varepsilon} \quad \text{in} \quad \mathbb{R}^N \setminus (1/h)A_0^{\delta_0}. \tag{5.8}$$

This implies $u_h^0 \in U_h^*$, and hence, since obviously $U_h^* \subset \widetilde{\Lambda}_h^*$,

$$\inf_{u\in\widetilde{\Lambda}_h^*} E_h(u) = \inf_{u\in U_h^*} E_h(u) = E_h(u_h^0) \equiv \widetilde{E}_h^*. \tag{5.9}$$

We begin by proving that we can choose $h_1 > 0$ sufficiently small so that for all $h \in (0, h_1)$ there results

$$E_h(u) > E_h(u_h^0) \quad \text{for all } u \in \partial U_h^*. \tag{5.10}$$

Crucial step in the proof of this fact is the following:

Claim. *For every $h > 0$ consider $\{u_k^h\}$ a generic minimizing sequence in $\widetilde{\Lambda}_h^*$ and let $u_h \in \Lambda_h$ its weak limit. Then*

$$\lim_{h\to 0^+} \sup_{x\notin(1/h)A_0^{\delta_0}} |u_h(x)| = 0. \tag{5.11}$$

Obviously it is sufficient to prove that, considered a generic sequence $h_n \to 0^+$, up to subsequence it is

$$\lim_{n\to\infty} \sup_{x\notin(1/h_n)A_0^{\delta_0}} |u_{h_n}(x)| = 0. \tag{5.12}$$

The proof of (5.12) can be found in Theorem 5.1 of [5] (see the steps 4 and 5 in particular) when A_0 consists of a single point; the general case is analogous.

Now let us go back to prove (5.10) and suppose by contradiction that for some $h > 0$ arbitrarily small there is $w_h \in \partial U_h^*$ such that $E_h(w_h) = \widetilde{E}_h^*$. Then it makes sense to consider a minimizing sequence in $\widetilde{\Lambda}_h^*$ weakly converging to w_h. But (5.11) would be in contradiction with the fact that $w_h \in \partial U_h^*$. Hence (5.10) follows.

Finally the object is to prove that a stronger version of (36) holds, i.e. for every $h \in (0, h_1)$:

$$\inf_{u\in\partial U_h^*} E_h(u) > E_h(u_h^0). \tag{5.13}$$

We proceed by contradiction and assume the existence of $h \in (0, h_1)$ and of a sequence $\{u_n\} \subset \partial U_h^*$ such that

$$E_h(u_n) \to E_h(u_h^0) \equiv \widetilde{E}_h^* \quad \text{as } n \to \infty. \tag{5.14}$$

The coercivity of E_h implies the boundedness of the sequence $\{u_n\}$ and then, up to subsequence,

$$u_n \rightharpoonup u \quad \text{weakly in } H_h, \tag{5.15}$$

with $u \in \Lambda_h$. By (iv) of Lemma 2.1 we have

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N \text{ as } n \to \infty, u_n \to u \quad \text{uniformly in } (1/h)A_0^{\delta_0} \text{ as } n \to \infty,$$

then

$$|u(x)| \le \overline{\varepsilon} \quad \text{for all } x \in \mathbb{R}^N \setminus (1/h)A_0^{\delta_0}.$$

Using Lemma 3.2, at least for large n, we get

$$\mathrm{ch}(u) = \mathrm{ch}(u_n) \neq 0,$$

by which $u \in \overline{U}_h^*$. This fact, combined with (5.9), (5.14) and the weakly lower semi-continuity of E_h, yields

$$\widetilde{E}_h^* \le E_h(u) \le \liminf_{n\to\infty} E_h(u_n) = \widetilde{E}_h^*,$$

i.e.

$$\lim_{n\to\infty} E_h(u_n) = E_h(u). \tag{5.16}$$

Observe that Fatou's lemma leads to

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} W(u_n)\,dx \ge \int_{\mathbb{R}^N} W(u)\,dx, \tag{5.17}$$

while the weakly lower semi-continuity yields

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V_h(x)|u_n|^2\right)\,dx \ge \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_h(x)|u|^2\right)\,dx \tag{5.18}$$

and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \ge \int_{\mathbb{R}^N} |\nabla u|^p\,dx. \tag{5.19}$$

Thus (5.16) assures that the equality must hold in (5.17)–(5.19). In particular

$$\|u_n\|_{Z_h} \to \|u\|_{Z_h} \quad \text{as } n \to \infty$$

if we denote by Z_h the space defined as

$$Z_h \equiv \left\{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1}) \;\middle|\; \int_{\mathbb{R}^N} V_h(x)|u|^2\,dx < \infty \right\}.$$

Z_h becomes an Hilbert space when endowed with the inner product

$$\langle u, v\rangle_{Z_h} \equiv \int_{\mathbb{R}^N} ((\nabla u|\nabla v) + V_h(x)uv)\,dx.$$

By the obvious continuous inclusion $H_h \subset Z_h$ and by (5.15) we get $u_n \rightharpoonup u$ weakly in Z_h; hence, using the uniform convexity of Hilbert spaces, there follows

$$u_n \to u \quad \text{in } Z_h, \tag{5.20}$$

and, consequently, taking into account of Lemma 2.2, possibly passing to a subsequence

$$u_n \to u \quad \text{in } L^2(\mathbb{R}^N, \mathbb{R}^{N+1}) \text{ and in } L^\infty(\mathbb{R}^N, \mathbb{R}^{N+1}).$$

Then we obtain

$$u_n \to u \quad \text{in } L^p(\mathbb{R}^N, \mathbb{R}^{N+1}).$$

But we also know that

$$\|\nabla u_n\|_{L^p} \to \|\nabla u\|_{L^p}$$

which implies

$$\|\nabla u_n\|_{W^{1,p}} \to \|\nabla u\|_{W^{1,p}}.$$

The uniform convexity of the Sobolev space $W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$ (see [1, p. 47, Theorem 3.5]) leads to $u_n \to u$ in $W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$; this, combined with (5.20) assures

$$u_n \to u \quad \text{in } H_h \text{ as } n \to \infty.$$

Then we conclude

$$u \in \partial U_h^*, \quad E_h(u) = \widetilde{E}_h^*$$

in contradiction with (5.10). □

Finally we provide a result which is concerned with the existence of paths connecting u_h^0 with some suitable function in the space Λ_h.

Lemma 5.3. *There exists $\overline{h} > 0$ such that for every $h \in (0, \overline{h})$ there is a continuous path $g_h : [0,1] \to \Lambda_h$ verifying*

$$g_h(0) = u_h^0, \quad g_h(1) = u_h^1,$$

where $u_h^1 \in \Lambda_h$ satisfies

$$E_h(u_h^1) \le E_h(u_h^0), \quad u_h^1 = 0 \text{ in } \mathbb{R}^N \setminus B\left(\frac{x_1}{h}, \frac{\delta_0}{h}\right), \quad \|u_h^1\|_{L^\infty} = \|u_h^0\|_{L^\infty}, \tag{5.21}$$

where x_1 has been defined in assumption (III). *Furthermore there results*

$$\sup_{t \in [0,1]} E_h(g_h(t)) \le M \quad \text{for all } h \in (0, \overline{h}) \tag{5.22}$$

for some $M > 0$. Finally, using the notation introduced in (4.5), *for all $h \in (0, \overline{h})$ and $t \in [0,1]$, there exists $c(h,t) \in \mathbb{R}^N$ such that:*

$$g_h(t)(x) = u_h^0(x + c(h,t)) \quad \text{for all } x \in \Upsilon(g_h(t), \overline{\varepsilon}). \tag{5.23}$$

PROOF. Fix $R > 0$ sufficiently large so that, according to Theorem 5.1, for h small enough

$$|u_h^0(x)| \le \min\{\overline{\varepsilon}, 1/2\} \quad \text{for all } x \in \mathbb{R}^N \setminus B_R(x_h^0). \tag{5.24}$$

Now consider $\eta_R \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ a cut-off function so that

$$\begin{aligned} \eta_R &\equiv 1 && \text{on } \partial B_{2R}(0), \\ \eta_R &\equiv 0 && \text{in } \mathbb{R}^N \setminus (B_{3R}(0) \setminus B_R(0)), \\ 0 \le \eta_R &\le 1, && |\nabla \eta_R| \le a/R \end{aligned}$$

where a is a constant independent of R. Then put $\eta_h^R(x) = \eta_R(x - x_h^0)$ and compute

$$\begin{aligned} 0 &= \langle E_h'(u_h^0),\ \eta_h^R u_h^0 \rangle \\ &= \int_{\mathbb{R}^N} (\nabla u_h^0 | \nabla(\eta_h^R u_h^0))\, dx + \int_{\mathbb{R}^N} V_h(x) |u_h^0|^2 \eta_h^R\, dx \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_h^0|^{p-2} (\nabla u_h^0 | \nabla(\eta_h^R u_h^0))\, dx + \int_{\mathbb{R}^N} W'(u_h^0) u_h^0\, \eta_h^R\, dx \\ &\ge \int_{B_{3R}(x_h^0) \setminus B_{2R}(x_h^0)} (|\nabla u_h^0|^2 + V_h(x)|u_h^0|^2 + |\nabla u_h^0|^p) \eta_h^R\, dx \\ &\quad + \int_{\mathbb{R}^N} \nabla u_h^0 \nabla \eta_h^R u_h^0\, dx + \int_{\mathbb{R}^N} |\nabla u_h^0|^{p-2} \nabla u_h^0 \nabla \eta_h^R u_h^0\, dx \\ &\quad + \int_{\mathbb{R}^N \setminus B_R(x_h^0)} W'(u_h^0) u_h^0 \eta_h^R\, dx. \end{aligned}$$

To proceed, note that by (1.5) and (5.24), we have

$$\int_{\mathbb{R}^N \setminus B_R(x_h^0)} W'(u_h^0) u_h^0\, \eta_h^R\, dx \ge 0,$$

hence we infer

$$\begin{aligned} \int_{B_{3R}(x_h^0) \setminus B_{2R}(x_h^0)} & (|\nabla u_h^0|^2 + V_h(x)|u_h^0|^2 + |\nabla u_h^0|^p) \eta_h^R\, dx \\ &\le \int_{\mathbb{R}^N} |\nabla u_h^0| \cdot |\nabla \eta_h^R| \cdot |u_h^0|\, dx + \int_{\mathbb{R}^N} |\nabla u_h^0|^{p-1} |\nabla \eta_h^R| \cdot |u_h^0|\, dx. \end{aligned}$$

From Hölder inequality it follows

$$\begin{aligned} \int_{B_{3R}(x_h^0) \setminus B_{2R}(x_h^0)} & (|\nabla u_h^0|^2 + V_h(x)|u_h^0|^2 + |\nabla u_h^0|^p) \eta_h^R\, dx \\ &\le \frac{a}{R} \bigg(\int_{\mathbb{R}^N} |\nabla u_h^0|^2\, dx \bigg)^{1/2} \bigg(\int_{\mathbb{R}^N} |u_h^0|^2\, dx \bigg)^{1/2} \\ &\quad + \frac{a}{R} \bigg(\int_{\mathbb{R}^N} |\nabla u_h^0|^p\, dx \bigg)^{(p-1)/p} \bigg(\int_{\mathbb{R}^N} |u_h^0|^p\, dx \bigg)^{1/p}. \end{aligned}$$

The boundedness of $\{u_h^0\}$ in the H_h-norm, which is guaranteed by (5.4), implies

$$\int_{B_{3R}(x_h^0)\setminus B_{2R}(x_h^0)} (|\nabla u_h^0|^2 + V_h(x)|u_h^0|^2 + |\nabla u_h^0|^p)\eta_h^R\,dx \to 0 \tag{5.25}$$

as $R \to \infty$ uniformly with respect to h sufficiently small. Now, since $0 \le \eta_R \le 1$, we easily compute

$$\begin{aligned}
\int_{B_{3R}(x_h^0)\setminus B_{2R}(x_h^0)} &(|\nabla(u_h^0\eta_h^R)|^2 + V_h(x)|u_h^0\eta_h^R|^2 + |\nabla(u_h^0\eta_h^R)|^p)\,dx \\
&\le 4\int_{B_{3R}(x_h^0)\setminus B_{2R}(x_h^0)} (|\nabla u_h^0|^2\eta_h^R + |\nabla\eta_h^R|^2|u_h^0|^2)\,dx \\
&\quad + \int_{B_{3R}(x_h^0)\setminus B_{2R}(x_h^0)} V_h(x)|u_h^0|^2\eta_h^R\,dx \\
&\quad + 2^p\int_{B_{3R}(x_h^0)\setminus B_{2R}(x_h^0)} (|\nabla u_h^0|^p\eta_h^R + |\nabla\eta_h^R|^p|u_h^0|^p)\,dx \\
&\le 2^p\int_{B_{3R}(x_h^0)\setminus B_{2R}(x_h^0)} (|\nabla u_h^0|^2 + V_h(x)|u_h^0|^2 + |\nabla u_h^0|^p)\eta_h^R\,dx \\
&\quad + \left(4\frac{a^2}{R^2} + 2^p\frac{a^p}{R^p}\right)\int_{\mathbb{R}^N} (|u_h^0|^2 + |u_h^0|^p)\,dx.
\end{aligned}$$

Once more the fact that $\{u_h^0\}$ is bounded in the L^2 and L^p-norm, together with (5.25), permits us to conclude

$$\int_{B_{3R}(x_h^0)\setminus B_{2R}(x_h^0)} (|\nabla(u_h^0\eta_h^R)|^2 + V_h(x)|u_h^0\eta_h^R|^2 + |\nabla(u_h^0\eta_h^R)|^p)\,dx \to 0, \tag{5.26}$$

as $R \to \infty$, and this decay to zero is uniform for h small enough as to satisfy (5.24). Therefore, if we set

$$\gamma_0 = \frac{1}{4}\text{meas}\,(B_{r_0}(0))\,(V_0 - V_1),$$

where r_0 verifies (5.5) and V_0, V_1 given by assumptions (I)–(III), it makes sense to choose $R_0 > r_0$ sufficiently large so as to satisfy (5.24) and $\overline{h} \in (0, h_0)$ small enough such that for every $h \in (0, \overline{h})$

$$\begin{aligned}
\int_{B_{3R_0}(x_h^0)\setminus B_{2R_0}(x_h^0)} \bigg(\frac{1}{2}(|\nabla(u_h^0\eta_h^{R_0})|^2 &+ V_h(x)|u_h^0\eta_h^{R_0}|^2) \\
&+ \frac{1}{p}|\nabla(u_h^0\eta_h^{R_0})|^p\bigg)\,dx \le \gamma_0.
\end{aligned} \tag{5.27}$$

Furthermore, since according to Theorem 5.1 it is $\text{dist}(hx_h^0, A_0) \to 0$ as $h \to 0^+$, without loss of generality we may assume

$$\text{dist}(x_h^0, (1/h)A_0) + 3R_0 < \delta_0/h$$

so as the following inclusions hold

$$B_{3R_0}(x_h^0) \subset (1/h)A_0^{\delta_0}, \quad B_{3R_0}(x_1/h) \subset B(x_1/h, \delta_0/h). \tag{5.28}$$

Now we are ready to construct the path satisfying the properties of the lemma. First define

$$\widetilde{u}_h^0(x) = \begin{cases} u_h^0(x) & \text{if } x \in B_{2R_0}(x_h^0), \\ \eta_h^{R_0} u_h^0(x) & \text{if } x \in \mathbb{R}^N \setminus B_{2R_0}(x_h^0), \end{cases} \qquad u_h^1(x) = \widetilde{u}_h^0\left(x - \frac{x_1}{h} + x_h^0\right).$$

For every $h \in (0, \overline{h})$ we define g_h in the following way:

$$g_h(t)(x) = \begin{cases} (1-2t)u_h^0(x) + 2t\widetilde{u}_h^0(x) & \text{if } x \in \mathbb{R}^N,\ t \in [0,1/2], \\ \widetilde{u}_h^0\left(x + (1-2t)\left(\dfrac{x_1}{h} - x_h^0\right)\right) & \text{if } x \in \mathbb{R}^N,\ t \in \,]1/2, 1]\,. \end{cases}$$

By construction we infer

$$g_h(0) = u_h^0, \quad g_h(1) = u_h^1.$$

Observe that, being $\eta_h^{R_0} \equiv 1$ on $\partial B_{2R_0}(x_h^0)$, it is obvious that $\widetilde{u}_h^0 \in H_h$. Furthermore, since the support of $\widetilde{u}_h^0$ is compact, all the functions obtained by translating it belong to H_h too, hence

$$g_h : [0,1] \to H_h.$$

(5.23) follows immediately from the construction: more precisely, the constants $c(h,t)$ are given by

$$c(h,t) = \begin{cases} 0 & \text{if } t \in [0,1/2], \\ (1-2t)\left(\dfrac{x_1}{h} - x_h^0\right) & \text{if } t \in \,]1/2, 1]\,. \end{cases}$$

The fact that each g_h takes values in Λ_h is a direct consequence of (5.23), since $u_h^0 \in \Lambda_h$ and $g_h(t) \neq \overline{\xi}$ in $\mathbb{R}^N \setminus \Upsilon(g_h(t), \overline{\varepsilon})$.

By definition it is obvious that g_h is continuous. Next object is to obtain a good estimate of $E_h(u_h^1)$. To this aim observe that, because of the choice of $\eta_h^{R_0}$, it is $u_h^1 = 0$ in $\mathbb{R}^N \setminus B_{3R_0}(x_1/h)$; thus, from assumption (III) and (5.28),

$$\begin{aligned} E_h(u_h^1) &= \int_{B_{3R_0}(x_1/h)} \left(\frac{1}{2}(|\nabla u_h^1|^2 + V_h(x)|u_h^1|^2) + \frac{1}{p}|\nabla u_h^1|^p + W(u_h^1)\right) dx \\ &= \int_{B_{3R_0}(x_h^0)} \left(\frac{1}{2}\left(|\nabla \widetilde{u}_h^0|^2 + V_h\left(x - x_h^0 + \frac{x_1}{h}\right)|\widetilde{u}_h^0|^2\right)\right. \\ &\qquad \left. + \frac{1}{p}|\nabla \widetilde{u}_h^0|^p + W(\widetilde{u}_h^0)\right) dx \\ &\le \int_{B_{3R_0}(x_h^0)} \left(\frac{1}{2}(|\nabla \widetilde{u}_h^0|^2 + V_h(x)|\widetilde{u}_h^0|^2) + \frac{1}{p}|\nabla \widetilde{u}_h^0|^p + W(\widetilde{u}_h^0)\right) dx \\ &\qquad - \frac{1}{2}(V_0 - V_1)\int_{B_{r_0}(x_h^0)} |\widetilde{u}_h^0|^2\, dx \le E_h(\widetilde{u}_h^0) - \gamma_0. \end{aligned} \tag{5.29}$$

The latter inequality follows from (5.5). On the other hand hypothesis (g) implies that W is nondecreasing with respect to the norm in the set $\{\xi \in \mathbb{R}^{N+1} \mid |\xi| \leq \overline{\varepsilon}\}$; hence (5.24) and the construction above yield

$$W(\widetilde{u}_h^0(x)) = W(\eta_h^{R_0} u_h^0(x)) \leq W(u_h^0(x)) \quad \text{for all } x \in B_{3R_0}(x_h^0) \setminus B_{2R_0}(x_h^0).$$

Now observe that using last inequality and taking into account of the definition of $\widetilde{u}_h^0$ we can achieve:

$$\begin{aligned}
E_h(\widetilde{u}_h^0) \leq \int_{B_{2R_0}(x_h^0)} \bigg(\frac{1}{2}(|\nabla u_h^0|^2 \\
&+ V_h(x)|u_h^0|^2) + \frac{1}{p}|\nabla u_h^0|^p \bigg)\, dx + \int_{\mathbb{R}^N} W(u_h^0)\, dx \\
&+ \int_{B_{3R_0}(x_h^0)\setminus B_{2R_0}(x_h^0)} \bigg(\frac{1}{2}(|\nabla (u_h^0 \eta_h^{R_0})|^2 + V_h(x)|u_h^0 \eta_h^{R_0}|^2) \\
&+ \frac{1}{p}|\nabla (u_h^0 \eta_h^{R_0})|^p \bigg)\, dx \\
\leq E_h(u_h^0) + \int_{B_{3R_0}(x_h^0)\setminus B_{2R_0}(x_h^0)} \bigg(\frac{1}{2}(|\nabla (u_h^0 \eta_h^{R})|^2 + V_h(x)|u_h^0 \eta_h^{R}|^2) \\
&+ \frac{1}{p}|\nabla (u_h^0 \eta_h^{R})|^p \bigg)\, dx.
\end{aligned} \tag{5.30}$$

Combining (5.30) with (5.27) and (5.29) we obtain: $E_h(u_h^1) \leq E_h(u_h^0)$. As regards the equality $\|u_h^1\|_{L^\infty} = \|u_h^0\|_{L^\infty}$, it follows from (5.23), since from (3.3) we have $\Upsilon(u_h^0, \overline{\varepsilon}) \neq \emptyset$. We can even say

$$\|g_h(t)\|_{L^\infty} = \|u_h^0\|_{L^\infty} \quad \text{for all } t \in [0,1], \text{and all } h \in (0, \overline{h}).$$

Then (5.21) is completely proved.

Finally it remains to prove the upper bound (5.22). Using (5.4), an easy computation shows that

$$\sup_{h \in (0,\overline{h})} \sup_{t \in [0,1]} \int_{\mathbb{R}^N} \bigg(\frac{1}{2}(|\nabla g_h(t)|^2 + |g_h(t)|^2) + \frac{1}{p}|\nabla g_h(t)|^p \bigg)\, dx < \infty. \tag{5.31}$$

Hence $\{g_h(t)\}$ is bounded in the H_0-norm for $h \in (0, \overline{h})$ and $t \in [0,1]$. Now choose $L > 0$ such that $A_{\delta_0}^0 \cup B_{\delta_0}(x_1) \subset B(0, L)$; notice that by construction there results:

$$|g_h(t)(x)| \leq |u_h^0(x)| \quad \text{for all } x \in \mathbb{R}^N \text{ if } t \in [0, 1/2],$$

while

$$\{x \in \mathbb{R}^N \mid g_h(t)(x) \neq 0\} \subset B(0, L/h) \quad \text{if } t \in [1/2, 1].$$

These two facts and straightforward calculations lead to

$$\int_{\mathbb{R}^N} \frac{1}{2} V_h(x)|g_h(t)|^2\,dx \tag{5.32}$$
$$\leq \begin{cases} \displaystyle\int_{\mathbb{R}^N} \frac{1}{2} V_h(x)|u_h^0|^2\,dx & \text{if } t \in [0,1/2], \\ \displaystyle\frac{1}{2} \sup_{x\in B(0,L/h)} V_h(x) \int_{\mathbb{R}^N} |g_h(t)|^2\,dx & \\ \quad \displaystyle= \frac{1}{2} \sup_{x\in B_L(0)} V(x) \int_{\mathbb{R}^N} |g_h(t)|^2\,dx & \text{if } t \in [1/2,1]. \end{cases}$$

Now analyse the term with W: from (5.23) we obtain

$$\int_{\mathbb{R}^N} W(g_h(t))\,dx = \int_{\Upsilon(g_h(t),\overline{\varepsilon})} W(u_h^0(\,\cdot\,+c(h,t)))\,dx + \int_{\mathbb{R}^N\setminus\Upsilon(g_h(t),\overline{\varepsilon})} W(g_h(t))\,dx.$$

On the other hand, since by construction $|g_h(t)(x)| \leq |u_h^0(x+c(h,t))|$ for all $x \in \mathbb{R}^N$, hypothesis (g) implies $W(g_h(t)) \leq W(u_h^0(\,\cdot\,+c(h,t))$ in $\mathbb{R}^N \setminus \Upsilon(g_h(t),\overline{\varepsilon})$, by which

$$\int_{\mathbb{R}^N} W(g_h(t))\,dx \leq \int_{\mathbb{R}^N} W(u_h^0(\,\cdot\,+c(h,t)))\,dx = \int_{\mathbb{R}^N} W(u_h^0)\,dx. \tag{5.33}$$

Combining (5.31)–(5.33) with (5.4) we conclude with (5.22). The proof of the lemma ends. □

Last lemma suggests us to modify the energy functional E_h in a suitable way as to assume a "mountain pass" geometry. This is what we will do in the next section.

6. Construction of the penalized functional

As already announced at the end of last section, the object is now to define a suitable modification of the energy functional for which we will find a critical point via an appropriate min-max scheme. This critical point will eventually be shown to be a solution of the original equation if it keeps away enough from the singularity, and, as we will see, this happens provided h is sufficiently small. First observe that, taking $\alpha = \beta = M+1$, with M given by (5.22), Lemma 4.1 provides the existence of $\overline{d} \in (0,1)$ such that for all $u \in H_0$, using the notations (4.5) and (4.6), there results:

$$E_i(u) \leq M+1, \quad \int_{\Pi(u,\overline{d}/2)} W(u)\,dx \leq M+1 \Rightarrow \min_{x\in\mathbb{R}^N} |u(x)-\overline{\xi}| > \overline{d}. \tag{6.1}$$

Without loss of generality we may assume $\overline{d} < 1-\overline{\varepsilon}$ so that for every $u \in H_0$ it is:

$$\mathbb{R}^N \setminus \Pi(u,\overline{d}) \subset \Upsilon(u,\overline{\varepsilon}). \tag{6.2}$$

Since we aim to apply the Mountain Pass Theorem, we want the penalized functional to satisfy some good properties; first of all we require it to be defined in all the Banach space H_h. To this aim we have to "stop up" in some sense the singularity $\overline{\xi}$ by modifying the nonlinear term: take $\tau : \mathbb{R}^+ \to \mathbb{R}$ the following auxiliary function

$$\tau = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ 1 - \exp\left(\dfrac{1}{|t-1|^4 - 1} + 1\right) & \text{if } 1 \le t < 2, \\ 1 & \text{if } t \ge 2. \end{cases}$$

Obviously it is $\tau \in C^2(\mathbb{R}^+, \mathbb{R})$. Now put

$$\widehat{W}(\xi) = \tau\left(\frac{8}{\overline{d}^2}|\xi - \overline{\xi}|^2\right) W(\xi).$$

As a consequence of the previous construction we have $\widehat{W} \in C^1(\mathbb{R}^{N+1}, \mathbb{R})$ and

$$\widehat{W}(\xi) = W(\xi) \quad \text{for all } \xi \in \mathbb{R}^{N+1} \text{ with } |\xi - \overline{\xi}| \ge \overline{d}/2. \tag{6.3}$$

Finally the modified functional $J_h : H_h \to \mathbb{R}$ is defined as

$$J_h(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}(|\nabla u|^2 + V_h(x)|u|^2) + \frac{1}{p}|\nabla u|^p + \widehat{W}(u)\right) dx.$$

According to (6.2) and (6.3) the set where $\widehat{W}(u)$ possibly differs from $W(u)$ is certainly contained in $\Upsilon(u, \overline{\varepsilon})$ which is compact because of (2.3), then we immediately get that J_h is well defined on the space H_h. Moreover, (1.5) yields

$$\widehat{W}'(\xi)\xi \ge 0 \quad \text{for all } \xi \in \mathbb{R}^{N+1} \text{ with } |\xi| \le \overline{\varepsilon}. \tag{6.4}$$

To begin, we point out the differentiability of J_h.

Lemma 6.1. *The modified energy functional J_h belongs to $C^1(H_h, \mathbb{R})$. Furthermore, for every $u, v \in H_h$, it is*

$$\begin{aligned} \langle J_h'(u), v\rangle = & \int_{\mathbb{R}^N} (\nabla u|\nabla v)\, dx + \int_{\mathbb{R}^N} V_h(x) u\, v\, dx \\ & + \int_{\mathbb{R}^N} |\nabla u|^{p-2}(\nabla u|\nabla v)\, dx + \int_{\mathbb{R}^N} \widehat{W}'(u) v\, dx, \end{aligned}$$

where $\widehat{W}'$ denotes the gradient of $\widehat{W}$.

The proof is identical to that of Lemma 4.1 in [4], so we omit it.

Obviously J_h is coercive in the H_h-norm, i.e.

$$\lim_{\|u\|_{H_h} \to \infty} J_h(u) = \infty. \tag{6.5}$$

We show next that J_h has good compactness properties, that is J_h satisfies the Palais–Smale condition. For the proof we need the inequality provided by the following theorem of elementary calculus.

THEOREM 6.2. *For every $p \geq 2$ and $n \geq 1$ there exists a constant $\omega = \omega(p,n) > 0$ such that, for all x, $y \in \mathbb{R}^n$:*

$$(|x|^{p-2}x - |y|^{p-2}y)\,(x-y) \geq \omega\,|x-y|^p.$$

For the proof we refer to [24, Lemma A.0.5, Appendix A]. By the previous lemma we deduce a sort of monotonicity of the operator $-\Delta_p$. Indeed, denoting by $\mathcal{M}_{N+1,N}$ the space of the $(N+1)\times N$ real matrix, and using the obvious identification $\mathcal{M}_{N+1,N} \approx \mathbb{R}^{(N+1)\times N}$, we get the existence of $\overline{\omega} = \overline{\omega}(N,p)$ such that, for all u, $v \in H_0$,

$$\begin{aligned}\langle -\Delta_p u + \Delta_p v, u - v\rangle &= \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v|\nabla u - \nabla v)\,dx \\ &\geq \overline{\omega}\int_{\mathbb{R}^N} |\nabla u - \nabla v|^p\,dx.\end{aligned}$$

This property will prove very useful in the proof of next result.

LEMMA 6.3. *Let $\{u_n\}$ be a sequence in H_h such that $\{J_h(u_n)\}$ is bounded and $J_h'(u_n) \to 0$. Then $\{u_n\}$ has a convergent subsequence.*

PROOF. Let us consider a sequence $\{u_n\} \subset H_h$ such that

(a) $\{J_h(u_n)\}$ is bounded,
(b) $J_h'(u_n) \to 0$ in $(H_h)'$.

By (6.5) we easily infer that $\{u_n\}$ is bounded in H_h. Then there exists a subsequence, still denoted by $\{u_n\}$, and a function $u \in H_h$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } H_h.$$

The object is to show that this convergence is strong. From Theorem 2.2 we can assume, up to subsequence,

$$u_n \to u \quad \text{uniformly in} \quad \mathbb{R}^N. \tag{6.6}$$

Now fix $R > 0$ such that, according to (6.6), for large n,

$$|u_n(x)| \leq \overline{\varepsilon} \quad \text{for all } x \in \mathbb{R}^N \setminus B_{R/2}(0). \tag{6.7}$$

Next choose η_R a cut-off function verifying

$$\eta_R \equiv 0 \quad \text{in } B_{R/2}(0), \qquad \eta_R \equiv 1 \quad \text{in } \mathbb{R}^N \setminus B_R(0), \quad 0 \leq \eta_R \leq 1, \quad |\nabla \eta_R| \leq a/R$$

where a is a constant independent of R.

Now straightforward calculations lead to

$$
\begin{aligned}
(6.8)\qquad \langle J_h'(u_n),\eta_R\, u_n\rangle &= \int_{\mathbb{R}^N} (\nabla u_n|\nabla(\eta_R\, u_n))\,dx + \int_{\mathbb{R}^N} V_h(x)|u_n|^2\,\eta_R\,dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}(\nabla u_n|\nabla(\eta_R\, u_n))\,dx + \int_{\mathbb{R}^N} \widehat{W}'(u_n)u_n\,\eta_R\,dx \\
&= \int_{\mathbb{R}^N} |\nabla u_n|^2\eta_R\,dx + \int_{\mathbb{R}^N} \nabla u_n\nabla\eta_R\,u_n\,dx \\
&\quad + \int_{\mathbb{R}^N} V_h(x)|u_n|^2\,\eta_R\,dx + \int_{\mathbb{R}^N} |\nabla u_n|^p\eta_R\,dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\eta_R\,u_n\,dx + \int_{\mathbb{R}^N} \widehat{W}'(u_n)u_n\,\eta_R\,dx.
\end{aligned}
$$

We notice that by (6.7) and by (6.4) we infer

$$
(6.9)\qquad \int_{\mathbb{R}^N\setminus B_{R/2}(0)} \widehat{W}'(u_n)u_n\,\eta_R\,dx \ge 0.
$$

Hence, combining (6.8) and (6.9) we deduce:

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_R(0)} (|\nabla u_n|^2 + V_h(x)|u_n|^2\,dx + |\nabla u_n|^p)\,dx &\le \langle J_h'(u_n),\ \eta_R\,u_n\rangle \\
&\quad - \int_{\mathbb{R}^N} \nabla u_n\nabla\eta_R\,u_n\,dx - \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\eta_R\,u_n\,dx.
\end{aligned}
$$

Applying Hölder inequality we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_R(0)} (|\nabla u_n|^2 + V_h(x)|u_n|^2 + |\nabla u_n|^p)\,dx &\le \langle J_h'(u_n),\eta_R\,u_n\rangle \\
&\quad + \frac{a}{R}\bigg(\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx\bigg)^{1/2}\bigg(\int_{\mathbb{R}^N} |u_n|^2\,dx\bigg)^{1/2} \\
&\quad + \frac{a}{R}\bigg(\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx\bigg)^{(p-1)/p}\bigg(\int_{\mathbb{R}^N} |u_n|^p\,dx\bigg)^{1/p}.
\end{aligned}
$$

Since $\{u_n\}$ is a bounded Palais–Smale sequence, there results $\langle J_h'(u_n),\eta_R\,u_n\rangle \to 0$ as $n\to\infty$. Finally the boundedness of $\{u_n\}$ in the H_h-norm and, consequently, in the L^p-norm, assures that, given $\delta>0$, there is $R>0$ such that

$$
\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} (|\nabla u_n|^2 + V_h(x)|u_n|^2 + |\nabla u_n|^p)\,dx \le \delta.
$$

In particular, taking into account of (6.6), this proves that

$$
u_n \to u \quad \text{in } L^2(\mathbb{R}^N,\mathbb{R}^{N+1}).
$$

Now we have

$$\begin{aligned}\langle J_h'(u_n) - J_h'(u), u_n - u\rangle = &\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2\,dx + \int_{\mathbb{R}^N} V_h(x)|u_n - u|^2\,dx \\ &+ \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}(\nabla u_n|\nabla(u_n - u))\,dx \\ &- \int_{\mathbb{R}^N} |\nabla u|^{p-2}(\nabla u|\nabla(u_n - u))\,dx \\ &+ \int_{\mathbb{R}^N} (\widehat{W}'(u_n) - \widehat{W}'(u))(u_n - u)\,dx.\end{aligned}$$

We claim that

$$\int_{\mathbb{R}^N} (\widehat{W}'(u_n) - \widehat{W}'(u))(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{6.10}$$

Assuming (6.10) and using Lemma 6.2 we immediately obtain

$$\begin{aligned}&\limsup_{n\to\infty}\biggl(\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2\,dx + \int_{\mathbb{R}^N} V_h(x)|u_n - u|^2\,dx + \int_{\mathbb{R}^N} |\nabla(u_n - u)|^p\,dx\biggr) \\ &\quad\le \limsup_{n\to\infty}\biggl(\int_{\mathbb{R}^N} (|\nabla(u_n - u)|^2 + V_h(x)|u_n - u|^2)\,dx \\ &\qquad + \frac{1}{\omega}\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u|\nabla u_n - \nabla u)\,dx\biggr) \\ &\quad\le \max\biggl\{1, \frac{1}{\omega}\biggr\}\limsup_{n\to\infty}\langle J_h'(u_n) - J_h'(u), u_n - u\rangle.\end{aligned}$$

Then, since $u_n \rightharpoonup u$ weakly and, by (2), $J_h'(u_n) \to 0$ in $(H_h)'$, we conclude

$$\lim_{n\to\infty}\biggl(\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2\,dx + \int_{\mathbb{R}^N} V_h(x)|u_n - u|^2\,dx + \int_{\mathbb{R}^N} |\nabla(u_n - u)|^p\,dx\biggr) = 0$$

and the thesis follows. It remains to prove (6.10). We use the differentiability of $\widehat{W}$ in 0:

$$\widehat{W}'(\xi) = \widehat{W}''(0)[\xi] + \omega(\xi) \tag{6.11}$$

where $\lim_{\xi\to 0}\omega(\xi)/|\xi| = 0$. Fix $\varepsilon > 0$ arbitrarily and choose $\varrho > 0$ such that, for every $\xi \in \Omega$,

$$|\xi| \le \varrho \Rightarrow |\omega(\xi)| \le \varepsilon|\xi|.$$

Applying (6.11) first with $\xi \equiv u_n(x)$ and then with $\xi \equiv u(x)$ and subtracting the two identities we get

$$\widehat{W}'(u_n(x)) - \widehat{W}'(u(x)) = \widehat{W}''(0)[u_n(x) - u(x)] + \omega(u_n(x)) - \omega(u(x)).$$

Now, using the fact that $u_n \to u$ in L^∞. by (2.3) it makes sense to choose $R > 0$ such that for large n

$$|u(x)| \le \varrho, \quad |u_n(x)| \le \varrho \quad \text{for all } x \in \mathbb{R}^N \setminus B_R(0).$$

Then there results

$$\begin{aligned}\Big(\int_{\mathbb{R}^N}(\widehat{W}'(u_n) - \widehat{W}'(u))(u_n-u)\,dx\Big)^2 \\ &\le \int_{\mathbb{R}^N}|\widehat{W}'(u_n)-\widehat{W}'(u)|^2\,dx\int_{\mathbb{R}^N}|u_n-u|^2\,dx \\ &\le 4|\widehat{W}''(0)|^2\int_{\mathbb{R}^N}|u_n-u|^2\,dx \\ &\quad + 4\Big(4\varepsilon\int_{\mathbb{R}^N\setminus B_R(0)}(|u_n|^2+|u|^2)\,dx \\ &\quad + \int_{B_R(0)}|\omega(u_n)-\omega(u)|^2\,dx\Big)\int_{\mathbb{R}^N}|u_n-u|^2\,dx.\end{aligned}$$

Since the set $\{u_n(x)\mid x\in\mathbb{R}^N\}\cup\{u(x)\mid x\in\mathbb{R}^N\}\subset\mathbb{R}^{N+1}$ is bounded and since ω is continuous, we obtain that $\omega(u_n)\to\omega(u)$ uniformly; hence we deduce

$$\Big(\int_{\mathbb{R}^N}(\widehat{W}'(u_n)-\widehat{W}'(u))(u_n-u)\,dx\Big)^2\le \text{const}\int_{\mathbb{R}^N}|u_n-u|^2\,dx\to 0 \quad \text{as } n\to\infty,$$

and we achieve the desired conclusion. □

Now one has in his hands all the instruments to capture the existence of solutions for equation (1.1). This will be the object of next section.

7. Local mountain pass

Lemma 6.3 makes possible to use Critical Point Theory to find solutions for equation (1.1) through the construction of appropriate min-max values of the functional J_h; we will use the following general version of the Mountain Pass Theorem.

Theorem 7.1 (Mountain Pass Theorem). *Let X be a Banach space and $f: X\to\mathbb{R}$ a C^1 functional. Assume that there exist x_0, $x_1\in X$ and a neighbourhood U of x_0 verifying*

$$x_1\in X\setminus U,\qquad \inf_{x\in\partial U} f(x)>\max\{f(x_0),f(x_1)\}.$$

Let us define $\Gamma=\{\gamma\in C([0,1],X)\mid \gamma(0)=x_0,\ \gamma(1)=x_1\}$ and

$$c=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} f(\gamma(t)).$$

If f satisfies the Palais–Smale condition, then c is a critical value for f and $c\ge\inf_{x\in\partial U} f(x)$.

See [3] for the proof.

In order to apply last theorem to the modified functional J_h, consider the open set

$$B_h \equiv \{u \in H_h \mid \inf_{x\in\mathbb{R}^N} |u(x) - \overline{\xi}| > \overline{d}\}$$

whose boundary is given by $\partial B_h \equiv \{u \in H_h \mid \inf_{x\in\mathbb{R}^N} |u(x) - \overline{\xi}| = \overline{d}\}$. Then the construction of J_h implies

$$E_h(u) = J_h(u) \quad \text{for all } u \in \overline{B}_h. \tag{7.1}$$

Now, by (5.22) and (6.3), taking into account of the definition of E_h, we immediately get

$$E_i(g_h(t)) \leq M + 1, \qquad \int_{\Pi(g_h(t),\overline{d}/2)} W(g_h(t))\,dx \leq M + 1,$$

and then (6.1) implies

$$\inf_{x\in\mathbb{R}^N} |g_h(t)(x) - \overline{\xi}| > \overline{d} \quad \text{for all } t \in [0,1], \tag{7.2}$$

i.e. $g_h(t) \in B_h$. In particular

$$u_h^0, u_h^1 \in B_h. \tag{7.3}$$

THEOREM 7.2. *For h sufficiently small the functional J_h possesses a critical point u_h, so that*

$$J_h(u_h) = \inf_{\gamma\in\Gamma_h} \sup_{t\in[0,1]} J_h(\gamma(t))$$

with the class Γ_h given by

$$\Gamma_h = \{\gamma \in C([0,1], H_h) \mid \gamma(0) = u_h^0,\ \gamma(1) = u_h^1\},$$

where u_h^0 and u_h^1 are the functions provided by Theorem 5.1 *and Lemma* 5.5. *Furthermore there results:*

$$J_h(u_h) > \max\{J_h(u_h^0), J_h(u_h^1)\}.$$

PROOF. Choose $h < \min\{h_0, \overline{h}\}$, where h_0 and $\overline{h}$ are provided respectively by Theorem 5.1 and Lemma 5.3 and assume, without loss of generality, that Lemma 5.2 holds. Then set

$$\widetilde{U}_h^* \equiv U_h^* \cap B_h$$

where U_h^* is the open set defined in (ustar). Now let

$$x_0 = u_h^0, \quad x_1 = u_h^1, \quad U = \widetilde{U}_h^*.$$

The object is to prove that with the above choices the geometrical hypotheses of the Mountain Pass Theorem are satisfied. To this aim observe that we already know from Lemma 5.2 that $u_h^0 \in U_h^*$; by (7.3) we infer $u_h^0 \in B_h$ too. Hence we have proved that $u_h^0 \in \widetilde{U}_h^*$, i.e. $\widetilde{U}_h^*$ is a neighbourhood of u_h^0. On the other hand it is immediate that $u_h^1 \notin \widetilde{U}_h^*$ since by (5.21) there exists $\overline{x} \in B(x_1/h, \delta_0/h)$ such that $|u_h^1(\overline{x})| = \|u_h^0\|_{L^\infty}$ and, by (3.3), $|u_h^1(\overline{x})| > 1 > \overline{\varepsilon}$, hence $u_h^1 \notin U_h^*$.

From (5.21), (7.1) and (7.3) we derive

$$J_h(u_h^1) = E_h(u_h^1) \le E_h(u_h^0) = J_h(u_h^0).$$

Finally we have to prove:

$$\inf_{u \in \partial \widetilde{U}_h^*} J_h(u) > J_h(u_h^0).$$

Lemma 5.2 and (7.1) lead to

$$\inf_{u \in \partial U_h^* \cap \overline{B}_h} J_h(u) = \inf_{u \in \partial U_h^* \cap \overline{B}_h} E_h(u) > E_h(u_h^0) = J_h(u_h^0). \tag{7.4}$$

Now take $u \in \partial B_h \cap \overline{U}_h^*$; then $\inf_{x \in \mathbb{R}^N} |u(x) - \overline{\xi}| = \overline{d}$. From (7.1) and (6.1) we easily compute

$$J_h(u) = E_h(u) \ge E_i(u) + \int_{\Pi(u,\overline{d}/2)} W(u)\,dx \ge M + 1$$

by which

$$\inf_{u \in \partial B \cap \overline{U}_h^*} J_h(u) \ge M + 1 > M \ge J_h(u_h^0). \tag{7.5}$$

The obvious inclusion $\partial \widetilde{U}_h^* \subset (\partial U_h^* \cap \overline{B}_h) \cup (\partial B_h \cap \overline{U}_h^*)$ combined with (7.4) and (7.5) implies

$$\inf_{u \in \partial \widetilde{U}_h^*} J_h(u) > J_h(u_h^0).$$

The validity of the Palais–Smale condition leads to a direct application of the Mountain Pass Theorem and consequently the definition of the min-max quantity provided in the statement of the theorem will yield a critical point for J_h. □

The function $u_h \in H_h$ obtained in last theorem solves the equation

$$-\Delta u_h + V_h(x) u_h - \Delta_p u_h + \widehat{W}'(u_h) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{7.6}$$

We want to show that actually u_h turns out to be a solution of the original equation (1.2) provided that h is sufficiently small. We notice that since for every $h > 0$ the set B_h is open, then because of (7.1) a critical point of J_h which lies in B_h will be critical for E_h and conversely. So the desired result will follow if we show that for all small h one has

$$u_h \in B_h, \tag{7.7}$$

and thus u_h will be the so called "local mountain pass" solution predicted in the introduction. In the first place we notice that it is

$$J_h(u_h) \le M. \tag{7.8}$$

Indeed, considered the path g_h constructed in Lemma 5.3, each g_h belongs to the class Γ_h which defines the critical value $J_h(u_h)$, then we obtain

$$J_h(u_h) \le \sup_{t\in[0,1]} J_h(g_h(t)).$$

By (7.1) and (7.2) we get $J_h(g_h(t)) = E_h(g_h(t))$; Lemma 5.3 permits us to obtain (7.8), by which

$$E_i(u_h) \le M.$$

Furthermore

$$\int_{\Pi(u_h,\overline{d}/2)} W(u_h)\,dx = \int_{\Pi(u_h,\overline{d}/2)} \widehat{W}(u_h)\,dx \le J_h(u_h) \le M.$$

Consequently (6.1) yields

$$\inf_{x\in\mathbb{R}^N} |u_h(x) - \overline{\xi}| > \overline{d}.$$

Then (7.7) is satisfied. We deduce that u_h is a critical point of E_h and hence provides a solution to equation (1.2). Next theorem sums up our main results for equation (1.1).

Theorem 7.3. *Under hypotheses* (a)–(g) *and* (I)–(III) *there exists* $h_1 > 0$ *such that for every* $h \in (0, h_1)$ *the function* u_h *obtained in Theorem* 7.2 *becomes a solution of equation* (1.2). *Furthermore, if for every* $h \in (0, h_1)$ *we put*

$$v_h(x) = u_h(x/h), \tag{7.9}$$

then v_h *obviously provides a solution of* (1.1). *Finally considered the sets*

$$S_h = \{x \in \mathbb{R}^N \mid W'(v_h(x))v_h(x) < 0\} \subset \mathbb{R}^N,$$

for all $\delta > 0$ *the family* $\{v_h\}$ *verifies*

$$\lim_{h\to 0^+} \sup_{x\in\mathbb{R}^N\setminus S_\delta} |v_h(x)| = 0, \tag{7.10}$$

where $S = \bigcup_{h\in(0,h_1)} S_h$ *and* $S_\delta = \{x \in \mathbb{R}^N \mid \operatorname{dist}(x, S) < \delta\}$. *In other words* v_h *decays uniformly to zero for* x *outside every neighbourhood of* S.

Proof. By rescaling we immediately obtain that each v_h related to u_h by (7.9) provides a solution to equation (1.1). Then it remains to prove (7.10). First observe how S_h can also be defined by the following relation:

$$\frac{1}{h} S_h = \{x \in \mathbb{R}^N \mid W'(u_h(x))u_h(x) < 0\}. \tag{7.11}$$

From (1.5) it immediately follows that $(1/h)S_h \subset \Upsilon(u_h, \overline{\varepsilon})$, in particular each set $(1/h)S_h$ is bounded. We begin by deducing that the family of sets $\{S_h\}$ is equibounded in $\mathbb{R}^N$, i.e. there exists $R > 0$ such that for small h

$$S_h \subset B(0, R). \tag{7.12}$$

For otherwise, there would exists a sequence $h_n \to 0^+$ and $x_{h_n} \in S_{h_n}$ verifying $|x_{h_n}| \to \infty$. From the inclusion $(1/h_n)S_{h_n} \subset \Upsilon(u_{h_n}, \overline{\varepsilon})$ we have $|u_{h_n}(x_{h_n}/h_n)| \geq \overline{\varepsilon}$. But (7.8) implies the equiuniform continuity of the sequence $\{u_{h_n}\}$; hence, following the same arguments we have used to prove (2.7) in Lemma 2.2, we analogously obtain

$$\exists \rho > 0 \quad \text{such that } B_\rho(x_{h_n}/h_n) \subset \{x \in \mathbb{R}^N \mid |u_{h_n}(x)| > \overline{\varepsilon}/2\}.$$

But then we would write

$$\begin{aligned}\int_{\mathbb{R}^N} V_{h_n}(x)|u_{h_n}|^2\,dx &\geq \int_{B_\rho(x_{h_n}/h_n)} V_{h_n}(x)|u_{h_n}|^2\,dx \\ &\geq \frac{\overline{\varepsilon}^2}{4}\int_{B_\rho(x_{h_n}/h_n)} V_{h_n}(x)\,dx \\ &= \frac{\overline{\varepsilon}^2}{4}\int_{B_\rho(0)} V(h_n x + x_{h_n})\,dx,\end{aligned}$$

and because of the coercivity of V the integral on the right side diverges as $n \to \infty$. Hence (7.12) holds and then S is bounded in $\mathbb{R}^N$.

Towards our aim fix $\delta > 0$ and consider $\eta \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$ a cut-off function verifying

$$\eta \equiv 0 \quad \text{in } S, \quad \eta \equiv 1 \quad \text{in } \mathbb{R}^N \setminus S_{\delta/2}, \quad 0 \leq \eta \leq 1, \quad |\nabla \eta| \leq a$$

where a is a constant.

Now, for $h \in (0, h_1)$, put $\eta_h(x) = \eta(hx)$; taking into account that each u_h is a critical point of E_h, straightforward calculations lead to

$$\begin{aligned}0 &= \langle E_h'(u_h), \eta_h u_h\rangle \\ &= \int_{\mathbb{R}^N} (\nabla u_h | \nabla(\eta_h u_h))\,dx + \int_{\mathbb{R}^N} V_h(x)|u_h|^2\,\eta_h\,dx \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_h|^{p-2}(\nabla u_h|\nabla(\eta_h u_h))\,dx + \int_{\mathbb{R}^N} W'(u_h)u_h\,\eta_h\,dx \\ &= \int_{\mathbb{R}^N} |\nabla u_h|^2 \eta_h\,dx + \int_{\mathbb{R}^N} \nabla u_h \nabla \eta_h\, u_h\,dx \\ &\quad + \int_{\mathbb{R}^N} V_h(x)|u_h|^2\,\eta_h\,dx + \int_{\mathbb{R}^N} |\nabla u_h|^p \eta_h\,dx \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_h|^{p-2}\nabla u_h \nabla \eta_h\, u_h\,dx + \int_{\mathbb{R}^N} W'(u_h)u_h\,\eta_h\,dx.\end{aligned} \tag{7.13}$$

Notice that the definition of S permits us to infer

$$\int_{\mathbb{R}^N \setminus (1/h)} SW'(u_h)u_h\eta_h\,dx \geq 0.$$

Inserting last inequality in (7.13) we deduce:

$$\int_{\mathbb{R}^N\setminus(1/h)S_{\delta/2}} (|\nabla u_h|^2 + V_h(x)|u_h|^2 + |\nabla u_h|^p)\,dx \le -\int_{\mathbb{R}^N} \nabla u_h \nabla\eta_h\, u_h\,dx - \int_{\mathbb{R}^N} |\nabla u_h|^{p-2}\nabla u_h\nabla\eta_h\,u_h\,dx.$$

Applying Hölder inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus(1/h)S_{\delta/2}} &(|\nabla u_h|^2 + V_h(x)|u_h|^2 + |\nabla u_h|^p)\,dx \\ &\le ah\bigg(\int_{\mathbb{R}^N} |\nabla u_h|^2\,dx\bigg)^{1/2}\bigg(\int_{\mathbb{R}^N} |u_h|^2\,dx\bigg)^{1/2} \\ &\quad + ah\bigg(\int_{\mathbb{R}^N} |\nabla u_h|^p\,dx\bigg)^{(p-1)/p}\bigg(\int_{\mathbb{R}^N} |u_h|^p\,dx\bigg)^{1/p}.\end{aligned}$$

The boundedness of $\{u_h\}$ in the H_h-norm which is guaranteed by (7.8) and, consequently, in the L^p-norm, assures that

$$\lim_{h\to 0^+}\int_{\mathbb{R}^N\setminus(1/h)S_{\delta/2}} (|\nabla u_h|^2 + V_h(x)|u_h|^2 + |\nabla u_h|^p)\,dx = 0. \tag{7.14}$$

We claim that

$$\lim_{h\to 0^+}\sup_{x\in\mathbb{R}^N\setminus(1/h)S_\delta} |u_h(x)| = 0. \tag{7.15}$$

We argue by contradiction and assume the existence of $\gamma > 0$, $h_n \to 0^+$ and $y_{h_n} \in \mathbb{R}^N \setminus (1/h_n)S_\delta$ such that $|u_{h_n}(x_{h_n})| \ge \gamma$. Repeating the same argument of the first part of the proof we obtain the existence of $\widetilde{r} > 0$ satisfying

$$B_{\widetilde{r}}(y_{h_n}) \subset \{x\in\mathbb{R}^N \mid |u_{h_n}(x)| > \gamma/2\}.$$

This fact yields

$$\int_{B(y_{h_n},\min\{\widetilde{r},\delta/2\})} |u_{h_n}(x)|^2\,dx \ge (\gamma^2/4)\operatorname{meas}(B(0,\min\{\widetilde{r},\delta/2\})).$$

On the other hand $B(y_{h_n},\min\{\widetilde{r},\delta/2\}) \subset \mathbb{R}^N\setminus(1/h_n)S_{\delta/2}$, so last inequality contradicts (7.14). Finally (7.10) follows from (7.15) by rescaling. □

Is is natural to wonder if the family of solutions $\{v_h\}$ to equation (1.1) exhibits some type of notable behaviour as $h \to 0$ and, in the second place, the question of locating their asymptotic spikes arises. Theorem 7.3 already provide a result with regards to this aim: the family $\{v_h\}$ decays uniformly to zero for x outside a bounded set. In particular they state that no points in $\mathbb{R}^N\setminus S$ is an eventual candidate for the concentration. We conjecture that it is possible to describe in a more precise way the asymptotic behaviour so as to localize exactly the concentration points, which will probably turn out to be critical for the potential V. In other words we think that it is possible to reproduce the same framework of the papers [4], [5], [15], [16] and [17]. However the approach the authors used in [4] and in [5] stops to work since the solutions provided there are minima for the energy functional and consequently the method employed are strongly bound to minimization techniques. On the other hand it is not possible to repeat the same arguments of the papers [15], [16] and [17]; indeed the results obtained by Del Pino–Felmer rely in an essential way on the properties of the positive solutions for the "limiting" equation

$$\Delta v - b\,v + f(v) = 0 \quad \text{in } \mathbb{R}^N,\ b > 0$$

where f satisfies some precise assumptions about its global behaviour. Such solutions are well known to be radially symmetric, to decay exponentially and to maximize at zero. However analogous results for equations involving the p-Laplacian operator or nonlinearities like W are unknown to be true. The difficulty with respect to the problem considered by Del Pino–Felmer consists essentially in the loss of such data, and this put an obstacle to determine the behaviour of the solutions v_h.

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