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EXISTENCE AND CONVERGENCE RESULTS FOR EVOLUTION HEMIVARIATIONAL INEQUALITIES

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Abstract. In the paper we examine nonlinear evolution hemivariational inequality defined on a Gelfand fivefold of spaces. First we show that the problem with multivalued and L-pseudomonotone operator and zero initial data has a solution. Then the existence result is established in the case when the operator is single valued of Leray–Lions type and the initial condition is nonzero. Finally, the asymptotic behavior of solutions of hemivariational inequality with operators of divergence form is considered and the result on upper semicontinuity of the solution set is given.

1. Introduction

In this paper we study the problem of existence of solutions for evolution hemivariational inequalities driven by multivalued coercive and pseudomonotone operators defined within the framework of an evolution triple of spaces. We also investigate the asymptotic behavior of solutions to parabolic hemivariational inequalities with single-valued nonlinear operators of divergence form.

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The problem under consideration is following

$$\begin{cases} u' + \mathcal{A}u + \partial J(u) \ni f, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is a nonlinear and multivalued operator between $\mathcal{V}$ and $2^{\mathcal{V}^*}$, $f \in \mathcal{V}^*$, $u_0 \in V$, ∂J denotes the Clarke subdifferential (cf. Clarke [4]) of a locally Lipschitz functional J defined on $\mathcal{X}$, $\mathcal{V} = L^p(0,T;V)$, $\mathcal{X} = L^p(0,T;X)$, V is a reflexive Banach space such that $V \subset X$ compactly, $0 < T < \infty$ and $2 \le p < \infty$ (see notation in Section 2). This problem can be considered as a nonlinear evolution inclusion with a nonmonotone multivalued perturbation.

The existence of solutions for hemivariational inequalities in the elliptic case has been investigated by many authors using different methods, see Panagiotopoulos [16], Naniewicz and Panagiotopoulos [15], Haslinger and Panagiotopoulos [6] and the literature therein. The parabolic hemivariational inequalities have been treated only recently by Miettinen [10] who used a regularization technique with the Galerkin method, by Carl [3] and Papageorgiou [19] who both combined the method of upper and lower solutions, the theory of pseudomonotone operators with truncation and penalization techniques. Moreover, Liu [9] has shown an existence result for parabolic hemivariational inequalities with a single-valued evolution operator of class (S_+) while Miettinen and Panagiotopoulos [11] and Migórski and Ochal [14] have studied the problem using a regularized approximating method.

In the present paper we generalize the results mentioned above and we prove a theorem on the existence of solutions to (1.1) using techniques of multivalued analysis and the theory of pseudomonotone operators. The idea of the proof goes back to Lions [8] who delivered a surjectivity result for evolution equations. For the preliminary version of our existence result see Migórski [13].

The second aim of the paper is to give a convergence result for the family (indexed by a parameter h) of parabolic hemivariational inequalities of type (1.1). The index appears in the time-dependent operators $A_h(t) : V \to V^*$, $t \in [0,T]$, $h \in \mathbb{N} \cup \{\infty\}$ of the form $A_h(t) = -\operatorname{div} a_h(x,t,D\,\cdot)$, in the functionals $\mathcal{J}_h : \mathcal{X} \to \mathbb{R}$, in the second member and in the initial condition. The mappings $a_h : \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ are supposed to be maximal monotone on $\mathbb{R}^N$ for a.e. $(x,t) \in \Omega \times (0,T)$ and satisfy suitable boundedness and coerciveness hypotheses. Being motivated by the potential applications to some problems in the homogenization theory, we are interested in a convergence result under the assumption that $a_h \xrightarrow{\mathrm{PG}} a_\infty$, as $h \to \infty$ in the sense of parabolic G-convergence of Svanstedt [22]. Under this hypothesis and other suitable conditions on the data we will show the upper semicontinuity of the solution set. To author's knowledge the related work on the dependence on parameters of the solution set of hemivariational inequalities is not yet seen.

It should be mentioned here that the inclusion (1.1) is of interest because it is a model for nonmonotone semipermeability problem:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \Delta u = f & \text{in } \Omega \times (0,T), \\
f = f_1 + f_2 & -f_1 \in \partial j(x,u) \text{ a.e. } \Omega \times (0,T), \\
u(0) = u_0, & \\
u = 0 & \text{on } \partial\Omega \times (0,T),
\end{cases}
\tag{1.2}
$$

(Ω being a bounded domain in $\mathbb{R}^3$) which arise in electrostatics, heat conduction problems and in the description of the flow of Bingham's fluids. The problem (1.2) can be written in the form (1.1) with $J(v) = \int_\Omega \int_0^T j(x, v(x,t))\,dx\,dt$.

The problems of type (1.2) were considered by Duvaut and Lions in [5], where semipermeability relations were assumed to be monotone and they lead to variational inequalities. The case of nonmonotone semipermeability relations were first studied by Panagiotopoulos in the stationary case in [17] under the name of hemivariational inequalities. For the description of temperature control problems related to (1.2) see Panagiotopoulos [16], Naniewicz and Panagiotopoulos [15], and Migórski [13]. For more motivation coming from nonsmooth/nonconvex mechanics and details concerning applications we refer to Panagiotopoulos [16] and [18], and the references mentioned there.

The outline of this paper is following. In Section 2 we recall basic notation, definitions and preliminary results. In Section 3 we first deal with the existence problem for (1.1) with zero initial data and then we study the case when the operator $\mathcal{A}$ is single-valued of classical Leray–Lions type. The discussion on the convergence of solutions to parabolic hemivariational inequalities is presented in Section 4.

2. Preliminaries

In this section we fix our notation, recall some basic definitions and facts from multivalued analysis and present auxiliary results.

Let V and X be two reflexive separable Banach spaces and let H be a real Hilbert space with $V \subset X \subset H$, where V is dense in X and X is dense in H. The embeddings are assumed to be continuous and V embeds compactly in X. Typically $V = W^{1,p}(\Omega;\mathbb{R}^N)$ or $V = W_0^{1,p}(\Omega;\mathbb{R}^N)$, $X = L^p(\Omega;\mathbb{R}^N)$, $H = L^2(\Omega;\mathbb{R}^N)$ with some $2 \le p < \infty$, Ω being an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary.

Let $2 \le p < \infty$ and $0 < T < \infty$. We introduce the following spaces $\mathcal{V} = L^p(0,T;V)$, $\mathcal{X} = L^p(0,T;X)$, $\mathcal{H} = L^2(0,T;H)$ and $\mathcal{W} = \{w \in \mathcal{V} : w' \in \mathcal{V}^*\}$, where the time derivative involved in the definition of $\mathcal{W}$ is understood in the sense of vector valued distributions. Clearly we have $\mathcal{W} \subset \mathcal{V} \subset \mathcal{X} \subset \mathcal{H} \approx \mathcal{H}^* \subset \mathcal{X}^* \subset \mathcal{V}^*$ with dense and continuous embeddings, $\mathcal{X}^* = L^q(0,T;X^*)$, $\mathcal{V}^* = L^q(0,T;V^*)$ and $1/p + 1/q = 1$. Recall that if for a Banach space Y, Y^* has the Radon–Nikodym property (in particular if Y is a reflexive Banach space or a separable dual Banach space), then we have $L^p(0,T;Y)^* = L^q(0,T;Y^*)$, $1/p + 1/q = 1$ (see for example Hu and Papageorgiou [7], Theorem A.3.98, p. 918). Equipped with the norm $||v||_{\mathcal{W}} = ||v||_{\mathcal{V}} + ||v'||_{\mathcal{V}^*}$ the space $\mathcal{W}$ becomes a separable, reflexive Banach space. The pairing for the pair $(\mathcal{V}, \mathcal{V}^*)$ is denoted by $\langle f, v\rangle_{\mathcal{V}} = \int_0^T \langle f(t), v(t)\rangle \, dt$ and the inner product on the Hilbert space $\mathcal{H}$ by $(f,v)_{\mathcal{H}} = \int_0^T (f(t), v(t))_H \, dt$. Evidently $\langle \cdot\, , \cdot \rangle|_{\mathcal{V}\times\mathcal{H}} = (\cdot\, , \cdot)_{\mathcal{H}}$.

The following lemma is needed in the sequel.

Lemma 2.1.

(a) *Every function from $\mathcal{W}$ is, after an eventual modification on a set of Lebesgue measure zero, continuous from $[0,T]$ into H; moreover, the embedding of $\mathcal{W}$ into $C([0,T];H)$ is continuous.*
(b) *If the embedding of V into X is compact, then so is the embedding of $\mathcal{W}$ into $\mathcal{X}$.*
(c) *If $\{v_n\}_{n\ge 1} \subset \mathcal{W}$, $v_n \xrightarrow{w} v$ in $\mathcal{W}$, then for every $t \in [0,T]$, $v_n(t) \xrightarrow{w} v(t)$ in H.*

Proof. The results in (a) and (b) are standard ones (see for example Lions [8] and Zeidler [23]). We show (c). From (a), we have $v_n \xrightarrow{w} v$ in $C([0,T];H)$. Fix $h \in H$, $t \in [0,T]$ and let $v^* \in C([0,T];H)^*$ be defined by $v^*(y) = (h, y(t))$. Then the result follows easily. □

Let Y be a reflexive Banach space and let $\langle \cdot\, , \cdot \rangle_Y$ denote the natural pairing between Y and its dual Y^*. In what follows we recall some definitions for a multivalued operator $T : Y \to 2^{Y^*}$ (see e.g. [2], [8] and [23]).

- An operator T is said to be pseudomonotone if it satisfies
  (a) for every $y \in Y$, Ty is a nonempty, convex and weakly compact set in Y^*,
  (b) T is u.s.c. from every finite dimensional subspace of Y into Y^* endowed with the weak topology,
  (c) if $y_n \xrightarrow{w} y$ in Y, $y_n^* \in Ty_n$ and $\limsup \langle y_n^*, y_n - y\rangle_Y \le 0$, then for each $z \in Y$ there exists $y^*(z) \in Ty$ such that
$$\langle y^*(z), y - z\rangle_Y \le \liminf \langle y_n^*, y_n - z\rangle_Y.$$
- An operator T is said to be generalized pseudomonotone if for every sequence $(y_n, y_n^*) \in \operatorname{Gr} T$ satisfying $y_n \xrightarrow{w} y$ in Y, $y_n^* \xrightarrow{w} y^*$ in Y^*

and $\limsup \langle y_n^*, y_n - y\rangle_Y \le 0$, we have $(y, y^*) \in \operatorname{Gr} T$ and $\langle y_n^*, y_n\rangle_Y \to \langle y^*, y\rangle_Y$.

- An operator T is said to be of type (M) if (a) and (b) hold, and (d) if $(y_n, y_n^*) \in \operatorname{Gr} T$, $y_n \xrightarrow{w} y$ in Y, $y_n^* \xrightarrow{w} y^*$ in Y^* and

$$\limsup \langle y_n^*, y_n\rangle_Y \le \langle y^*, y\rangle_Y,$$

then $(y, y^*) \in \operatorname{Gr} T$.

Let $L : D(L) \subset Y \to Y^*$ be a linear densely defined maximal monotone operator.

- An operator T is said to be generalized pseudomonotone with respect to $D(L)$ if and only if (a) and (b) hold and
  (e) if $\{y_n\} \subset D(T) \cap D(L)$ is such that $y_n \xrightarrow{w} y$ in Y, $Ly_n \xrightarrow{w} Ly$ in Y^*, $y_n^* \in T(y_n)$, $y_n^* \xrightarrow{w} y^*$ in Y^* and $\limsup \langle y_n^*, y_n\rangle_Y \le \langle y^*, y\rangle_Y$, then $(y, y^*) \in \operatorname{Gr} T$ and $\langle y_n^*, y_n\rangle_Y \to \langle y^*, y\rangle_Y$.
- An operator T is called surjective if for each $f \in Y^*$ there exists an element $y \in D(T)$ such that $f \in Ty$, i.e. $R(T) = Y^*$.
- An operator T is said to be coercive if there exists a function $c : \mathbb{R}^+ \to \mathbb{R}$ with $c(r) \to \infty$ as $r \to \infty$ such that $\langle y^*, y\rangle_Y \ge c(||y||_Y)||y||_Y$ for every $(y, y^*) \in \operatorname{Gr} T$ or equivalently (see [20], p. 115)

$$\frac{\inf\{\langle y^*, y\rangle : y^* \in Ty\}}{||y||_Y} \to \infty, \quad \text{as } ||y|| \to \infty,\ y \in D(T).$$

- An operator T is said to be bounded if it maps bounded sets into bounded sets. T is quasi-bounded if to every $M > 0$ there corresponds a constant $C > 0$ such that whenever $(y, y^*) \in \operatorname{Gr} T$, $\langle y^*, y\rangle \le M||y||_Y$ and $||y||_Y \le M$, it follows that $||y^*||_{Y^*} \le C$.
- A single-valued operator $T : Y \to Y^*$ is said to be hemicontinuous if it is weakly continuous on straight lines in Y, i.e. if $s \mapsto \langle T(u + sv), w\rangle$ is continuous for all $u, v, w \in Y$. It is said to be demicontinuous if it is continuous from Y to Y^* endowed with weak topology.

The following results will be used in the sequel.

Proposition 2.1. *Let $T : Y \to Y^*$ be a pseudomonotone operator. If $y_n \xrightarrow{w} y$ in Y and $Ty_n \xrightarrow{w} \eta$ in Y^* with $\limsup \langle Ty_n, y_n\rangle_Y \le \langle \eta, y\rangle_Y$, then $\eta = Ty$.*

Proof. Let $z \in Y$. Since

$$\begin{aligned}\langle Ty, y - z\rangle &\le \liminf \langle Ty_n, y_n - z\rangle \le \limsup \langle Ty_n, y_n - z\rangle \\ &\le \limsup \langle Ty_n, y_n\rangle - \lim \langle Ty_n, z\rangle \le \langle \eta, y\rangle - \langle \eta, z\rangle = \langle \eta, y - z\rangle,\end{aligned}$$

we have $\langle Ty, y-z\rangle \le \langle \eta, y-z\rangle$ for all $z \in Y$. Taking $z = y \pm v$, $v \in Y$, we get $Ty = \eta$. □

The following surjectivity result for mappings of type (M) can be found in Chapter III.5, p. 156 of Pascali and Sburlan [20] or Hu and Papageorgiou [7], Corollary 6.29, p. 372.

Theorem 2.1. *If $T : Y \to 2^{Y^*}$ is quasi-bounded coercive operator of type* (M)*, then T is surjective.*

Finally, we recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $g : E \to \mathbb{R}$, where E is a Banach space (see Clarke [4]). The generalized directional derivative of g at x in the direction v, denoted by $g^0(x; v)$, is defined by

$$g^0(x;v) = \limsup_{y\to x,\ t\downarrow 0} \frac{g(y+tv) - g(y)}{t}.$$

The generalized gradient of g at x, denoted by $\partial g(x)$, is a subset of a dual space E^* given by $\partial g(x) = \{\zeta \in E^* : g^0(x;v) \ge \langle \zeta, v\rangle_{E\times E^*}$ for all $v \in E\}$.

3. Existence of solutions

The goal of this section is to investigate the existence of solutions to an abstract evolution inclusion which can be considered as a multivalued version of a parabolic hemivariational inequality.

Let $L : D(L) \subset \mathcal{V} \to \mathcal{V}^*$ be the operator defined by $Lv = v'$ with $D(L) = \{v \in \mathcal{V} : v' \in \mathcal{V}^*, v(0) = 0\}$. It is well known (see e.g. Proposition 32.10, p. 855 of Zeidler [23]) that L is linear densely defined and maximal monotone operator.

Let $\mathcal{J} : \mathcal{X} \to \mathbb{R}$ be a locally Lipschitz function and let $\mathcal{J}^0(\,\cdot\,,\,\cdot\,)$ and $\partial\mathcal{J}(\,\cdot\,)$ denote, respectively, the generalized directional derivative and the generalized gradient of $\mathcal{J}$ in the sense of Clarke [4].

The evolution hemivariational inequality under consideration is following: find $u \in D(L)$ such that

$$\langle Lu + \mathcal{A}u - f, v - u\rangle_{\mathcal{V}} + \mathcal{J}^0(u; v-u) \ge 0 \quad \text{for all } v \in \mathcal{V},$$

where $\mathcal{A}$ is a multivalued map from $\mathcal{V}$ to $2^{\mathcal{V}^*}$. By using the definition of the generalized gradient, this problem can be formulated as follows: find $u \in \mathcal{W}$ such that

$$\begin{cases} Lu + \mathcal{A}u + \partial\mathcal{J}(u) \ni f, \\ u(0) = 0. \end{cases} \tag{3.1}$$

Our hypotheses on the data of (3.1) are following.

$\underline{H(\mathcal{A})}$: $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{V}^*}$ is an operator which is bounded, coercive and generalized pseudomonotone with respect to $D(L)$.

<u>$H(\mathcal{J})$</u>: $\mathcal{J} : \mathcal{X} \to \mathbb{R}$ is a function which is Lipschitz continuous on each bounded subset of $\mathcal{X}$ and there exists $k \geq 0$ such that

$$\mathcal{J}^0(v; -v) \leq k(1 + ||v||_{\mathcal{X}}) \quad \text{for all } v \in \mathcal{X}. \tag{3.2}$$

By a solution of (3.1) we mean a function $u \in \mathcal{W}$ such that $Lu + \rho + w = f$, $u(0) = 0$ with $\rho \in \mathcal{A}u$ and $w \in \partial\mathcal{J}(u)$. We have the following existence result concerning problem (3.1).

THEOREM 3.1. *If hypotheses* H($\mathcal{A}$) *and* H($\mathcal{J}$) *hold and* $f \in \mathcal{V}^*$, *then problem* (3.1) *has at least one solution.*

PROOF. We introduce on $D(L)$, which is a linear subspace of $\mathcal{V}$, the graph norm by $||u||_{D(L)} = ||u||_{\mathcal{V}} + ||Lu||_{\mathcal{V}^*}$ for $u \in D(L)$. Equipped with this norm $D(L)$ is a reflexive Banach space and the embedding $D(L) \subset \mathcal{V}$ is dense and continuous.

Let $F : \mathcal{V} \to \mathcal{V}^*$ be the duality map. It is known that it has many nice properties (see Zeidler [23], Proposition 32.22); it is single-valued, demicontinuous, bijective, strictly monotone, bounded and $F^{-1} : \mathcal{V}^* \to \mathcal{V} = \mathcal{V}^{**}$ is equal to the duality map of the dual space. Moreover for all $u \in \mathcal{V}$, we have $||Fu||_{\mathcal{V}^*} = ||u||_{\mathcal{V}}$ and $\langle Fu, u\rangle = ||u||^2_{\mathcal{V}}$.

For every positive ε, we define two operators

$$\begin{cases} M_\varepsilon : D(L) \to D(L)^*, \\ \langle M_\varepsilon u, v\rangle_{D(L)} = \varepsilon\langle F^{-1}Lu, Lv\rangle_{\mathcal{V}} + \langle Lu, v\rangle_{\mathcal{V}}, \end{cases}$$

for $u, v \in D(L)$ and

$$\begin{cases} P_\varepsilon : D(L) \to 2^{D(L)^*}, \\ P_\varepsilon u = M_\varepsilon u + \mathcal{A}u + \partial\mathcal{J}(u), \end{cases}$$

for $u \in D(L)$. First we establish some properties of operators M_ε and P_ε. From the estimate

$$\begin{aligned} |\langle M_\varepsilon u, v\rangle_{D(L)}| &\leq \varepsilon||F^{-1}Lu||_{\mathcal{V}}||Lv||_{\mathcal{V}^*} + ||Lu||_{\mathcal{V}^*}||v||_{\mathcal{V}} \\ &\leq \varepsilon||Lu||_{\mathcal{V}^*}(||Lv||_{\mathcal{V}^*} + ||v||_{\mathcal{V}}) \leq \varepsilon||u||_{D(L)}||v||_{D(L)}, \end{aligned}$$

we have $||M_\varepsilon u||_{D(L)^*} \leq \varepsilon||u||_{D(L)}$, so M_ε is a bounded operator. Exploiting the monotonicity of F^{-1} we can show that M_ε is also monotone. It is easy to see that M_ε is demicontinuous (recall that F^{-1} is such). Since M_ε is defined on the whole space $D(L)$ and monotone, we know (cf. Pascali and Sburlan [20]) that M_ε is hemicontinuous. Therefore Proposition 27.6 of Zeidler [23] tells us that M_ε is pseudomonotone operator (being monotone and hemicontinuous).

Claim 1. For every fixed positive ε, the operator P_ε is of type (M).

From H($\mathcal{A}$) and from the fact that the values of $\partial\mathcal{J}$ are nonempty, weakly compact and convex subsets of $\mathcal{X}^*$ (cf. Clarke [4]), we easily get that for all $v \in D(L)$, $P_\varepsilon v$ is a nonempty, weakly compact and convex subset of $D(L)^*$.

Next, since the mapping $\partial\mathcal{J} : \mathcal{X} \to 2^{\mathcal{X}^*}$ has a sequentially closed graph in $\mathcal{X} \times \mathcal{X}^*_{\mathrm{weak}}$ topology and it is a locally relatively weakly compact map, by Lemma 7.10 of Phelps [21] (or Proposition 2.23, p. 43 of Hu and Papageorgiou [7]) we have that $\partial\mathcal{J}$ is also u.s.c. in this topology. From $H(\mathcal{A})$ again and the demicontinuity of M_ε, we obtain that P_ε is u.s.c. from every finite dimensional subspace of $D(L)$ into $D(L)^*_{\mathrm{weak}}$.

Subsequently, assume that $(u_k, p_k) \in \operatorname{Gr} P_\varepsilon$, $u_k \xrightarrow{w} u$ in $D(L)$ (i.e. $u_k \xrightarrow{w} u$ in $\mathcal{V}$ and $Lu_k \xrightarrow{w} Lu$ in $\mathcal{V}^*$), $p_k \xrightarrow{w} p$ in $D(L)^*$ and $\limsup \langle p_k, u_k\rangle_{D(L)} \le \langle p, u\rangle_{D(L)}$. We will show that $(u,p) \in \operatorname{Gr} P_\varepsilon$. To this end, let $\rho_k \in \mathcal{A}u_k$ and $w_k \in \partial\mathcal{J}(u_k)$ be such that

$$p_k = M_\varepsilon u_k + \rho_k + w_k. \tag{3.3}$$

In view of the boundedness of $\mathcal{A}$ and $\partial\mathcal{J}$, we have that $\{\rho_k\}$ and $\{w_k\}$ remain in bounded subsets of $\mathcal{V}^*$ and $\mathcal{X}^*$, respectively. We can assume that $\rho_k \xrightarrow{w} \rho$ in $\mathcal{V}^*$ (which entails also weakly in $D(L)^*$) and $w_k \xrightarrow{w} w$ in $\mathcal{X}^*$. Using the compactness of the embedding $\mathcal{W} \subset \mathcal{X}$, we suppose that $u_k \to u$ in $\mathcal{X}$. Then by using again the sequential closedness of the graph of $\partial\mathcal{J}$, we have $w \in \partial\mathcal{J}(u)$. We also have immediately $u \in D(L)$ (recall that by Mazur's theorem, $D(L)$ is weakly closed since it is closed and convex subset of $\mathcal{W}$).

We will show now that

$$\limsup_k \langle \rho_k + w_k, u_k - u\rangle_{D(L)} \le 0. \tag{3.4}$$

Suppose (3.4) does not hold. Thus, we can find $d > 0$ and a subsequence of $\{u_k\}$, which is identified for simplicity of notation with $\{u_k\}$, such that

$$\lim_k \langle \rho_k + w_k, u_k - u\rangle_{D(L)} = d > 0.$$

Hence

$$\begin{aligned}\limsup_k \langle M_\varepsilon u_k, u_k - u\rangle_{D(L)} = {} & \limsup_k \langle p_k, u_k - u\rangle_{D(L)} \\ & - \lim_k \langle \rho_k + w_k, u_k - u\rangle_{D(L)} \le -d < 0.\end{aligned}$$

From the pseudomonotonicity of M_ε it follows that

$$\langle M_\varepsilon u, u - v\rangle_{D(L)} \le \liminf_k \langle M_\varepsilon u_k, u_k - v\rangle_{D(L)} \quad \text{for all } v \in D(L).$$

In particular, for $v = u$, we obtain

$$0 \le \liminf_k \langle M_\varepsilon u_k, u_k - u\rangle_{D(L)} \le \limsup_k \langle M_\varepsilon u_k, u_k - u\rangle_{D(L)} \le -d < 0,$$

which gives a contradiction. The proof of (3.4) is completed.

From (3.4) and the preceding convergences, we get

$$\limsup_k \langle \rho_k, u_k\rangle_{D(L)} = \limsup_k \langle \rho_k + w_k, u_k - u\rangle_{D(L)} - \lim_k \langle w_k, u_k - u\rangle_{\mathcal{X}} + \lim_k \langle \rho_k, u\rangle_{D(L)} \le \langle \rho, u\rangle_{D(L)}.$$

From the fact that $\mathcal{A}$ is generalized pseudomonotone with respect to $D(L)$, we have $(u,\rho) \in \operatorname{Gr}\mathcal{A}$ and

$$\langle \rho_k, u_k\rangle_{D(L)} \to \langle \rho, u\rangle_{D(L)}. \tag{3.5}$$

Since M_ε is a bounded operator, by passing to a subsequence if necessary, we may assume that there is $\eta \in D(L)^*$ such that

$$M_\varepsilon u_k \xrightarrow{w} \eta \quad \text{in } D(L)^*, \text{ as } k \to \infty. \tag{3.6}$$

Moreover, from (3.3), (3.5) and (3.6), we have

$$\begin{aligned}\limsup_k \langle M_\varepsilon u_k, u_k\rangle_{D(L)} &= \limsup_k \langle p_k, u_k - u\rangle_{D(L)} + \lim_k \langle \rho_k, u\rangle_{D(L)} \\ &\quad - \lim_k \langle \rho_k, u_k\rangle_{D(L)} + \lim_k \langle w_k, u - u_k\rangle_{\mathcal{X}} \\ &\quad + \lim_k \langle M_\varepsilon u_k, u\rangle_{D(L)} \le \langle \eta, u\rangle_{D(L)}.\end{aligned}$$

Hence and from (3.6), by Proposition 2.1, we deduce that $\eta = M_\varepsilon u$. Now using the above convergences and passing to the limit in (3.3), we obtain

$$p = M_\varepsilon u + \rho + w$$

which together with $\rho \in \mathcal{A}u$ and $w \in \partial\mathcal{J}(u)$ implies that $(u,p) \in \operatorname{Gr} P_\varepsilon$. Hence the operator P_ε is of type (M), as claimed.

Claim 2. For every fixed ε, operator P_ε is bounded and coercive.

The boundedness of P_ε follows from the one of $\mathcal{A}$ and $\partial\mathcal{J}$. We show the coerciveness. By the monotonicity of L and the properties of the duality map, for any $v \in D(L)$, we have

$$\langle M_\varepsilon v, v\rangle_{D(L)} \ge \varepsilon \langle F^{-1}Lv, Lv\rangle_{\mathcal{V}} = \varepsilon ||Lv||^2_{\mathcal{V}^*}.$$

From the hypothesis $H(\mathcal{J})$, the coercivity of $\mathcal{A}$ and the inequality $||\cdot||_{\mathcal{X}} \le c_0 ||\cdot||_{\mathcal{V}}$ we get the following estimate

$$\langle \rho + w, v\rangle_{D(L)} \ge c(||v||_{\mathcal{V}})||v||_{\mathcal{V}} - k(1 + c_0||v||_{\mathcal{V}}) \ge \widetilde{c}(||v||_{\mathcal{V}})||v||_{\mathcal{V}} - k$$

for all $v \in D(L)$, $\rho \in \mathcal{A}v$ and $w \in \partial\mathcal{J}(v)$, where a function $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}$ satisfies $\widetilde{c}(r) \to \infty$, as $r \to \infty$. In consequence, for any $v \in D(L)$ and $p \in P_\varepsilon v$, we obtain

$$\langle p, v\rangle_{D(L)} \ge \widetilde{c}(||v||_{\mathcal{V}})||v||_{\mathcal{V}} + \varepsilon ||Lv||^2_{\mathcal{V}^*} - k. \tag{3.7}$$

Hence

$$\frac{\langle p, v\rangle_{D(L)}}{||v||_{D(L)}} \to \infty \quad \text{as } ||v|| \to \infty,$$

which ensures the coercivity of P_ε.

Claims 1 and 2 allows us to apply Theorem 2.1 and deduce that for any $\varepsilon > 0$ operator P_ε is surjective. This implies that for every $f \in \mathcal{V}^* \subset D(L)^*$, there exists $u_\varepsilon \in D(L)$ such that $f \in P_\varepsilon u_\varepsilon$.

Then, observing that $F^{-1}Lu_\varepsilon \in D(L^*)$ (cf. Lions [8], p. 317) (L^* is the adjoint operator to L) and $\langle F^{-1}Lu_\varepsilon, Lv\rangle = \langle L^*(F^{-1}Lu_\varepsilon), v\rangle$, the problem $f \in P_\varepsilon u_\varepsilon$ is equivalent to

$$
\begin{cases} \varepsilon L^*(F^{-1}Lu_\varepsilon) + Lu_\varepsilon + \rho_\varepsilon + w_\varepsilon = f, \quad \varepsilon > 0, \\ u_\varepsilon \in D(L),\ \rho_\varepsilon \in \mathcal{A}u_\varepsilon,\ w_\varepsilon \in \partial\mathcal{J}(u_\varepsilon). \end{cases} \tag{3.8}
$$

For the a priori estimate, we observe that from (3.7), we have

$$
\widetilde{c}(||u_\varepsilon||_{\mathcal{V}})||u_\varepsilon||_{\mathcal{V}} - k \le ||f||_{\mathcal{V}^*}||u_\varepsilon||_{\mathcal{V}}.
$$

Hence $\{u_\varepsilon\}$ is bounded in $\mathcal{V}$ uniformly with respect to ε. It follows from the boundedness of $\mathcal{A}$ and $\partial\mathcal{J}$ that $\{\rho_\varepsilon\}$, $\rho_\varepsilon \in \mathcal{A}u_\varepsilon$ and $\{w_\varepsilon\}$, $w_\varepsilon \in \partial\mathcal{J}(u_\varepsilon)$ remain in bounded subsets of $\mathcal{V}^*$ and $\mathcal{X}^*$, respectively. From (3.8), we have

$$
\begin{aligned} &\varepsilon\langle L^*(F^{-1}Lu_\varepsilon), F^{-1}Lu_\varepsilon\rangle_{D(L)} + \langle Lu_\varepsilon, F^{-1}Lu_\varepsilon\rangle_{\mathcal{V}} \\ &\qquad + \langle \rho_\varepsilon, F^{-1}Lu_\varepsilon\rangle_{D(L)} + \langle w_\varepsilon, F^{-1}Lu_\varepsilon\rangle_{D(L)} = \langle f, F^{-1}Lu_\varepsilon\rangle_{D(L)}. \end{aligned}
$$

The first term on the left hand side is nonnegative since L^* is monotone (see e.g. Zeidler [23], Theorem 32.L, p. 897). The second term, by the property of F^{-1}, is equal to $||Lu_\varepsilon||^2_{\mathcal{V}^*}$. Therefore, we obtain

$$
||Lu_\varepsilon||^2_{\mathcal{V}^*} \le (||f||_{\mathcal{V}^*} + ||\rho_\varepsilon||_{\mathcal{V}^*} + ||w_\varepsilon||_{\mathcal{V}^*})||Lu_\varepsilon||_{\mathcal{V}^*}
$$

which obviously implies that $\{Lu_\varepsilon\}$ is bounded in $\mathcal{V}^*$ independently of ε.

From the previous steps of the proof, we deduce that $\{u_\varepsilon\}$, $\{\rho_\varepsilon\}$ and $\{w_\varepsilon\}$ lie in bounded sets of $D(L)$, $\mathcal{V}^*$ and $\mathcal{X}^*$, respectively. So we can extract subsequences such that

$$
\begin{cases} u_\varepsilon \to u & \text{weakly in } D(L) \text{ and in } \mathcal{X}, \\ \rho_\varepsilon \xrightarrow{w} \rho & \text{in } \mathcal{V}^*, \\ w_\varepsilon \xrightarrow{w} w & \text{in } \mathcal{X}^*, \text{ as } \varepsilon \to 0 \end{cases} \tag{3.9}
$$

(recall here that $D(L) \subset \mathcal{X}$ compactly). Again by the closedness properties of $D(L)$ and $\mathrm{Gr}\,\partial\mathcal{J}$, we get $u \in D(L)$ and $w \in \partial\mathcal{J}(u)$.

We also have $\rho \in \mathcal{A}u$. Indeed, by the estimate

$$
\langle F^{-1}Lu_\varepsilon, Lu_\varepsilon - Lu\rangle_{\mathcal{V}} \le ||Lu_\varepsilon||_{\mathcal{V}^*}(||Lu_\varepsilon||_{\mathcal{V}^*} + ||Lu||_{\mathcal{V}^*}) \le \widehat{c}
$$

from (3.8) we obtain

$$\begin{aligned}
&\langle \rho_\varepsilon, u_\varepsilon - u\rangle_{D(L)} \\
&\qquad \le \langle f - w_\varepsilon, u_\varepsilon - u\rangle_{\mathcal{V}} - \varepsilon\langle F^{-1}Lu_\varepsilon, Lu_\varepsilon - Lu\rangle_{\mathcal{V}} - \langle Lu_\varepsilon, u_\varepsilon - u\rangle_{\mathcal{V}} \\
&\qquad \le \langle f - w_\varepsilon, u_\varepsilon - u\rangle_{\mathcal{V}} + \varepsilon\widehat{c} - \langle L(u_\varepsilon - u), u_\varepsilon - u\rangle_{\mathcal{V}} - \langle Lu, u_\varepsilon - u\rangle_{\mathcal{V}} \\
&\qquad \le \langle f - Lu, u_\varepsilon - u\rangle_{\mathcal{V}} - \langle w_\varepsilon, u_\varepsilon - u\rangle_{\mathcal{X}} + \varepsilon\widehat{c}.
\end{aligned}$$

From (3.9), we have $\limsup_{\varepsilon\to 0}\langle \rho_\varepsilon, u_\varepsilon\rangle_{\mathcal{V}} \le \langle \rho, u\rangle_{\mathcal{V}}$. Because $\mathcal{A}$ is generalized pseudomonotone with respect to $D(L)$, we have $\rho \in \mathcal{A}u$ and $\langle \rho_\varepsilon, u_\varepsilon\rangle_{\mathcal{V}} \to \langle \rho, u\rangle_{\mathcal{V}}$, as $\varepsilon \to 0$.

Finally let $v \in D(L)$. From (3.8) again, by taking the duality with v and using the fact $\lim_{\varepsilon\to 0}\langle F^{-1}Lu_\varepsilon, Lv\rangle_{\mathcal{V}} = 0$, we get

$$\lim_{\varepsilon\to 0}(\langle Lu_\varepsilon, v\rangle_{\mathcal{V}} + \langle \rho_\varepsilon, v\rangle_{\mathcal{V}} + \langle w_\varepsilon, v\rangle_{\mathcal{X}}) = \langle f, v\rangle_{\mathcal{V}}.$$

Applying (3.9) the last equality gives

$$\langle Lu + \rho + w, v\rangle_{\mathcal{V}} = \langle f, v\rangle_{\mathcal{V}} \quad \text{for all } v \in D(L).$$

As $D(L)$ is dense in $\mathcal{V}$, we have $\langle Lu+\rho+w-f, v\rangle_{\mathcal{V}} = 0$ for all $v \in \mathcal{V}$. Consequently $u \in D(L)$ satisfies $Lu + \rho + w = f$ with $\rho \in \mathcal{A}u$ and $w \in \partial\mathcal{J}(u)$. This means that u solves (3.1) as required. □

REMARK 3.1. We should point out that Theorem 3.1 remains valid if the operator $\mathcal{A} : \mathcal{V} \to 2^{\mathcal{V}^*}$ satisfies H($\mathcal{A}$) with the coercivity condition with a function $c(r) \approx r$ as $r \to \infty$, and the function $\mathcal{J} : \mathcal{X} \to \mathbb{R}$ satisfies H($\mathcal{J}$) with a weaker growth condition than (3.2), namely

$$\mathcal{J}^0(v; -v) \le k(1 + ||v||_{\mathcal{X}}^\sigma) \quad \text{for all } v \in \mathcal{X},$$

with $k \ge 0$ and $1 \le \sigma < p$.

In conjunction with (3.1) we consider the following problem:

$$\text{(3.10)} \qquad \begin{cases} \text{find } u \in \mathcal{W} \text{ such that } u(0) = 0 \text{ and} \\ \displaystyle\left\langle \frac{du}{dt}, v\right\rangle_{\mathcal{V}} + \langle \rho, v\rangle_{\mathcal{V}} + \int_0^T j^0(t, u; v)\, dt \ge \langle f, v\rangle_{\mathcal{V}}, \\ \text{for every } \rho \in \mathcal{A}u \text{ and } v \in \mathcal{V}. \end{cases}$$

We admit the following hypothesis:

<u>$H(j)$</u> : $j : (0,T) \times X \to \mathbb{R}$ is a function such that

(1) $t \mapsto j(t,x)$ is measurable on $(0,T)$, for each $x \in X$,
(2) $x \mapsto j(t,x)$ is locally Lipschitz on X for each $t \in (0,T)$ and $j(\,\cdot\,,x) \in L^1(0,T)$,
(3) for any $x \in X$ and $t \in (0,T)$ and for any $\eta \in \partial_x j(t,x)$ we have $||\eta||_{X^*} \le c(1 + ||x||_X^{p-1})$ with a constant $c \ge 0$ independent of $x \in X$.

Corollary 3.1. *Assume that* H($\mathcal{A}$) *holds and* $f \in \mathcal{V}^*$. *Let the function* $j : (0,T) \times X \to \mathbb{R}$ *satisfy* H(j) *and the generalized sign condition*

$$j^0(t, x; -x) \le \alpha(t)(1 + ||x||_X) \quad \text{for all } x \in X \tag{3.11}$$

with a nonnegative function $\alpha \in L^q(0,T)$. *Then the problem* (3.10) *has at least one solution.*

Proof. Consider the functional $\mathcal{J} : \mathcal{X} = L^p(0,T;X) \to \mathbb{R}$ of the form

$$\mathcal{J}(v) = \int_0^T j(t, v(t))\, dt \quad \text{for } v \in \mathcal{X}.$$

From Theorem 2.7.5 of Clarke [4], it follows that the functional $\mathcal{J}$ is well defined, Lipschitz continuous on every bounded subset of $\mathcal{X}$, and for $v \in \mathcal{X}$ and $\chi \in \mathcal{X}^*$ such that $\chi \in \partial \mathcal{J}(v)$, we have $\chi(t) \in \partial_x j(t, v(t))$ a.e. $t \in (0,T)$. Moreover, by Fatou's lemma, we easily get

$$\mathcal{J}^0(u; v) \le \int_0^T j^0(t, u(t); v(t))\, dt \quad \text{for } u, v \in \mathcal{X}. \tag{3.12}$$

Hence and by (3.11) and the Hölder inequality, we obtain

$$\begin{aligned} \mathcal{J}(v; -v) &\le \int_0^T j^0(t, v(t); -v(t))\, dt \\ &\le \int_0^T \alpha(t)(1 + ||v(t)||_X)\, dt \le c||\alpha||_{L^q}(1 + ||v||_{\mathcal{X}}) \end{aligned}$$

with $c > 0$ and for each $v \in \mathcal{X}$. Therefore $\mathcal{J}$ satisfies $H(\mathcal{J})$. Invoking Theorem 3.1, we know that problem (3.1) has a solution, i.e. there exists $u \in \mathcal{W}$, $u(0) = 0$ such that $u' + \mathcal{A}u + \partial \mathcal{J}(u) \ni f$. Let $\rho \in \mathcal{A}(u)$. By the definition of generalized gradient and (3.12), we get

$$\langle f - Lu - \rho, v \rangle_{\mathcal{X}} \le \mathcal{J}^0(u, v) \le \int_0^T j^0(t, u(t); v(t))\, dt$$

for $v \in \mathcal{X}$. Taking $v \in \mathcal{V}$, we have

$$\langle f - u' - \rho, v \rangle_{\mathcal{V}} \le \int_0^T j^0(t, u(t); v(t))\, dt$$

which means that u solves (3.10). □

In what follows we will be dealing with the problem (3.1) with a non zero initial data and single-valued operator $\mathcal{A}$. We shall prove an existence result for the problem

$$\begin{cases} u' + \mathcal{A}u + \partial \mathcal{J}(u) \ni f, \\ u(0) = u_0. \end{cases} \tag{3.13}$$

The hypothesis on the operator $\mathcal{A}$ is following. Let $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ be the Nemitsky operator corresponding to a family of operators $A(\cdot)$, i.e. $(\mathcal{A}v)(t) = A(t)v(t)$.

$\underline{H(A)}$: $A(t) : V \to V^*$ is an operator such that

(i) $t \mapsto A(t)v$ is measurable on $(0, T)$ for all $v \in V$,
(ii) $v \mapsto A(t)v$ is demicontinuous and pseudomonotone, for all $t \in (0, T)$,
(iii) $||A(t)v||_{V^*} \le \beta(t) + c_1||v||_V^{p-1}$, a.e. $t \in (0, T)$ with $2 \le p < \infty$, $\beta \in L^q(0, T)$, $1/p + 1/q = 1$,
(iv) $\langle A(t)v, v\rangle_V \ge c||v||_V^p - a||v||_V^r - \gamma(t)$, $1 \le r \le p-1$, $a, c > 0$, $\gamma \in L^1(0, T)$.

THEOREM 3.2. *If hypotheses* H(A), H$(\mathcal{J})$ *hold,* $f \in \mathcal{V}^*$ *and* $u_0 \in V$, *then problem* (3.13) *has a solution.*

PROOF. We transform (3.13) into an equivalent evolution inclusion which is solved in much the same way as in Theorem 3.1. We define the operators $\overline{A}(t) : V \to V^*$ and $\widehat{\partial\mathcal{J}} : \mathcal{X} \to 2^{\mathcal{X}^*}$ by $\overline{A}(t)u = A(t)(u + u_0)$ and $\widehat{\partial\mathcal{J}}(v)(\,\cdot\,) = \partial\mathcal{J}(v(\,\cdot\,) + u_0)$, respectively. By hypothesis H(A), we have that $\overline{A}(\,\cdot\,)v$ is measurable, $\overline{A}(t)(\,\cdot\,)$ is demicontinuous and pseudomonotone, and

$$\begin{aligned} ||\overline{A}(t)v||_{V^*} &\le \overline{\beta}(t) + \overline{c_1}||v||_V^{p-1}, && \text{a.e. } t \text{ with } \overline{\beta} \in L^q \text{ and } \overline{c_1} \ge 0,\\ \langle \overline{A}(t)v, v\rangle_V &\ge \widehat{c}||v||^p - \widehat{a}||v||^r - \overline{\gamma}(t), && \text{a.e. } t \text{ with } \overline{\gamma} \in L^1 \text{ and } \widehat{c}, \widehat{a} > 0. \end{aligned}$$

This means that $\overline{A}$ inherits all properties of A. By applying Proposition 1 of Papageorgiou [19] or Theorem 2(b) of Berkovits and Mustonen [1], we know that the Nemitsky operator $\overline{\mathcal{A}} : \mathcal{V} \to \mathcal{V}^*$ corresponding to $\overline{A}(t)(\,\cdot\,)$ is generalized pseudomonotone with respect to $D(L)$. Clearly $\overline{\mathcal{A}}$ is bounded and coercive.

Next, by $H(\mathcal{J})$ we deduce that $\widehat{\partial\mathcal{J}}$ is a bounded mapping with a sequentially closed graph in $\mathcal{X} \times \mathcal{X}^*_{\text{weak}}$, its values are nonempty, weakly compact and convex subsets of $\mathcal{X}^*$. Moreover, using $H(\mathcal{J})$ and the property that the function $h \mapsto \mathcal{J}^0(v; h)$ is subadditive (see Clarke [4]), for every $v \in \mathcal{X}$ and $v^* \in \widehat{\partial\mathcal{J}}(v) \subset \mathcal{X}^*$, we have

$$\begin{aligned} -\langle v^*, v\rangle_{\mathcal{X}} &\le \mathcal{J}^0(v + u_0; -v)\\ &\le \mathcal{J}^0(v + u_0; -(v + u_0)) + \mathcal{J}^0(v + u_0; u_0)\\ &\le k(1 + ||v + u_0||_{\mathcal{X}}) + c||u_0|| \end{aligned}$$

with a suitable $c > 0$. Hence $\widehat{\partial\mathcal{J}}$ is subcoercive, that is,

$$\langle v^*, v\rangle_{\mathcal{X}} \ge -k_1(1 + ||v||_{\mathcal{X}} + ||u_0||_V) - c||u_0||_V.$$

Consider the evolution inclusion:

$$\begin{cases} Lz + \overline{\mathcal{A}}z + \widehat{\partial\mathcal{J}}(z) \ni f,\\ z(0) = 0. \end{cases} \tag{3.14}$$

Using the analogous argument as in the proof of Theorem 3.1, the problem (3.14) has a solution $z \in \mathcal{W}$. It is clear now that $u(t) = z(t) + u_0$ is a solution of problem (3.13). $\square$

4. A convergence result for parabolic hemivariational inequality

In this section we present a convergence result concerning the upper semicontinuity of the solution set for the parabolic hemivariational inequality of the form: find $u_h \in \mathcal{W}$ such that

$$\begin{cases} u_h' + \mathcal{A}_h u_h + \partial \mathcal{J}_h(u_h) \ni f_h, \\ u_h(0) = u_0^h, \end{cases} \tag{4.1}$$

where $\mathcal{A}_h : \mathcal{V} \to \mathcal{V}^*$ are Nemitsky operators corresponding to $A_h(t) : V \to V^*$, $A_h(t)v = -\operatorname{div} a_h(x, t, Dv)$, i.e. $(\mathcal{A}_h v)(t) = A_h(t)v(t)$ for $v \in \mathcal{V}$, $h \in \mathbb{N}$. The maps a_h are supposed to be monotone and to satisfy coerciveness and boundedness hypotheses uniformly with respect to $h \in \mathbb{N}$. Throughout this section we put $V = W_0^{1,p}(\Omega)$, $X = L^p(\Omega)$ with $2 \le p < \infty$ and $H = L^2(\Omega)$, where Ω is an open bounded subset of $\mathbb{R}^N$. Let $0 < T < \infty$ and $Q = \Omega \times (0, T)$.

Following Svanstedt [22], we admit the following

Definition 4.1. Let m_0, m_1, m_2 be three positive real constants and $0 < \alpha \le 1$. By $S = S(m_0, m_1, m_2, \alpha)$ we denote the class of functions $a : Q \times \mathbb{R}^N \to \mathbb{R}^N$ which satisfy the following conditions:

(i) $|a(x, t, 0)| \le m_0$, a.e. in Q,
(ii) $a(\,\cdot\,, \,\cdot\,, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^N$,
(iii) $|a(x, t, \xi) - a(x, t, \eta)| \le m_1(1 + |\xi| + |\eta|)^{p-1-\alpha} |\xi - \eta|^\alpha$, a.e. in Q and for every $\xi, \eta \in \mathbb{R}^N$,
(iv) $(a(x, t, \xi) - a(x, t, \eta), \xi - \eta) \ge m_2 |\xi - \eta|^p$, a.e. in Q and for all $\xi, \eta \in \mathbb{R}^N$.

Remark 4.1. *If $a \in S(m_0, m_1, m_2, \alpha)$, then the following inequalities hold*

$$|a(x, t, \xi)| \le \operatorname{const}(1 + |\xi|)^{p-1}, \quad |\xi|^p \le \operatorname{const}(1 + (a(x, t, \xi), \xi)).$$

for every $\xi \in \mathbb{R}^N$ and a.e. in Q.

Definition 4.2. A sequence of maps $\{a_h\}_{h \in \mathbb{N}} \subset S(m_0, m_1, m_2, \alpha)$ is said to PG-*converge* to a map a, and we write $a_h \xrightarrow{\mathrm{PG}} a$, as $h \to \infty$, if for every $g \in L^q(0, T; W^{-1,q}(\Omega))$, the sequence $\{y_h\} \subset \mathcal{W}$ of unique solutions of the following problems

$$\begin{cases} y_h' - \operatorname{div} a_h(x, t, Dy_h) = g \quad \text{in } Q, \\ y_h(0) = 0, \end{cases}$$

satisfies

$$y_h \xrightarrow{w} y \quad \text{in } \mathcal{W}, \qquad a_h(x, t, Dy_h) \xrightarrow{w} a(x, t, Dy) \quad \text{in } L^q(Q; \mathbb{R}^N),$$

where $y \in \mathcal{W}$ is the solution of the problem

$$\begin{cases} y' - \operatorname{div} a(x,t,Dy) = g & \text{in } Q, \\ y(0) = 0. \end{cases}$$

It was shown recently by Svanstedt [22] (cf. also Migórski [12]) that the class $S(m_0, m_1, m_2, \alpha)$ is compact with respect to PG-convergence. Namely, we have

Proposition 4.1. *Let $\{a_h\}_{h\in\mathbb{N}}$ be a sequence in $S(m_0, m_1, m_2, \alpha)$. Then there exists a subsequence of $\{a_h\}$ which PG-converges to a map a of the class $S(\widetilde{m_0}, \widetilde{m_1}, m_2, \widetilde{\alpha})$. The positive constants $\widetilde{m_0}, \widetilde{m_1}$ depend on the constants p, m_0, m_1, m_2, α and $\widetilde{\alpha} = \alpha/(p-\alpha)$.*

With every map $a_h \in S(m_0, m_1, m_2, \alpha)$ (with constants m_0, m_1, m_2, α independent of h) we associate a nonlinear operator $A_h(t) : V \to V^*$ of the form

$$\langle A_h(t)u, v\rangle = \int_\Omega (a_h(x,t,Du), Dv)\,dx. \tag{4.2}$$

Hypotheses:

$\underline{H(A)_1}$: $\mathcal{A}_h : \mathcal{V} \to \mathcal{V}^*$ is a sequence of operators corresponding to $A_h(t)$ of the form (4.2) with $\{a_h\} \subset S$ and $a_h \xrightarrow{\text{PG}} a$, as $h \to \infty$.

$\underline{H(J)_1}$: $\mathcal{J}_h, \mathcal{J} : \mathcal{X} \to \mathbb{R}$ are functions which are Lipschitz continuous on each bounded subset of $\mathcal{X}$ and satisfy the condition (3.2) in $H(\mathcal{J})$ uniformly with respect to h, and

$$\limsup_{h\to\infty} \operatorname{graph} \partial\mathcal{J}_h \subset \operatorname{graph} \partial\mathcal{J} \tag{4.3}$$

in $\mathcal{X} \times \mathcal{X}^*_{\text{weak}}$ topology.

$\underline{(H_0)}$: $u_0^h, u_0 \in V$, $u_0^h \xrightarrow{w} u_0$ in V.

$\underline{H(f)}$: $f_h, f \in \mathcal{V}^*$, $f_h \to f$ in $\mathcal{V}^*$ or $f_h, f \in \mathcal{X}^*$, $f_h \xrightarrow{w} f$ in $\mathcal{X}^*$.

We denote by $\mathcal{N}(\mathcal{A}_h, \mathcal{J}_h, f_h, u_0^h)$ the solution set of (4.1) corresponding to data $\mathcal{A}_h$, $\mathcal{J}_h$, f_h, u_0^h.

Theorem 4.1. *Under the hypotheses* H$(A)_1$, H$(J)_1$, (H$_0$) *and* H(f), *for every $\{u_h\} \subset \mathcal{N}(\mathcal{A}_h, \mathcal{J}_h, f_h, u_0^h)$ there exists a subsequence $\{u_{h_k}\}$ such that $u_{h_k} \xrightarrow{w} u$ in $\mathcal{W}$ and $u \in \mathcal{N}(\mathcal{A}, \mathcal{J}, f, u_0)$.*

Proof. We observe that under H$(A)_1$ for every $h \in \mathbb{N}$ the operator $A_h(t)$ is bounded, monotone, hemicontinuous (so also demicontinuous) and coercive (cf. Remark 4.1). So from Theorem 3.1 it follows that $\mathcal{N}_h = \mathcal{N}(\mathcal{A}_h, \mathcal{J}_h, f_h, u_0^h)$ and $\mathcal{N} = \mathcal{N}(\mathcal{A}, \mathcal{J}, f, u_0)$ are nonempty. Suppose $u_h \in \mathcal{N}_h$. We first show a priori bounds for u_h. The problem (4.1) is equivalent to the following operator equation: find $u_h \in \mathcal{W}$ such that

$$
(4.4) \qquad \begin{cases} u_h' + \mathcal{A}_h u_h + w_h = f_h, \\ w_h \in \partial \mathcal{J}_h(u_h) & \text{with } w_h \in \mathcal{X}^*, \\ u_h(0) = u_0^h. \end{cases}
$$

From the integration by parts formula for functions in $\mathcal{W}$ (see e.g. Zeidler [23], Proposition 23.23, pp. 422–423) it follows that $\langle u_h', u_h \rangle_{\mathcal{V}} = |u_h(T)|_H^2/2 - |u_0^h|_H^2/2$. Exploiting the subcoercivity of $\partial \mathcal{J}_h$ (i.e. $\langle v^*, v \rangle_{\mathcal{X}} \geq -k(1 + c_0 ||v||_{\mathcal{V}})$ for each $v^* \in \partial \mathcal{J}_h(v)$) and the coercivity of $\mathcal{A}_h$, from the equality

$$
\langle u_h', u_h \rangle_{\mathcal{V}} + \langle \mathcal{A}_h u_h, u_h \rangle_{\mathcal{V}} + \langle w_h, u_h \rangle_{\mathcal{X}} = \langle f_h, u_h \rangle_{\mathcal{V}},
$$

we have

$$
\frac{1}{2}|u_h(T)|_H^2 - \frac{1}{2}|u_0^h|_H^2 + c||u_h||_{\mathcal{V}}^2 - k(1 + c_0||u_h||_{\mathcal{V}}) \leq ||f_h||_{\mathcal{V}^*} ||u_h||_{\mathcal{V}}.
$$

Hence

$$
c||u_h||_{\mathcal{V}}^2 \leq \frac{1}{2}|u_0^h|_H^2 + k + (kc_0 + ||f_h||_{\mathcal{V}^*})||u_h||_{\mathcal{V}}.
$$

This implies that $\{u_h\}$ is bounded in $\mathcal{V}$ uniformly with respect to h. Next, since $\mathcal{A}_h$ and $\partial \mathcal{J}_h$ are bounded operators, from $u_h' = f_h - \mathcal{A}_h u_h - w_h$, we infer that $\{u_h'\}$ is bounded in $\mathcal{V}^*$. Thus we have shown that $\{u_h\}$ is bounded in $\mathcal{W}$. Due to the weak compactness of a ball in the reflexive Banach space $\mathcal{W}$, by passing to a subsequence if necessary, we assume that there exists $u \in \mathcal{W}$ such that $u_h \xrightarrow{w} u$ in $\mathcal{W}$. Now it remains to prove that $u \in \mathcal{N}$.

In view of the boundedness of $\partial \mathcal{J}_h$, we have that $\{w_h\}$ lies in a bounded subset of $\mathcal{X}^*$. We can assume that

$$
(4.5) \qquad w_h \xrightarrow{w} w \quad \text{in } \mathcal{X}^*.
$$

Using the compactness of the embedding $\mathcal{W} \subset \mathcal{X}$, we have

$$
(4.6) \qquad u_h \to u \quad \text{in } \mathcal{X} = L^p(Q; \mathbb{R}^N).
$$

By (4.3), we get $(u, w) \in \operatorname{graph} \partial \mathcal{J}$, i.e.

$$
(4.7) \qquad w \in \partial \mathcal{J}(u).
$$

From Lemma 2.1(c), it follows that $u_h(0) \xrightarrow{w} u(0)$ in H. So using (H_0), we pass to the limit in the initial condition and we have

$$
(4.8) \qquad u(0) = u_0.
$$

From Remark 4.1, we may assume, possibly passing to a subsequence that

$$
(4.9) \qquad a_h(x, t, Du_h) \xrightarrow{w} b(x, t) \quad \text{in } \mathcal{X}^* = L^q(Q; \mathbb{R}^N),
$$

with some $b \in \mathcal{X}^*$. Let now $\xi \in \mathbb{R}^N$, let Ω_0 be an open set such that $\Omega_0 \Subset \Omega$, and let Δ be an open interval with $\Delta \Subset (0,T)$. Let $\Phi \in C_0^\infty(\Omega)$, $\Psi \in C_0^\infty((0,T))$ be such that $\Phi|_{\Omega_0} = 1$ and $\Psi|_\Delta = 1$. We define

$$v(x,t) = \Phi(x)\Psi(t)(\xi, x)_{\mathbb{R}^N}.$$

Let us consider a sequence $\{v_h\} \subset \mathcal{W}$ of solutions to the following auxiliary problems

$$v_h'(t) - \operatorname{div} a_h(x,t,Dv_h) = v'(t) - \operatorname{div} a(x,t,Dv), \tag{4.10}$$

$$v_h(0) = 0. \tag{4.11}$$

By $\mathrm{H}(A)_1$ we have

$$\begin{cases} v_h \xrightarrow{w} v & \text{in } \mathcal{W}, \\ a_h(x,t,Dv_h) \xrightarrow{w} a(x,t,\xi) & \text{in } L^q(\Omega_0 \times \Delta; \mathbb{R}^N). \end{cases} \tag{4.12}$$

Let $\varphi \in C_0^\infty(\Omega_0 \times \Delta)$, $\varphi \geq 0$. The monotonicity implies

$$\int_{\Omega_0 \times \Delta} (a_h(x,t,Du_h) - a_h(x,t,Dv_h), Du_h - Dv_h)\varphi \, dxdt \geq 0. \tag{4.13}$$

From (4.4) and (4.10) we have

$$u_h' - v_h' - \operatorname{div}(a_h(x,t,Du_h) - a_h(x,t,Dv_h)) + w_h = f_h - v' + \operatorname{div} a(x,t,Dv).$$

Multiplying the last equation by $(u_h - v_h)\varphi$ and integrating by parts, we obtain

$$\begin{aligned} &\int_{\Omega_0 \times \Delta} (a_h(x,t,Du_h) - a_h(x,t,Dv_h), Du_h - Dv_h)\varphi \, dx \, dt \\ &\quad = \langle u_h' - v_h', (u_h - v_h)\varphi\rangle_{\Omega_0 \times \Delta} - \langle w_h, (u_h - v_h)\varphi\rangle_{L^p(\Omega_0 \times \Delta; \mathbb{R}^N)} \\ &\qquad - \int_{\Omega_0 \times \Delta} (a_h(x,t,Du_h) - a_h(x,t,Dv_h), (u_h - v_h)D\varphi) \, dx \, dt \\ &\qquad + \langle f_h - v' + \operatorname{div} a(x,t,Dv), (u_h - v_h)\varphi\rangle_{L^p(\Delta; W_0^{1,p}(\Omega_0))}. \end{aligned} \tag{4.14}$$

Claim. We have

$$\langle u_h' - v_h', (u_h - v_h)\varphi\rangle_{\mathcal{V}} \to \langle u' - v', (u - v)\varphi\rangle_{\mathcal{V}}, \quad \text{as } h \to \infty.$$

To show this convergence, let $z_h = u_h - v_h$, $z = u - v$. We know that $z_h \to z$ weakly in $\mathcal{W}$ and also in $\mathcal{H}$. Since

$$\langle z_h', z_h\varphi\rangle_{\mathcal{V}} = -\frac{1}{2}(z_h, z_h\varphi')_{\mathcal{H}}$$

and

$$|(z_h, z_h\varphi')_{\mathcal{H}} - (z, z\varphi')_{\mathcal{H}}| \leq ||z_h||_{\mathcal{H}}||(z_h - z)\varphi||_{\mathcal{H}} + ||z_h - z||_{\mathcal{H}}||z\varphi'||_{\mathcal{H}} \to 0$$

we have $\langle z_h', z_h\varphi\rangle_{\mathcal{V}} \to \langle z', z\varphi\rangle_{\mathcal{V}}$ which proves the Claim. Furthermore, from (4.9), (4.11), (4.12) and (4.6) we get

$$\int_{\Omega_0\times\Delta}(a_h(x,t,Du_h) - a_h(x,t,Dv_h),(u_h - v_h)D\varphi)\,dx\,dt \\ \to \int_{\Omega_0\times\Delta}(b(x,t) - a(x,t,\xi),(u-v)D\varphi)\,dx\,dt,$$

as $h\to\infty$. Using (4.5), (4.6), (4.12), $H(f)$ and Claim, we pass to the limit in (4.14) and we obtain

$$\begin{aligned}\int_{\Omega_0\times\Delta}&(a_h(x,t,Du_h) - a_h(x,t,Dv_h), Du_h - Dv_h)\varphi\,dx\,dt\\ &\to \langle u'-v',(u-v)\varphi\rangle_{\Omega_0\times\Delta} - \int_{\Omega_0\times\Delta}(b(x,t)-a(x,t,\xi),(u-v)D\varphi)\,dx\,dt\\ &\quad - \langle w,(u-v)\varphi\rangle_{\Omega_0\times\Delta} + \langle f - v' + \operatorname{div} a(x,t,Dv),(u-v)\varphi\rangle_{\Omega_0\times\Delta}\\ &= \langle -u' - w + f + \operatorname{div} a(x,t,Dv),(u-v)\varphi\rangle_{\Omega_0\times\Delta}\\ &\quad - \int_{\Omega_0\times\Delta}(b(x,t)-a(x,t,\xi),(u-v)D\varphi)\,dx\,dt = I_1 + I_2.\end{aligned}$$

On the other hand, taking the limit in $\mathcal{V}^*_{\text{weak}}$ topology, from the equation $u_h' - \operatorname{div} a_h(x,t,Du_h) + w_h = f_h$ we immediately get $u' - \operatorname{div} b(x,t) + w = f$. Inserting the last equality into I_1, we have

$$\begin{aligned}I_1 = &\int_{\Omega_0\times\Delta}(b(x,t)-a(x,t,\xi),(u-v)D\varphi)\,dx\,dt\\ &+ \int_{\Omega_0\times\Delta}(b(x,t)-a(x,t,\xi),(Du-Dv)\varphi)\,dx\,dt.\end{aligned}$$

This allows us to pass to the limit in (4.13) and we get

$$\int_{\Omega_0\times\Delta}(b(x,t)-a(x,t,\xi), Du-\xi)\varphi\,dx\,dt \ge 0.$$

Since φ can be chosen arbitrarily, we have

$$(b(x,t)-a(x,t,\xi), Du(x,t)-\xi)\ge 0 \quad \text{a.e. in } \Omega_0\times\Delta, \text{ for all } \xi\in\mathbb{R}^N,$$

and hence also a.e. in Q, for every $\xi\in\mathbb{R}^N$. Recalling that $a(x,t,\cdot)$ is continuous (by the definition of class $S(m_0,m_1,m_2,\alpha)$), by applying the Minty lemma, it follows that

$$(b(x,t)-a(x,t,Du(x,t)), Du(x,t)-\xi)\ge 0$$

a.e. in Q and for all $\xi\in\mathbb{R}^N$. From the arbitrariety of $\xi\in\mathbb{R}^N$, we deduce that $b(x,t) = a(x,t,Du(x,t))$ a.e. in Q. By (4.4) and (4.9), letting $h\to\infty$, we have $u' - \operatorname{div} b(x,t) + w = f$. This together with (4.7) and (4.8) gives $u\in\mathcal{N}(\mathcal{A},\mathcal{J},f,u_0)$. The proof is completed. □

Corollary 4.1. *Under the hypotheses of Theorem* 4.1*, we have*

$$\limsup_{h\to\infty} \mathcal{N}(\mathcal{A}_h, \mathcal{J}_h, f_h, u_0^h) \subset \mathcal{N}(\mathcal{A}, \mathcal{J}, f, u) \quad \textit{in } \mathcal{W}_{\mathrm{weak}} \textit{ topology.}$$

Remark 4.2. It should be mentioned that the sufficient conditions for the convergence (4.3) of Clarke's generalized gradients have been found by Zolezzi [24]. The key conditions imposed on the sequence $\mathcal{J}_h$ were Γ-convergence, local equi-boundedness and equi-lower semidifferentiability.

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