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A SHORT PROOF OF THE CONVERSE TO THE CONTRACTION PRINCIPLE AND SOME RELATED RESULTS

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Dedicated to the memory of Juliusz P. Schauder

Abstract. We simplify a proof of Bessaga's theorem given in the monograph of Deimling. Moreover, our argument let us also obtain the following result.

Let F be a selfmap of an arbitrary set Ω and $\alpha \in (0,1)$. Then F is an α-similarity with respect to some complete metric d for Ω (that is, $d(Fx, Fy) = \alpha d(x, y)$ for all $x, y \in \Omega$) if and only if F is injective and F has a unique fixed point.

Finally we present that the converse to the Contraction Principle for bounded spaces is independent of the Axiom of Choice.

1. Introduction

In 1959 Bessaga [2] proved the following converse of the Banach Contraction Principle. We quote it in a slightly more general form as is done in the monograph of Deimling [3, Theorem 17.5].

Theorem 1 (Bessaga). *Let $\Omega \neq \emptyset$ be an arbitrary set, $F : \Omega \mapsto \Omega$ and $\alpha \in (0,1)$. Then*

(a) *If F^n has at most one fixed point for every $n \in \mathbb{N}$, then there exists a metric d such that $d(Fx, Fy) \le \alpha\, d(x, y)$ for all $x, y \in \Omega$.*

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(b) *If, in addition, some* F^n *has a fixed point, then there is a complete metric* d *such that* $d(Fx,Fy)\le \alpha d(x,y)$ *for all* $x,y\in\Omega$.

There are at least three different proofs of this result. The first one, due to Bessaga [2], uses a special form of the Axiom of Choice (abbr., AC). Its original version is not long, however, some statements are left to the reader for verifying. Actually, Bessaga proved part (b) of Theorem 1.

The second proof, from Deimling's book [3, pp. 191–192], is a special case of that given by Wong [11], and it uses the Kuratowski–Zorn Lemma. In fact, Wong extended Bessaga's theorem to a finite family of commuting maps.

The third proof, due to Janos [7], is based on combinatorial techniques with a use of Ramsey's theorem. Actually, the existence of a separable metric is shown here (under the assumption that Ω has at most continuum many elements), though this metric need not be complete.

Our purpose here is to give possibly the simplest proof of Bessaga's theorem by modifying the proof presented in Deimling [3]. Moreover, our argument enables us to extend part (a) of Theorem 1 and it let us also answer the question: When is F a similarity? That is, when does F satisfy the condition $d(Fx,Fy)=\alpha d(x,y)$ for all $x,y\in\Omega$?

Finally we show that the converse to the Contraction Principle for bounded spaces is independent of the AC.

2. A proof of Bessaga's theorem

It is worth noticing that according to the following lemma, the problem of the existence of a metric d for Ω can be simplified — it suffices to prove the existence of a function φ of *one* variable, which would satisfy the Schröder functional inequality for all $x\in\Omega$:

$$\varphi(Fx)\le \alpha\,\varphi(x). \tag{1}$$

Lemma 1. *Let* F *be a selfmap of a set* Ω *and* $\alpha\in(0,1)$. *The following statements are equivalent:*

(i) *there exists a complete metric* d *for* Ω *such that*

$$d(Fx,Fy)\le \alpha d(x,y)\quad \textit{for all } x,y\in\Omega,$$

(ii) *there exists a function* $\varphi:\Omega\mapsto\mathbb{R}_+$ *such that* $\varphi^{-1}(\{0\})$ *is a singleton and inequality* (1) *holds on* Ω.

Proof. (i)⇒(ii) By the Banach theorem F has a fixed point z. Then it suffices to set $\varphi(x):=d(x,z)$.

(ii)⇒(i) Define d by $d(x,y):=\varphi(x)+\varphi(y)$ if $x\ne y$ and $d(x,x):=0$. It is easily seen that d is a metric for Ω and by (1) F is α-contractive. Let (x_n) be a

Cauchy sequence. We may assume that the set $\{x_n : n \in \mathbb{N}\}$ is infinite (otherwise (x_n) contains a constant subsequence and then (x_n) converges). Then there is a subsequence (x_{k_n}) of distinct elements so

$$d(x_{k_n}, x_{k_m}) = \varphi(x_{k_n}) + \varphi(x_{k_m}) \quad \text{for } n \neq m$$

and hence $\varphi(x_{k_n}) \to 0$. By (ii), $\varphi(z) = 0$ for some $z \in \Omega$. Then $d(x_{k_n}, z) \to 0$, which implies that also (x_n) converges to z. (This clarifies an argument in [3, Step 6, p. 192], where it is claimed that each Cauchy sequence converges to z.)□

Now we give a proof of part (b) of Theorem 1. By hypothesis, some F^n has a unique fixed point z. By uniqueness, also $z = Fz$. Hence and by (a) z is a unique fixed point of each iterate of F. With a help of the Kuratowski–Zorn Lemma we will show that (1) has a solution $\varphi : \Omega \mapsto \mathbb{R}_+$ such that $\varphi^{-1}(\{0\}) = \{z\}$. Define (in a slightly different way than in [3, Step 4, p. 192])

$$\Phi := \{\varphi : D_\varphi \mapsto \mathbb{R}_+ \mid \{z\} \subseteq D_\varphi \subseteq \Omega,\ \varphi^{-1}(\{0\}) = \{z\},\ F(D_\varphi) \subseteq D_\varphi \text{ and (1) holds on } D_\varphi\}.$$

Then Φ is nonempty since if we set $D_{\varphi_*} := \{z\}$ and $\varphi_*(z) := 0$ then $\varphi_* \in \Phi$. We equip Φ with the following partial ordering:

$$\varphi_1 \preceq \varphi_2 \Leftrightarrow D_{\varphi_1} \subseteq D_{\varphi_2} \quad \text{and} \quad \varphi_2|_{D_{\varphi_1}} = \varphi_1.$$

If Φ_0 is a chain in $(\Phi, \preceq)$, then the set $D := \bigcup_{\varphi \in \Phi_0} D_\varphi$ is F-invariant and a function ψ defined on D by $\psi(x) := \varphi(x)$ if $x \in D_\varphi$, is an upper bound for Φ_0. By the Kuratowski–Zorn Lemma, there exists a maximal element $\varphi_0 : D_0 \mapsto \mathbb{R}_+$ in $(\Phi, \preceq)$. It suffices to show that $D_0 = \Omega$. Suppose, on the contrary, that there is an $x_0 \in \Omega \setminus D_0$. Set $O(x_0) := \{F^{n-1}(x_0) : n \in \mathbb{N}\}$. Now we will simplify significantly an argument from Steps 2, 3 and 5 in [3, pp. 191–192].

Step 1. Suppose that $O(x_0) \cap D_0 = \emptyset$. Then the elements $F^{n-1}(x_0)$ for $n \in \mathbb{N}$ are distinct (otherwise, we would get that $z \in O(x_0)$, which yields a contradiction since $z \in D_0$). Define

$$D_\varphi := O(x_0) \cup D_0, \quad \varphi|_{D_0} := \varphi_0 \quad \text{and} \quad \varphi(F^{n-1}x_0) := \alpha^{n-1} \quad \text{for } n \in \mathbb{N}.$$

Then $\varphi \in \Phi$, $\varphi \neq \varphi_0$ and φ dominates φ_0, a contradiction. Therefore we infer that $O(x_0) \cap D_0 \neq \emptyset$.

Step 2. By Step 1 we may define $m := \min\{n \in \mathbb{N} : F^n x_0 \in D_0\}$. Then $F^{m-1}x_0 \notin D_0$. Define $D_\varphi := \{F^{m-1}x_0\} \cup D_0$. Then

$$F(D_\varphi) = \{F^m x_0\} \cup F(D_0) \subseteq D_0 \subset D_\varphi,$$

so D_φ is F-invariant. We will define a function $\varphi : D_\varphi \mapsto \mathbb{R}_+$. Set $\varphi|_{D_0} := \varphi_0$. The following two cases are possible.

(2a) $F^m x_0 = z$. Then set $\varphi(F^{m-1}x_0) := 1$.

(2b) $F^m x_0 \neq z$. Then set $\varphi(F^{m-1}x_0) := \varphi_0(F^m x_0)/\alpha$.

In both these cases $\varphi \in \Phi$, $\varphi \neq \varphi_0$ and $\varphi_0 \preceq \varphi$, which yields a contradiction. Therefore we may infer that $D_0 = \Omega$. Applying Lemma 1 completes the proof of part (b) of Theorem 1.

3. An extension of part (a) of Theorem 1

LEMMA 2. *Let F be a selfmap of a set Ω and $\alpha \in (0,1)$. The following statements are equivalent:*

(i) *F has no periodic points,*

(ii) *the Schröder equation $\varphi(Fx) = \alpha\varphi(x)$ has a solution $\varphi : \Omega \mapsto (0,\infty)$.*

Lemma 2 can be derived from Kuczma's Theorem 1.10 [9] proved with a use of the Axiom of Choice. However our argument of the previous section can also be used here with some slight modifications.

PROOF OF LEMMA 2. To prove (i)⇒(ii) define

$$\Phi := \{\varphi : D_\varphi \mapsto (0,\infty) \mid D_\varphi \neq \emptyset,\ D_\varphi \subseteq \Omega,\ F(D_\varphi) \subseteq D_\varphi \text{ and (2) holds on } D_\varphi\}.$$

To see that Φ is nonempty fix an $x_0 \in \Omega$ and set $D_{\varphi_*} := O(x_0)$. Then by (i) the elements $F^{n-1}x_0$ ($n \in \mathbb{N}$) are distinct and we may define a function φ_* by setting

$$\varphi_*(F^{n-1}x_0) := \alpha^{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Then $\varphi_* \in \Omega$. Now we may repeat our argument used in the proof of part (b) of Theorem 1. Observe that functions φ defined in Step 1 and case (2b) satisfy (2) and have positive values, whereas case (2a) cannot happen here.

To prove (ii)⇒(i) suppose, on the contrary, that $x_0 = F^k x_0$ for some $x_0 \in \Omega$ and $k \in \mathbb{N}$. By (2), $\varphi(x_0) = \varphi(F^k x_0) = \alpha^k \varphi(x_0)$ and hence $\varphi(x_0) = 0$, a contradiction. □

Under the assumptions of Theorem 1 assume that (b) does not hold. That means F has no periodic point. So, by Lemma 2, (2) has a solution φ with positive values and if we define a metric d as in the proof of (ii)⇒(i) of Lemma 1, then it is easily seen that F satisfies the condition: $Fx \neq Fy$ implies that $d(Fx,Fy) = \alpha d(x,y)$. Moreover, for each $x \in \Omega$, the open ball $K(x,\varphi(x))$ is a singleton. Thus we proved the following

THEOREM 2. *Let F be a selfmap of a set Ω and $\alpha \in (0,1)$. If F has no periodic points, then there exists a metric d, which induces the discrete topology for Ω and such that for all $x,y \in \Omega$, $Fx \neq Fy$ implies that $d(Fx,Fy) = \alpha d(x,y)$. In particular, F is α-contractive.*

4. When is F a similarity?

Obviously, if F satisfies the condition

$$d(Fx, Fy) = \alpha d(x, y) \quad \text{for all } x, y \in \Omega$$

and some $\alpha \in (0, 1)$, then necessarily F is injective and F has at most one periodic point. The converse of this result is also true according to the following two theorems. The first of them is an immediate consequence of Theorem 2.

Theorem 3. *Let F be a selfmap of a set Ω and $\alpha \in (0, 1)$. The following statements are equivalent:*

(i) *F is a periodic-point free injection,*
(ii) *F is a fixed-point free α-similarity with respect to some metric d for Ω.*

Theorem 4. *Let F be a selfmap of a set Ω and $\alpha \in (0, 1)$. The following statements are equivalent.*

(i) *F is injective and F has a unique periodic point,*
(ii) *F is an α-similarity with respect to some complete metric d.*

Proof. To prove that (i) implies (ii) we use again an argument similar to that presented in Section 2. Let $z = Fz$. It suffices to show that the Schröder equation (2) has a solution $\varphi : \Omega \mapsto \mathbb{R}_+$ such that $\varphi^{-1}(\{0\}) = \{z\}$ and then define d as in the proof of (ii)$\Rightarrow$(i) of Lemma 1. This time we set

$$\Phi := \{\varphi : D_\varphi \mapsto \mathbb{R}_+ \mid \{z\} \subseteq D_\varphi \subseteq \Omega,\ \varphi^{-1}(\{0\}) = \{z\},\ F(D_\varphi) \subseteq D_\varphi \text{ and (2) holds on } D_\varphi\}$$

and we may repeat an argument used in the proof of part (b) of Theorem 1. Observe that case (2a) cannot happen here since F is injective, whereas functions φ, defined in Step 1 and case (2b), belong to Φ defined above. □

5. A converse to the Contraction Principle for bounded spaces

We emphasize that, according to another result of Bessaga [2], Theorem 1 is *equivalent* to some form of the AC. Consequently Theorem 1 cannot be proved in an elementary way. However, in some particular cases, it is possible to construct an appropriate metric without using any choice as is done in the proof of the following theorem.

Theorem 5. *Let F be a selfmap of a set Ω and $\alpha \in (0, 1)$. The following statements are equivalent:*

(i) *the intersection $\bigcap_{n \in N} F^n(\Omega)$ is a singleton,*
(ii) *the Schröder inequality (1) has a bounded solution $\varphi : \Omega \mapsto \mathbb{R}_+$ such that $\varphi^{-1}(\{0\})$ is a singleton,*

(iii) *there exists a complete and bounded metric d for Ω such that F is α-contractive with respect to d.*

PROOF. (i)⇒(ii). Let $\bigcap_{n\in\mathbb{N}} F^n(\Omega) = \{z\}$. For $x \neq z$ define

$$n(x) := \sup\{n \in \mathbb{N} \cup \{0\} : x \in F^n(\Omega)\}.$$

Since the sequence $(F^n(\Omega))_{n=1}^{\infty}$ is decreasing, condition (i) implies that $n(x)$ is finite. Define a function φ by

$$\varphi(z) := 0 \quad \text{and} \quad \varphi(x) := \alpha^{n(x)} \quad \text{for } x \neq z.$$

Clearly φ is bounded and $\varphi^{-1}(\{0\}) = \{z\}$. Fix an $x \in \Omega$. If $Fx = z$ then (1) holds. So let $Fx \neq z$. Then $n(Fx) \geq n(x) + 1$ and hence

$$\varphi(Fx) = \alpha^{n(Fx)} \leq \alpha^{n(x)+1} = \alpha\, \varphi(x).$$

Thus (ii) holds.

(ii)⇒(iii). It suffices to define d in the same way as in the proof of Lemma 1 ((ii) ⇒ (i)).

(iii)⇒(i). By the Banach theorem $\bigcap_{n\in\mathbb{N}} F^n(\Omega)$ is nonempty. That this set is a singleton follows from the fact that $\operatorname{diam} F^n(\Omega) \to 0$. □

THEOREM 6. *Let F be a selfmap of a set Ω and $\alpha \in (0,1)$. The following statements are equivalent:*

(i) *the intersection $\bigcap_{n\in N} F^n(\Omega)$ is empty,*

(ii) *the Schröder inequality* (1) *has a bounded solution $\varphi : \Omega \mapsto (0,\infty)$,*

(iii) *there exists a bounded metric d for Ω such that F is a fixed-point free α-contraction with respect to d.*

PROOF. (i)⇒(ii). Let $x_0 \notin \Omega$ and $\Omega_0 := \Omega \cup \{x_0\}$. Let $F_0 x_0 := x_0$ and $F_0|_\Omega := F$. By Theorem 5 ((i)⇒(ii)), there is a solution $\varphi_0 : \Omega \mapsto \mathbb{R}_+$ of (1) such that $\varphi_0^{-1}(\{0\})$ is a singleton. Since $\varphi_0(x_0) = 0$, $\varphi := \varphi_0|\Omega$ is a function we need.

(ii)⇒(iii). Define d as in the proof of Lemma 1 ((ii)⇒(i)). Then F is α-contractive with respect to d and F is fixed-point free; otherwise, if $x_0 = Fx_0$ then by (1) $\varphi(x_0) = 0$, which violates (ii).

(iii)⇒(i). By (iii), diam $F^n(\Omega) \to 0$ so $\operatorname{card}\big(\bigcap_{n=0}^{\infty} F^n(\Omega)\big) \leq 1$. Suppose that $\{x_0\} = \bigcap_{n=0}^{\infty} F^n(\Omega)$. Then

$$\{Fx_0\} = F\bigg(\bigcap_{n=0}^{\infty} F^n(\Omega)\bigg) \subseteq \bigcap_{n=0}^{\infty} F^{n+1}(\Omega) = \{x_0\},$$

which yields a contradiction. Thus we may infer that (i) holds. □

REMARK. Using the same trick as in the proof of Theorem 6 ((i)⇒(ii)), one can also derive immediately part (a) of Theorem 1 from its part (b) as well as Theorem 3 from Theorem 4.

A natural question arises whether we can improve the implication (i)⇒(iii) of Theorem 5 by showing the existence of a compact metric d. It follows from Archangielski's theorem (cf. Engelking [4, Theorem 3.1.29]) that in general this is not possible unless Ω has at most continuum many points. So we should restrict to consider sets Ω with $\operatorname{card}(\Omega) \leq c$. In fact, the above question for such sets Ω was posed by de Groot [1] in 1968 and was answered in the negative by A. Kubena. Kubena's argument is presented in the recent paper of Janos [8]. On the other hand, dropping the cardinality restriction on Ω, Iwanik [5] has proved that there exists a compact Hausdorff topology τ (not necessarily metrizable) for Ω such that F is τ-continuous. If, moreover, τ is metrizable, then there exists an equivalent metric d for Ω such that F is a Banach contraction with respect to d. This result was proved by Janos (cf. [6] or [3, p. 192–193]).

6. Fryszkowski's problem

We close the paper with the following problem formulated by Professor Andrzej Fryszkowski of the Technical University of Warsaw after my talk during the 2nd Symposium on Nonlinear Analysis in Toruń, September 13–17, 1999.

Problem (A. Fryszkowski). Let Ω be an arbitrary nonempty set, 2^Ω be the family of all nonempty subsets of Ω and $F : \Omega \mapsto 2^\Omega$ be a set-valued mapping. Find necessary and (or) sufficient conditions for the existence of a (complete) metric d for Ω such that given $\alpha \in (0,1)$, F would be a Nadler [10] set-valued α-contraction with respect to d, that is,

$$H(Fx, Fy) \leq \alpha d(x,y) \quad \text{for all } x, y \in \Omega,$$

where H denotes the Hausdorff metric generated by d.

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