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SETS OF SOLUTIONS OF NONLINEAR INITIAL-BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we deal with the general initial-boundary value problem for a second order nonlinear nonstationary evolution equation. The associated operator equation is studied by the Fredholm and Nemitskiĭ operator theory. Under local Hölder conditions for the nonlinear member we observe quantitative and qualitative properties of the set of solutions of the given problem. These results can be applied for the different mechanical and natural science models.

Introduction

The generic properties of solutions of the second order ordinary differential equations was studied by L. Brüll and J. Mawhin in [2], J. Mawhin in [16] and by V. Šeda in [21]. Such questions were solved for nonlinear diffusional type problems with the especial Dirichlet, Neumann and Newton type conditions in the papers [9]–[10].

In the present paper we study the set structure of classic solutions, bifurcation points and the surjectivity of an associated operator to a general second order nonlinear evolution problem, by the Fredholm operator theory. The present results allows us to search the generic properties of non-parabolic models which decribe mechanical, physical, reaction-diffusion and ecology processes.

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1. The formulation of problem and basic notions

Throughout this paper we assume that the set $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$. The real number T is positive and $Q := (0,T] \times \Omega$, $\Gamma := (0,T] \times \partial\Omega$.

We use the notation D_t for $\partial/\partial t$ and D_i for $\partial/\partial x_i$ and D_{ij} for $\partial^2/\partial x_i \partial x_j$, where $i,j = 1,\dots,n$ and $D_0 u$ for u. The symbol $\operatorname{cl} M$ means the closure of set M in $\mathbb{R}^n$.

We consider the nonlinear differential equation (possibly a non-parabolic type)

$$D_t u - A(t,x,D_x)u + f(t,x,u,D_1 u,\dots,D_n u) = g(t,x) \tag{1.1}$$

for $(t,x) \in Q$, where the coefficients a_{ij}, a_i, a_0 for $i,j = 1,\dots,n$ of the second order linear operator

$$A(t,x,D_x)u = \sum_{i,j=1}^{n} a_{ij}(t,x) D_{ij} u + \sum_{i=1}^{n} a_i(t,x) D_i u + a_0(t,x)u$$

are continuous functions from the space $C(\operatorname{cl} Q, \mathbb{R})$. The function f is from the space $C(\operatorname{cl} Q \times \mathbb{R}^{n+1}, \mathbb{R})$ and $g \in C(\operatorname{cl} Q, \mathbb{R})$.

Together with the equation (1.1) we consider the following general homogeneous boundary condition

$$B_3(t,x,D_x)u|_\Gamma := \sum_{i=1}^{n} b_i(t,x) D_i u + b_0(t,x)u|_\Gamma = 0, \tag{1.2}$$

where the coefficients b_i for $i = 1,\dots,n$ and b_0 are continuos functions from $C(\operatorname{cl}\Gamma, \mathbb{R})$.

Furthermore we require for the solution of (1.1) to satisfy the homogeneous initial condition

$$u|_{t=0} = 0 \quad \text{on } \operatorname{cl}\Omega. \tag{1.3}$$

Remark 1.1. In the case, if $b_i = 0$ for $i = 1,\dots,n$ and $b_0 = 1$ in (1.2) we get the Dirichlet problem studied in [9].

If we consider the vector function $\nu := (0,\nu_1,\dots,\nu_n) : \operatorname{cl}\Gamma \to \mathbb{R}^{n+1}$ which the value $\nu(t,x)$ means the unit inner normal vector to $\operatorname{cl}\Gamma$ at the point $(t,x) \in \operatorname{cl}\Gamma$ and we put $b_i = \nu_i$ for $i = 1,\dots,n$ on $\operatorname{cl}\Gamma$, then the problem (1.1)–(1.3) represents the Newton or Neuman problem investigated in [10].

Our considerations are concerned to a broad class of nonparabolic operators. However let us remind the definition of the uniform parabolicity for a operator of the type $D_t - A(t,x,D_x)$ (see [14, p. 12]), which we need in Proposition 2.2.

Definition 1.1 (The uniform parabolicity condition (P)). We say that the differential operator

$$D_t - A(t, x, D_x)$$

is *uniform parabolic on* $\operatorname{cl} Q$ in the sense of I. G. Petrovskiĭ with the constant δ or shortly, the operator satisfies *the parabolicity condition* (P) if there is a constant $\delta > 0$ such that for all $(t, x) \in \operatorname{cl} Q$ and each $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$ the inequality

$$\sum_{i,j=1}^{n} a_{ij}(t, x)\sigma_i\sigma_j \geq \delta\Big[\sum_{i=1}^{n} \sigma_i^2\Big]$$

holds.

In the following definitions we shall use the notations

$$\langle u\rangle^{s}_{t,\mu,Q} := \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{|u(t, x) - u(s, x)|}{|t - s|^{\mu}}, \tag{1.4}$$

$$\langle u\rangle^{y}_{x,\nu,Q} := \sup_{\substack{(t,x),(t,y)\in \operatorname{cl} Q\\ x\neq y}} \frac{|u(t, x) - u(t, y)|}{|x - y|^{\nu}}, \tag{1.5}$$

$$\begin{aligned} \langle f\rangle^{s,y,v}_{t,x,u} &:= |f(t, x, u_0, \ldots, u_n) - f(s, y, v_0, \ldots, v_n)|,\\ \langle f\rangle^{s,y,v(s,y)}_{t,x,u(t,x)} &:= |f[t, x, u(t, x), D_1u(t, x), \ldots, D_nu(t, x)]\\ &\qquad - f[s, y, v(s, y), D_1v(s, y), \ldots, D_nv(s, y)]|, \end{aligned}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are from $\mathbb{R}^n$ and $|x - y| = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2}$ and $\mu, \nu \in \mathbb{R}$.

The concept of a domain with a locally smooth boundary is given in the following definition.

Definition 1.2. Let $r \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that *the boundary* $\partial\Omega$ *belongs to the class* C^r, $r \geq 1$ if:

(i) there exists a tangential space to $\partial\Omega$ in any point from boundary $\partial\Omega$,

(ii) assume $y \in \partial\Omega$ and let $(y; z_1, \ldots, z_n)$ be a local orthonormal coordinate system with the center y and with the axis z_n oriented like the inner normal to $\partial\Omega$ at the point y. Then there exists a number $b > 0$ such that for every $y \in \partial\Omega$ there exists a neighbourhood $O(y) \subset \mathbb{R}^n$ of the point y and a function $F \in C^r(\operatorname{cl} B, \mathbb{R})$ such that the part of boundary

$$\partial\Omega \cap O(y) = \{(z', F(z')) \in \mathbb{R}^n,\ z' = (z_1, \ldots, z_{n-1}) \in B\},$$

where $B = \{z' \in \mathbb{R}^{n-1} \mid |z'| < b\}$.

Here $C^r(\operatorname{cl} B,\mathbb{R})$ is a vector space of the functions $u\in C^l(\operatorname{cl} B,\mathbb{R})$ for $l=[r]$ with the finite norm

$$\|u\|_{l+\alpha}=\sum_{0\le k\le l}\sup_{x\in\operatorname{cl} B}|D^k_x u(x)|+\sum_{k=l}\langle D^k_x u\rangle^y_{x,\alpha,B},$$

whereby $\alpha=r-[r]\in[0,1)$ and $r=l+\alpha$.

Further, we shall need the following Hölder spaces (see [6, p. 147]).

Definition 1.3. Let $\alpha\in(0,1)$.

(1) By the symbol $C^{(1+\alpha)/2,1+\alpha}_{t,x}(\operatorname{cl} Q,\mathbb{R})$ we denote the vector space of continuous functions $u:\operatorname{cl} Q\to\mathbb{R}$ which have continuous derivatives $D_i u$ for $i=1,\dots,n$ on $\operatorname{cl} Q$ and the norm

$$\begin{aligned}(1.6)\qquad \|u\|_{(1+\alpha)/2,1+\alpha,Q}:=\sum_{i=0}^n\sup_{(t,x)\in\operatorname{cl}Q}|D_i u(t,x)|+\langle u\rangle^s_{t,(1+\alpha)/2,Q}+\\ +\sum_{i=1}^n\langle D_i u\rangle^s_{t,\alpha/2,Q}+\sum_{i=1}^n\langle D_i u\rangle^y_{x,\alpha/2,Q}\end{aligned}$$

is finite.

(2) The symbol $C^{(2+\alpha)/2,2+\alpha}_{(t,x)}(\operatorname{cl} Q,\mathbb{R})$ means the vector space of continuous functions $u:\operatorname{cl} Q\to\mathbb{R}$ for which there exist continuous derivatives $D_t u$, $D_i u$, $D_{ij}u$ on $\operatorname{cl} Q$, $i,j=1,\dots,n$ and the norm

$$\begin{aligned}(1.7)\quad \|u\|_{(2+\alpha)/2,2+\alpha,Q}=\sum_{i=0}^n\sup_{(t,x)\in\operatorname{cl} Q}|D_i u(t,x)|+\sup_{(t,x)\in\operatorname{cl} Q}|D_t u(t,x)|\\ +\sum_{i,j=1}^n\sup_{(t,x)\in\operatorname{cl} Q}|D_{ij}u(t,x)|+\sum_{i=1}^n\langle D_i u\rangle^s_{t,(1+\alpha)/2,Q}+\langle D_t\rangle u^s_{t,\alpha/2,Q}\\ +\sum_{i,j=1}^n\langle D_{ij}u\rangle^s_{t,\alpha/2.Q}+\langle D_t u\rangle^y_{x,\alpha,Q}+\sum_{i,j=1}^n\langle D_{ij}u\rangle^y_{x,\alpha,Q}\end{aligned}$$

is finite.

(3) The symbol $C^{(3+\alpha)/2,3+\alpha}_{t,x}(\operatorname{cl} Q,\mathbb{R})$ means the vector space of continuous functions $u:\operatorname{cl} Q\to\mathbb{R}$ for which the derivatives $D_t, D_i u, D_t D_i u, D_{ij}u, D_{ijk}u$, $i,j,k=1,\dots,n$ are continuous on $\operatorname{cl} Q$ and the norm

$$\begin{aligned}(1.8)\ \|u\|_{(3+\alpha)/2,3+\alpha,Q}:=\sum_{i=0}^n\sup_{(t,x)\in\operatorname{cl}Q}|D_i u(t,x)|+\sum_{i,j=1}^n\sup_{(t,x)\in\operatorname{cl}Q}|D_{ij}u(t,x)|\\ +\sum_{i=0}^n\sup_{(t,x)\in\operatorname{cl} Q}|D_t D_i u(t,x)|+\sum_{i,j,k=1}^n\sup_{(t,x)\in\operatorname{cl} Q}|D_{ijk}u(t,x)|\\ +\langle D_t u\rangle^s_{t,(1+\alpha)/2,Q}+\sum_{i,j=1}^n\langle D_{ij}u\rangle^s_{t,(1+\alpha)/2,Q}+\sum_{i=1}^n\langle D_t D_i u\rangle^s_{t,\alpha/2,Q}\end{aligned}$$

$$+\sum_{i,j,k=1}^{n}\langle D_{ijk}u\rangle^{s}_{t,\alpha/2,Q}+\sum_{i=1}^{n}\langle D_tD_iu\rangle^{y}_{x,\alpha,Q}+\sum_{i,j,k=1}^{n}\langle D_{ijk}u\rangle^{y}_{x,\alpha,Q}$$

is finite.

The above defined norm spaces are Banach ones.

Now we can define the Hölder space of functions defined on the manifold $\operatorname{cl}\Gamma$ (see [14, p. 10]).

Definition 1.4. Let the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$ belong to C^r for $r \geq 1$ (see Definition 1.2). We put $S_y := \partial\Omega \cap O(y)$ and $\Gamma_y = (0,T] \times S_y$ for $y \in \partial\Omega$, where $O(y)$ is a neighbourhood of the point y from Definition 1.2.

The symbol $C^{(2+\alpha)/2,2+\alpha}_{t,x}(\operatorname{cl}\Gamma,\mathbb{R})$ means the vector space of continuous functions $u : \operatorname{cl}\Gamma \to \mathbb{R}$ for which there exist continuous derivates $D_tu, D_iu, D_{ij}u$ on $\operatorname{cl}\Gamma$, $i,j = 1,\dots,n$ and the norm

$$\|u\|_{(2+\alpha)/2,2+\alpha,\Gamma} = \sup_{y\in\partial\Omega}\|u\|_{(2+\alpha)/2,2+\alpha,\Gamma_y}$$

is finite. Here $\alpha \in (0,1)$ and the norm on the right hand side of the last equality is defined by the formula (1.7) in which we write Γ_y instead of Q.

Definition 1.5. (The smoothness condition $(S^{1+\alpha}_3)$). Let $\alpha \in (0,1)$. We say that the differential operator $A(t,x,D_x)$ from (1.1) and $B_3(t,x,D_x)$ from (1.2), respectively satisfies *the smoothness condition* $(S^{1+\alpha}_3)$ if

- (i) the coefficients a_{ij}, a_i, a_0 from (1.1) for $i,j = 1,\dots,n$ belong to the space $C^{(1+\alpha)/2,1+\alpha}_{t,x}(\operatorname{cl}Q,\mathbb{R})$ and $\partial\Omega \in C^{3+\alpha}$ and
- (ii) the coefficients b_i from (1.2) for $i = 1,\dots,n$ belong to the space $C^{(2+\alpha)/2,2+\alpha}_{t,x}(\operatorname{cl}\Gamma,\mathbb{R})$.

Definition 1.6 (The complementary condition (C)). If at least one of the coefficients b_i for $i = 1,\dots,n$ of the differential operator $B_3(t,x,D_x)$ in (1.2) is not zero we say that $B_3(t,x,D_x)$ satisfies the *complementary condition* (C).

Now we are prepared to formulate hypotheses for the deriving of fundamental lemmas.

Definition 1.7. (1) Fredholm conditions:

(A_3.1) Consider the operator $A_3 : X_3 \to Y_3$, where

$$A_3u = D_tu - A(t,x,D_x)u,\ u \in X_3$$

and the operators $A(t,x,D_x)$ and $B_3(t,x,D_x)$ satisfy the smoothness condition $(S^{1+\alpha}_3)$ for $\alpha \in (0,1)$ and the complementary condition (C).

Here we consider the vector spaces

$$D(A_3) := \{u \in C^{(3+\alpha)/2,3+\alpha}_{t,x}(\operatorname{cl} Q, \mathbb{R}) \mid B_3(t,x,D_x)u|_\Gamma = 0,$$
$$u|_{t=0}(x) = 0 \text{ for } x \in \operatorname{cl} Q\},$$
$$H(A_3) := \{v \in C^{(1+\alpha)/2,1+\alpha}_{t,x}(\operatorname{cl} Q, \mathbb{R}) \mid B_3(t,x,D_x)v(t,x)|_{t=0, x\in\partial\Omega} = 0\}$$

and Banach subspaces (of the given Hölder spaces)

$$X_3 = (D(A_3), \|\cdot\|_{(3+\alpha)/2,3+\alpha,Q}) \quad \text{and} \quad Y_3 = (H(A_3), \|\cdot\|_{(1+\alpha)/2,1+\alpha,Q}).$$

(A_3.2) There is a second order linear homeomorphism $C_3 : X_3 \to Y_3$ with

$$C_3 u = D_t u - C(t,x,D_x)u, \quad u \in X_3,$$

where

$$C(t,x,D_x)u = \sum_{i,j=1}^{n} c_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} c_i(t,x)D_i u + c_0(t,x)u$$

satisfying the smoothness condition ($S_3^{1+\alpha}$). The operator C_3 is not necessary parabolic one.

(2) Local Hölder and compatibility conditions:

Let $f := f(t,x,u_0,\dots,u_n) : \operatorname{cl} Q \times \mathbb{R}^{n+1} \to \mathbb{R}$, $\alpha \in (0,1)$ and let p, q, p_r for $r = 0,\dots,n$ be nonnegative constants. Here, D represents any compact subset of $(\operatorname{cl} Q) \times \mathbb{R}^{n+1}$. For f we need the following assumtions:

(N_3.1) Let $f \in C^1(\operatorname{cl} Q \times \mathbb{R}^{n+1}, \mathbb{R})$ and let the first derivatives $\partial f/\partial x_i$, $\partial f/\partial u_j$ be locally Hölder continuous on $\operatorname{cl} Q \times \mathbb{R}^{n+1}$ such that

$$\left.\begin{aligned} &\langle \partial f/\partial x_i \rangle^{s,y,v}_{t,x,u} && (1.9)\\ &\langle \partial f/\partial u_j \rangle^{s,y,v}_{t,x,u} && (1.10)\end{aligned}\right\} \le p|t-s|^{\alpha/2} + q|x-y|^\alpha + \sum_{r=0}^{n} p_r|u_r - v_r|$$

for $i = 1,\dots,n$ and $j = 0,\dots,n$ and any D.

(N_3.2) Let $f \in C^3(\operatorname{cl} Q \times \mathbb{R}^{n+1}, \mathbb{R})$ and let the local growth conditions for the third derivatives of f hold on any D:

$$\left.\begin{aligned} &\langle \partial^3 f/\partial\tau\partial x_i\partial u_j \rangle^{t,x,v}_{t,x,u} && (1.11)\\ &\langle \partial^3 f/\partial\tau\partial u_j\partial u_k \rangle^{t,x,v}_{t,x,u} && (1.12)\\ &\langle \partial^3 f/\partial x_i\partial x_l\partial u_j \rangle^{t,x,v}_{t,x,u} && (1.13)\\ &\langle \partial^3 f/\partial x_i\partial u_j\partial u_k \rangle^{t,x,v}_{t,x,u} && (1.14)\\ &\langle \partial^3 f/\partial u_j\partial u_k\partial u_r \rangle^{t,x,v}_{t,x,u} && (1.15)\end{aligned}\right\} \le \sum_{s=0}^{n} p_s|u_s - v_s|^{\beta_s}$$

where $\beta_s > 0$ for $s = 0,\dots,n$ and $i,l = 1,\dots,n$, $j,k,r = 0,\dots,n$.

(N_3.3) The equality of compatibility

$$\sum_{i=1}^{n} b_i(t,x)D_i f(t,x,0,\dots,0) + b_0(t,x)f(t,x,0,\dots,0)|_{t=0,\ x\in S} = 0$$

holds.

(3) Almost coercive condition:

Let for any bounded set $M_3 \subset Y_3$ there be the number $K > 0$ such that for all solutions $u \in X_3$ of the problem (1.1), (1.2), (1.3) with the right hand side $g \in M_3$, the following alternative holds:

(F_3.1) Either

(α_3) $\|u\|_{(1+\alpha)/2,1+\alpha,Q} \le K$, $f := f(t,x,u_0) : \operatorname{cl} Q \times \mathbb{R} \to \mathbb{R}$ and the coefficients of the operators A_3 and C_3 (see (1.1) and (A_3.2)) satisfy the equations

$$a_{ij} = c_{ij},\ a_i = c_i \quad \text{for } i,j = 1,\dots,n,\ a_0 \ne c_0 \text{ on } \operatorname{cl} Q$$

or

(β_3) $\|u\|_{(2+\alpha)/2,2+\alpha,Q} \le K$, $f : f(t,x,u_0,\dots,u_n) : \operatorname{cl} Q \times \mathbb{R}^{n+1} \to \mathbb{R}$ and the coefficients of the operators A_3 and C_3 satisfy the relations

$$a_{ij} = c_{ij} \quad \text{for } i,j = 1,\dots,n \quad \text{and} \quad a_i \ne c_i \text{ for at least one } i = 1,\dots,n$$

on $\operatorname{cl} Q$.

Remark 1.2. (1) Especialy, the condition (A_3.2) is satisfied for the diffusion operator

$$C_3 u = D_t u - \triangle u, \quad u \in X_3$$

or for any uniformly parabolic operator C_3 with sufficiently smooth coefficients (see Definition 1.1 and Proposition 2.2). However the operator C_3 is not necessarily uniform parabolic.

(2) The local Hölder condition in (N_3.1) and (N_3.2) admit sufficiently strong growths of f in the last variables $u_0,\dots,u_n$. For example, it includes exponential and power type growths.

Definition 1.8.

(1) A couple $(u,g) \in X_3 \times Y_3$ will be called *the bifurcation point of the mixed problem* (1.1)–(1.3) if u is a solution of that mixed problem and there exists a sequence $\{g_k\} \subset Y_3$ such that $g_k \to g$ in Y_3 as $k \to \infty$ and the problem (1.1)–(1.3) for $g = g_k$ has at least two different solutions u_k, v_k for each $k \in N$ and $u_k \to u$, $v_k \to u$ in X_3 as $k \to \infty$.

(2) The set of all solutions $u \in X_3$ of (1.1)–(1.3) (or the set of all functions $g \in Y_3$) such that (u,g) is a bifurcation point of the problem (1.1)–(1.3)

will be called *the domain of bifurcation* (*the bifurcation range*) of that problem.

Recall some other notions in the following definitions:

Definition 1.9. Let X and Y be two Banach spaces either both real or both complex.

(1) The mapping $F : X \to Y$ is *proper* (resp. *σ-proper*) if for each compact $K \subset Y$, the set $F^{-1}(K)$ is compact (resp. is a countable union of compact sets).
(2) The mapping $F : X \to Y$ is *closed* if for each closed set $S \subset X$, the set of image $f(S)$ is closed in Y.
(3) We call $F : X \to Y$ a *coercive mapping* if for each bounded set $S \subset Y$, the set $F^{-1}(S)$ is bounded in X.

Definition 1.10. Let M_1, M_2 be two metric spaces.

(1) The mapping $F : M_1 \to M_2$ is said *locally injective at a point* $u_0 \in M_1$ if there is a neighbourhood $U(u_0)$ of u_0 such that F is injective in $U(u_0)$. F is *injective in* M_1 if it is locally injective at each points $u \in M_1$.
(2) Let the mapping $F : M_1 \to M_2$ be continuous. Then F is said *locally invertible at a point* $u_0 \in M_1$ if there is neighbourhood $U(u_0)$ of u_0 and a neighbourhood $U_1(F(u_0))$ of $F(u_0)$ such that F is a homeomorphism of $U(u_0)$ onto $U_1(F(u_0))$. F je *locally invertible in* M_1 if it is locally invertible at each point $u \in M_1$.
(3) Let the mapping $F : X \to Y$ be continuous (X, Y are Banach spaces, $F \in C(X, Y)$). We denote by Σ the set of all points $u \in X$ for which F is not locally invertible.

Definition 1.11. We say that $G = I - g : X \to X$ is *strict solvable field,* if it is a condensing field and there is a sequence $r_k \to \infty$ as $k \to \infty$ such that the degree of the mapping G $\deg(G, U(0, r_k), 0) \neq 0$, where $U(0, r_k) \subset X$ is the sphere with the center 0 and the radius r_k for $k = 1, 2, \ldots$

Definition 1.12.

(1) If $D \subset X$ is a nonempty open set and $F : (D \subset X) \to Y$ is a Frèchet differentiable mapping, then $u_0 \in D$ is called *a regular point of* F if the Frèchet derivative $F'(u_0)$ is a linear homeomorphism of X onto Y ($F'(u_0) : X \to Y$ is bijective). The point $u_1 \in D$ is called *a critical point of* F, if the equation $F'(u_1)h = 0 \in Y$ has a nontrivial solution $h \in X$.
(2) If $u_0 \in D$ is not regular point of F, then it is called a *singular point of* F.

(3) The image by F of a singular point is called a *singular value of* F. If S is the set of all singular point of $F : X \to Y$, then $F(S)$ is called the set of *all singular values of* F and $Y - F(S)$ is the set of *all regular values of* F.

(4) A subset of a topological space Z is *residual*, when it is a countable intersection of dense open subset of Z.

Definition 1.13. The mapping $F : X \to Y$ is called *a local C^1-diffeomorphism at* u_0, if there exists a neighbourhood $U(u_0)$ of u_0 and $U_1(F(u_0))$ of $F(u_0)$ such that F bijectively maps $U_1(u_0)$ onto $U_2(F(u_0))$ and both F and F^{-1} are C^1-maps.

Remark 1.2.

(1) The set $X - \Sigma$ is open. Hence Σ is closed subset of X.

(2) It is clear that if F is locally invertible at u_0, then F is locally injective at u_0.

(3) By the Baire theorem, if Z is a complete metric space or if Z is a locally compact Hausdorff topological space, then a residual set is dense in Z.

2. General results

The following results will be used to prove fundamental lemmas and main results for the nonlienar problem (1.1)–(1.3). Here X and Y are Banach spaces either both real or complex.

Proposition 2.1 (S. M. Nikol'skiĭ, see [25, p. 233]). *A linear bounded operator $A : X \to Y$ is Fredholm of the zero index if and only if $A = C + T$, where $C : X \to Y$ is a linear homeomorphism and $T : X \to Y$ is a linear completly continuous operator.* (*For the definition of a linear and nonlinear Fredholm operator* (*see* [27, p. 365–366]))

The following proposition deals with the solution of a linear parabolic problem (see [14, p. 21], or [11]).

Proposition 2.2. *Let the operator A be from* (1.1) *and the assumptions* (P), (C), $(S_3^{1+\alpha})$ *satisfy. The necessary and sufficient condition for the existence and uniqueness of the solution $u \in C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\, Q, \mathbb{R})$ of the linear parabolic problem for the equation*

$$D_t u - A(t, x, D_x)u = f(t, x) \quad on\ \Omega$$

with the data (1.2), (1.3) *is that for $f \in C_{t,x}^{(1+\alpha)/2,\ 1+\alpha}(\mathrm{cl}\, Q, \mathbb{R})$ and the compatibility condition from* (N_3.3)

$$\sum_{i=1}^{n} b_i(t, x) D_i f(t, x) + b_0(t, x) f(t, x)|_{t=0,\, x\in\partial\Omega} = 0$$

holds. Then moreover, there exists a constant $K > 0$ *independent of* f *such that*

$$K^{-1}\|f\|_{(1+\alpha)/2,1+\alpha,Q} \leq \|u\|_{(3+\alpha)/2,3+\alpha,Q} \leq K\|f\|_{(1+\alpha)/2,1+\alpha,Q}.$$

Proposition 2.3 ([21, Proposition 2.1]). *Let* $F : X \to Y$ *be a coutinuous mapping. If* F *is proper, then* F *is a nonconstant closed mapping. Conversely if* $\dim X = \infty$ *and* $F : X \to Y$ *is a nonconstant closed mapping, then* F *is proper.*

Proposition 2.4 ([21, Proposition 2.2]). *Let* $F : X \to Y$, $F = F_1 + F_2$, *where* $F_1 : X \to Y$ *is a coutinuous proper mapping and* $F_2 : X \to Y$ *is a completely continuous one. Then*

(j) *The restriction of the mapping* F *to an arbitrary bounded closed set in* X *is a proper mapping.*
(jj) *If moreover,* F *is coercive, then* F *is a proper mapping.*

The relation between the local invertibility and homeomorphism of X on Y gives R. Caccioppoli in [3]; see [27, p. 174].

Proposition 2.5 (The global inverse mapping theorem). *Let* $F \in C(X, Y)$ *be locally invertible mapping in* X, *then* F *is a homeomorphism of* X *onto* Y *if and only if* F *is proper.*

Proposition 2.6 (The Ambrosetti theorem [1, p. 216]). *Let* $F \in C(X, Y)$ *be a proper mapping. Then the cardinal number card* $F^{-1}(\{q\})$ *of the set* $F^{-1}(\{q\})$ *is constant and finite* (*it may be zero*) *for each* q *taken from the same* (*connected*) *component of the set* $Y - F(\Sigma)$.

Proposition 2.7 ([21, Theorem 3.2, Corollary 3.3, Remark 3.1]). *Let the assumptions:*

(i) $F = I - f : X \to X$ *is a condensing and coercive map,*
(ii) *there exists a strictly solvable field* $G = I - g : X \to X$ *and* $K > 0$ *such that for all solution* $u \in X$ *of the equation*

$$F(u) = kG(u) \quad \text{and for all } k < 0,$$

the estimate $||u||_X < K$ *holds,*

or the assumptions:

(i') $F = A + N : X \to Y$ *is a coercive mapping, where* $A = C + T : X \to Y$ *and* C *is linear homeomorphism of* X *onto* Y, $T : X \to Y$ *is a linear completely continuous operator and* $N : X \to Y$ *is completely continuous,*
(ii') *there is a strictly solvable field* $G = I - g : X \to X$ *and* $K > 0$ *such that for all solution* $u \in X$ *of the equation*

$$F(u) = kC \circ G(u) \quad \text{and for all } k < 0,$$

the estimate $\|u\|_X < K$ *holds,*

be satisfied, respectively. Then the following statements are true:

(j) F *is a proper map,*

(jj) F *is surjective.*

The following proposition gives the important theorem for nonlinear Fredholm mapping.

Proposition 2.8 (S. Smale [19], F. Quinn [17]). *If* $F : X \to Y$ *is a Fredholm mapping of class* C^q, $q > \max(\operatorname{ind} F, 0)$ *and either* X *has a countable basis* (*Smale*) *or* F *is* σ*-proper* (*Quinn*), *then the set* R_F *of all regular values of* F *is residual in* Y. *If* F *is proper, then* R_F *is open and dense in* Y.

Proposition 2.9 ([27, p. 172]). *Let* $F : (U(u_0) \subset X) \to Y$ *be a* C^1*-mapping. Then* F *is a local* C^1*-diffeomorphism at* u_0 *if and only if* u_0 *is a regular point of* F.

Proposition 2.10 ([18, Corollary 2.3.14, p. 89]). *Let* $\dim Y \geq 3$ *and let* $F : X \to Y$ *be a Fredholm mapping of the zero index. If* u_0 *is an isolated singular point of* F, *then the mapping* F *is localy invertibly at* u_0.

3. Fundamental lemmas

Lemma 3.1. *Let the conditions* (A_3.1) *and* (A_3.2) *hold* (*see Definition* 1.7). *Then*

(j) $\dim X_3 = \infty$.

(jj) *The operator* $A_3 : X_3 \to Y_3$ *is a linear bounded Fredholm operator of the zero index.*

Proof. (j) To prove the first part of this lemma we use the decomposition theorem from [24, p. 139]:

Let X be linear space and $x^* : X \to \mathbb{R}$ be a linear functional on X such that $x^* \neq 0$. Further put $M = \{x \in X \mid x^*(x) = 0\}$ and $x_0 \in X - M$. Then every element $x \in X$ can be expressed by the formula

$$x = \left[\frac{x^*(x)}{x^*(x_0)}\right] x_0 + m \quad \text{for } m \in M,$$

i.e. there is a one-dimensional subspace L_1 of X such that $X = L_1 \oplus M$.

If we put now

$$M_1 := \{u \in C^{(3+\alpha)/2, 3+\alpha}_{t,x}(\operatorname{cl} Q, \mathbb{R}) =: H^{3+\alpha} \mid B_3(t, x, D_x)u|_\Gamma = 0\},$$

which is the linear subspace of $H^{3+\alpha}$, then there exists a linear subspace L_1 of $H^{3+\alpha}$ with $\dim L_1 = 1$ such that $H^{3+\alpha} = L_1 \oplus M_1$. Similar, if we take $M_2 := \{u \in M_1 \mid u|_{t=0} = 0 \text{ on } \operatorname{cl} Q\}$, then there is a subspace L_2 of M_1 with

$\dim L_2 = 1$ such that $M_1 = L_2 \oplus M_2$. Hence, we have $H^{3+\alpha} = L_1 \oplus L_2 \oplus D(A_3)$. Since $\dim H^{3+\alpha} = \infty$ we get that $\dim X_3 = \infty$.

(jj) 1. In the first step we prove the boundedness of the linear operator A_3. For this aim we observe the norm $\|A_3 u\|_{(1+\alpha)/2,1+\alpha,Q}$ for $u \in D(A_3)$. From the assumption $(S_3^{1+\alpha})$ we get for $k = 0, 1, \dots, n$

$$\sup_{(t,x)\in \operatorname{cl} Q} |D_k A_3 u(t,x)| \leq K_1 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad \text{for } K_1 > 0. \tag{3.1}$$

Applying again the smoothness assumption $(S_3^{1+\alpha})$, the mean value theorem for the function u and $D_i u$ and the boundedness of Q we obtain for the second member of the above mentioned norm the following estimation:

$$\begin{aligned}\langle A_3 u\rangle^s_{t,(1+\alpha)/2,Q} &= \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{|A_3 u(t,x) - A_3 u(s,x)|}{|t-s|^{(1+\alpha)/2}} \\ &\leq K_2 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad \text{for } K_2 > 0.\end{aligned} \tag{3.2}$$

The third member of the norm (1.6) we estimate for $k = 1, \dots, n$ as follows:

$$\begin{aligned}\langle D_k A_3 u\rangle^s_{t,\alpha/2,Q} &= \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{|D_k A_3 u(t,x) - D_k A_3 u(s,x)|}{|t-s|^{\alpha/2}} \\ &\leq K_3 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad \text{for } K_3 > 0.\end{aligned} \tag{3.3}$$

An estimation of the last member in (1.6) for $A_3 u$ is given by the following inequality for $k = 1, \dots, n$

$$\begin{aligned}\langle D_k A_3 u\rangle^y_{x,\alpha/2,Q} &= \sup_{\substack{(t,x),(t,y)\in \operatorname{cl} Q\\ x\neq y}} \frac{|D_k A_3 u(t,x) - D_k A_3 u(t,y)|}{|x-y|^{\alpha/2}} \\ &\leq K_4 \|u\|_{(3+\alpha)/2,3+\alpha,Q} \quad \text{for } K_4 > 0.\end{aligned} \tag{3.4}$$

From the estimations (3.1)–(3.4) we can conclude that

$$\|A_3 u\|_{Y_3} = \|A_3 u\|_{(1+\alpha)/2,1+\alpha,Q} \leq K(n, T, \alpha, \Omega, a_{ij}, a_i, a_0)\|u\|_{X_3}.$$

2. To prove that A_3 is a Fredholm operator with the zero index we express it in the form

$$A_3 u = C_3 u + [C(t,x,D_x) - A(t,x,D_x)]u =: C_3 u + T_3 u,$$

where C_3 is the linear homeomorphism and C is the linear operator from $(A_3.2)$. By the decomposition Nikol'skiĭ theorem from Proposition 2.1, it is sufficient to show that $T_3 : X_3 \to Y_3$ is the linear completely continuous operator.

The complete continuity of T_3 can be proved by the Ascoli–Arzela theorem (see [23, p. 141]).

From $(S_3^{1+\alpha})$ the uniform boundedness of the operator

$$T_3u = \sum_{i,j=1}^{n}[c_{ij}(t,x) - a_{ij}(t,x)]D_{ij}u + \sum_{i=1}^{n}[c_i(t,x) - a_i(t,x)]D_iu + [c_0(t,x) - a_0(t,x)]u$$

follows by the same way as the boundedness of operator A_3 in the previous part 1. Thus for all $u \in M \subset X_3$, where M is a bounded set by the constant $K_1 > 0$, we obtain the estimate

$$\|T_3u\|_{Y_3} \le K(n, \alpha T, \Omega, a_{ij}, c_{ij}, a_i, c_i, a_0, c_0)\|u\|_{X_3} \le KK_1.$$

Using the smoothness condition of the operators A and C we get inequalities:

$$\begin{aligned}|T_3u(t,x) - T_3u(s,y)| \le{} & \sum_{i,j=1}^{n} |[c_{ij} - a_{ij}](t,x) - [c_{ij} - a_{ij}](s,y)||D_{ij}u(t,x)| \\ & + \sum_{i,j=1}^{n} |c_{ij}(s,y) - a_{ij}(s,y)||D_{ij}u(t,x) - D_{ij}u(s,y)| \\ & + \sum_{i=1}^{n} |[c_i - a_i](t,x) - [c_i - a_i](s,y)||D_iu(t,x)| \\ & + \sum_{i=1}^{n} |c_i(s,y) - a_i(s,y)||D_iu(t,x) - D_iu(s,y)| \\ & + |[c_0 - a_0](t,x) - [c_0 - a_0](s,y)||u(t,x)| \\ & + |c_0(s,y) - a_0(s,y)||u(t,x) - u(s,y)| \\ \le{} & 4K_1Kn^2[|t-s|^{\alpha/2} + |x-y|^{\alpha}] \\ & + 2K_1Kn[(|t-s|^{\alpha/2} + |x-y|^{\alpha}) + (|t-s|^{(1+\alpha)/2} + |x-y|)] \\ & + 2K_1K[(|t-s|^{\alpha/2} + |x-y|^{\alpha}) + (|t-s| + |x-y|)],\end{aligned}$$

where K_1, K are positive constants. Hence the equicontinuity of $T_3M \subset Y_3$ follows. This finishes the proof of Lemma 3.1. □

The Lemma 3.1 implies the following alternative.

Corollary 3.1. *Let L mean the set of all second order linear differential operators*

$$A_3 = D_t - A(t,x,D_x) : X_3 \to C_{t,x}^{(1+\alpha)/2,1+\alpha}(\operatorname{cl} Q, \mathbb{R})$$

satifying the condition (C) *and* $(S_2^{1+\alpha})$. *Then for each $A_3 \in L$ the mixed homogeneous problem $A_3u = 0$ on Q,* (1.2), (1.3) *has a nontrivial solution or any $A_3 \in L$ is a linear bounded Fredholm operator of the zero index mapping X_3 onto Y_3.*

The following lemma establishes the complete continuity of the Nemitskiĭ operator from the nonlinear part of the equation (1.1).

Lemma 3.2. *Let the assumptions* (N_3.1) *and* (N_3.3) *satisfy. Then the Nemitskiĭ operator* $N_3 : X_3 \to Y_3$ *defined by*

$$(N_3u)(t,x) = f[t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)] \tag{3.5}$$

for $u \in X_3$ *and* $(t,x) \in \operatorname{cl} Q$ *is completely continuous.*

Proof. Let $M_3 \subset X_3$ be a bounded set. By the Ascoli–Arzela theorem it is sufficient to show that the set $N_3(M_3)$ is uniform bounded and equicontinuous. The assumption (N_3.3) we use to prove the inclusion $N_3(M_3) \subset Y_3$.

Take $u \in M_3$. According to the assumption (N_3.1) we obtain the local boundedness of the function f and its derivatives $\partial f/\partial x_i$ on $(\operatorname{cl} Q) \times \mathbb{R}^{n+1}$ for $i = 1,\dots,n$. Hence and from the equation

$$D_i(N_3u)(t,x) = \{D_if[\,\cdot\,] + \sum_{l=0}^{n} \frac{\partial f}{\partial u_l}[\,\cdot\,]D_iD_lu\}[\,\cdot\,,\,\cdot\,,u,D_1u,\dots,D_nu](t,x)$$

we have the estimation

$$\sup_{(t,x)\in \operatorname{cl} Q} |D_i(N_3u)(t,x)| \le K_1$$

for $i = 0,\dots,n$ with a positive sufficiently large constant K_1 not depending on $u \in M_3$.

Using the differentiability of f and the mean value theorem in the variable t for the difference of the derivatives of u we can write

$$\langle N_3u\rangle^s_{t,(1+\alpha)/2,Q} \le K_1.$$

Similarly, by (1.9) and (1.10), we have

$$\langle D_iN_3u\rangle^s_{t,\alpha/2,Q} \le K_1 \quad \text{and} \quad \langle D_iN_3u\rangle^y_{x,\alpha,Q} \le K_1$$

for $i = 1,\dots,n$ and $u \in M_3$. The previous estimations yield the inequality

$$\|N_3u\|_{Y_3} \le K_1 \quad \text{for all } u \in M_3.$$

With respect to (N_3.1) for any $u \in M_3$ and $(t,x),(s,y) \in \operatorname{cl} Q$ such that $|t-s|^2 + |x-y|^2 < \delta^2$ with a sufficiently small $\delta > 0$ we have

$$|N_3u(t,x) - N_3u(s,y)| < \varepsilon, \quad \varepsilon > 0,$$

which is the equicontinuity of $N_3(M_3)$. This finishes the proof of Lemma 3.2. □

Lemma 3.3. *Let the assumptions* (A_3.1), (A_3.2), (N_3.1), (N_3.3) *and* (F_3.1) *hold. Then the operator* $F_3 = A_3 + N_3 : X_3 \to Y_3$ *is coercive.*

Proof. We need prove that if the set $M_3 \subset Y_3$ is bounded in Y_3, then the set of arguments $F_3^{-1}(M_3) \subset X_3$ is bounded in X_3.

In the both cases (α_3) and (β_3) we get for all $u \in F_3^{-1}(M_3)$

$$\|N_3 u\|_{(1+\alpha)/2,1+\alpha,Q} \le K_1,$$

where $K_1 > 0$ is a sufficiently large constant. Hence $\|A_3 u\|_{Y_3} \le K_1$ for any $u \in F_3^{-1}(M_3)$.

The hypothesis (A_3.2) ensures the existence and uniqueness of the solution $u \in X_3$ of the linear equation $C_3 u = y$ and for any $y \in Y_3$

$$\|u\|_{X_3} \le K_1 \|y\|_{Y_3} \tag{3.6}$$

If we write

$$C_3 u = A_3 u + \sum_{i,j=1}^{n} [a_{ij}(t,x) - c_{ij}(t,x)] D_{ij} u + \sum_{i=1}^{n} [a_i(t,x) - c_i(t,x)] D_i u + [a_0(t,x) - c_0(t,x)] u,$$

then in the both cases and for each $u \in F_3^{-1}(M_3)$ we obtain

$$\|y\|_{Y_3} \le \|C_3 u\|_{Y_3} \le K_1$$

whence by the inequality (3.6) we can conclude that the operator F_3 is coercive.□

Lemma 3.4. *Let the Nemitskiĭ operator* $N_3 \colon X_3 \to Y_3$ *from* (3.5) *satisfy the conditions* (N_3.2), (N_3.3). *Then the operator* N_3 *is continuously Fréchet differentiable, i.e.* $N_3 \in C^1(X_3, Y_3)$ *and it is completely continuous.*

Proof. From (N_3.2) we obtain (N_3.1) which implies by Lemma 3.2 the complete continuity of N_3. To obtain the first part of the assertion of this lemma we need prove that the Fréchet derivative $N_3' \colon X_3 \to L(X_3, Y_3)$ defined by the equation

$$N_3'(u)h(t,x) = \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}(t, x, u(t,x), D_1 u(t,x), \dots, D_n u(t,x)] D_j h(t,x)$$

for $u, h \in X_3$ is continuous on X_3. Thus we must prove for every $v \in X_3$:

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon, v) > 0 \quad \forall u \in X_3, \ \|u - v\|_{X_3} < \delta \colon \sup_{h \in X_3, \|h\|_{X_3} \le 1} \|N_3'(u) - N_3'(v)h\|_{Y_3} < \varepsilon. \tag{3.7}$$

Using the norms (1.6), (1.8) and the estimation $\|u-v\|_{X_3} < \delta$ we have for the first term of (3.7) by the mean value theorem:

$$\begin{aligned}
&\sum_{i=0}^{n} \sup_{(t,x)\in \operatorname{cl} Q} |D_i[N_3'(u)-N_3'(v)]h(t,x)| \\
&\quad \leq \sum_{i,j=0}^{n} \sup_{(t,x)\in \operatorname{cl} Q} \Big[\langle \partial^2 f/\partial x_i \partial u_j\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} |D_j h(t,x)| \\
&\quad\quad + \sum_{k=0}^{n} \langle \partial^2 f/\partial u_j \partial u_k\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} |D_{ik}u|.|D_j h|(t,x) \\
&\quad\quad + \sum_{k=0}^{n} |\partial^2 f/\partial u_j \partial u_k(t,x,v(t,x),\dots)||D_{ik}u - D_{ik}v||D_j h|(t,x) \\
&\quad\quad + \langle \partial f/\partial u_j\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} |D_{ij}h(t,x)|\Big] < K\delta \quad \text{for } K>0.
\end{aligned}$$

The second term of (3.7) we estimate as follows:

$$\begin{aligned}
&\langle [N_3'(u)-N_3'(v)]h\rangle_{t,(1+\alpha)/2,Q}^{s} \\
&\quad \leq \sum_{j=0}^{n} \sup_{\operatorname{cl} Q, t\neq s} |t-s|^{-(1+\alpha)/2} \bigg[\bigg|\int_s^t D_\tau \langle \partial f/\partial u_j\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}\, d\tau\bigg| |D_j h(t,x)| \\
&\quad\quad + \langle \partial f/\partial u_j\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} \bigg|\int_s^t D_\tau D_j h(\tau,x)\, d\tau\bigg|\bigg] \leq K\delta \quad \text{for } K>0.
\end{aligned}$$

Here we have used the mean value theorem for $\partial^2 f/\partial\tau\partial u_j, \partial^2 f/\partial u_j \partial u_k$ and $\partial f/\partial u_j$ for $j,k=0,\dots,n$.

The third term of (3.7) gives by (1.11), (1.12), (1.14), (1.15):

$$\begin{aligned}
&\sum_{i=1}^{n} \langle D_i\{[N_3'(u)-N_3'(v)]h\}\rangle_{t,\alpha/2,Q}^{s} \\
&\quad \leq \sum_{i=1}^{n}\sum_{j=0}^{n} \sup_{\operatorname{cl} Q, t\neq s} |t-s|^{-\alpha/2} \bigg\{\bigg|\int_s^t D_\tau \langle \partial^2 f/\partial x_i \partial u_j\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}\, d\tau\bigg| |D_j h(t,x)| \\
&\quad\quad + \langle \partial^2 f/\partial x_i \partial u_j\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} \bigg|\int_s^t D_\tau D_j h(\tau,x)\, d\tau\bigg| \\
&\quad\quad + \sum_{k=0}^{n} \bigg[\bigg|\int_s^t D_\tau \langle \partial^2 f/\partial u_j \partial u_k\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}\, d\tau\bigg| |D_{ik}u||D_j h|(t,x) \\
&\quad\quad + \bigg|\int_s^t D_\tau[\partial^2 f/\partial u_j \partial u_k(\tau,x,v,\dots)\, d\tau\bigg| |D_{ik}u(t,x) - D_{ik}v(t,x)||D_j h(t,x)| \\
&\quad\quad + \langle \partial^2 f/\partial u_j \partial u_k\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} |D_{ik}u(t,x) - D_{ik}u(s,x)||D_j h(t,x)| \\
&\quad\quad + |\partial^2 f/\partial u_j \partial u_k(s,x,v,\dots)||D_{ik}u(t,x) - D_{ik}v(t,x) \\
&\quad\quad - [D_{ik}u(s,x) - D_{ik}v(s,x)]||D_j h(t,x)|
\end{aligned}$$

$$
\begin{aligned}
&+ \langle \partial^2 f/\partial u_j \partial u_k \rangle_{s,x,u(s,x)}^{s,x,v(s,x)} \, |D_{ik}u(s,x)| \left| \int_s^t D_\tau D_j h(\tau,x)\, d\tau \right| \\
&+ |\partial^2 f/\partial u_j \partial u_k (s,x,v,\dots)| |D_{ik}u(s,x) - D_{ik}v(s,x)| \left| \int_s^t D_\tau D_j h(\tau,x)\, d\tau \right| \\
&+ \left| \int_s^t D_\tau \langle \partial f/\partial u_j \rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)} d\tau \right| |D_{ij}h(t,x)| \\
&+ \langle \partial f/\partial u_j \rangle_{s,x,u(s,x)}^{s,x,v(s,x)} |D_{ij}h(t,x) - D_{ij}h(s,x)| \Big] \Big\} \\
\le{}& K \bigg(\sum_{s=0}^{n} \delta^{\beta_s} + \delta \bigg) \quad \text{for } K > 0.
\end{aligned}
$$

Making the corresponding changes the last term of (3.7)

$$
\sum_{i=1}^{n} \langle D_i \{ [N_3'(u) - N_3'(v)]h \} \rangle_{x,\alpha,Q}^{y}
$$

by the condition (N_3.2) gives the required estimation. This finishes the proof of Lemma 3.4. □

The results of Lemmas 3.1–3.4 we can sum up in

Theorem 3.1. *The following implications are true:*

(1) (A_3.1), (A_3.2) *imply that the operator* $A_3 : X_3 \to Y_3$ *is linear bounded Fredholm operator of the zero index.*
(2) (N_3.1), (N_3.3) *imply that the Nemitskiĭ operator* $N_3 : X_3 \to Y_3$ *is completely continuous.*
(3) (A_3.1), (A_3.2), (N_3.1), (N_3.3), (F_3.1) *imply that the operator* $F_3 = A_3 + N_3 : X_3 \to Y_3$ *is coercive.*
(4) (N_3.2), (N_3.3) *imply that* $N_3 \in C^1(X_3, Y_3)$ *and completely continuous.*

4. Generic properties for continuous operators

On a mutual equivalence between the solution of the given initial-boundary value problem and an operator equation says the following lemma.

Lemma 4.1. *Let* $A_3 : X_3 \to Y_3$ *be the linear operator from Lemma* 3.1 *and let* $N_3 : X_3 \to Y_3$ *be the Nemitskiĭ operator from Lemma* 3.2 *and* $F_3 = A_3 + N_3 : X_3 \to Y_3$. *Then*

(j) *the function* $u \in X_3$ *is a solution of the initial-boundary value problem* (1.1)–(1.3) *for* $g \in Y_3$ *if and only if* $F_3 u = g$,
(jj) *the couple* $(u, g) \in X_3 \times Y_3$ *is the bifurcation point of the initial-boundary value problem* (1.1)–(1.3) *if and only if* $F_3(u) = g$ *and* $u \in \Sigma$, *where* Σ *means the set of all points of* X_3 *at which* F_3 *is not locally invertible* (*see Definition* 1.10).

Proof. (j) The first equivalence directly follows from the definition of operator F_3 and the mixed problem (1.1)–(1.3).

(jj) If (u, g) is a bifurcation point of the mixed problem (1.1)–(1.3) and u_k, v_k and g_k for $k = 1, 2\dots$ have the same meaning as in Definition 1.8, then with respect to (j) we have $F_3(u) = g, F_3(u_k) = g_k = F_3(v_k)$. Thus F_3 is not locally injective at u. Hence, F_3 is not locally invertible at u, i.e. $u \in \Sigma$. Conversely, if F_3 is not locally invertible at u and $F_3(u) = g$, then F_3 is not locally injective at u. Indirectly, from Definition 1.8, we see that the couple (u, g) is a bifurcation point of (1.1)–(1.3). □

Lemma 4.2. *Let*

(i) *the operator* $A(t, x, D_x) \neq 0$ *from* (1.1) *and the operator* $B_3(t, x, D_x)$ *from* (1.2) *satisfy the smoothness condition* $(S_3^{1+\alpha})$,

(ii) *the nonlinear part* f *of the equation* (1.1) *belong to* $C(\mathrm{cl}\, Q \times \mathbb{R}^{n+1}, \mathbb{R})$,

(iii) *the operator* $A_3 + N_3 : X_3 \to Y_3$ *be nonconstant.*

Then for any compact set of the right hand sides $g \in Y_3$ *from* (1.1), *the set of all solutions of problem* (1.1)–(1.3) *is compact* (*possibly empty*).

Proof. Following the proof of Lemma 3.1 we see that $\dim X_3 = \infty$ and the linear operator $A_3 : X_3 \to Y_3$ is continuous and accordingly closed. From the hypothesis (ii) the Nemitskiĭ operator $N_3 : X_3 \to Y_3$ given in (3.5) is closed, too. By the Proposition 2.3 the operator $F_3 = A_3 + N_3 : X_3 \to Y_3$ is proper and with respect to Definition 1.9 and Lemma 4.1 we get our assertion. □

Theorem 4.1. *Under the assumptions* (A_3.1), (A_3.2) *and* (N_3.1), (N_3.3) *the following statements hold for the problem* (1.1)–(1.3):

(a) *the operator* $F_3 = A_3 + N_3 : X_3 \to Y_3$ *is continuous,*

(b) *for any compact set of the right hand sides* $g \in Y_3$ *from* (1.1), *the corresponding set of all solutions is a countable union of compact sets,*

(c) *for* $u_0 \in X_3$ *there exists a neighbourhood* $U(u_0)$ *of* u_0 *and* $U(F_3(u_0))$ *of* $F_3(u_0) \in Y_3$ *such that for each* $g \in U(F_3(u_0))$ *there is a unique solution of* (1.1)–(1.3) *if and only if the operator* F_3 *is locally injective at* u_0.

Moreover, if (F_3.1) *is assumed, then*

(d) *for each compact set of* Y_3 *the corresponding set of all solutions is compact* (*possibly empty*).

Proof. Assertion (a) is evident by Lemma 3.1 and Lemma 3.2.

Using the Nikol'skiĭ theorem (Proposition 2.1) for A_3 we can write

$$F_3 = C_3 + (T_3 + N_3), \tag{4.1}$$

where $C_3 : X_3 \to Y_3$ is a continuous homeomorpism and is proper (see Proposition 2.3) and $T_3 + N_3 : X_3 \to Y_3$ is a completely continuous mapping. Now take the compact set $K \subset Y_3$ and $F_3^{-1}(K)$. Then there exists a sequence of the closed and bounded sets $M_n \subset F_3^{-1}(K) \subset X_3$ for $n = 1, 2, \dots$ such that $\bigcup_{n=1}^{\infty} M_n = F_3^{-1}(K)$

According to Proposition 2.4(j) the restrictions $F_3|_{M_n}$ for $n = 1, 2, \dots$ are proper mappings and $(F_3|_{M_n})^{-1}(K) = M_n$ is compact set. Hence, the operator F_3 is σ-proper, which gives the result (b).

The assertion (d) is a direct consequence of Proposition 2.4(jj).

Suppose, now, that F_3 is injective in a neighbourhood $U(u_0)$ of $u_0 \in X_3$. From decomposition (4.1) the mapping

$$C_3^{-1} F_3 = I + C_3^{-1}(T_3 + N_3),$$

where $I : X \to Y$ is the identity, is completely continuous and injective in $U(u_0)$. On the basis of the Schauder domain invariance theorem (see [5, p. 66]) the set $C_3^{-1} F_3(U(u_0))$ is open in X_3 and the restriction $C_3^{-1} F_3|_{U(u_0)}$ is a homeomorphism of $U(u_0)$ onto $C_3^{-1} F_3(U(u_0))$. Therefore F_3 is locally invertible. From the Definition 1.10.2 and Lemma 4.1 we obtain (c).

The most important properties of the mapping F_3, whereby A_3 is linear bounded Fredholm operator of zero index, N_3 is copletely continuous and F_3 is coercive, gives the following theorem. □

Theorem 4.2. *If the hypotheses* (A_3.1), (A_3.2), (N_3.1), (N_3.3) *and* (F_3.1) *are satysfied, then for the initial-boundary value problem* (1.1)–(1.3) *the following statements hold:*

- (e) *For each $g \in Y_3$ the set S_{3g} of all solutions is compact* (*possibly empty*).
- (f) *The set $R(F_3) = \{g \in Y_3$; there exists at least one solution of the given problem$\}$ is closed and connected in Y_3.*
- (g) *The domain of bifurcation D_{3b} is closed in X_3 and the bifurcation range R_{3b} is closed in Y_3. $F_3(X_3 - D_{3b})$ is open in Y_3.*
- (h) *If $Y_3 - R_{3b} \neq \emptyset$, then each component of $Y_3 - R_{3b}$ is a nonempty open set* (*i.e. a domain*).

The number n_{3g} of solutions is finite, constant (*it may be zero*) *on each component of the set $Y_3 - R_{3b}$, i.e. for every g belonging to the same component of $Y_3 - R_{3b}$.*

- (i) *If $R_{3b} = 0$, then the given problem has a unique solution $u \in X_3$ for each $g \in Y_3$ and this solution continuously depends on g as a mapping from Y_3 onto X_3.*
- (j) *If $R_{3b} \neq \emptyset$, then the boundary of the F_3-image of the set of all points from X_3 in which the operator F_3 is locally invertible, is a subset of the F_3-image of all points from X_3 in which F_3 is not locally invertible, i.e.*

$$\partial F_3(X_3 - D_{3b}) \subset F_3(D_{3b}) = R_{3b}.$$

Proof. The statement (e) follows immediately from Theorem 4.1(d).

(f) Let the sequence $\{g_n\}_{n\in N} \subset R(F_3) \subset Y_3$ converge to $g \in Y_3$ as $n \to \infty$. By Theorem 4.1(d) there is a compact set of all solutions $\{u_\gamma\}_{\gamma\in I} \subset X_3$ (I is a index set) of the equations $F_3(u) = g_n$ for all $n = 1, 2, \ldots$ Then there exists a sequence $\{u_{n_k}\}_{k\in N} \subset \{u_\gamma\}_{\gamma\in I}$ converging to $u \in X_3$ for which $F_3(u_{n_k}) = g_{n_k} \to g$. Since, the operator F_3 is proper (Theorem 4.1(d)), whence it is closed (Proposition 2.3), such we have $F_3(u) = g$. Hence $g \in R(F_3)$ and $R(F_3)$ is a closed set.

The connectedness of $R(F_3) = F_3(X_3)$ follows from the fact that $R(F_3)$ is a conditinuous image of the connected set X_3.

(g) According to Lemma 4.1(jj) $D_{3b} = \Sigma_3$ and $R_{3b} = F_3(D_{3b})$. Since $X_3 - \Sigma_3$ is an open set, D_{3b} and its coutinuous image R_{3b} are the closed sets in X_3 and Y_3, respectively.

Since $X_3 - D_{3b}$ is a set of all points in which the mapping F_3 is locally invertible, the Definition 1.10.2 ensures that to each $u_0 \in X_3 - D_{3b}$ there is a neighborhood $U_1(F_3(u_0)) \subset F_3(X_3 - D_{3b})$ which means that the set $F_3(X_3 - D_{3b})$ is open.

(h) The set $Y_3 - R_{3b} = Y_3 - F_3(D_{3b}) \neq 0$ is open in Y_3, then each its component is noempty and open.

The second part of (h) follows from A. Ambrosetti theorem (Proposition 2.6).

(i) Since $R_{3b} = \emptyset$, the mapping F_3 is a locally invertible in X_3. From Proposition 2.4(jj) we get that F_3 is a proper mapping. Then The Global Inverse Mapping Theorem (Proposition 2.5) proves this statement.

(j) By (f) and (g), we have ($\Sigma_3 = D_{3b}$)

$$F_3(X_3) = F_3(\Sigma_3) \cup F_3(X_3 - \Sigma_3) = F_3(\Sigma_3) \cup \overline{F_3(X_3 - \Sigma_3)} = \overline{F(X_3)}. \tag{4.2}$$

Furter $\partial F_3(X_3 - \Sigma_3) = \overline{F(X_3 - \Sigma_3)} - F(X_3 - \Sigma_3)$ and thus the previous equality implies the assertion (j). □

Theorem 4.3. *Under the assumption* (A_3.1), (A_3.2), (N_3.1), (N_3.3) *and* (F_3.1) *each of the following conditions is sufficient for the solvability of problem* (1.1)–(1.3) *for each* $g \in Y_3$*:*

(k) *For each* $g \in R_{3b}$ *there is a solution* u *of* (1.1)–(1.3) *such that* $u \in X_3 - D_{3b}$.

(l) *The set* $Y_3 - R_{3b}$ *is connected and there is a* $g \in R(F_3) - R_{3b}$.

Proof. First of all we see that the conditions (k) and (l) are mutualy equivalent to the conditions:

(k') $F_3(D_{3b}) \subset F_3(X_3 - D_{3b})$,

(l') $Y_3 - R_{3b}$ is a connected set and

$$F_3(X_3 - D_{3b}) - R_{3b} \neq \emptyset, \tag{4.3}$$

respectively ($D_{3b} = \Sigma_3$). Then it is sufficient to show that the conditions (k') and (l'), respectively are sufficient for the surjectivity of the operator $F_3 : X_3 \to Y_3$.

(k') From the first equality of (4.2) we obtain $F_3(X_3) = F_3(X_3 - D_{3b})$. Hence $R(F_3)$ is an open as well as closed subset of the connected space Y_3. Thus $R(F_3) = Y_3$.

(l') By (h) of Theorem 4.2 card $F_3^{-1}(\{q\}) = \text{const} =: k \geq 0$ for every $q \in Y_3 - R_{3b}$.

If $k = 0$, then $F_3(X_3) = R_{3b}$ and $F_3(X_3 - D_{3b}) \subset R_{3b}$. This is a contradiction to (4.3). Then $k > 0$ and $R(F_3) = Y_3$. □

The other surjecivity theorem is true:

THEOREM 4.4. *Let the hypotheses* (A_3.1), (A_3.2), (N_3.1), (N_3.3), (F_3.1) *and*

(i) *there exists a constant $K > 0$ such that all solutions $u \in X_3$ of the initial-boundary value problem for the equation*

$$C_3u + \mu[A_3u - C_3u + N_3u] = 0, \quad \mu \in (0,1) \tag{4.4}$$

with data (1.2), (1.3) *fulfill one of the conditions* (α_3) *or* (β_3) *of the almost coercive condition* (F_3.1). *Then*

(m) *the problem* (1.1)–(1.3) *has at least one solution for each $g \in Y_3$,*

(n) *the number n_{3g} of solutions* (1.1)–(1.3) *is finite, constant and different from zero on each component of the set $Y_3 - R_{3b}$ (for all g belonging to the same component of $Y_3 - R_{3b}$).*

PROOF. (m) It is sufficient to prove the surjectivity of the mapping $F_3 : X_3 \to Y_3$. From Lemma 3.1 we can write

$$F_3 = A_3 + N_3 = C_3 + (T_3 + N_3),$$

where $C_3 : X_3 \to Y_3$ is a linear homeomorphism X_3 onto Y_3 and $T_3 + N_3 : X_3 \to Y_3$ is a completely continuous operator. Then the operator

$$C_3^{-1}F_3 = I + C_3^{-1}(T_3 + N_3) : X_3 \to X_3$$

is a completely continuous and condensing (see [27, p. 496]). The set $\Sigma_3 = D_{3b}$ is the set of all points $u \in X_3$ where $C_3^{-1}F_3$, as well as F_3, is not locally invertible.

Denote $S_1 \subset X_3$ a bounded set. Then $C_3(S_1) =: S$ is bounded in Y_3 and by Lemma 3.3 $F_3^{-1}(S) = F_3^{-1}(C_3(S_1)) = (C_3^{-1} \circ F_3)^{-1}(S_1)$ is a bounded set in X_3. Thus the operator $C_3^{-1} \circ F_3$ is coercive.

Now we show that the condition (i) implies (ii) in Proposition 2.7 for $F(u) = C_3^{-1} \circ F_3(u)$ and $C(u) = G(u) = u$, $u \in X_3$.

In fact, as $C_3^{-1} \circ F_3(u) = ku$ if and only if $F_3(u) = kC_3(u)$ we get for $k < 0$

$$C_3u + (1-k)^{-1}[A_3u - C_3u + N_3u] = 0, \tag{4.5}$$

where $(1-k)^{-1} \in (0,1)$. This implies by (i): in the case (α_3), there is a constant $K > 0$ such that, for all solution $u \in X_3$ of (4.5), $||u||_{(1+\alpha)/2,1+\alpha,Q} \le K$ and, in the case (β_3), $||u||_{(2+\alpha)/2,2+\alpha,Q} \le K$. Further, by the same method as in Lemma 3.3 we get the estimation $||u||_{X_3} < K_1$, $K_1 > 0$ for all solution $u \in X_3$ of $C_3^{-1} \circ F_3 u = ku$. By Proposition 2.7 we get the surjectivity of F_3 and thus (m).

(n) From the Theorem 4.2(h) and the surjectivity of F_3 it follows that there is $n_{3g} \neq 0$. This finishes the proof of Theorem 4.4. □

5. Generic properties for C^1-differentiable operator

In the case, if the Nemitskiĭ operator $N_3 \in C^1(X,Y)$, we get stronger results, than in the Section 4.

THEOREM 5.1. *Assume that the hypotheses* (A_3.1), (A_3.2), (N_3.2), (N_3.3) *hold. Then the open set $Y_3 - R_{3b}$ is dense in Y_3 and thus the range of bifurcation R_{3b} of initial-boundary value problem* (1.1)–(1.3) *is nowhere dense in Y_3.*

PROOF. The Theorem 3.1 ensures that the operator A_3 is a linear Fredholm operator of the zero index, the Nemitskiĭ operator $N_3 : X_3 \to Y_3$ is completely continuous and $N_3 \in C^1(X_3, Y_3)$.

Since $N_3'(u) : X_3 \to Y_3$ is complete continuous, by Proposition 2.1 the operator $F_3'(u) = = A_3 + N_3'(u) : X_3 \to Y_3$ is a linear Fredholm operator of the zero index for each $u \in X_3$ and $F_3 \in C^1(X_3, Y_3)$ is also a Fredholm operator of the zero index (see [27, p. 366]).

$F_3'(u)$ is a linear homeomorphism if and only if it is bijective. Since $F_3'(u)$ is a Fredholm mapping of zero index so $F_3'(u)$ is bijective if and only if it is injective. Thus $u \in X_3$ is a singular point of Fredholm operator F_3 if and only if u is a critical point of F_3. Since Σ_3 is a subset of all critical points of F_3 (see Proposition 2.9), then evidently Σ_3 is a subset of all singular points S_3 of F_3, i.e. $\Sigma_3 \subset S_3$. Hence, the open set of the regular values of F_3

$$R_{F_3} = Y_3 - F_3(S_3) \subset Y_3 - F_3(\Sigma_3) \subset Y_3 - R_{3b}.$$

By Theorem 4.1(b) and Proposition 2.8, R_{F_3} is a residual set in Y_3. From Proposition 2.3 the operator F_3 is proper. Then again from Proposition 2.8 the set R_{F_3} is dense in Y_3. Applying Lemma 4.1 we get our assertion. □

Recall that the point $u \in X_3$ means *a singular* or *critical* or *regular solution of the mixed problem* (1.1)–(1.3) if it is singular or critical or regular point of the operator F_3, respectively. Also we shall investigate the linear problem in $h \in X_3$ for some $u \in X_3$

$$A_3 h(t,x) + \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}\,[t,x,u(t,x),D_1 u(t,x),\dots D_n u(t,x)]\,D_j h(t,x) = g(t,x) \tag{5.1}$$

with the conditions (1.2), (1.3).

Theorem 5.2. *Assume that the hypotheses* (A_3.1), (A_3.2), (N_3.2), (N_3.3) *and* (F_3.1) *hold. Then*

(a) *For any compact set of* Y_3 (*of the right hand sides* $g \in Y_3$ *of the equation* (1.1)) *the set of all corresponding solutions of the initial-boundary value problem* (1.1)–(1.3) *is compact.*

(b) *The number of solutions of* (1.1)–(1.3) *is constant and finite* (*it may be zero*) *on each connected component of the open set* $Y_3 - F(S_3)$, *i.e. for any* g *belonging to the same connected component of* $Y_3 - F_3(S_3)$. *Here* S_3 *means the set of all critical points of problem* (1.1)–(1.3).

(c) *Let* $u_0 \in X_3$ *is regular solutions of* (1.1)–(1.3) *with the right hand side* $g_0 \in Y_3$. *Then there exists a neighbourhood* $U(g_0) \subset Y_3$ *of* g_0 *such that for any* $g \in U(g_0)$ *the initial-boundary value problem* (1.1)–(1.3) *has one and only one solution* $u \in X_3$. *This solution continuously depends on* g.

The associated linear problem (5.1), (1.2), (1.3) *for* $u = u_0$ *has a unique solution* $h \in X_3$ *for any* g *from a neighbourhood* $U(g_0)$ *of* $g_0 = F_3(u_0)$. *This solution continuously depends on* g.

(d) *Denote by* G_3 *the set of all right hand side* $g \in Y_3$ *of equation* (1.1) *for which the corresponding solutions* $u \in X_3$ *of the problem* (1.1)–(1.3) *are its critical solutions. Then* G_3 *is closed and nowhere dense in* Y_3.

(e) *If the singular points set of the initial-boundary value problem* (1.1)–(1.3) *is empty, then this problem has unique solution* $u \in X_3$ *for each* $g \in Y_3$. *It continuously depends of the right hand side* g.

Proof. By the given hypotheses we obtain the assertions (1)–(4) from Theorem 3.1.

With respect to assertion (jj) of Proposition 2.4 the operator F_3 is proper, what implies (a).

In the proof of Theorem 5.1 we have showed that the set of all singular points of F_3 is equal to the set of all critical points of F_3. Then the assertion (b) follows from Proposition 2.6 (Ambrosetti).

(c) Since $u_0 \in X_3 - S_3$, where S_3 is a set of all singular (under our assumptions all critical) points, then according to Proposition 2.9 the mapping F_3 is a local homeomorphism at u_0, which proves the first part of (c).

However, F_3 is a local C^1-diffeomorphism. Thus $F_3' \in C(X_3, Y_3)$, where

$$F_3'(u)h = A_3 h + \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}[t, x, u, D_1 u, \dots D_n u] D_j h$$

and $(F_3^{-1})' \in C(Y_3, X_3)$, where $(F_3^{-1})'(F_3 u) = [F_3'(u)]^{-1}$ for every $u \in X_3$ (see [8, p. 115]). Hence the linear problem (5.1), (1.2), (1.3) for $u = u_0$ has a unique solution $h \in X_3$ for any g from a neighbourhood $U(g_0)$ of $g_0 = F_3(u_0)$. This solution continuously depends of the right hand side g. The proof of (c) is completed.

(d) In our case the set of all singular points S_3 of F_3 is equal to the set of all critical point F_3 and $G_3 = F_3(S_3)$. We get (d) from the Proposition 2.8 (Smale, Quinn).

(e) By Proposition 2.9, the operator $F_3 : X_3 \to Y_3$ is locally C^1-diffeomorphism at any point $u \in X_3$, i.e. it is C^1-deffeomorphism on X_3. Hence we get the last assertion. □

Corollary 5.1. *Let the hypothesis of Theorem* 5.2 *hold and*

(i) *the linear homogeneous problem* (5.1), (1.2), (1.3) (*for* $g = 0$) *has only zero solution* $h = 0 \in X_3$ *for any* $u \in X_3$.

Then the initial-boundary value nonlinear problem (1.1)–(1.3) *has a unique solution* $u \in X_3$ *for any* $g \in Y_3$. *This solution* u *is continuously depend of* g. *Moreover, linear problem* (5.1), (1.2), (1.3) *has a unique solution* $h \in X_3$ *for any* $u \in X_3$ *and right hand side* $g \in Y_3$ *of* (5.1) *and this solution continuously depends on* g.

The proof of Corollary 5.1 follows by (c) of Theorem 5.2.

Corollary 5.2. *Let the hypothesis of Theorem* 5.2 *hold. Then*

(f) *If* $S_3 \neq \emptyset$, *then* $\partial F_3(X_3 - S_3) \subset F_3(S_3)$.

(g) *If* $F_3(S_3) \subset F_3(X_3 - S_3)$ *then the problem* (1.1)–(1.3) *has the solution* $u \in X_3$ *for any* $g \in Y_3$, *i.e.* $R(F_3) = Y_3$, (F_3 *is a surjectivity* X_3 *onto* Y_3).

(h) *If* $Y_3 - F_3(S_3)$ *is connected and* $X_3 - S_3 \neq \emptyset$, *then* $R(F_3) = Y_3$ (*the surjectivity of* F_3 *or the solvability of* (1.1)–(1.3) *for any* $g \in Y_3$).

Proof. By (f) of Theorem 4.2 and (d) of Theorem 5.2 the sets $F_3(X_3)$ and $F_3(S_3)$ are closed and $F_3(X_3 - S_3)$ is open. Hence we have the relation

$$F_3(X_3) = F_3(S_3) \cup F_3(X_3 - S_3) = F_3(S_3) \cup \overline{F_3(X_3 - S_3)} = \overline{F_3(X_3)} \tag{5.2}$$

which is similar to (4.2).

(f) Since $F \in C^1(X_3, Y_3)$, such as in Theorem 5.1 we get $\Sigma_3 \subset S_3$. Hence and from Theorem 4.2(j)

$$\partial F(X_3 - S_3) \subset \partial F(X_3 - \Sigma_3) \subset F(\Sigma_3) \subset F(S_3).$$

(g) From the first equation of (5.2) we have $F_3(X_3) = F_3(X_3 - S_3)$ and so $R(F_3)$ is an open as well as a closed subset of the connected space Y_3. Thus $R(F_3) = Y_3$.

(h) Since $Y_3 - F_3(S_3)$ is connected, then by Amhnesetti theorem (Proposition 2.6) we obtain that card $F_3^{-1}(\{g\})$ =const=: $k \geq 0$ for each $g \in Y_3 - F_3(S_3)$.

If $k = 0$, then $F_3(X_3) = F_3(S_3)$ and $F(X_3 - S_3) \subset F(S_3)$. This is a contradiction with $X_3 - S_3 \neq \emptyset$. Hence $k > 0$. Then $R(F_3) = Y_3$. □

Theorem 5.3. *Suppose that the hypotheses* (A_3.1), (A_3.2), (N_3.2), (N_3.3) *and* (F_3.1) *hold together with the condition*

(i) *Each point $u \in X_3$ is either a regular point or an isolated critical point of problem* (1.1)–(1.3).

Then to each $g \in Y_3$ there exists one solution $u \in X_3$ of the problem (1.1)–(1.3) *and it is continuously depends on g.*

Proof. The associated operator $F_3 : X_3 \to Y_3$ is a proper C^1-Fredholm mapping of the zero index. By Proposition 2.9 and 2.10 F_3 is a locally homeomorphic mapping of X_3 into Y_3, and Proposition 2.5 (the global inversion theorem) implies the statement of this theorem. □

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