**T**opological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 15, 2000, 191–202

# COINCIDENCE AND FIXED POINT THEOREMS WITH APPLICATIONS

Qamrul H. Ansari — Adam Idzik — Jen-Chih Yao

*Dedicated to the memory of Juliusz P. Schauder*

Abstract. In this paper, we first establish a coincidence theorem under the noncompact settings. Then we derive some fixed point theorems for a family of functions. We apply our fixed point theorem to study nonempty intersection problems for sets with convex sections and obtain a social equilibrium existence theorem. We also introduce a concept of a quasi-variational inequalities and prove an existence result for a solution to such a system.

## 1. Introduction and preliminaries

In 1952, Debreu [7] introduced the concept of the generalized the Nash equilibrium which extends the classical concept of Nash equilibrium for a noncooperative game [18]. Since then, it is widely studied by using some kinds of fixed point theorems, see for example [6], [9], [10], [12], [13], [16], [17], [20]–[23], and references therein. The remaining part of this section deals with preliminaries. In Section 2, we establish a coincidence theorem under the noncompact setting. Then we derive some fixed point theorems for a family of functions which generalize earlier results of Lan and Webb [14]. In Section 3, we study nonempty intersection problems for sets with convex sections. A social equilibrium existence theorem which is applied to results on saddle points, minimax theorems

2000 *Mathematics Subject Classification.* 54H25, 47H10, 49J40, 52A07, 49J35.
*Key words and phrases.* Coincidence, fixed point, convex section, equilibrium, quasi-variational inequality.
This research was supported by the National Science Council of the Republic of China.

and Nash equilibria, is obtained in Section 4. In the last section, we introduce a concept of a system of quasi-variational inequalities which includes the system of variational inequalities studied in [1], [3], [5], [19], as a special case. We also derive existence results for such a system of quasi-variational inequalities.

We shall use the following notation and definitions. Let $A$ be a nonempty set. We shall denote by $2^A$ the family of all subsets of $A$. If $A$ and $B$ are two nonempty subsets of a topological vector space $X$ such that $B \subseteq A$, we shall denote by $\mathrm{int}_A B$ the interior of $B$ in $A$. If $A$ is a subset of a vector space, $\mathrm{co}\, A$ denotes the convex hull of $A$.

Let $X$ and $Y$ be two topological vector spaces and $\varphi : X \to 2^Y$ be a multivalued map. Then $\varphi$ is said to have a *local intersection property* [24] if for each $x \in X$ with $\varphi(x) \neq \emptyset$, there exists an open neighbourhood $N(x)$ of $x$ such that $\bigcap_{z \in N(x)} \varphi(z) \neq \emptyset$.

A multivalued map $\varphi$ is said to be *transfer open-valued* [4] if for any $x \in X$, $y \in \varphi(x)$ there exists a $z \in X$ such that $y \in \mathrm{int}_Y \varphi(z)$.

A *graph of* $\varphi$, denoted by $\mathrm{gr}\, \varphi$, is

$$\{(x, z) \in X \times Y : x \in X,\ z \in \varphi(x)\}.$$

An *inverse* of $\varphi$, denoted by $\varphi^{-1}$, is the multivalued map from the range of $\varphi$ to $X$ defined by

$$x \in \varphi^{-1}(z) \text{ if and only if } z \in \varphi(x).$$

We mention recent results of Ding [8] and Lin [15], Yu [25] and the well known Berge's theorem [2] which will be used in the sequel.

Lemma 1.1 ([8], [15]). *Let $X$ and $Y$ be two topological vector spaces and $\varphi : X \to 2^Y$ be a multivalued map with nonempty values. Then the following statements are equivalent:*

(i) *$\varphi^{-1}$ is transfer open-valued,*
(ii) *$\varphi$ has the local intersection property,*
(iii) $X = \bigcup_{y \in Y} \mathrm{int}_X \varphi^{-1}(y)$.

Lemma 1.2 ([25]). *Let $X$ and $Y$ be two Hausdorff topological vector spaces and $Y$ be compact. Let $f : X \times Y \to \mathbb{R}$ be a function such that*

(i) *$f$ is upper semicontinuous on $X \times Y$, and*
(ii) *for each fixed $y \in Y$, $x \mapsto f(x, y)$ is lower semicontinuous on $X$.*

*Then the function $\Phi : X \to \mathbb{R}$ defined by*

$$\Phi(x) = \max_{u \in Y} f(x, y) \quad \textit{for all } x \in X$$

*is continuous on $X$.*

Lemma 1.3 ([2]). *Let $X$ and $Y$ be topological vector spaces, $f : X \times Y \to \overline{\mathbb{R}}$ an extended real-valued function, $\varphi : X \to 2^Y$ a multivalued map, and*

$$\widehat{f}(x) = \sup_{y \in \varphi(x)} f(x, y) \quad \text{for all } x \in X.$$

(i) *If $f$ is upper semicontinuous and $\varphi$ is upper semicontinuous with compact values, then $\widehat{f}$ is upper semicontinuous.*
(ii) *If $f$ is lower semicontinuous and $\varphi$ is lower semicontinuous, then $\widehat{f}$ is lower semicontinuous.*

## 2. Coincidence and fixed point theorems

Let $I$ be an index set and for each $i \in I$, let $E_i$ be a Hausdorff topological vector space. Let $\{K_i\}_{i \in I}$ be a family of nonempty convex subsets with each $K_i$ in $E_i$. Let $K = \prod_{i \in I} K_i$ and $K^i = \prod_{j \in I,\ j \neq i} K_j$ and, we write $K = K^i \times K_i$. For each $x \in K$, $x_i \in K_i$ denotes the $i$th coordinate and $x^i \in X^i$ the projection of $x$ on $X^i$ and we also write $x = (x^i, x_i)$. We use this denotation throughout our paper.

Theorem 2.1. *For each $i \in I$, let $\varphi_i : K_i \to 2^{K_i}$ and $\psi_i : K^i \to 2^{K_i}$ be two multivalued maps. Assume that the following conditions hold:*

(i) *For each $i \in I$ and each $x^i \in K^i$, $\varphi_i^{-1}(\psi_i(x^i))$ is nonempty and convex.*
(ii) *For each $i \in I$, $K^i = \bigcup\{\operatorname{int}_{K^i} \psi_i^{-1}(\varphi_i(x_i)) : x_i \in K_i\}$.*
(iii) *If $K^i$ is not compact, assume that there exist a nonempty compact convex subset $B_i$ of $K_i$ and a nonempty compact subset $D^i$ of $K^i$ such that for each $x^i \in K^i \backslash D^i$ there exists $\widetilde{y}_i \in B_i$ such that $x^i \in \operatorname{int}_{K^i} \psi_i^{-1}(\varphi_i(\widetilde{y}_i))$.*

*Then there exists $\overline{x} \in K$ such that $\psi_i(\overline{x}^i) \cap \varphi_i(\overline{x}_i) \neq \emptyset$, for each $i \in I$.*

Proof. Although it is based on one given in [1] for the fixed points of the family of functions, we include it for the sake of completeness of the paper. For each $i \in I$, we define a multivalued map $\phi_i : K_i \to 2^{K^i}$ by

$$\phi_i(x_i) = \{x^i \in K^i : x^i \notin \operatorname{int}_{K^i} \psi_i^{-1}(\varphi_i(x_i))\} = K^i \setminus \operatorname{int}_{K^i} \psi_i^{-1}(\varphi_i(x_i)).$$

Then $\phi_i$ satisfies the following conditions:

(a) For each $x_i \in K_i$, $\phi_i(x_i)$ is closed in $K^i$.
(b) For each $i \in I$, then $\bigcap_{x_i \in B_i} \phi_i(x_i)$ is compact in $K^i$.
Indeed, if $K^i$ is compact, $\bigcap_{x_i \in B_i} \phi_i(x_i)$ is compact since $\bigcap_{x_i \in B_i} \phi_i(x_i)$ is closed in $K^i$ by (a). If $K^i$ is not compact,

$$\bigcap_{x_i \in B_i} \phi_i(x_i) = \bigcap_{x_i \in B_i} \{x^i \in K^i : x^i \notin \operatorname{int}_{K^i} \psi_i^{-1}(\varphi_i(x_i))\} \subset D^i$$

by (iii) and thus is compact.

(c) Since for each $i \in I$, $K^i = \bigcup\{\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(x_i)) : x_i \in K_i\}$, we have $\bigcap_{x_i\in K_i}\phi_i(x_i) = \bigcap_{x_i\in K_i}\{K^i \setminus \mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(x_i))\} = \emptyset$, for each $i \in I$.

Now, we will show that there exist $a_{i1}, \ldots, a_{il_i} \in K_i$ such that

$$\Big(\bigcap_{x_i\in B_i}\phi_i(x_i)\Big) \cap \Big(\bigcap_{k=1}^{l_i}\phi_i(a_{ik})\Big) = \emptyset. \tag{2.1}$$

Suppose that (2.1) is not true, then for every finite set $\{y_1, \ldots, y_n\} \subset K_i$, we have

$$\Big(\bigcap_{x_i\in B_i}\phi_i(x_i)\Big) \cap \Big(\bigcap_{j=1}^{n}\phi_i(y_j)\Big) \neq \emptyset.$$

Let $\chi(y) = \big(\bigcap_{x_i\in B_i}\phi_i(x_i)\big)\cap\big(\phi_i(y)\big)$ for $y \in K_i$. Then the family $\{\chi(y) : y \in K_i\}$ has the finite intersection property. Note that $\chi(y)$ is compact in $K$ for each $y \in K_i$ because $\bigcap_{x_i\in B_i}\phi_i(x_i)$ is compact and $\phi_i(y)$ is closed in $K^i$. It follows that $\bigcap_{y\in K_i}\chi(y) \neq \emptyset$ and thus $\bigcap_{y\in K_i}\phi_i(y) \neq \emptyset$ which is a contradiction with (c).

By (2.1), we have

$$\Big(\bigcup_{x_i\in B_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(x_i))\Big) \cup \Big(\bigcup_{k=1}^{l_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(a_{ik}))\Big) = K^i. \tag{2.2}$$

Let $F_i = \mathrm{co}(B_i \bigcup\{a_{i1}, \ldots, a_{il_i}\})$. Then $F_i$ is compact in $K_i$. Let $F^i = \prod_{j\in I, j\neq i} F_j$, then $F^i$ is a compact subset of $K^i$. By (2.2), we have

$$F^i \subset \Big(\bigcup_{x_i\in B_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(x_i))\Big) \cup \Big(\bigcup_{k=1}^{l_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(a_{ik}))\Big).$$

Since $F^i$ is compact, there exist $b_{i1}, \ldots, b_{it_i} \in B_i$ such that

$$F^i \subset \Big(\bigcup_{j=1}^{t_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(b_{ij}))\Big) \cup \Big(\bigcup_{k=1}^{l_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(a_{ik}))\Big). \tag{2.3}$$

Let $\{c_{i1}, \ldots, c_{in_i}\} = \{a_{i1}, \ldots, a_{il_i}, b_{i1}, \ldots, b_{it_i}\}$. We rewrite (2.3) as follows

$$F^i \subset \bigcup_{k=1}^{n_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(c_{ik})).$$

Let $X_i = \mathrm{co}\{c_{i1}, \ldots, c_{in_i}\}$ and $X^i = \prod_{j\in I, j\neq i} X_j$. We denote by $\Delta_i$ the vector subspace of $E_i$ generated by $X_i$. Then $\Delta_i$ is a finite dimensional subspace. We note that $X^i$ is a compact set in $\prod_{j\in I, j\neq i}\Delta_j$, and $X^i \subset F^i \subset \bigcup_{k=1}^{n_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(c_{ik}))$. Therefore

$$X^i \subset \Big(\bigcup_{k=1}^{n_i}\mathrm{int}_{K^i}\psi_i^{-1}(\varphi_i(c_{ik}))\Big) \cap X^i \subseteq \bigcup_{k=1}^{n_i}\mathrm{int}_{X^i}\psi_i^{-1}(\varphi_i(c_{ik})) \subset X^i$$

and hence $X^i = \bigcup_{k=1}^{n_i}\mathrm{int}_{X^i}\psi_i^{-1}(\varphi_i(c_{ik}))$.

Since $X^i$ is compact, there exists a partition of unity $\{g_{i1}, \dots, g_{in_i}\}$ subordinated to this finite subcovering such that:

(a) for each $k = 1, \dots, n_i$, $g_{ik} : X^i \to [0,1]$ is continuous,
(b) for each $k = 1, \dots, n_i$, $g_{ik}(x^i) = 0$, for $x^i \notin \text{int}_{X^i}\psi_i^{-1}(\varphi_i(c_{ik}))$,
(a) for each $x^i \in X^i$, $\sum_{k=1}^{n_i} g_{ik}(x^i) = 1$.

For each $i \in I$, we define a map $f_i : X^i \to X_i$ by $f_i(x^i) = \sum_{k=1}^{n_i} g_{ik}(x^i)c_{ik}$, for all $x^i \in X^i$. Obviously, for each $i \in I$, $f_i$ is continuous. For each $x^i \in X^i$ and each $k$ with $g_{ik}(x^i) \neq 0$, we have $x^i \in \text{int}_{X^i}\psi_i^{-1}(\varphi_i(c_{ik})) \subset \psi_i^{-1}(\varphi_i(c_{ik}))$ and so that $c_{ik} \in \varphi_i^{-1}(\psi_i(x^i))$ for each $i \in I$. Because $f_i(x^i)$ is a convex combination of $c_{i1}, \dots, c_{ik_i}$ and because $\varphi_i^{-1}(\psi_i(x^i))$ is convex by (i), we have for each $i \in I$, $f_i(x^i) \in \varphi_i^{-1}(\psi_i(x^i))$, for all $x^i \in X^i$.

Define a map $h : X \to X$ by $h(x) = (f_i(x^i))_{i\in I}$. Since for each $x \in X$, we have $x^i \in X^i$ and $f_i(x^i) \in X_i$, it follows that $h$ is well-defined and continuous. By Tychonoff's fixed point theorem, $h$ has a fixed point $\overline{x} = (f_i(\overline{x}^i))_{i\in I} \in X$. This implies that $\overline{x}_i = f_i(\overline{x}^i)$ for each $i \in I$. Hence $\overline{x}_i = f_i(\overline{x}^i) \in \varphi_i^{-1}(\psi_i(\overline{x}^i))$ and therefore $\psi_i(\overline{x}^i) \cap \varphi_i(\overline{x}_i) \neq \emptyset$, for each $i \in I$. □

When $\varphi(x_i) = \{x_i\}$, we have the following result on fixed points for a family of multivalued maps.

THEOREM 2.2. *For each $i \in I$, let $\psi_i : K^i \to 2^{K_i}$ be a multivalued map. Assume that the following conditions hold:*

(i) *For each $i \in I$ and each $x^i \in K^i$, $\psi_i(x^i)$ is nonempty and convex.*
(ii) *For each $i \in I$, $K^i = \bigcup\{\text{int}_{K^i}\psi_i^{-1}(x_i) : x_i \in K_i\}$.*
(iii) *If $K^i$ is not compact, assume that there exist a nonempty compact convex subset $B_i$ of $K_i$ and a nonempty compact subset $D^i$ of $K^i$ such that for each $x^i \in K^i \setminus D^i$ there exists $\widetilde{y}_i \in B_i$ such that $x^i \in \text{int}_{K^i}\psi_i^{-1}(\widetilde{y}_i)$.*

*Then there exists $\overline{x} \in K$ such that $\overline{x}_i \in \psi_i(\overline{x}^i)$, for each $i \in I$.*

REMARK 2.3.

(a) Theorems 2.1 and 2.2 are non-compact version of Theorems 3 and 4 in [10], respectively.
(b) If for each $x_i \in K_i$, $\psi_i^{-1}(x_i)$ is open in $K^i$, then by assumption (i) in Theorem 2.2, $K^i = \bigcup\{\text{int}_{K^i}\psi_i^{-1}(x_i) : x_i \in K_i\}$. Hence Theorem 2.2 contains Theorem 2.1 in [14].
(c) In view of Lemma 1.1, assumption (ii) in Theorem 2.2 can be replaced by any one of the following conditions:
(ii)' for each $i \in I$, $\psi_i^{-1}$ is transfer open-valued,
(ii)" for each $i \in I$, $\psi_i$ has the local intersection property.

The following result is a consequence of Theorem 2.2 and generalizes Theorem 2.2 in [14].

Theorem 2.4. *For each $i \in I$, let $\phi_i : K^i \to 2^{K_i}$ be a multivalued map. Assume that the following conditions hold:*

(i) *For each $i \in I$ and each $x^i \in K^i$, $\phi_i(x^i)$ is nonempty.*
(ii) *For each $i \in I$, $K^i = \bigcup\{\mathrm{int}_{K^i}\phi_i^{-1}(x_i) : x_i \in K_i\}$.*
(iii) *If $K^i$ is not compact, assume that there exist a nonempty compact convex subset $B_i$ of $K_i$ and a nonempty compact subset $D^i$ of $K^i$ such that for each $x^i \in K^i \setminus D^i$ there exists $\widetilde{y}_i \in B_i$ such that $x^i \in \mathrm{int}_{K^i}\mathrm{co}\,\phi_i^{-1}(\widetilde{y}_i)$.*

*Then there exists $\overline{x} \in K$ such that $\overline{x}_i \in \mathrm{co}\,\phi_i(\overline{x}^i)$, for each $i \in I$.*

Proof. For each $i \in I$, we define a multivalued map $\psi_i : K^i \to 2^{K_i}$ by $\psi_i(x^i) = \mathrm{co}\,\phi_i(x^i)$. Then it is easy to verify that for each $i \in I$, $\psi_i$ satisfies all the conditions of Theorem 2.2. □

## 3. Intersection theorems for sets with convex sections

Let $Y$ be a topological space. A family $\{A_i\}_{i\in I}$ of subsets in $Y$ is said to be *open transfer complete* (respectively, *closed transfer complete*) if $y \in A_i$ (respectively, $y \notin A_i$), there exists $j \in I$ such that $y \in \mathrm{int}_Y A_j$ (respectively, $y \notin \mathrm{cl}_Y A_j$), where $\mathrm{cl}_Y A$ denotes the closure of $A$ in $Y$ for any subset $A$ of $Y$.

For $A \subset K$, $x^i \in K^i$ and $x_i \in K_i$, we define $A[x_i] = \{x^i \in K^i : (x^i, x_i) \in A\}$ and $A[x^i] = \{x_i \in K_i : (x^i, x_i) \in A\}$.

Now we extend Lemma 2.1 in [4] as follows:

Lemma 3.1. *Let $\{A_i\}_{i\in I}$ be a family of subsets of $K$. Then the following conditions hold:*

(i) *for each $i \in I$, the family $\{A_i[x^i] : x^i \in K^i\}$ is closed transfer complete if and only if*

$$\bigcap_{x^i\in K^i} A_i[x^i] = \bigcap_{x^i\in K^i} \mathrm{cl}_{K_i} A_i[x^i],$$

(ii) *for each $i \in I$, the family $\{A_i[x^i] : x^i \in K^i\}$ is open transfer complete if and only if*

$$\bigcup_{x^i\in K^i} A_i[x^i] = \bigcup_{x^i\in K^i} \mathrm{int}_{K_i} A_i[x^i],$$

(iii) *if for each $i \in I$, $A_i[x^i]$ is nonempty and the family $\{A_i[x_i] : x_i \in K_i\}$ is open transfer complete, then $K^i = \bigcup_{x_i\in K_i} \mathrm{int}_{K^i} A_i[x_i]$.*

Since the proof of this lemma is similar to the proof of Lemma 2.1 in [4], we omit it.

From Theorem 2.4, we obtain the following results on sets with convex sections:

THEOREM 3.2. *Let $\{A_i\}_{i\in I}$ be a family of subsets of $K$. Assume that the following conditions hold:*

(i) *for each $i \in I$ and each $x^i \in K^i$, $A_i[x^i]$ is nonempty,*
(ii) *for each $i \in I$, $K^i = \bigcup_{x_i\in K_i} \operatorname{int}_{K^i} A_i[x_i]$,*
(iii) *if $K^i$ is not compact, assume that there exist a nonempty compact convex subset $B_i$ of $K_i$ and a nonempty compact subset $D^i$ of $K^i$ such that for each $x^i \in K^i \setminus D^i$ there exists $\widetilde{y}_i \in B_i$ such that $x^i \in \operatorname{int}_{K^i} \operatorname{co} A_i[\widetilde{y}_i]$.*

*Then there exists $\overline{x} \in K$ such that $\overline{x}_i \in \operatorname{co} A_i[\overline{x}^i]$, for each $i \in I$.*

PROOF. For each $i \in I$, we define a multivalued map $\phi_i : K^i \to 2^{K_i}$ by

$$\phi_i(x^i) = A_i[x^i], \quad \text{for all } x^i \in K^i.$$

It is easy to verify that for each $i \in I$, $\phi_i$ satisfies all the conditions of Theorem 2.4. Hence there exists $\overline{x} \in K$ such that $\overline{x}_i \in \operatorname{co} A_i[\overline{x}^i]$, for each $i \in I$. □

THEOREM 3.3. *Let $\{A_i\}_{i\in I}$ and $\{\widetilde{A}_i\}_{i\in I}$ be two families of subsets of $K$. Assume that the following conditions hold:*

(i) *for each $i \in I$ and each $x^i \in K^i$, $A_i[x^i]$ is nonempty,*
(ii) *for each $x \in K$, there exists a subset $I(x) \subset I$ such that for $i \in I(x)$, $\operatorname{co} A_i[x^i] \subset \widetilde{A}_i[x^i]$,*
(iii) *for each $i \in I$, $K^i = \bigcup_{x_i\in K_i} \operatorname{int}_{K^i} A_i[x_i]$,*
(iv) *if $K^i$ is not compact, assume that there exist a nonempty compact convex subset $B_i$ of $K_i$ and a nonempty compact subset $D^i$ of $K^i$ such that for each $x^i \in K^i \setminus D^i$ there exists $\widetilde{y}_i \in B_i$ such that $x^i \in \operatorname{int}_{K^i} \operatorname{co} A_i[\widetilde{y}_i]$.*

*Then there exists $\overline{x} \in K$ such that $\bigcap_{i\in I(\overline{x})} \widetilde{A}_i \neq \emptyset$.*

PROOF. By Theorem 3.2, there exists $\overline{x} \in K$ such that $\overline{x}_i \in \operatorname{co} A_i[\overline{x}^i]$, for each $i \in I$. From assumption (ii), we have $\overline{x}_i \in \widetilde{A}_i[\overline{x}^i]$ for $i \in I(\overline{x})$. This implies that $\overline{x} \in \widetilde{A}_i$, for each $i \in I(\overline{x})$. □

REMARK 3.4. Theorems 3.2 and 3.3 generalize Theorems 2.3 and 2.4, respectively, in [14].

In view of Lemma 3.1, we have the following

REMARK 3.5. The assumption (ii) in Theorem 3.2 and the assumption (iii) in Theorem 3.3 can be replaced by the following condition:

(0) *For each $i \in I$, the family $\{A_i[x_i] : x_i \in K_i\}$ is open transfer complete.*

## 4. Equilibrium existence theorems

For $S \subset K$, $x^i \in K^i$ and $x_i \in K_i$, let $S(x^i) = \{y_i \in K_i : (x^i, y_i) \in S\}$.

From Theorem 2.2, we obtain the following social equilibrium existence theorem (cf. [7]):

Theorem 4.1. *Let $\{K_i\}_{i\in I}$ be a family of nonempty compact convex subsets with each $K_i$ in $E_i$. For each $i \in I$, let $S_i : K^i \to 2^{K_i}$ be an upper semicontinuous multivalued map with nonempty compact convex values such that $S_i^{-1}(x_i)$ is open in $K^i$, for all $x_i \in K_i$. For each $i \in I$, let $f_i : K \to \mathbb{R}$ satisfy the following conditions:*

(i) *for each $i \in I$, $f_i$ is upper semicontinuous on $\operatorname{gr} S_i$,*
(ii) *for each $i \in I$, $\widehat{f}_i(x^i) = \max_{z\in S_i(x^i)} f_i(x^i, z)$ is a lower semicontinuous function,*
(iii) *for each $i \in I$ and for each fixed $y_i \in K_i$, $x^i \mapsto f_i(x^i, y_i)$ is lower semicontinuous on $K^i$,*
(iv) *for each $i \in I$ and for each fixed $x^i \in K^i$, $y_i \mapsto f_i(x^i, y_i)$ is quasi-concave on $K_i$.*

*Then there exists an equilibrium point $\overline{x} \in \operatorname{gr} S_i$ for each $i \in I$; that is, $\overline{x}_i \in S_i(\overline{x}^i)$ and $f_i(\overline{x}) = \max_{x_i\in S_i(\overline{x}^i)} f_i(\overline{x}^i, x_i)$, for each $i \in I$.*

Proof. For each $i \in I$ and each $n = 1, 2, \ldots$, we define a multivalued map $\psi_{(i,n)} : K^i \to 2^{K_i}$ by

$$\psi_{(i,n)}(x^i) = \{x_i \in S_i(x^i) : f_i(x^i, x_i) > \max_{z\in S_i(x^i)} f_i(x^i, z) - 1/n\}, \quad \text{for all } x^i \in K^i.$$

Since $S_i(x^i)$ is compact and $f_i$ is upper semicontinuous, we have $\psi_{(i,n)}(x^i)$ is nonempty for each $i \in I$ and $x^i \in K^i$. By the assumption (iv), for each $i \in I$ and $x^i \in K^i$, $\psi_{(i,n)}(x^i)$ is convex.

Now for each $i \in I$ and $x_i \in S_i(x^i)$, we have

$$\begin{aligned}\psi^{-1}_{(i,n)}(x_i) &= \{x^i \in K^i : x_i \in S_i(x^i) \text{ and } f_i(x^i, x_i) > \max_{z\in S_i(x^i)} f_i(x^i, z) - 1/n\}\\ &= S_i^{-1}(x_i) \cap \{x^i \in K^i : f_i(x^i, x_i) > \max_{z\in S_i(x^i)} f_i(x^i, z) - 1/n\}.\end{aligned}$$

By our assumptions and Lemma 1.3, the set

$$\{x^i \in K^i : f_i(x^i, x_i) > \max_{z\in S_i(x^i)} f_i(x^i, z) - 1/n\}$$

is open in $K^i$. Since $S_i^{-1}(x_i)$ is open in $K^i$ for any $x_i \in K_i$, $\psi^{-1}_{(i,n)}(x_i)$ is open in $K^i$, for all $x_i \in K_i$. Since for each $i \in I$, $\psi_{(i,n)}(x^i)$ is nonempty and $\psi^{-1}_{(i,n)}(x_i)$ is open in $K^i$, we have

$$K^i = \bigcup_{x_i\in K_i} \psi^{-1}_{(i,n)}(x_i) = \bigcup_{x_i\in K_i} \operatorname{int}_{K^i} \psi^{-1}_{(i,n)}(x_i).$$

By Theorem 2.2, there exists $\widehat{x}_n = (\widehat{x}^{(i,n)}, \widehat{x}_{(i,n)}) \in K$ such that $\widehat{x}_{(i,n)} \in \psi_{(i,n)}(\widehat{x}^{(i,n)})$, for each $i \in I$ and, for each $n = 1, 2, \ldots$, that is,

$$\widehat{x}_{(i,n)} \in S_i(\widehat{x}^{(i,n)}) \ : \ f_i(\widehat{x}^{(i,n)}, \widehat{x}_{(i,n)}) > \max_{z\in S_i(\widehat{x}^{(i,n)})} f_i(\widehat{x}^{(i,n)}, z) - 1/n,$$

for each $n = 1, 2, \ldots$. Since $K_i$ is compact, without loss of generality, we may assume that $\widehat{x}_n \to \overline{x} \in K$, that is, $\widehat{x}^{(i,n)} \to \overline{x}^i \in K^i$ and $\widehat{x}_{(i,n)} \to \overline{x}_i \in K^i$. Since for each $i \in I$, $S_i$ is compact-valued and upper semicontinuous, the graph of $S_i$ is closed and therefore $\overline{x}_i \in S_i(\overline{x}^i)$. By assumptions (i) and (ii), we have

$$\begin{aligned} f_i(\overline{x}^i, \overline{x}_i) &\geq \overline{\lim_{n\to\infty}} f_i(\widehat{x}^{(i,n)}, \widehat{x}_{(i,n)}) \geq \overline{\lim_{n\to\infty}} [\max_{z\in S_i(\widehat{x}^{(i,n)})} f_i(\widehat{x}^{(i,n)}, z) - 1/n] \\ &\geq \underline{\lim_{n\to\infty}} [\max_{z\in S_i(\widehat{x}^{(i,n)})} f_i(\widehat{x}^{(i,n)}, z) - 1/n] \geq \max_{z\in S_i(\overline{x}^i)} f_i(\overline{x}^i, z). \end{aligned}$$

Hence $f_i(\overline{x}^i, \overline{x}_i) = \max_{z\in S_i(\overline{x}^i)} f_i(\overline{x}^i, z)$. □

Remark 4.2.

(a) In the proof of Theorem 4.1 we used in fact the nets (the sets $K_i$ need not be metrizable).
(b) We notice that Theorem 5.2 in [16] is not correct in the present form. We need one more assumption that for each $i = 1, \ldots, n$, $G_i^{-1}(z)$ is open in $K$, where $G_i$ is defined as in Theorem 5.2 in [16]. Theorem 4.1 corrects and generalizes this theorem in the sense that the index set need not be finite.
(c) Similar results to Theorem 4.1 were obtained by Idzik [10] (see Theorem 7) and Idzik and Park [12] (see Theorem 3.2) with the inequalities for equilibrium points instead the equalities.

From Theorem 4.1, we have the following saddle point and minimax theorems:

Theorem 4.3. *Let $X$ and $Y$ be two compact convex subset of a Hausdorff topological vector space $E$. Let $f : X \times Y \to \mathbb{R}$ be an upper semicontinuous function on $X \times Y$ such that*

(i) *for each fixed $y \in Y$, $x \mapsto f(x, y)$ is lower semicontinuous and quasi-convex on $X$, and*
(ii) *for each fixed $x \in X$, $y \mapsto f(x, y)$ is quasi-concave on $Y$.*

*Then $f$ has a saddle point $(\overline{x}, \overline{y}) \in X \times Y$, that is*

$$\min_{y\in Y} f(\overline{x}, y) = f(\overline{x}, \overline{y}) = \max_{x\in X} f(x, \overline{y}).$$

Proof. It is similar to the proof of Theorem 3.3 in [12]. □

Theorem 4.4. *Under the hypothesis of Theorem 4.3, we have the following minimax inequality*

$$\max_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \max_{x\in X} f(x, y).$$

Proof. It is similar to the proof of Theorem 3.4 in [12]. □

Remark 4.5. In Theorems 4.3 and 4.4, we have neither assumed that $X$ and $Y$ are convexly totally bounded (see [11] for the definition) nor $f$ is continuous on $X \times Y$ as it is assumed in Theorems 3.3 and 3.4 in [12] and hence Theorems 4.3 and 4.4 generalize Theorems 3.3 and 3.4, respectively, in [12].

When $S_i(x^i) = K_i$ for each $x^i \in K^i$, we obtain the following generalization of the Nash equilibrium theorem (the condition (ii) of Theorem 4.1 is fulfilled by Lemma 1.2:

Theorem 4.6. *Let $\{K_i\}_{i \in I}$ be a family of nonempty compact convex subset with each $K_i$ in $E_i$. For each $i \in I$, let $f_i : K \to \mathbb{R}$ satisfy the following conditions:*

(i) *for each $i \in I$, $f_i$ is upper semicontinuous,*
(ii) *for each $i \in I$ and for each fixed $y_i \in K_i$, $x^i \mapsto f_i(x^i, y_i)$ is lower semicontinuous on $K^i$,*
(iii) *for each $i \in I$ and for each fixed $x^i \in K^i$, $y_i \mapsto f_i(x^i, y_i)$ is quasi-concave on $K_i$.*

*Then there exists a point $\overline{x} \in K$ such that, for each $i \in I$,*

$$f_i(\overline{x}) = \max_{y_i \in K_i} f_i(\overline{x}^i, y_i).$$

Remark 4.7. Theorem 4.6 is an infinite version of Theorem 3.2 in [25] and it generalizes Theorem 5 in [21] in the following ways:

(a) $K$ need not be convexly totally bounded [11],
(b) for each $i \in I$, $f_i$ need not be continuous.

## 5. The system of quasi-variational inequalities

For each $i \in I$, let $E_i$ be a locally convex Hausdorff topological vector space with its dual $E_i^*$. For each $i \in I$, let $\theta_i : K^i \to E_i^*$ be an operator and $\sigma_i : K^i \to 2^{K_i}$ be a multivalued map. We consider the *system of quasi-variational inequalities* (in short, SQVI) which is to find $\overline{x} \in K$ such that for each $i \in I$,

$$\overline{x}_i \in \sigma_i(\overline{x}^i) : \langle \theta_i(\overline{x}^i), \overline{x}_i - y_i \rangle \leq 0 \quad \text{for all } y_i \in \sigma_i(\overline{x}^i),$$

where $\langle \cdot\, , \cdot \rangle$ denotes the pairing between $E_i^*$ and $E_i$.

In the case each $i \in I$ and $x^i \in K^i$, $\sigma_i(x^i) = K_i$, we have the *system of variational inequalities* (SVI), that is, to find $\overline{x} \in K$ such that for each $i \in I$,

$$\langle \theta_i(\overline{x}^i), \overline{x}_i - y_i \rangle \leq 0 \quad \text{for all } y_i \in K_i.$$

SVI was considered by Pang [19] with applications in equilibrium problems. Later, it has also been studied by Ansari and Yao [1], Bianchi [3] and Cohen and Chaplais [5].

Now from Theorem 4.1, we derive the following existence result for the SQVI:

Theorem 11. *Let $\{K_i\}_{i\in I}$ be a family of nonempty compact convex subsets with each $K_i$ in $E_i$. For each $i \in I$, let $\sigma_i : K^i \to 2^{K_i}$ be an upper semicontinuous multivalued map with nonempty compact convex values such that $\sigma_i^{-1}(x_i)$ is open in $K^i$, for all $x_i \in K_i$. Let $\theta_i : K^i \to E_i^*$ be a continuous operator on $K^i$. Then there exists a solution to the* SQVI.

Proof. Taking $f_i(x^i, y_i) = \langle \theta_i(x^i), x_i - y_i\rangle$ in Theorem 4.1, we obtain the result. □

## References


[1] Q. H. Ansari and J. C. Yao, *A fixed point theorem and its applications to the system of variational inequalities*, Bull. Austral. Math. Soc. **59** (1999), 433–442.

[2] C. Berge, *Topological Spaces*, Oliver and Boyd, Edinburgh, Scotland, 1963.

[3] M. Bianchi, *Pseudo P-monotone operators and variational inequalities*, Universita Cattolica del Sacro Cuore **6** (1993).

[4] S. S. Chang, B. S. Lee, X. Wu, Y. J. Cho and G. M. Lee, *On the generalized quasi-variational inequality problems*, J. Math. Anal. Appl. **203** (1996), 686–711.

[5] G. Cohen and F. Chaplais, *Nested monotony for variational inequalities over product of spaces and convergence of iterative algorithms*, J. Optim. Theory Appl. **59** (1988), 360–390.

[6] P. Cubiotti, *Existence of Nash equilibria for generalized games without upper semicontinuity*, Internat. J. Game Theory **26** (1997), 267–273.

[7] G. Debreu, *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 886–893; *Twenty Papers of Gerald Debreu*, Mathematical Economics, Cambridge Univ. Press, Cambridge, 1983.

[8] X. P. Ding, *Coincidence theorems involving composites of acyclic mappings in contractible spaces*, Appl. Math. Lett. **11** (1998), 85–89.

[9] A. Idzik, *Remarks on Himmelberg's fixed point theorems*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **26** (1978), 909–912.

[10] ———, *Fixed point theorems for family of functions*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **26** (1978), 913–916.

[11] ———, *Almost fixed point theorems*, Proc. Amer. Math. Soc. **104** (1988), 779–784.

[12] A. Idzik and S. Park, *Leray–Schauder type theorem and equilibrium existence theorems*, Lecture Notes in Nonlinear Analysis, vol. 2; J. Andres, L. Górniewicz and P. Nistri (eds.), Differential Inclusions Optim. Control, Juliusz Schauder Center for Nonlinear Studies, Nicolas Copernicus University, Toruń, Edinburgh, Scotland, 1998, pp. 191–197.

[13] T. Kaczyǹski and V. Zeidan, *An application of Ky-Fan point theorem to an optimization problem*, Nonlinear Anal. **13** (1989), 259–261.

[14] K. Q. Lan and J. Webb, *New fixed point theorems for a family of mappings and applications to problems on sets with convex sections*, Proc. Amer. Math. Soc. **126** (1998), 1127–1132.

[15] L. J. Lin, *Applications of a fixed point theorem in G-convex space*, Nonlinear Anal., 2000 (to appear).

[16] E. Marchi and J.-E. Martìnez–Legaz, *Some results on approximate continuous selections, fixed points and minimax inequalities*, Fixed Point Theory and Applications (M. A. Thèra and J.-B. Baillon, eds.), Longman Scientific & Technical, 1991, pp. 327–342.

[17] E. Marchi and J.-E. Martinez-Legaz, *A generalization of Fan–Browder's fixed point theorem and its applications*, Topol. Methods Nonlinear Anal. **2** (1993), 277–291.

[18] J. Nash, *Non-cooperative games*, Ann. Math. **54** (1951), 286–293.

[19] J. S. Pang, *Asymmetric variational inequality problems over product sets: applications and iterative methods*, Math. Programming **31** (1985), 206–219.

[20] S. Park, *Remarks on a social equilibrium existence theorem of G. Debreu*, Appl. Math. Lett. **11** (1998), 51–54.

[21] S. Park and J. A. Park, *The Idzik type quasivariational inequalities and noncompact optimization problems*, Colloq. Math. **71** (1996), 287–295.

[22] D. I. Rim and W. K. Kim, *A fixed point theorem and existence of equilibrium for abstract economies*, Bull. Austral. Math. Soc. **45** (1992), 385–394.

[23] G. Tian and J. Zhou, *The maximum theorem and the existence of Nash equilibrium of (generalized) games without lower semicontinuities*, J. Math. Anal. Appl. **166** (1992), 351–364.

[24] X. Wu and S. Shen, *A further generalization of Yannelis–Prabhakar's continuous selection theorem and its applications*, J. Math. Anal. Appl. **196** (1996), 61–74.

[25] J. Yu, *Essential equilibria of n-person noncooperative games*, J. Math. Econom. **31** (1999), 361–372.

*Manuscript received December 1, 1999*

Qamrul H. Ansari and Jen-Chih Yao
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, TAIWAN 804, ROC

*E-mail address*: ansari@math.nsysu.edu.tw, yaojc@math.nsysu.edu.tw

Adam Idzik
Institute of Computer Science
Polish Academy of Sciences
Ordona 21
01-237 Warsaw, POLAND

*E-mail address*: adidzik@ipipan.waw.pl