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STABILITY OF PRINCIPAL EIGENVALUE OF THE SCHRÖDINGER TYPE PROBLEM FOR DIFFERENTIAL INCLUSIONS

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Abstract. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Denote by $\lambda_1(m)$ the principal eigenvalue of the Schrödinger operator $L_m(u) = -\nabla^2 u - mu$ defined on $H_0^1(\Omega) \cap W^{2,1}(\Omega)$. We prove that $\lambda_1 : L^{3/2}(\Omega) \to \mathbb{R}$ is continuous.

Consider differential inclusion

$$(*) \qquad \begin{cases} -\nabla^2 x \in \mathcal{F}(t,x), \\ x|_{\partial\Omega} = 0, \end{cases}$$

where t runs over bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\Gamma = \partial\Omega$, ∇^2 is Laplace operator in Ω and $\mathcal{F}$ is a Lipschitzean multifunction with a constant $m \in L^p(\Omega)$, i.e.

$$\operatorname{dist}_H(\mathcal{F}(t,x), \mathcal{F}(t,y)) \leq m(t)|x-y|.$$

By a solution $(*)$ we mean a function $x \in H_0^1(\Omega) \cap W^{2,1}(\Omega)$ such that

$$-\nabla^2 x \in \mathcal{F}(t, x(t))$$

for a.e. $t \in \Omega$. In the paper [2] we examined the case $n = 1$ i.e.

$$(**) \qquad \begin{cases} -x'' \in \mathcal{F}(t,x) \qquad \text{for } t \in [0;\pi], \\ x(0) = 0 = x(\pi). \end{cases}$$

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We have proved that if m is sufficiently small then the set of solutions of $(**)$ is an absolute retract. The main tool used in [2] were the spectral properties of the operator $L_m = -\nabla^2 - m$ extended to Sobolev space H_0^1 and in particular the stability property of the principal eigenvalue of the operator L_m, $m \in L^1$. Having this property we were able to renorm L^1, in such a way that the solution set of $(**)$ is the set fixed of points of certain multivalued contraction and then apply the B-C-F theorem [3], [6] on properties of the set of fixed points. Transfering of these methods to the case of $\mathbb{R}^n$ it seems to be possible however it demands thorough study of spectral properties of the operator

$$L_m x = -\nabla^2 x - m \cdot x \quad \text{for } x \in H_0^1.$$

In particular, we need to examine the stability properties of the principal eigenvalue of the operator L_m in dependence on $m \in L^p$ with properly chosen p. We should point out that spectral properties of the operator L_m are well known, in case m is a sufficiently smooth function. The results, known in the literature, concerning the stability of the principal (or others) eigenvalue of the operator L_m seem not to cover our case $m \in L^p$. In this paper we deal with L_m for $t \in \Omega \subset \mathbb{R}^3$, where Ω is bounded domain and $m \in L^{3/2}$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary Γ and $H_0^1(\Omega)$ be a Sobolev space i.e. a completion in the norm

$$\| u \| = (\|\nabla u\|_2 + \|u\|_2)^{1/2},$$

of the space $C_0^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \mid \operatorname{supp} u \subset \Omega\}$ of infinitely many times differentiable functions where $\|u\|_p = \big(\int_\Omega |u|^p\big)^{1/p}$, is a norm in L^p with obvious modification for $p = \infty$. Moreover,

$$W^{2,p} = \{u \in L^p \mid \partial_i \partial_j u \in L^p,\ i,j = 1,2,3\}.$$

Then H_0^1 can be continuously embedded in L^6 and compactly embedded in L^2. The latter means in particular that there exists a constant S such that

$$\left(\int_\Omega |u(t)|^6 \, dt\right)^{1/6} \leq S \| u \| \tag{1}$$

for $u \in H_0^1$. Moreover, for $m \in L^{3/2}$ the space H_0^1 can be continuously embedded in $L^2(m) = \{u \mid u^2 m \in L^1\}$, because from (1) and the Hölder inequality we have

$$\int_\Omega m u^2 \leq \left(\int_\Omega m^{3/2}\right)^{2/3} \left(\int_\Omega |u^2|^3\right)^{1/3} \leq \| m \|_{3/2}^{2/3} S^2 \| u \|^2. \tag{2}$$

Consider a quadratic form

$$D_m[u] = \int_\Omega (|\nabla u|^2 - m u^2) \, dt \tag{3}$$

and let

$$D_m[u,v] = \int_\Omega (\nabla u \nabla v - muv)\, dt \tag{4}$$

be the corresponding bilinear form. It generates the operator L_m by the formula $\langle L_m u, v\rangle$. The previous remarks mean, in particular, that the domain of L_m contains H_0^1. Let

$$H[u] = \int_\Omega u^2(t)\, dt. \tag{5}$$

In case, when $m = 0$ it is known [5], that there exists a number

$$\lambda_1 = \lambda_1(\Omega) := \inf_{0 \neq u \in H_0^1} \frac{D_0[u]}{H[u]} \tag{6}$$

and it is the principal eigenvalue of the Laplace operator $L_0 u = -\nabla^2 u$, for $u \in H_0^1$. Therefore, there exists an eigenfunction $u_1 \in H_0^1$ such that

$$-\nabla^2 u_1 = \lambda_1(\Omega) u_1 \quad \text{for } u_1|_\Gamma = 0. \tag{7}$$

Moreover, there exists $\lambda_1(\Omega) < \lambda_2 \leq \lambda_3 \leq \dots$, such that $\lim_{n\to\infty} \lambda_n = \infty$ and $\lambda_1, \dots, \lambda_n$ are consequtive eigenvalues of L_0. The relation (6) means, in particular, that for arbitrary $u \in H_0^1$ we have

$$\int_\Omega |\nabla u|^2 \geq \lambda_1(\Omega) \int_\Omega |u|^2 \tag{8}$$

and

$$\int_\Omega |\nabla u_1|^2 = \lambda_1(\Omega) \int_\Omega |u_1|^2. \tag{9}$$

We shall show that the operator L_m posses analogous properties for $m \in L^{3/2}$. The most methods used are based on the monograph [5]. We begin with

PROPOSITION 1. *Let $m_n \to m_0$ in $L^{3/2}$. Then for arbitrary $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ such that*

$$\int_\Omega m_n u^2(t)\, dt \leq \varepsilon \int_\Omega |\nabla u|^2\, dt + K_\varepsilon \int_\Omega |u(t)|^2\, dt. \tag{10}$$

PROOF. Let S be a constant such that

$$\left\{\int_\Omega u^6(t)\, dt\right\}^{1/6} \leq S \left\{\int_\Omega [|\nabla u|^2 + |u(t)|^2]\, dt\right\}^{1/2}. \tag{11}$$

Fix $\varepsilon > 0$ and pick N such that

$$\|m_n - m_0\|_{3/2} < \varepsilon/2S^2 \quad \text{for } n > N.$$

Observe that the function $\max_{0\le i\le N}|m_i(t)| \in L^{3/2}$ and therefore

$$\lim_{K\to\infty} \mu\{t \mid \max_{0\le i\le N} |m_i(t)| > K-\varepsilon\} = 0.$$

Applying the Vitali–Hahn–Saks Theorem we conclude that there exists a constant K_ε such that on the set

$$\Omega_\varepsilon = \{t \mid \max_{0\le i\le N} |m_i(t)| > K_\varepsilon - \varepsilon\}$$

is satisfied the following inequality

$$\int_{\Omega_\varepsilon} |m_n(t)|^{3/2}\,dt \le \frac{\varepsilon^{3/2}}{2\sqrt{2}S^3}$$

for $n = 0,1,\ldots$ To see that (10) holds we have to consider two cases.

(a) If $n \le N$, then

$$\begin{aligned}\int_\Omega m_n u^2 &= \int_{\Omega\setminus\Omega_\varepsilon} m_n u^2 + \int_{\Omega_\varepsilon} m_n u^2\\ &\le (K_\varepsilon-\varepsilon)\int_{\Omega\setminus\Omega_\varepsilon} u^2 + \left\{\int_{\Omega_\varepsilon} m_n^{3/2}\right\}^{2/3}\left\{\int_{\Omega_\varepsilon}(u^2)^3\right\}^{1/3}\\ &\le (K_\varepsilon-\varepsilon)\int_\Omega u^2 + \frac{\varepsilon}{2S^2}S^2\left\{\int_\Omega [(u^2)+|\nabla u|^2]\right\}\\ &= \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \left(K_\varepsilon - \frac{\varepsilon}{2}\right)\int_\Omega |u|^2.\end{aligned}$$

(b) For $n > N$ we have

$$\begin{aligned}\int_\Omega m_n u^2 &\le \int_\Omega |m_n - m_0|u^2 + \int_\Omega m_0 u^2\\ &\le \left\{\int_\Omega |m_n-m_0|^{3/2}\right\}^{2/3}\left\{\int_\Omega |u^2|^3\right\}^{1/3} + \int_\Omega m_0 u^2\\ &\le \|m_n - m_0\|_{3/2}\left\{\int_\Omega u^6\right\}^{2\times 1/6} + \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + (K_\varepsilon - \varepsilon)\int_\Omega u^2\end{aligned}$$

and from (a) it can be estimated by

$$\le \frac{\varepsilon}{2S^2}S^2\int_\Omega |\nabla u|^2 + u^2 + \left(K_\varepsilon - \frac{\varepsilon}{2}\right)\int_\Omega u^2 = \varepsilon\int_\Omega |\nabla u|^2 + K_\varepsilon\int_\Omega u^2. \qquad \square$$

Proposition 2. *Let $m_n \to m_0$ in $L^{3/2}$. Then there exists a constant $K > 0$ such that*

$$\int_\Omega m_n u^2(t)\,dt \le \frac{1}{2}\int_\Omega |\nabla u|^2\,dt + K\int_\Omega |u(t)|^2\,dt \tag{12}$$

for $n = 0, 1, \ldots$ *and* $u \in H_0^1$,

$$\| u \|^2 \leq 2D_{m_n}[u] + (2K+1)H[u]; \tag{13}$$

$$\frac{D_{m_n}[u]}{H[u]} \geq \frac{1}{2}\lambda_1(\Omega) - K. \tag{14}$$

Proof. To obtain (12) put in Proposition 1 $\varepsilon = 1/2$ and $K = K_{1/2}$. Observe that from (12) we have

$$\begin{aligned} \frac{1}{2}\| u \|^2 &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 = D_{m_n}[u] + \int_\Omega m_n u^2 - \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}H[u] \\ &\leq D_{m_n}[u] + (K + \frac{1}{2})H[u] \end{aligned}$$

and thus (13).

To prove (14) observe that from (12) it follows that

$$\begin{aligned} D_{m_n}[u] &= \int_\Omega |\nabla u|^2 - \int_\Omega m_n |u|^2 \\ &\geq \int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla u|^2 - K\int_\Omega |u|^2 = \frac{1}{2}D_0[u] - KH[u] \end{aligned}$$

and so

$$D_m[u] \geq \left[\frac{1}{2}\lambda_1(\Omega) - K\right] H[u].$$

Now we divide the last inequlity by $H[u] > 0$ and this yields (14). □

Corollary 1. *Let* $m_n \to m_0$ *in* $L^{3/2}$. *Then there exists constant* $K > 0$ *such that for all* $n = 0, 1, \ldots$

$$\inf_{0 \neq u \in H_0^1} \frac{D_{m_n}[u]}{H[u]} \geq \frac{1}{2}\lambda_1(\Omega) - K > -\infty.$$

Corollary 2. *There exists a constant* C *such that for arbitrary* $u \in H_0^1$ *we have*

$$\langle L_m u, u\rangle \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - C\int_\Omega |u|^2.$$

Corollary 3. *The operator* L_m *is continuous on* H_0^1 *since the quadratic form* D_m *is continuous.*

Proof. Let $u_k \to u_0$ in H_0^1. Then from (12) we have

$$\int_\Omega m(t)(u_k(t) - u_0(t))^2\, dt \to 0,$$

i.e. $u_k \to u_0$ in $L^2(m)$. Therefore $\int_\Omega m(t)(u_k(t))^2\, dt \to \int_\Omega m(t)(u_0(t))^2\, dt$, and hence

$$D_m[u_k] \to D_m[u_0].$$

Similarly $\langle L_m u, v\rangle = D_m[u, v]$ is a continuous bilinear form on H_0^1. □

Proposition 3. *Assume that there exists a function $u_1 \in H_0^1$ such that*

$$\inf_{0\neq u\in H_0^1} \frac{D_m[u]}{H[u]} = \frac{D_m[u_1]}{H[u_1]} = \lambda_1.$$

Then λ_1 and u_1 are, respectively, an eigenvalue and an eigenfunction of L_m, i.e. $-\nabla^2 u_1 \in L^{6/5}$ and

$$(-\nabla^2 - m)u_1 = \lambda_1 u_1.$$

Proof. For any $0 \neq u \in H_0^1$ denote by $F[u] = D_m[u]/H[u]$. Then

$$\inf_{0\neq u\in H_0^1} F[u] = F[u_1] = \lambda_1. \tag{15}$$

Fix $\varphi \in H_0^1$. Then taking into account (15) we have, for arbitrary $\varepsilon \in \mathbb{R}$, an inequality

$$F[u_1 + \varepsilon\varphi] \geq F[u_1]. \tag{16}$$

We shall show that there exists the variation $\frac{d}{d\varepsilon}F[u_1+\varepsilon\varphi]|_{\varepsilon=0}$, it is represented by the Gateaux derivative and it vanishes. Denote by $h(\varepsilon) = F[u_1 + \varepsilon\varphi]$ and notice that h is a differentiable function, since it is a composition of an affine function and the quotient of continuous quadratic forms on H_0^1 (and therefore Frechet's differentiable). It assumes the minimum at $t = 0$, i.e. $h(0) \leq h(\varepsilon)$. Thus from Fermat's Lemma we get $h'(0) = 0$. One can easily check that

$$h'(0) = \{H[u_1]\}^{-1} 2 \left\{ \int_\Omega (\nabla u_1 \nabla\varphi - m u_1 \varphi - \lambda_1 u_1 \varphi) \right\}. \tag{17}$$

Hence

$$D_m[u_1, \varphi] = \lambda_1 \int_\Omega u_1\varphi \quad \text{for all } \varphi \in H_0^1,$$

i.e. $L_m u_1 = \lambda_1 u_1$ and $-\nabla^2 u_1 = \lambda_1 u_1 + m u_1 \in L^6 + L^{3/2}L^6 \subset L^{6/5}$. Indeed,

$$\begin{aligned} DF[u_1]\cdot\varphi &= \{H[u_1]\}^{-1} D_m[u_1,\varphi] - \{H[u_1]\}^{-2} H[u_1,\varphi] D_m[u_1] \\ &= \{H[u_1]\}^{-1}\left\{ D_m[u_1,\varphi] - \lambda_1 \int_\Omega u_1\varphi \right\} \end{aligned}$$

for $u_1 \neq 0$, and so

$$\begin{aligned} \frac{d}{d\varepsilon} h(\varepsilon)\bigg|_{\varepsilon=0} &= DF[u_1+\varepsilon\varphi]\cdot \frac{d}{d\varepsilon}(u_1+\varepsilon\varphi)\bigg|_{\varepsilon=0} \\ &= \{H[u_1]\}^{-1}\left\{ D_m[u_1,\varphi] - \lambda_1 \int_\Omega u_1\varphi \right\}. \end{aligned}$$

□

Proposition 4. *Denote by* $K = \{\varphi \in H_0^1 \mid H[\varphi] = 1\}$*. Then there exists a function* $u_1 \in K$ *such that for every* $u \in K$ *the following inequality*

$$D_m[u_1] \leq D_m[u]$$

holds.

Proof. Put $\lambda_1 = \inf_{u \in K} D_m[u]$. From Proposition 1 we can conclude that $D_m[u] \geq \lambda_1(\Omega)/2 - C$ for arbitrary $u \in K$. Hence

$$\lambda_1 \geq \lambda_1(\Omega)/2 - C > -\infty. \tag{18}$$

Consider a sequence $\{\varphi_k\} \subset K$, $k = 1, \ldots$ such that

$$D_m[\varphi_k] \to \lambda_1 \quad \text{for } k \to \infty. \tag{19}$$

Since from (13) in Proposition 2 we have $\| \varphi_k \|^2 \leq 2D_m[\varphi_k] + (2C + 1)$ then from (17) one can conclude that the sequence $\{\varphi_k\} \subset K$ is bounded in H_0^1. Hence it relatively compact in $L^2(\Omega)$ and, passing to a subsequence, we may assume that $\varphi_k \to u_1$ in L^2. The latter, in particular, means that

$$H[\varphi_k - u_1] \to 0 \quad \text{for } k \to \infty. \tag{20}$$

We shall check that u_1 is a required function. Let us notice that

$$H\left[\frac{\varphi_k - \varphi_l}{2}\right] + H\left[\frac{\varphi_k + \varphi_l}{2}\right] = \frac{1}{2}H[\varphi_k] + \frac{1}{2}H[\varphi_l] = 1 \tag{21}$$

and

$$D_m\left[\frac{\varphi_k - \varphi_l}{2}\right] + D_m\left[\frac{\varphi_k + \varphi_l}{2}\right] = \frac{1}{2}D_m[\varphi_k] + \frac{1}{2}D_m[\varphi_l]. \tag{22}$$

From (21) and (20) we see that

$$H\left[\frac{\varphi_k - \varphi_l}{2}\right] \to 0 \quad \text{when } k, l \to \infty \tag{23}$$

and $H[(\varphi_k + \varphi_l)/2] \to 1$ as $k, l \to \infty$. From the definition of λ_1 one can easily see that for arbitrary k, l we have

$$D_m\left[\frac{\varphi_k + \varphi_l}{2}\right] \geq H\left[\frac{\varphi_k + \varphi_l}{2}\right]\lambda_1.$$

Thus, from (22), we have

$$\liminf_{k,l \to \infty} D_m\left[\frac{\varphi_k + \varphi_l}{2}\right] \geq \lambda_1.$$

Fix $\varepsilon > 0$. Then there exist n_0 such that for $k, l \geq n_0$ the following inequalities

$$H\left[\frac{\varphi_k - \varphi_l}{2}\right] \leq \varepsilon,$$
$$D_m\left[\frac{\varphi_k + \varphi_l}{2}\right] \geq \lambda_1 - \varepsilon,$$
$$D_m[\varphi_k], D_m[\varphi_l] \leq \lambda_1 + \varepsilon$$

hold. Then from (23) and (22) we may see that

$$D_m\left[\frac{\varphi_k - \varphi_l}{2}\right] \leq \lambda_1 + \varepsilon - \lambda_1 + \varepsilon = 2\varepsilon.$$

But this, taking into account (13) in Proposition 1, means that

$$\| \varphi_k - \varphi_l \|^2 \leq (5 + 4C)\varepsilon.$$

Hence $\{\varphi_k\}$ is a sequence Cauchy in H_0^1, so $\varphi_k \to u_1$ in H_0^1 and $u_1 \in H_0^1$. Moreover, from the continuity of D_m (Corollary 3) and (18) we see that $D_m[\varphi_k] \to D_m[u_1] = \lambda_1$. □

Assume that functions $u_1, \dots, u_{k-1} \in H_0^1$ are such that $H[u_j] = 1$, $j = 1, \dots, k-1$ and for $\varphi \in H_0^1$

$$\int_\Omega (\nabla u_j \nabla \varphi - (m(t) + \lambda_j) u_j \varphi) \, dt = 0.$$

Consider the space $L(k) = \operatorname{span}\{u_1, \dots, u_{k-1}\}^\perp$, i.e.

$$L(k) = \left\{ v \in H_0^1 \,\middle|\, \int_\Omega v(t) u_j(t) \, dt = 0, \; j = 1, \dots, k-1 \right\}.$$

Proposition 5. *Denote by $K(k)$ the set $K(k) = \{\varphi \in L(k) \mid H[\varphi] = 1\}$. Then $K(k)$ is closed in H_0^1 and there exists $u_k \in K(k)$ such that*

$$\inf_{0 \neq u \in K(k)} D_m[u] = \lambda_k = D_m[u_k].$$

Moreover, every λ_k, u_k is, respectively, an eigenvalue and a corresponding eigenfunction of the operator L_m, i.e.

$$-\nabla^2 u_k - m u_k = \lambda_k u_k.$$

Proof. The closedness of $K(k)$ in H_0^1 follows from a fact that the convergence in H_0^1 implies the convergence in L^2 and $L(k)$ is closed in the norm topology in L^2. Let $\varphi_l \in K(k)$ be such a sequence that $D_m[\varphi_l] \to_{l \to \infty} \lambda_k$. Similarly as in Proposition 4 we may show that $\{\varphi_l\}$ contains a converging in H_0^1 sebsequence. With no loss of generality we may assume that

$$\varphi_l \to u_k \quad \text{for} \quad l \to \infty$$

in H_0^1. But $H[\varphi_l] = 1$. Then also $H[u_k] = 1$ and

$$D_m[\varphi_l] \xrightarrow[l\to\infty]{} D_m[u_k] = \lambda_k.$$

Analogously, as in Proposition 3, we have, for arbitrary $\varphi \in L(k)$,

$$\frac{dF}{d\varphi}[u_k] = \{H[u_k]\}^{-1} \left\{ \int_\Omega (\nabla u_k \nabla \varphi - m u_k \varphi - \lambda_k u_k \varphi)\, dt \right\} = 0 \tag{24}$$

and thus, for $\varphi \in L(k)$,

$$\int_\Omega -\nabla^2 u_k \varphi = \int_\Omega (m + \lambda_k) u_k \varphi. \tag{25}$$

The latter means that

$$-\nabla^2 u_k - m u_k - \lambda_k u_k \in L(k)^\perp \subset \operatorname{span}\{u_1, \ldots, u_{k-1}\}.$$

Therefore there exist constants $C_1, \ldots, C_{k-1}$ such that

$$L_m u_k - \lambda_k u_k = C_1 u_1 + \ldots + C_{k-1} u_{k-1}. \tag{26}$$

Multiplying both sides of (26) by $u_j \in M(k)$, $j = 1, \ldots, k-1$ and then integrating we get

$$C_j = \int_\Omega (-\nabla^2 u_k - m u_k - \lambda_k u_k) u_j = \langle L_m u_j, u_k \rangle = 0, \quad j = 1, \ldots, k-1.$$

Hence $C_1 = \ldots = C_{k-1} = 0$ and $-\nabla^2 u_k - m u_k - \lambda_k u_k = 0$. □

Now we shall show that the oprator L_m has infinitely many eigenvalues.

Theorem 1. *There exist an nondecreasing sequence of reals* $\lambda_1 \le \ldots \le \lambda_k \le \ldots$ *and a sequence of functions* $u_1, \ldots, u_k, \ldots \in H_0^1$ *such that* $\lim_{k\to\infty} \lambda_k = \infty$ *and for an arbitrary* $\varphi \in H_0^1$ *we have*

$$\int_\Omega (\nabla u_k \nabla \varphi - m u_k \varphi - \lambda_k u_k \varphi)\, dt = 0, \tag{27}$$

$$\int_\Omega u_k u_l\, dt = 0 \quad \text{for } k \ne l, \tag{28}$$

$$\int_\Omega u_k^2 u_l\, dt = 1 \quad \text{for } k \ge 1, \tag{29}$$

$$\int_\Omega [\nabla u_k \nabla u_l - m u_k u_l]\, dt = 0 \quad \text{for } k \ne l, \tag{30}$$

$$\int_\Omega [|\nabla u_k|^2 - m u_k^2]\, dt = \lambda_k \quad \text{for } k \ge 1, \tag{31}$$

Proof. It follows from Proposition 4 that there exist u_1, λ_1 such that for any $\varphi \in H_0^1$ we have the relations

$$\int_\Omega \{\nabla u_1 \nabla \varphi - m u_1 \varphi - \lambda_1 u_1 \varphi\}\, dt = 0,$$

$$H[u_1] = 1, \quad D_m[u_1] = \lambda_1 = \inf_{u \in K_1} \frac{D_m[u]}{H[u]}.$$

Let $S_1 = \operatorname{span}\{u_1\}$. Proposition 5 guarantees the existence of λ_2 and $u_2 \in M_2$ such that for every $\varphi \in S_1^{\perp}$

$$\int_\Omega \{\nabla u_2 \nabla \varphi - m u_2 \varphi - \lambda_2 u_2 \varphi\}\, dt = 0,$$

$$H[u_2] = 1, \quad D_m[u_2] = \lambda_2 = \inf_{u \in S_1^{\perp}} \frac{D_m[u]}{H[u]} \quad \text{and} \int_\Omega u_1(t) u_2(t)\, dt = 0.$$

Denote by $S_2 = \operatorname{span}\{u_1, u_2\}$. Continuing inductively this procedure we have the existence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ and eigenfunctions of the operator L_m and they are orthonormal in L^2.

We shall observe that $\lambda_k \to \infty$ for $k \to \infty$. Assume to a contrary that there is an A such that $\lambda_k \leq A$, for $k \geq 1$. Since $D_m[u_k] = \lambda_k$ then from Proposition 13 we would have

$$\| u_k \| \leq \sqrt{2A + 2C + 1} < \infty$$

for every $k \in \mathbb{N}$. Hence $\{u_k\}$ would be a bounded in H_0^1 sequence, and therefore compact in L^2. Passing to a subsequence we may require that $u_k \to u_0$ in L^2 and so

$$\|u_k - u_l\|_2 \to 0 \quad \text{as } k, l \to \infty.$$

But this is impossible since u_k and u_l are ortonormal in L^2 and

$$\|u_k - u_l\|_2{}^2 = \|u_k\|_2{}^2 + \|u_l\|_2{}^2 = 2 \not\to 0 \quad \text{as} \quad k, l \to \infty.$$

Thus $\lambda_k \to \infty$ for $k, l \to \infty$. □

Remark 1. It easy to observe that for $\{m_n\} \subset L^\infty$, $n = 0, 1, \ldots,$ and $m_n \to m_0$ in L^∞

$$\lambda_1(m_n) \to \lambda_1(m_0).$$

Proof. To see this let u_n be the first eigenfunction corresponding to the eigenvalue $\lambda_1(m_n)$ of L_{m_n} with $\|u_n\|_2 = 1$, $n = 0, 1, \ldots$ Then, from Theorem 1, we have

$$\begin{aligned}\lambda_1(m_n) - \lambda_1(m_0) &\leq \frac{D_{m_n}[u_0]}{H[u_0]} - \lambda_1(m_0) = \frac{D_{m_n}[u_0] - \lambda_1(m_0) H[u_0]}{H[u_0]} \\ &= \frac{D_{m_n}[u_0] - D_{m_0}[u_0]}{H[u_0]} = \int_\Omega (m_n - m_0) u_0^2\, dt \\ &\leq \|m_n - m_0\|_\infty \to 0\end{aligned}$$

for $n \to \infty$. Similarly

$$\lambda_1(m_0) - \lambda_1(m_n) \leq \|m_0 - m_n\|_\infty \|u_n\|_2 \to 0 \quad \text{for } n \to \infty.$$ □

Much more difficult is the case when $\{m_n\} \subset L^{3/2}$, $n = 0, 1, \ldots$ and $m_n \to m_0$ in $L^{3/2}$. The previous way of reasoning demands the boundedness of norms

$\{\|u_n\|\}$. Indeed, let u_n be the first eigenfunction corresponding to the first eigenvalue $\lambda_1(m_n)$ of L_{m_n}, $n = 0, 1, \ldots$ Then, from Theorem 1, we conclude that

$$\begin{aligned}|\lambda_1(m_n)-\lambda_1(m_0)| &\le \sup\left\{\frac{D_{m_n}[u_0]-\lambda_1(m_0)H[u_0]}{H[u_0]}, \frac{D_{m_0}[u_m]-\lambda_1(m_n)H[u_m]}{H[u_m]}\right\}\\ &= \sup\left\{\int_\Omega (m_n - m_0)u_0^2\,dt, \int_\Omega (m_n - m_0)u_n^2\,dt\right\}\\ &\le \sup\{\|m_n - m_0\|_{3/2}\|u_0\|_6, \|m_0 - m_n\|_{3/2}\|u_n\|_6\}.\end{aligned}$$

So boundedness of the $\{\| u_n \|\}$ is needed.

From Proposition 1, see also Corollary 1, it follows that if $m_n \to m_0$ in $L^{3/2}$ then there exists a constant C_0 such that for $n = 0, 1, \ldots$ we have

$$\lambda_1(m_n) \ge C_0,$$

and, for every $0 \ne u \in H_0^1$,

$$D_{m_n}[u] = \int_\Omega (|\nabla u|^2 - m_n u^2)\,dt \ge C_0 H[u].$$

It means that for all $0 \ne u \in H_0^1$ and for all $1 > \varepsilon > 0$

$$\int_\Omega (|\nabla u|^2 - (m_n + C_0 - \varepsilon)u^2)\,dt \ge \varepsilon \int_\Omega u^2\,dt$$

and equivalently

$$\int_\Omega ((1+\varepsilon)|\nabla u|^2 - (m_n + C_0 - \varepsilon)u^2)\,dt \ge \varepsilon \int_\Omega (u^2 + |\nabla u|^2)\,dt$$

or

$$\int_\Omega \left(|\nabla u|^2 - \left(\frac{m_n + C_0 - \varepsilon}{1+\varepsilon}\right)u^2\right)dt \ge \frac{\varepsilon}{1+\varepsilon}\| u \|^2. \tag{32}$$

Denote by

$$p_{n,\varepsilon} = \frac{m_n + C_0 - \varepsilon}{1+\varepsilon} \in L^{3/2}$$

and by

$$D_{n,\varepsilon}[u, v] = \int_\Omega (\nabla u \nabla v - p_{n,\varepsilon})uv\,dt.$$

The inequality (32) means that for every $u \in H_0^1$ we have

$$D_{n,\varepsilon}[u, u] \ge \frac{\varepsilon}{1+\varepsilon}\| u \|^2, \tag{33}$$

hence the form $D_{n,\varepsilon}$ is positively defined. Moreover, $D_{n,\varepsilon}$ are continuous bilinear forms on $H_0^1 \times H_0^1$. Therefore, from the Lax–Milgram theorem, for every $x^\star \in H^{-1}$ and $n = 0, 1, \ldots$ there exists a unique $u_n \in H_0^1$ such that

$$D_{n,\varepsilon}[u_n, v] = \langle x^\star, v\rangle \tag{34}$$

for all $v \in H_0^1$. Notice that since H_0^1 embedds in L^6 thus $L^{6/5}$ embedds in H^{-1}. To see this let us consider, for $f \in L^{6/5}$, a functional $x^\star$ by $\langle x^\star, v\rangle = \int_\Omega f(t)v(t)\,dt$. Observe that for every $v \in H_0^1$, the following inequalities

$$|\langle x^\star, v\rangle| \le \|f\|_{6/5}\|v\|_6 \le S\|f\|_{6/5}\|\, v\,\|$$

hold. The latter implies that $x^\star$ is continuous on H_0^1. Hence, from (34) one can conclude that for every $f \in L^{6/5}$ and $n = 0, 1, \dots$ there exists unique $u_n \in H_0^1$ such that

$$D_{n,\varepsilon}[u_n, v] = \langle f, v\rangle \tag{35}$$

for all $v \in H_0^1$. Moreover, from (32) and (35),

$$\|\, u_n\,\|^2 \le \frac{1+\varepsilon}{\varepsilon} D_{n,\varepsilon}[u_n, u_n] = \frac{1+\varepsilon}{\varepsilon}\langle f, u_n\rangle \le \frac{1+\varepsilon}{\varepsilon}\|f\|_{6/5} S\|\, u_n\,\|.$$

So we get

$$\|\, u_n\,\| \le \frac{1+\varepsilon}{\varepsilon} S\|f\|_{6/5}. \tag{36}$$

Denote by $T_n : L^{6/5} \to H_0^1 \hookrightarrow L^6$ the mapping $T_n f = u_n$, such that for all $v \in H_0^1$ we have

$$D_{n,\varepsilon}[T_n f, v] = \langle f, v\rangle.$$

Obviously all T_n's are linear and inequality (36) means that

$$\|\, T_n\,\| \le (1+\varepsilon)S/\varepsilon, \quad n = 0, 1, \dots$$

Thus operators $T_n : L^{6/5} \to L^6$ are uniformly bounded.

We shall show the following

Lemma 1. *There exists a constant C such that*

$$\|\, T_n f - T_0 f\,\| \le C\|m_n - m_0\|_{3/2}\|f\|_{6/5}$$

for every $f \in L^{6/5}$ and $n = 1, \dots$

Proof. Let us recall that from (35) for all $n = 0, 1, \dots$ it follows that

$$\int_\Omega (\nabla(u_n - u_0)\nabla v - (p_{n,\varepsilon} u_n - p_{0,\varepsilon} u_0)v)dt = 0.$$

Equivalently

$$\int_\Omega (\nabla(u_n - u_0)\nabla v - (1+\varepsilon)^{-1}[(m_n - m_0)u_n + (m_0 + c_0 - \varepsilon)(u_n - u_0)]v)\,dt = 0$$

or

$$D_{0,\varepsilon}[u_n, v] = \langle (1+\varepsilon)^{-1}(m_n - m_0)u_n, v\rangle$$

i.e.

$$u_n - u_0 = T_0((1+\varepsilon)^{-1}(m_n - m_0)u_n).$$

From (36) we conclude that

$$\begin{aligned}\| u_n - u_0 \| &\le \frac{1+\varepsilon}{\varepsilon} S \|(1+\varepsilon)^{-1}(m_n - m_0)u_n\|_{6/5} \\ &\le \frac{1}{\varepsilon} S\|m_n - m_0\|_{3/2}\|u_n\|_6 \le \frac{1+\varepsilon}{\varepsilon^2} S^3 \|m_n - m_0\|_{3/2}\|f\|_{6/5}.\end{aligned}$$

Finally

$$\| T_n f - T_0 f \| \le C\|m_n - m_0\|_{3/2}\|f\|_{6/5},$$

where $C = (1+\varepsilon)S^3/\varepsilon^2$. □

From Lemma 1 it follows that

$$\| T_n - T_0 \| \le C\|m_n - m_0\|_{3/2}$$

and therefore the operators $T_n : L^{6/5} \to H_0^1$ tend to $T_0 : L^{6/5} \to H_0^1$. Moreover, the operators

$$T_n|_{L^2} : L^2 \to L^2$$

are compact. Thus from Lemma VII.6.3 in [4] it follows that

$$\sigma(T_n) \to \sigma(T_0)$$

in the Hausdorff metric. But $\sigma(T_n) \subset [0,\infty)$ and $\sup \sigma(T_n)$ tend to

$$\left(\lambda_1\left(\frac{m_n}{1+\varepsilon}\right) - \frac{C_0 - \varepsilon}{1+\varepsilon}\right)^{-1}.$$

Thus $\lambda_1(m_n/(1+\varepsilon))$ tends to $\lambda_1(m_0/(1+\varepsilon))$ for every $0 < \varepsilon < 1$. Therefore, we have proved the following:

THEOREM 2. *Let $\{m_n\} \subset L^{3/2}$, $n = 0, 1, \ldots$ and $m_n \to m_0$ in $L^{3/2}$. Then*

$$\lambda_1(m_n) \to \lambda_1(m_0).$$

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