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GLOBAL EXISTENCE AND BLOW-UP RESULTS FOR AN EQUATION OF KIRCHHOFF TYPE ON $\mathbb{R}^N$

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Abstract. We discuss the asymptotic behaviour of solutions for the nonlocal quasilinear hyperbolic problem of Kirchhoff Type

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t = |u|^a u, \quad x \in \mathbb{R}^N,\ t \geq 0,$$

with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, in the case where $N \geq 3$, $\delta \geq 0$ and $(\phi(x))^{-1} = g(x)$ is a positive function lying in $L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. When the initial energy $E(u_0,u_1)$, which corresponds to the problem, is non-negative and small, there exists a unique global solution in time. When the initial energy $E(u_0,u_1)$ is negative, the solution blows-up in finite time. A combination of the modified potential well method and the concavity method is widely used.

1. Introduction

In this work we study the following degenerate nonlocal quasilinear wave equation of Kirchhoff type with a weak dissipative term

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \Delta u + \delta u_t = |u|^a u, \quad x \in \mathbb{R}^N,\ t \geq 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

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with initial conditions u_0, u_1 in appropriate function spaces, $N \geq 3$, and $\delta \geq 0$. Throughout the paper we assume that the functions ϕ and $g : \mathbb{R}^N \to \mathbb{R}$ satisfy the following condition:

$(\mathcal{G})$ $\quad \phi(x) > 0,$ for all $x \in \mathbb{R}^N$ and $(\phi(x))^{-1} =: g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

The original equation is

$$ph\frac{\vartheta^2 u}{\vartheta t^2} + \delta\frac{\vartheta u}{\vartheta t} = \left\{p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\vartheta u}{\vartheta x}\right)^2 dx\right\}\frac{\vartheta^2 u}{\vartheta x^2} + f, \tag{1.3}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, p the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modulus and f the external force. When $p_0 = 0$ the equation is considered to be of *degenerate type*, otherwise it is of *nondegenerate type*. When $\delta = f = 0$, the equation was introduced by G. Kirchhoff [13] in the study of oscillations of stretched strings and plates. That's why equation (1.3) is called the *Kirchhoff string*.

In the case of *bounded domain*, when $\delta = 0$ and $f \neq 0$, the global existence is rather well studied in the class of analytic function spaces (e.g. see [6], [31]). H. Crippa [4] has proved local in time solvability in the class of usual Sobolev spaces (see also [33]). A. Arosio and S. Garavaldi [1] have shown the existence of a unique local solution in the case of mildly degenerate type. For $\delta \geq 0$ and $f(u) = 0$, in the degenerate case, the global existence of solutions has been shown by K. Nishihara and Y. Yamada [25], when the initial data are small enough. When $\delta > 0$ and $f(u) = 0$, M. Nakao [18] has derived decay estimates for the solutions (see also [17], [22], [29]). In particular, T. Kobayashi [14] constructed a unique weak solution by a Faedo-Galerkin method for a quasilinear wave equation with strong dissipation (see also [5], [20]). K. Nishihara [23] has derived a decay estimate from below of the potential of solutions. In the case of $\delta \geq 0$ and $f \neq 0$, M. Hosoya and Y. Yamada [8] have studied the non-degenerate case with linear dissipation and proved the global existence of a unique solution under small initial data. Concerning decay properties of solutions, K. Nishihara and K. Ono [24] studied cases of non-degenerate and degenerate type. Also R. Ikehata [9] has shown that for sufficiently small initial data, global existence can be obtained, even when the influence of the source terms is stronger than that of the damping terms. Finally, K. Ono [26]–[28], for $\delta \geq 0$, has proved global existence, decay estimates and blow up results for a (mildly) degenerate non-linear wave equation of Kirchhoff type with strong dissipation.

In the case of *unbounded domain*, P. D'Ancona and S. Spagnolo [7] have shown the global existence of a unique C^∞ solution for the non-degenerate type with small C_0^∞ initial data. Recently, G. Todorova [32] studied the global existence and nonexistence of solutions both in bounded and unbounded domains with nonlinear damping and small enough C_0^∞ initial data. Finally, N. Karahalios and N. Stavrakakis [10]–[12] have studied global existence, blow-up and asymptotic behaviour of solutions for some semilinear wave equations with weak damping on all $\mathbb{R}^N$.

The presentation of this paper has as follows: In Section 2, in order to overcome difficulties on non-compactness arising from the unboundedness of the domain, we discuss properties of the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and some weighted L^p spaces. In Section 3, we show the existence of a unique local weak solution of the problem (1.1)–(1.2) with $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\delta > 0$, by applying the Banach contraction mapping principle. In Section 4, we are able, (only for $N = 3$) to construct a unique global (weak) solution for the problem (1.1)–(1.2) and derive decay properties of it, when $\delta > 0$ and the initial energy is non-negative and small. To this end we use a *modified potential well technique*. In Section 5, by exploring a *concavity argument*, we show blowing up of the local solution of (1.1)–(1.2) under the assumption that the initial energy is negative.

Notation. We denote by B_R the open ball of $\mathbb{R}^N$ with center 0 and radius R. Sometimes for simplicity we use the symbols C_0^∞, $\mathcal{D}^{1,2}$, L^p, $1 \leq p \leq \infty$, for thespaces $C_0^\infty(\mathbb{R}^N)$, $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$, respectively; $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$, where in case of $p = 2$ we may omit the index.

2. Preliminary results

In this section, we briefly mention some facts, notation and results, which will be used later in this paper. The space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the *energy norm* $\|u\|_{\mathcal{D}^{1,2}} =: \int_{\mathbb{R}^N} |\nabla u|^2 dx$. It is known that

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2N/(N-2)}(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\}$$

and $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is embedded continuously in $L^{2N/(N-2)}(\mathbb{R}^N)$, that is, there exists $k > 0$ such that

$$\|u\|_{2N/(N-2)} \leq k\|u\|_{\mathcal{D}^{1,2}}. \tag{2.1}$$

We shall frequently use the following version of the *generalized Poincaré's inequality*

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \alpha \int_{\mathbb{R}^N} g u^2 dx, \tag{2.2}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha =: k^{-2}\|g\|_{N/2}^{-1}$ (see [3, Lemma 2.1] It is shown that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. The space $L_g^2(\mathbb{R}^N)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product

$$(u,v)_{L^2_g(\mathbb{R}^N)} =: \int_{\mathbb{R}^N} guv\,dx. \tag{2.3}$$

It is clear that $L^2_g(\mathbb{R}^N)$ is a separable Hilbert space. Moreover, we have the following compact embedding.

Lemma 2.1. *Let $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then the embedding $\mathcal{D}^{1,2} \subset L^2_g$ is compact.*

Proof. For the proof we refer to [2] (see also [12, Lemma 2.1]). □

The following lemmas will be proved to be useful in the sequel. For the proofs we refer to [12].

Lemma 2.2. *Let $g \in L^{2N/(2N-pN+2p)}(\mathbb{R}^N)$. Then the following continuous embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^p_g(\mathbb{R}^N)$ is valid, for all $1 \le p \le 2N/(N-2)$.*

Remark 2.3. The assumption of Lemma 2.2 is satisfied under the hypothesis $(\mathcal{G})$, if $p \ge 2$.

Lemma 2.4. *Let g satisfy condition $(\mathcal{G})$. If $1 \le q < p < p* = 2N/(N-2)$, then the following weighted inequality*

$$\|u\|_{L^p_g} \le C_0 \|u\|^{1-\theta}_{L^q_g} \|u\|^\theta_{\mathcal{D}^{1,2}} \tag{2.4}$$

is valid, for all $\theta \in (0,1)$, for which $1/p = (1-\theta)/q + \theta/p^*$, and $C_0 = k^\theta$.

To study the properties of the operator $-\phi\Delta$, we consider the equation

$$-\phi(x)\Delta u(x) = \eta(x), \quad x \in \mathbb{R}^N, \tag{2.5}$$

without boundary conditions. Since for every u, $v \in C_0^\infty(\mathbb{R}^N)$ we have

$$(-\phi\Delta u, v)_{L^2_g} = \int_{\mathbb{R}^N} \nabla u \nabla v\,dx, \tag{2.6}$$

we may consider equation (2.5) as an operator equation of the form

$$A_0 u = \eta,\ \ A_0 : D(A_0) \subseteq L^2_g(\mathbb{R}^N) \to L^2_g(\mathbb{R}^N), \quad \eta \in L^2_g(\mathbb{R}^N). \tag{2.7}$$

Relation (2.6) implies that the operator $A_0 = -\phi\Delta$ with domain of definition $D(A_0) = C_0^\infty(\mathbb{R}^N)$, is symmetric. From (2.2) and equation (2.6) we have that

$$(A_0 u, u)_{L^2_g} \ge \alpha \|u\|^2_{L^2_g} \quad \text{for all } u \in D(A_0). \tag{2.8}$$

So the operator $A_0 = -\phi\Delta$ is a symmetric, strongly monotone operator on $L^2_g(\mathbb{R}^N)$. Hence, Friedrich's extension theorem Theorem 19.C [34] is applicable. The *energy scalar product* given by (2.6) is

$$(u,v)_E = \int_{\mathbb{R}^N} \nabla u \nabla v\,dx$$

and the *energy space* is the completion of $D(A_0)$ with respect to $(u,v)_E$. It is obvious that the energetic space X_E is the homogeneous Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The *energy extension* $A_E = -\phi\Delta$ of A_0,

$$-\phi\Delta : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.9}$$

is defined to be the duality mapping of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. We define $D(A)$ to be the set of all solutions of equations (2.5), for arbitrary $\eta \in L^2_g(\mathbb{R}^N)$. *Friedrich's extension* A of A_0 is the restriction of the energetic extension A_E to the set $D(A)$. The operator $A = -\phi\Delta$ is self-adjoint and therefore graph-closed. Its domain $D(A)$, is a Hilbert space with respect to the graph scalar product

$$(u,v)_{D(A)} = (u,v)_{L^2_g} + (Au, Av)_{L^2_g} \quad \text{for all } u, v \in D(A).$$

The norm induced by the scalar product is

$$\|u\|_{D(A)} = \left\{ \int_{\mathbb{R}^N} g|u|^2\,dx + \int_{\mathbb{R}^N} \phi|\Delta u|^2\,dx \right\}^{1/2},$$

which is equivalent to the norm

$$\|Au\|_{L^2_g} = \left\{ \int_{\mathbb{R}^N} \phi|\Delta u|^2\,dx \right\}^{1/2}.$$

So we have established the evolution triple

$$D(A) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^2_g(\mathbb{R}^N) \subset \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{2.10}$$

where all the embeddings are dense and compact. Finally, for later use, it is necessary to remind that the eigenvalue problem

$$-\phi(x)\Delta u = \mu u, \quad x \in \mathbb{R}^N, \tag{2.11}$$

has a complete system of eigensolutions $\{w_n, \mu_n\}$ satisfying the following properties

$$\begin{cases} -\phi\Delta w_j = \mu_j w_j, & j = 1,2,\ldots, \quad w_j \in \mathcal{D}^{1,2}(\mathbb{R}^N), \\ 0 < \mu_1 \le \mu_2 \le \ldots, & \mu_j \to \infty, \quad \text{as } j \to \infty. \end{cases} \tag{2.12}$$

In order to clarify the kind of solutions we are going to obtain for the problem (1.1)–(1.2), we give the definition of the *weak solution* for this problem.

Definition 2.5. A *weak solution* of the problem (1.1)–(1.2) is a function u such that

(i) $u \in L^2[0,T; D(A)]$, $u_t \in L^2[0,T;\mathcal{D}^{1,2}(\mathbb{R}^N)]$, $u_{tt} \in L^2[0,T; L^2_g(\mathbb{R}^N)]$,

(ii) for all $v \in C_0^\infty([0,T] \times (\mathbb{R}^N))$, satisfies the generalized formula

$$\int_0^T (u_{tt}(\tau), v(\tau))_{L^2_g} d\tau + \int_0^T \bigg(\|\nabla u(t)\|^2 \int_{\mathbb{R}^N} \nabla u(\tau) \nabla v(\tau)\, dx\, d\tau \bigg) + \delta \int_0^T (u_t(\tau), v(\tau))_{L^2_g} d\tau - \int_0^T (f(u(\tau)), v(\tau))_{L^2_g} d\tau = 0, \tag{2.13}$$

where $f(s) = |s|^a s$, and

(iii) satisfies the initial conditions

$$u(x,0) = u_0(x) \in D(A), \quad u_t(x,0) = u_1(x) \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

3. Existence results

In order to obtain a local existence result for the problem (1.1)–(1.2), we need information concerning the solvability of the corresponding nonhomogeneous linearized problem around the function v, where $(v, v_t) \in C(0,T; D(A) \times \mathcal{D}^{1,2})$ is given, restricted in the sphere B_R.

$$\begin{aligned} &u_{tt} - \phi(x)\|\nabla v(t)\|^2 \Delta u + \delta u_t = |v|^a v, && (x,t) \in B_R \times (0,T), \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x \in B_R, \\ &u(x,t) = 0, && (x,t) \in \partial B_R \times (0,T) \\ &v \in C(0,T; D(A)), && v_t \in C(0,T; \mathcal{D}^{1,2}). \end{aligned} \tag{3.1}$$

Proposition 3.1. *Assume that $u_0 \in D(A)$, $u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $0 \le a \le 4/(N-2)$, then the linear wave equation (3.1) has a unique solution such that*

$$u \in C(0,T; D(A)) \quad \text{and} \quad u_t \in C(0,T; \mathcal{D}^{1,2}).$$

Proof. The proof follows the lines of [12, Proposition 3.1]. The Galerkin method is used, based on the information taken from the eigenvalue problem (2.11). □

Next, we will prove the following theorem

Theorem 3.2. *Assume that $f(u) = |u|^a u$ is a nonlinear C^1-function such that $|f'(u)| \le k_2|u|^a$ and $0 \le \alpha \le 4/(N-2)$, $N \ge 3$. If $(u_0, u_1) \in D(A) \times \mathcal{D}^{1,2}$ and satisfy the nondegenerate condition*

$$\|\nabla u_0\|^2 > 0,$$

then there exists $T = T(\|u_0\|_{D(A)}, \|\nabla u_1\|^2) > 0$ such that problem (1.1)–(1.2) admits a unique local weak solution u satisfying

$$u \in \mathcal{C}(0,T; D(A)), \quad u_t \in \mathcal{C}(0,T; \mathcal{D}^{1,2}).$$

Moreover, at least one of the following statements holds true, either

(i) $T = \infty$, *or*
(ii) $\lim e(u(t)) \equiv \lim(\|u_t(t)\|^2_{\mathcal{D}^{1,2}} + \|u(t)\|^2_{D(A)}) = \infty$, as $t \to T_-$.

Proof. To apply Banach contraction mapping principle, we introduce the two parameter space of solutions

$$X_{T,R} =: \{v \in \mathcal{C}(0,T;D(A)) : v_t \in \mathcal{C}(0,T;\mathcal{D}^{1,2}), v(0) = u_0, \\ v_t(0) = u_1,\ e(v(t)) \le R^2,\ \text{for all } t \in [0,T]\},$$

for any given $T > 0$ and $R > 0$. It is easy to see that $X_{T,R}$ is a complete metric space under the distance

$$d(u,v) =: \sup_{0 \le t \le T} e_1(u(t) - v(t)),\ \text{where } e_1(v) =: \|v_t\|^2_{L^2_g} + \|v\|^2_{\mathcal{D}^{1,2}}.$$

Next, we introduce the non-linear mapping S in the following way. Given $v \in X_{T,R}$ we define $u = Sv$ to be the unique solution of the linear wave equation (3.1). In the sequel we shall show that there exist $T > 0$, $R > 0$ such that the following two conditions are valid

$$\text{(i) } S \text{ maps } X_{T,R} \text{ into itself,} \tag{3.2}$$

$$\text{(ii) } S \text{ is a contraction with respect to the metric } d(\,\cdot\,,\,\cdot\,). \tag{3.3}$$

We set $2M_0 =: \|\nabla u_0\|^2 > 0$ and denote by

$$T_0 =: \sup\{t \in [0,\infty) : \|\nabla v(s)\|^2 > M_0,\ \text{for } 0 \le s \le t\}.$$

Then we have

$$T_0 > 0 \text{ and } \|\nabla v(t)\|^2 \ge M_0 \quad \text{for all } t \in [0,\, T_0]. \tag{3.4}$$

(i) To check (3.2), we multiply (3.1) by $-2\Delta u_t$ (in the sense of the inner product in the space L^2) and integrate over $\mathbb{R}^N$, to get

$$-2\int_{\mathbb{R}^N} \Delta u_t u_{tt}\,dx + 2\|\nabla v\|^2 \int_{\mathbb{R}^N} \phi(x)\Delta u_t \Delta u\,dx \\ - 2\delta \int_{\mathbb{R}^N} \Delta u_t u_t = -2\int_{\mathbb{R}^N} f(v)\Delta u_t. \tag{3.5}$$

So

$$\frac{d}{dt} e^*_2(u) + 2\delta\|\nabla u_t\|^2 = \left(\frac{d}{dt}\|\nabla v\|^2\right)\|u\|^2_{D(A)} - 2(f(v), \Delta u_t), \tag{3.6}$$

where we set

$$e^*_2(u(t)) =: \|\nabla u_t(t)\|^2 + \|\nabla v(t)\|^2 \|u(t)\|^2_{D(A)}.$$

Note that

$$e^*_2(u) \ge \|\nabla u_t\|^2 + M_0\|u\|^2_{D(A)} \ge c_1^{-2} e(u), \tag{3.7}$$

with $c_1 =: (\max\{1, M_0^{-1}\})^{1/2}$. To proceed further, we observe that

$$(3.8)\qquad \bigg(\frac{d}{dt}\|\nabla v\|^2\bigg)\|u\|^2_{D(A)} = 2\int_{\mathbb{R}^N} \Delta v v_t \phi(x) g(x)\,dx \|u\|^2_{D(A)} \\ \leq 2(\|v\|^2_{D(A)})^{1/2}(\|v_t\|^2_{L^2_g})^{1/2}\|u(t)\|^2_{D(A)} \\ \leq c_2 R^2 e_2^*(u),$$

with $c_2 =: 2kc_1^2$, where k is the constant of the embedding $\mathcal{D}^{1,2} \subset L^2_g$. We also have that

$$(3.9)\quad -2(f(v)\Delta u_t) = 2\int_{\mathbb{R}^N} f'(v)\nabla v \nabla u_t\,dx \leq 2k_2\|v\|^a_{L^{Na}}\|\nabla v\|_{L^{2N/(N-2)}}\|\nabla u_t\|,$$

where we used Hölder inequality, with $p^{-1} = 1/N$, $q^{-1} = (N-2)/2N$ and $r^{-1} = 1/2$. Then, from Lemma 2.2 and the embeddings (2.10) we obtain

$$(3.10)\quad \|v\|^a_{L^{Na}} \leq R^a,\ \|\nabla v\|_{L^{2N/(N-2)}} \leq \|v\|_{D(A)} \leq R \quad\text{and}\quad \|\nabla u_t\| \leq e(u)^{1/2}.$$

Using estimates (3.8)–(3.10), we get from equation (3.6) that

$$\frac{d}{dt}e_2^*(u(t)) \leq c_2R^2 e_2^*(u(t)) + c_3R^{a+1}e_2^*(u(t))^{1/2},$$

with $c_3 =: 2k_2c_1$. Hence, from Gronwall's inequality, we get

$$e_2^*(u(t)) \leq \{e_2^*(u(0))^{1/2} + c_3R^{a+1}T\}^2 e^{c_2R^2T}.$$

Thus from estimation (3.7) we obtain

$$(3.11)\qquad e(u) \leq c_1^2\{e_2^*(u(0))^{1/2} + c_3R^{a+1}T\}^2 e^{c_2R^2T} =: C^*_{T,R},$$

for any $t \in [0,T]$, with $T \leq T_0$. Therefore, if we assume that

$$(3.12)\qquad C^*_{T,R} < R^2,$$

then the statement (3.2) is valid.

(ii) We take $v_1, v_2 \in X_{T,R}$ and denote by $u_1 = Sv_1$, $u_2 = Sv_2$. Hereafter we suppose that (3.12) is valid, i.e., $u_1,\ u_2 \in X_{T,R}$. We set $w = u_1 - u_2$. The function w satisfies the following relation

$$w_{tt} - \phi\|\nabla v_1\|^2\Delta w + \delta w_t = \phi\{\|\nabla v_1\|^2 - \|\nabla v_2\|^2\}\Delta u_2 + f(v_1) - f(v_2) \\ w(0) = 0, \quad w_t(0) = 0.$$

Multiplying the previous equation by $2gw_t$ and integrating over $\mathbb{R}^N$ we have

$$(3.13)\quad 2\int_{\mathbb{R}^N} gw_tw_{tt}\,dx - 2\int_{\mathbb{R}^N}\|\nabla v_1\|^2\Delta w w_t\,dx + 2\delta\int_{\mathbb{R}^N} gw_t^2\,dx \\ = 2\{\|\nabla v_1\|^2 - \|\nabla v_2\|^2\}\int_{\mathbb{R}^N}\Delta u_2 w_t\,dx + 2\int_{\mathbb{R}^N} g(f(v_1) - f(v_2))w_t\,dx.$$

Therefore we have

$$\frac{d}{dt}e^*_{v_1}(w) + 2\delta\|w_t\|^2_{L^2_g} = \frac{d}{dt}\|\nabla v_1\|^2\|\nabla w\|^2 + 2\{\|\nabla v_1\|^2 - \|\nabla v_2\|^2\} \times (\Delta u_2, w_t) + 2(f(v_1) - f(v_2), w_t)_{L^2_g} \equiv I_1(t) + I_2(t) + I_3(t), \tag{3.14}$$

where we set $e^*_{v_1}(w(t)) =: \|w_t(t)\|^2_{L^2_g} + \|v_1(t)\|^2_{\mathcal{D}^{1,2}}\|w(t)\|^2_{\mathcal{D}^{1,2}}$. Note that the following estimations are valid

$$e^*_{v_1}(w) \geq \|w_t\|^2_{L^2_g} + M_0\|w\|^2_{\mathcal{D}^{1,2}} \geq c_1^{-2}e_1(w). \tag{3.15}$$

As in (3.8), we observe that

$$I_1(t) \leq c_2R^2e^*_{v_1}(w), \tag{3.16}$$

$$I_2(t) \leq 2(R+R)e(v_1 - v_2)^{1/2}\int_{\mathbb{R}^N}|\Delta u_2||w_t|\,dx. \tag{3.17}$$

For the last term of (3.17), from estimation (3.15), we have that

$$\int_{\mathbb{R}^N}|\Delta u_2||w_t|\phi^{1/2}\phi^{1/2}g\,dx \leq (\|u_2(t)\|^2_{D(A)})^{1/2}(\|w_t(t)\|^2_{L^2_g})^{1/2} < Re_1(w(t))^{1/2} < Rc_1e^*_{v_1}(w)^{1/2}. \tag{3.18}$$

Thus, from (3.17) and (3.18), we derive that

$$I_2(t) \leq c_4R^2e_1(v_1 - v_2)^{1/2}e^*_{v_1}(w(t))^{1/2}, \tag{3.19}$$

where $c_4 =: 4c_1$. Applying the generalized Poincaré's inequality (2.2) and the embeddings (2.10), we obtain

$$\begin{aligned} I_3(t) &\leq 2k_0\alpha^{-1}(\|\nabla v_1\|^a + |\nabla v_2\|^a)\|\nabla(v_1 - v_2)\|\|w_t\|_{L^2_g} \\ &\leq c_6R^ae_1(v_1 - v_2)^{1/2}e^*_{v_1}(w)^{1/2}, \end{aligned} \tag{3.20}$$

where $c_6 =: 4k_0\alpha^{-1}c_1$ and k_0 is a constant derived from the formula of f. From estimates (3.16), (3.19) and (3.20) we obtain the following estimate for the relation (3.14)

$$\frac{d}{dt}e^*_{v_1}(w) \leq c_2R^2e^*_{v_1}(w) + (c_4R^2 + c_6R^a)e_1(v_1 - v_2)^{1/2}e^*_{v_1}(w)^{1/2}.$$

Gronwall's inequality and the fact that $e^*_{v_1}(w(0)) = 0$ imply

$$e^*_{v_1}(w) \leq (c_4R^2 + c_6R^a)^2T^2e^{c_2R^2T}\sup_{0\leq t\leq T}e_1(v_1(t) - v_2(t)). \tag{3.21}$$

Therefore, from (3.11) and (3.21), we get

$$d(u_1, u_2) \leq C_{T,R}d(v_1, v_2), \tag{3.22}$$

where

$$C_{T,R} =: 4\max\left\{1, \frac{\|\nabla u_0\|^{-2}}{2}\right\}R^4T^2(1 + k_0k^2\|g\|_{N/2}R^{a-2})^2e^{2kc_1^2R^2T},$$

by substituting c_1, c_2, c_4, c_6. Therefore, the map S is a contraction if

$$C_{T,R} < 1. \tag{3.22}$$

Let us note that the two inequalities (3.12) and (3.22) are justified at the same time, if the parameter R is sufficiently large and T is sufficiently small. Finally, applying the Banach's fixed point theorem, we obtain the local existence result.

The second statement of Theorem 3.2 is proved by a standard continuation argument. Indeed, let $[0, T)$ be the maximal existence interval on which the solution of the problem (1.1)–(1.2) exists. Suppose that $T < \infty$ and $\lim_{t\to T-} e(u(t)) < \infty$. Then there exists a sequence $\{t_n,\ n = 1, 2, \ldots\}$ and a constant $K > 0$, such that $t_n \to T_-$, as $n \to +\infty$ and $e(u(t_n)) \leq K$, $n = 1, 2, \ldots$ As we have already shown above, for each $n \in \mathcal{N}$ there exists a unique solution of the problem (1.1), (1.2) with initial data $\Big\{u(t_n), u_t(t_n)\Big\}$ on $[t_n, t_n + T^*]$, where $T^* > 0$ depending on K and independent of $n \in \mathcal{N}$. Thus, we canget $T < t_n + T^*$, for $n \in \mathcal{N}$ large enough. This contradicts the maximality of T and the proof of Theorem 3.2 is completed. □

4. Global existence and energy decay

In this section we consider the global existence and energy decay questions of the solution for the initial value problem (1.1), (1.2). First, we multiply equation (1.1) by $2gu_t$ and integrate over $\mathbb{R}^N$ to get

$$\frac{d}{dt}\bigg\{\|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{2}{a+2}\|u(t)\|^{a+2}_{L^{a+2}_g}\bigg\} + 2\delta\|u_t(t)\|^2_{L^2_g} = 0. \tag{4.1}$$

We define as *the energy* of the problem (1.1), (1.2) the quantity

$$E(t) =: E(u(t), u_t(t)) =: \|u_t(t)\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{2}{a+2}\|u(t)\|^{a+2}_{L^{a+2}_g}. \tag{4.2}$$

So equation (4.1) becomes

$$\frac{d}{dt}E(t) + 2\delta\|u_t(t)\|^2_{L^2_g} = 0. \tag{4.3}$$

Also we introduce *the potential* of the problem (1.1), (1.2), as

$$\mathcal{J}(u) =: \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{2}{a+2}\|u(t)\|^{a+2}_{L^{a+2}_g}. \tag{4.4}$$

Hence from equation (4.1) and the definitions (4.2) and (4.4) we have the relation

$$E(t) = \|u_t(t)\|^2_{L^2_g} + \mathcal{J}(u). \tag{4.5}$$

Finally, we introduce a modified version of the *modified potential well* used in [12] (see also [21] and [30]), by

$$\mathcal{W} =: \{u \in D(A) : \mathcal{K}(u) = \|u(t)\|^4_{\mathcal{D}^{1,2}} - \|u(t)\|^{a+2}_{L^{a+2}_g} > 0\} \cup \{0\}. \tag{4.6}$$

Now we give two auxiliary lemmas

Lemma 4.1. *If* $2 < a < 4/(N-2)$*, then* $\mathcal{W}$ *is an open neighborhood of* 0 *in the space* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof. Indeed, since $2 < a < 4/(N-2)$, by Lemma 2.4 and inequality (2.2) we have that

$$\begin{aligned} (4.7)\quad \|u\|_{L_g^{a+2}}^{a+2} &\leq C_0\|u\|_{L_g^2}^{(1-\theta)(a+2)}\|u\|_{\mathcal{D}^{1,2}}^{\theta(a+2)} \\ &\leq C_0\|u\|_{L_g^2}^{(1-\theta)(a+2)}\|u\|_{\mathcal{D}_{1,2}}^{\theta(a+2)-4}\|u\|_{\mathcal{D}^{1,2}}^4 \leq \frac{C_0}{\alpha}\|u\|_{\mathcal{D}^{1,2}}^{a-2}\|u\|_{\mathcal{D}^{1,2}}^4. \end{aligned}$$

Hence, by (4.7), we get

$$\mathcal{K} = \|u\|_{\mathcal{D}^{1,2}}^4 - \|u\|_{L_g^{a+2}}^{a+2} \geq \left(1 - \frac{C_0}{\alpha}\|u\|_{\mathcal{D}^{1,2}}^{a-2}\right)\|u\|_{\mathcal{D}^{1,2}}^4. \tag{4.8}$$

Therefore, if

$$\|u\|_{\mathcal{D}^{1,2}} \leq (k^{-\theta-2}\|g\|_{N/2}^{-1})^{1/(a-2)},$$

then $\mathcal{K}(u) \geq 0$ and 0 is in $\mathcal{W}$. □

Let us note that condition $2 < a < 4/(N-2)$ implies that N may be equal to 3 only.

Lemma 4.2. *If* $u \in \mathcal{W}$, $N = 3$ *and* $a > 2$*, then we have*

$$0 \leq \frac{a-2}{2(a+2)}\|u\|_{\mathcal{D}^{1,2}}^4 \leq \mathcal{J}(u) \leq E(u, u_t). \tag{4.9}$$

Proof. Since $\alpha > 2$, from the definitions (4.4) and (4.6), for any $u \in W$, we have that

$$\begin{aligned} \mathcal{J}(u) &= \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u\|_{L_g^{a+2}}^{a+2} \\ &\geq \frac{1}{2}\|u\|_{\mathcal{D}^{1,2}}^4 - \frac{2}{a+2}\|u\|_{\mathcal{D}^{1,2}}^4 = \frac{a-2}{2(a+2)}\|u\|_{\mathcal{D}^{1,2}}^4. \end{aligned}$$ □

Concerning the time behaviour of the energy we have the following remarks. Integrate equation (4.3) over $[0, t]$, to get

$$E(t) + 2\delta\int_0^t \|u_t(t)\|_{L_g^2}^2\,dx = E(0). \tag{4.10}$$

Let us note that, if $u \in \mathcal{W}$, then $E(u, u_t) \geq 0$. Whereas, if $E(u, u_t) < 0$, then $u \notin \overline{\mathcal{W}}$. From equation (4.3) and definition (4.2) we obtain that

$$(d/dt)E(u, u_t) = -2\delta\|u_t(t)\|_{L_g^2}^2 \leq 0. \tag{4.11}$$

Therefore the energy $E(t)$ is a nonincreasing function of t. Hence we have that

$$E(t) \leq E(0) \quad \text{for every } t \in [0, T). \tag{4.12}$$

The next theorem deals with the global existence and the energy decay properties of the problem (1.1), (1.2).

Theorem 4.3. *Assume that* $N = 3$, $8/3 < a < 4$, $u_0 \in \mathcal{W}(\subset D(A))$ *and* $u_1 \in \mathcal{D}^{1,2}$. *Also suppose that the following inequality holds true*

$$E(0) \leq \left(\frac{1}{C_0\mu_0^{p_1}}\right)^{1/p_2} \quad \text{if } \frac{8}{3} < a < 4 \text{ and } p_2 > 0. \tag{4.13}$$

Then

(a) *for* $p_1 =: (2(a+2)-3a)/2$ and $p_2 =: (3a-8)/8$ *there exists a unique global solution* $u \in \mathcal{W}$ *of the problem* (1.1), (1.2) *satisfying*

$$u \in C([0,\infty); D(A)) \quad \textit{and} \quad u_t \in C([0,\infty); \mathcal{D}^{1,2}(\mathbb{R}^N)). \tag{4.14}$$

(b) *Moreover, this solution obeys the following estimate*

$$\|u_t\|^2_{L^2_g} + d_*^{-1}\|\nabla u\|^4 \leq E(u,u_t) \leq \{E(u_0,u_1)^{-1/2} + d_0^{-1}[t-1]^+\}^{-2}, \tag{4.15}$$

where $d_* =: 2(a+2)/(a-2)$ *and* $d_0 \geq 1$, *that is,*

$$\|\nabla u\|^4 \leq C_*(1+t)^{-1}, \tag{4.16}$$

where C_* *is some constant depending on* $\|u_0\|^4_{\mathcal{D}^{1,2}}$ and $\|u_1\|_{L^2_g}$.

Proof. (a) To show that the local solution given by Theorem 3.2, remains in the modified potential well $\mathcal{W}$, as long as it exists, we shall argue by contradiction. Assume that there exists time $T^* > 0$, such that $u(t) \in \mathcal{W}$, where $0 \leq t < T^*$ and $u(T^*) \in \partial\mathcal{W}$. Then $\mathcal{K}(u(T^*)) = 0$ and $u(T^*) \neq 0$. We multiply equation (1.1) by gu and integrate over $\mathbb{R}^N$ to get the equation

$$\frac{d}{dt}(u(t),u_t(t))_{L^2_g} - \|u_t(t)\|^2_{L^2_g} + \frac{\delta}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_g} + \|u(t)\|^4_{\mathcal{D}^{1,2}} - \int_{\mathbb{R}^N} g(x)|u(t)|^{a+2}\,dx = 0. \tag{4.17}$$

We integrate (4.17) over $[0,t]$, for some $t \in [0,T)$ and get the inequality

$$\delta\|u(t)\|^2_{L^2_g} \leq \delta\|u(0)\|^2_{L^2_g} + 2\left(\frac{\delta}{4}\|u(t)\|^2_{L^2_g} + \frac{1}{\delta}\|u_t(t)\|^2_{L^2_g}\right) + 2(u_0,u_1)_{L^2_g} + 2\int_0^t \|u_t(s)\|^2_{L^2_g}\,ds, \tag{4.18}$$

where we used Young's inequality for $\varepsilon = \delta/2$ in the first term of (4.17). Since $u(t)$ is in $\mathcal{W}$, we integrate equation (4.1) over $[0,t]$ to get

$$\|u_t(s)\|^2_{L^2_g} - \|u_1\|^2_{L^2_g} + \frac{1}{2}\|u(t)\|^4_{\mathcal{D}^{1,2}} - \frac{1}{2}\|u_0\|^4_{\mathcal{D}^{1,2}} + 2\delta\int_0^t \|u_t(s)\|^2_{L^2_g}ds - \frac{2}{a+2}\|u(t)\|^{a+2}_{L^{a+2}_g} + \frac{2}{a+2}\|u_0\|^{a+2}_{L^{a+2}_g} = 0.$$

Therefore, we have the following estimate

$$\frac{1}{2}\|u_t(s)\|^2_{L^2_g} + \delta\int_0^t \|u_t(s)\|^2_{L^2_g}\, ds \le E(0). \tag{4.19}$$

From relations (4.18), (4.19) we get that

$$\|u(t)\|^2_{L^2_g} \le \frac{2}{\delta}\left\{\delta\|u(0)\|^2_{L^2_g} + 2(u_0,u_1)_{L^2_g} + \frac{4}{\delta}E(0)\right\} =: \mu_0^2. \tag{4.20}$$

Using Lemma 2.4 and relation (4.20) we obtain the inequality

$$\begin{aligned}\|u(t)\|^{a+2}_{L^{a+2}_g} &\le C_0\mu_0^{(a+2)(1-\theta)}\|u(t)\|^{(a+2)\theta}_{\mathcal{D}^{1,2}}\\ &\le C_0\mu_0^{(a+2)(1-\theta)}\mathcal{J}(u)^{(a+2)\theta/4-1}\|u(t)\|^4_{\mathcal{D}^{1,2}}\\ &\le C_0\mu_0^{(a+2)(1-\theta)}E(0)^{(a+2)\theta/4-1}\|u(t)\|^4_{\mathcal{D}^{1,2}},\end{aligned} \tag{4.21}$$

where, according to Lemma 2.4, the constants are

$$\begin{aligned}\theta &=: \frac{3a}{2(a+2)},\\ p_1 &=: (a+2)(1-\theta) = \frac{2(a+2)-3a}{2},\\ p_2 &=: \frac{(a+2)\theta}{4} - 1 = \frac{3a-8}{8}.\end{aligned}$$

Thus we have that,

$$\|u(t)\|^{a+2}_{L^{a+2}_g} \le C_0\,\mu_0^{p_1}\,E(0)^{p_2}\,\|u(t)\|^4_{\mathcal{D}^{1,2}}. \tag{4.22}$$

Let the hypothesis (4.13) is satisfied. Then we get that $E(0)^{p_2}\,C_0\,\mu_0^{p_1} \le 1$. Setting $\delta_1 =: E(0)^{p_2}\,C_0\,\mu_0^{p_1}$, for $t = T^*$, the inequality (4.21) implies

$$\begin{aligned}\mathcal{K}(u(T^*)) &= \|u(T^*)\|^4_{\mathcal{D}^{1,2}} - \|u(T^*)\|^{a+2}_{L^{a+2}_g}\\ &\ge \|u(T^*)\|^4_{\mathcal{D}^{1,2}} - \delta_1\,\|u(T^*)\|^4_{\mathcal{D}^{1,2}} = (1-\delta_1)\,\|u(T^*)\|^4_{\mathcal{D}^{1,2}} > 0,\end{aligned} \tag{4.23}$$

and the contradiction is achieved.

(b) To show the decay property of the energy $E(t)$ associated with equation (1.1), for simplicity we assume that $\delta = 1$. Integrating equation (4.3) over $[t, t+1]$, we have

$$2\int_t^{t+1}\|u_t(s)\|^2_{L^2_g}\, ds = E(t) - E(t+1)\ (=: 2D^2(t)). \tag{4.24}$$

Then there exist $t_1 \in [t, t+1/4],\ t_2 \in [t+3/4, t+1]$ such that

$$\|u_t(t_i)\|_{L^2_g} \le 2D(t) \quad \text{for } i = 1,2 \tag{4.25}$$

(see also Proposition 2.3 in [27]). Multiplying equation (1.1) by u_t and integrating over $\mathbb{R}$, we have that

$$\mathcal{K} = \|u_t(t)\|^2_{L^2_g} - \frac{d}{dt}(u(t), u_t(t))_{L^2_g} - (u(t), u_t(t))_{L^2_g}. \tag{4.26}$$

Integrating (4.26) over $[t_1, t_2]$, it follows from (4.23), (4.24) and (4.25) that

$$\begin{aligned}\frac{1}{2}\int_{t_1}^{t_2} \|u(s)\|^4_{\mathcal{D}^{1,2}}\, ds &\le \int_{t_1}^{t_2} \mathcal{K}(u(s))\, ds \le \int_t^{t+1} \|u_t(s)\|^2_{L^2_g}\, ds \\ &+ \bigg\{\bigg(\int_t^{t+1} \|u_t(s)\|^2_{L^2_g}\, ds\bigg)^{1/2} + \sum_{i=1}^{2} \|u_t(t_i)\|_{L^2_g}\bigg\} \sup_{t\le s\le t+1} \|u(s)\|_{L^2_g} \\ &\le D^2(t) + 5D(t)\alpha^{-1}(d_* E(t))^{1/4},\end{aligned} \tag{4.27}$$

where $d_* =: 2(a+2)/(a-2)$ and the Lemma 4.2 is used in the last inequality. Then we have from (4.5), (4.24) and (4.27) that

$$\begin{aligned}\int_{t_1}^{t_2} E(s)\, ds &\le \int_{t_1}^{t_2} \{\|u_t(s)\|^2_{L^2_g}\, ds + \|u(s)\|^4_{\mathcal{D}^{1,2}}\, ds\} \\ &\le D^2(t) + 2(D^2(t) + 5D(t)\alpha^{-1}(d_* E(t))^{1/4}) \\ &= 3D^2(t) + 10\alpha^{-1}(d_*)^{1/4} D(t) E(t)^{1/4}.\end{aligned} \tag{4.28}$$

On the other hand, integrating (4.3) over $[t, t_2]$, from (4.24) and (4.28) we obtain that

$$\begin{aligned}E(t) &= E(t_2) + 2\int_t^{t_2} \|u_t(s)\|^2_{L^2_g}\, ds \\ &\le 2\int_{t_1}^{t_2} E(s)\, ds + 2\int_t^{t+1} \|u_t(s)\|^2_{L^2_g}\, ds \\ &\le 2(3D^2(t) + 10\alpha^{-1}(d_*)^{1/4} D(t) E(t)^{1/4}) + 2D^2(t) \\ &\le 8D^2(t) + \frac{\varepsilon_1}{2} d_*^2 (20\alpha^{-1} D(t))^{4/3} + (2\varepsilon_1)^{-1} E(t),\end{aligned}$$

where Young's inequality is used for $p^{-1} = 3/4$ and $q^{-1} = 1/4$. Hence

$$E(t) \le 2(8D^{2/3}(t) + d_*^2(20\alpha^{-1})^{4/3}) D^{4/3}(t). \tag{4.29}$$

Since $2D^2(t) = E(t) - E(t+1) \le E(t) \le E(0) (\le 1)$, it follows from (4.29) that

$$E(t) \le 2\{8(E(0)/2)^{1/3} + d_*^2(20\alpha^{-1})^{4/3}\} D^{4/3}(t) = C_5 D^{4/3}(t), \tag{4.30}$$

where $C_5 =: 2\{8(E(0)/2)^{1/3} + d_*^2(20\alpha^{-1})^{4/3}\}$. Also from relation (4.24) we have that

$$D^{4/3}(t) = 2^{-2/3}(E(t) - E(t+1))^{2/3}. \tag{4.31}$$

Thus from (4.31), relation (4.30) becomes

$$E^{3/2}(t) \le 2^{-1} C_5^{3/2} \{E(t) - E(t+1)\}. \tag{4.32}$$

Next we will use the following Lemma (for the proof see Lemma 2.2 in [27]) and [19].

Lemma 4.3. *Let φ be a non-increasing and non-negative function on* $[0,\infty)$ *satisfying*

$$\sup_{t\le s\le t+1}\varphi(s)^{1+r}\le k\{\varphi(t)-\varphi(t+1)\},$$

for $r>0$ and $k>0$. *Then*

$$\varphi(t)\le\{\varphi(0)^{-r}+rk^{-1}[t-1]^{+}\}^{-1/r},\quad \text{for } r\ge 0.$$

Thus, applying Lemma 4.3 we can get the decay estimate of the energy $E(t)$, such that

$$E(t)\le\{E(0)^{-1/2}+d_0^{-1}[t-1]^{+}\}^{-2}, \tag{4.33}$$

with some positive constant d_0 given by

$$d_0=:2^{3/2}[8(E(0)/2)^{1/3}+d_*^2(20\alpha^{-1})^{4/3}]^{3/2}(\ge 1). \tag{4.34}$$

Hence,

$$\|\nabla u\|^4\le C_*(1+t)^{-1}, \tag{4.35}$$

with some constant $C_*\ge 1$ depending on $\|u_0\|^4_{\mathcal{D}^{1,2}}$ and $\|u_1\|_{L^2_g}$. The proof of Theorem 4.3 is now completed. □

Blow-up results

In this section we consider the blowing-up property of the solution of the initial value problem (1.1)–(1.2). To show blow-up of the solution, we adapt to our case the *concavity method*, introduced by Levine in [15] and [16]. The concavity method is based on the constructionand the properties of the two functionals $P(t)$ and $R(t)$.

$$P(t)=:\|u(t)\|^2_{L^2_g}+\delta\left\{\int_0^t\|u(s)\|^2_{L^2_g}\,ds+(T_0-t)\|u_0\|^2_{L^2_g}\right\}+r(t+\tau)^2, \tag{5.1}$$

$$\begin{aligned}R(t)=:&\left\{\|u(t)\|^2_{L^2_g}+\delta\int_0^t\|u(s)\|^2_{L^2_g}\,ds+r(t+\tau)^2\right\}\\&\times\left\{\|u_t(t)\|^2_{L^2_g}+\delta\int_0^t\|u_t(s)\|^2_{L^2_g}\,ds+r\right\}\\&-\left\{(u(t),u_t(t))_{L^2_g}+\delta\int_0^t(u(s),u_t(s))_{L^2_g}\,ds+r(t+\tau)\right\}^2,\end{aligned} \tag{5.2}$$

where $t\in[0,T_0]$ and T_0,r,τ are positive constants, to be specified latter. Then we have that $P(t)>0$ and

$$P'(t) = 2\bigg\{(u(t), u_t(t))_{L^2_g} + \delta\int_0^t (u(s), u_t(s))_{L^2_g}\,ds + r(t+\tau)\bigg\}, \tag{5.3}$$

$$P''(t) = 2\{(u(t), u_{tt}(t))_{L^2_g} + \|u_t(t)\|^2_{L^2_g} + \delta(u(t), u_t(t))_{L^2_g} + r\}. \tag{5.4}$$

If u is a solution of equation (1.1), then multiplying (1.1) by gu and integrating over $\mathbb{R}^N$ we have

$$(u(t), u_{tt}(t))_{L^2_g} = -\|\nabla u(t)\|^4 - \delta\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_g} + \|u(t)\|^{a+2}_{L^{a+2}_g}. \tag{5.5}$$

Thus combining relations (5.4) and (5.5) we have that

$$P''(t) = 2\{-\|\nabla u(t)\|^4 + \|u(t)\|^{a+2}_{L^{a+2}_g} + \|u_t(t)\|^2_{L^2_g} + r\}. \tag{5.6}$$

On the other hand we observe that $R(t) \geq 0$ and from relations (5.2), (5.3) we get

$$\begin{aligned} R(t) = {} & \{P(t) - \delta(T_0 - T)\|u_0\|^2_{L^2_g}\} \\ & \times \bigg\{\|u_t(t)\|^2_{L^2_g} + \delta\int_0^t \|u_t(s)\|^2_{L^2_g}\,ds + r\bigg\} - \frac{1}{4}P'(t)^2, \end{aligned}$$

or

$$\begin{aligned} P'(t)^2 = 4\bigg[& \{P(t) - \delta(T_0 - t)\|u_0\|^2_{L^2_g}\} \\ & \times \bigg\{\|u_t(t)\|^2_{L^2_g} + \delta\int_0^t \|u_t(s)\|^2_{L^2_g}\,ds + r\bigg\} - R(t)\bigg]. \end{aligned} \tag{5.7}$$

Hence from relation (5.7) we get

$$\begin{aligned} & P(t)P''(t) - \left(\frac{a}{4} + 1\right)P'(t)^2 \geq \\ & \qquad P(t)\bigg[P''(t) - (a+4) \times \bigg\{\|u_t\|^2_{L^2_g} + \delta\int_0^t \|u_t\|^2_{L^2_g}\,ds + r\bigg\}\bigg]. \end{aligned} \tag{5.8}$$

From relations (4.10) and (5.6) we observe that

$$\begin{aligned} & P''(t) - (a+4)\bigg\{\|u_t\|^2_{L^2_g} + \delta\int_0^t \|u_t\|^2_{L^2_g}\,ds + r\bigg\} \\ & \quad \geq -(a+2)\bigg\{\|u_t\|^2_{L^2_g} + E(0) - E(t) - \frac{2}{a+2}\|u\|^{a+2}_{L^{a+2}_g} + r\bigg\} - 2\|\nabla u\|^4 \\ & \quad = -(a+2)\{E(0) + r\} + \frac{a-2}{2}\|\nabla u(t)\|^4. \end{aligned} \tag{5.9}$$

Fixing $r = -E(0) > 0$ relation (5.9) becomes

$$\begin{aligned} & P''(t) - (a+4)\bigg\{\|u_t(t)\|^2_{L^2_g} + \delta\int_0^t \|u_t(s)\|^2_{L^2_g}\,ds + (-E(0))\bigg\} \\ & \qquad \geq \frac{a-2}{2}\|\nabla u(t)\|^4 =: \mathcal{Q}(t). \end{aligned} \tag{5.10}$$

Then, from relations (5.8) and (5.10), we obtain

$$P(t)P''(t) - \left(\frac{a}{4}+1\right)P'(t)^2 \geq P(t)\mathcal{Q}(t) \geq 0, \tag{5.11}$$

which implies the concavity character of the functional $P(t)$, i.e.,

$$(P(t)^{-a/4})'' = -\frac{a}{4}P(t)^{-a/4-2}\left\{P(t)P''(t) - \left(\frac{a}{4}+1\right)P'(t)^2\right\} \leq 0. \tag{5.12}$$

After all these calculations we are ready to state and prove the blow-up result.

Theorem 5.1. *Suppose that $a \geq 2$, $N \geq 3$ and the initial energy $E(u_0,u_1)$ is negative. Then there exists a time T, where*

$$\begin{aligned} 0 < T \leq a^{-2}(-E(u_0,u_1))^{-1}[\{(2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g})^2 \\ + a^2(-E(u_0,u_1))\|u_0\|^2_{L^2_g}\}^{1/2} + 2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g}], \end{aligned} \tag{5.13}$$

such that the (unique) solution of the problems (1.1) and (1.2) *blows-up at time T, i.e.,*

$$\lim_{t\to T-}\|u(t)\|^2_{L^2_g} = \infty. \tag{5.14}$$

Proof. We choose T_0 such that

$$\frac{4P(0)}{aP'(0)} \leq T_0. \tag{5.15}$$

Let us note that $P(0) > 0$ and from (5.1), (5.3) choosing τ sufficiently large, we have $P'(0) > 0$. Since the graph of a concave function always lies below any tangent line of it, we obtain that

$$P(t) \geq \left\{\frac{4P(0)^{a/4+1}}{4P(0) - aP'(0)t}\right\}^{4/a}. \tag{5.16}$$

Therefore, there exists some $T \in (0,T_0]$, such that

$$\lim_{t\to T-}\left\{\|u\|^2_{L^2_g} + \delta\int_0^t \|u\|^2_{L^2_g}\,ds\right\} = \infty, \quad \text{i.e. } \lim_{t\to T-}\|u\|^2_{L^2_g} = \infty,$$

which proves relation (5.14). Finally, we find an upper bound for the blow-up time. To this end, using relations (5.1), (5.3), (taken at $t = 0$) and inequality (5.15) we get,

$$T(\tau) \equiv \frac{2\{\|u_0\|^2_{L^2_g} + (-E(0))\tau^2\}}{a\{(u_0,u_1)_{L^2_g} + (-E(0))\tau\} - 2\delta\|u_0\|^2_{L^2_g}} \leq T_0. \tag{5.17}$$

The suitable value τ_0 of τ for the blow-up, corresponds to the minimum value of $T(\tau)$. Since

$$T'(\tau) = \frac{2aE^2(0)\tau^2 + 4E(0)\tau[2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g}] + 2aE(0)\|u_0\|^2_{L^2_g}}{[a\{(u_0,u_1)_{L^2_g} + (-E(0))\tau\} - 2\delta\|u_0\|^2_{L^2_g}]^2},$$

we get that $T(\tau)$ takes the minimum value on the interval $(0,\infty)$ at the value $\tau=\tau_0$, where

$$\tau_0 \equiv a^{-2}(-E(0))^{-1}[\{(2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g})^2 + a^2(-E(0))\|u_0\|^2_{L^2_g}\}^{1/2} + 2\delta\|u_0\|^2_{L^2_g} - a(u_0,u_1)_{L^2_g}],$$

and the proof of Theorem 5.1 is completed. □

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