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A COINCIDENCE THEORY INVOLVING FREDHOLM OPERATORS OF NONNEGATIVE INDEX

Dorota Gabor — Wojciech Kryszewski

Dedicated to the memory of Juliusz P. Schauder

Abstract. We construct a homotopy invariant appropriate for studying the existence of coincidence points of Fredholm operators of nonnegative index and multivalued admissible maps. Cohomotopy methods are used as a more suitable tool than homological ones. Both finite and infinite dimensional cases are investigated.

1. Introduction

The need of an algebraic homotopy invariant responsible for the existence of coincidence points of continuous maps $f, g : X \to Y$, where X, Y are topological spaces, is clear and stems from its possible applications.

In the present paper we shall deal with a bit more general situation. Namely consider a diagram of continuous maps

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "F"'] & & Y \end{tikzcd}$$

where Γ is a space and p is a surjection. Intuitively and vaguely speaking one expects that if maps $F \circ p$ and q meet a transversality condition of sorts, then their graphs must intersect yielding a coincidence point of either $F \circ p$ and q or

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F and the set-valued map $X \ni x \mapsto q(p^{-1}(x)) \subset Y$ which is stable with respect to small perturbations of the considered maps.

In the simplest situation when $\Gamma = D^m(0,1)$ is the closed unit disc in the Euclidean space $\mathbb{R}^m$ $(m \geq 1)$, $X = \Gamma$, $p = \mathrm{id}$ is the identity, $Y = \mathbb{R}^m$ and $F = j : D^m(0,1) \to \mathbb{R}^m$ is the inclusion, then the Brouwer degree $\deg_B(j-q, D^m(0,1), 0)$ (if defined) provides an algebraic measure of the geometric situation between the graph of q and the diagonal. If, however $Y = \mathbb{R}^n$ $(n \neq m)$ and F is no longer the inclusion but, say, a linear map, then the situation changes dramatically. If $n > m$, then there are arbitrarily small perturbations of q without coincidence points with F; if $n < m$, then q may still have homotopically stable coincidences with F but their existence cannot be detected by the behaviour of the homology class of the cycle $q(S^{m-1})$ since it is trivial. The passage to cohomology would hardly help if this dimension defect occurs.

The approach we shall present relies on the cohomotopy methods rather and provides an invariant taking values in the $(m-n)$th stable homotopy group of spheres. We shall apply this altitude to an infinite dimensional setting, too. Namely, we shall assume that X, Y are Banach spaces, F is a Fredholm operator of index $i(F) > 0$ (observe, that above $i(F) = m - n$) and $p : \Gamma \to X$ is a Vietoris map (that is, so to say, a proper surjection with acyclic fibres). A coincidence index to be defined constitutes an algebraic count of solutions to the inclusion (multivalued equation) $F(x) \in \varphi(x) := q(p^{-1}(x))$, $x \in X$, as well as of coincidence points of $F \circ p$ and q. Regarding the set-valued setting move natural and appropriate for our purposes, we thus obtain an index which generalize invariants introduced in e.g. [18] where φ was single–valued and $i(F) = 0$; [19] where φ had compact convex values and again $i(F) = 0$; and those implicitly contained in [8] where $i(F) \geq 0$ but φ was single–valued and compact (in this context see also [22], [3]).

All topological spaces considered in the paper are *metric* and single–valued maps are *continuous*.

If V is a subset of a space, then $\mathrm{cl}\, V$, $\mathrm{int}\, V$, and $\mathrm{bd}\, V$ denote the *closure*, the *interior* and the *boundary* of V, respectively. If V is a subset of a Banach space, then $\mathrm{conv}\, V$ stands for its *convex hull* and $\overline{\mathrm{conv}}\, V = \mathrm{cl}\,\mathrm{conv}\, V$. For $z \in \mathbb{R}^n$, $\varepsilon > 0$, let $B^n(z,\varepsilon) = \{x \in \mathbb{R}^n \mid \|x - z\| < \varepsilon\}$, $D^n(z,\varepsilon) = \mathrm{cl}\, B^n(z,\varepsilon)$, $B^n = B^n(0,1)$ and $D^n = D^n(0,1)$. Similarly if z belongs to a Banach space E, then $B^E(z,\varepsilon) = \{x \in E \mid \|x - z\| < \varepsilon\}$, $D^E(z,\varepsilon) = \mathrm{cl}\, B^E(z,\varepsilon)$, $B^E = B^E(0,1)$ and $D^E = D^E(0,1)$.

2. Preliminaries

Let (S^n, s_0), $n \geq 0$, be the unit sphere in $\mathbb{R}^{n+1}$ with a chosen base point s_0, e.g. put $s_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. Given a space X and its *closed* subset A, the set $[X, A; S^n, s_0]$ of all homotopy classes $[w]$ of maps $w : (X, A) \to (S^n, s_0)$ is usually denoted by $\pi^n(X, A)$[1]. If $f : (X, A) \to (Y, B)$, where B is closed in a space Y, then the *induced transformation* $f^\# : \pi^n(Y, B) \to \pi^n(X, A)$ is defined via $f^\#[w] = [w \circ f]$ for any $[w] \in \pi^n(Y, B)$. It is clear that homotopic maps induce the same transformation while homotopy equivalences induce bijections. It is well-known (see [1]), that the map f on X which collapses A to a point, i.e. $f : (X, A) \to (X/A, \{A\})$ induces a bijection $f^\# : \pi^n(X/A, \{A\}) \to \pi^n(X, A)$. Therefore $\pi^n(D^m, S^{m-1})$ is equal (up to bijection) to $\pi^n(S^m)$. Hence if $n = m$, one may identify $\pi^n(S^n)$ with the group of integers and distinguish the class, denoted further by $\mathbf{1}$, represented by the identity $S^n \to S^n$; if $n > m$, then $\pi^n(S^m)$ is trivial and contains only one homotopy class, denoted $\mathbf{0}$, represented by the constant map $S^m \to s_0$; and if $n < m$, then the set $\pi^n(S^m)$ is nontrivial in general. By the Freudenthal suspension theorem (see [21]), if $n \le m < 2n-1$, then $\pi^n(S^m) = \pi_m(S^n) = \Pi_{m-n}$ is the $(m-n)$th *stable homotopy group of spheres* (i.e. $\Pi_{m-n} := \varinjlim_{k\to\infty} \pi_{m-n+k}(S^k)$).

If A, B are closed subspaces of X, then the coboundary operator $\delta : \pi^n(A, A \cap B) \to \pi^{n+1}(X, A)$ is defined. Moreover, it is straightforward to get the following *excision property.*

Proposition 2.1. *The transformation $l^\# : \pi^n(A \cup B, B) \to \pi^n(A, A \cap B)$, induced by the inclusion $l : (A, A \cap B) \to (A \cup B, B)$, is bijective.*

In particular, $\pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B(z, \varepsilon)) \cong \pi^n(D^m, S^{m-1}) \cong \pi^n(S^m)$.

For further reasons we shall need the notion of the Mayer–Vietoris coboundary operator of a relative triad. Suppose $X = X_+ \cup X_-$ where $X_\pm$ are closed, $A_\pm := A \cap X_\pm$ and let $X_0 := X_+ \cap X_-$, $A_0 := A_+ \cap A_- = A \cap X_0$. We are going to define a transformation $\Delta : \pi^{n-1}(X_0, A_0) \to \pi^n(X, A)$, $n \ge 1$. Although it admits a rigorous algebraic definition (see [8] or [15]) we give here only its simple geometric description. The space $\mathbb{R}^n$ treated as a subspace of $\mathbb{R}^{n+1}$ cuts it into two closed halfspaces denoted, according to the given orientation by $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$, respectively. Treating S^{n-1} as an equator in S^n, $\mathbb{R}^{n+1}_\pm$determine the north and the south hemispheres $S^n_\pm$ in S^n. Given $\alpha \in \pi^{n-1}(X_0, A_0)$ represented by $w_0 : (X_0, A_0) \to (S^{n-1}, s_0)$, we take its arbitrary extension $w : (X, A) \to (S^n, s_0)$ such that $w(X_\pm) \subset \mathbb{R}^{n+1}_\pm$ (since both $S^n_\pm$ are contractible, this extension does exist) and put $\Delta(\alpha) = [w]$.

It is well–known that if X is compact, the covering dimension $\dim X < \infty$ and the Čech cohomology (with integer coefficients) $\check{H}^q(X, A) = 0$ for $q \ge 2m-1$ $(m \ge 1)$, then $\pi^n(X, A)$ admits the structure of an abelian group by the usual Borsuk method (see [17], [20] and [11]) for $n \ge m$. It holds, in particular if $\dim X < 2m - 1$. Essentially by the same methods one may introduce the

[1] In case $A = \emptyset$, $\pi^n(X) = [X; S^n]$.

group structure to $\pi^n(X,A)$ if X is only paracompact but still $\check{H}^q(X,A)=0$ for $q\geq 2m-1$ and $n\geq m$. In this case δ and the induced transformations are homomorphisms.

Let X, Y be spaces. By a set-valued map φ from X to Y we understand an *upper semicontinuous* transformation which assigns to a point $x\in X$ a *compact nonempty* set $\varphi(x)\subset Y$ (with regard generalities on set-valued maps — see [10]). Observe that $\varphi : X \multimap Y$ may be represented by the formula

$$\varphi(x)=q_\varphi(p_\varphi^{-1}(x)),\quad x\in X,$$

where $X\xleftarrow{p_\varphi}\Gamma(\varphi)\xrightarrow{q_\varphi}Y$, $\Gamma(\varphi)=\{(x,y)\in X\times Y\mid y\in\varphi(x)\}$ is the *graph* of φ and p_φ, q_φ are respective projections onto X and into Y, respectively. Note that, in view of the upper semicontinuity of φ, p_φ is a proper surjection as a closed map with compact fibers $p_\varphi^{-1}(x)$, $x\in X$.

Clearly φ may admit other factorizations of the form $X\xleftarrow{p}\Gamma\xrightarrow{q}Y$, where Γ is a space and p, q are no longer projections, but p is still a proper surjection.

On the other hand any pair (p,q) of that type, determines a set-valued map $X\ni x\mapsto q(p^{-1}(x))=\varphi(x)$, but without additional assumptions concerning p and q we have no sufficient information about the structure of φ.

Definition 2.2. We say that a pair (p,q) of maps from the diagram

$$X\xleftarrow{p}\Gamma\xrightarrow{q}Y,$$

is *admissible* if p is a *Vietoris map* i.e.:

(i) p is a *proper surjection* with fibres acyclic with respect to the Čech cohomology (i.e. $\check{H}^*(p^{-1}(x))=\check{H}^*(pt)$, where pt is a one-point space);
(ii) $\dim p:=\sup\dim p^{-1}(x)<\infty$.[2]

Remark 2.3. (i) Observe that if $\dim\Gamma<\infty$, then condition (ii) above holds automatically; conversely if p is a Vietoris map and $\dim X<\infty$, then $\dim\Gamma<\infty$, too.

(ii) The values of the set-valued map $\varphi(x)=q(p^{-1}(x))$, $x\in X$, determined by an admissible pair (p,q), are continuous images of acyclic sets. Such maps are also called *admissible* and have been studied intensively in many papers (see [10], [12] and others). The class of admissible maps is large and closed under compositions (see [10]). For instance, it contains compact convex–valued maps as well as those with acyclic or contractible values.

One of the main reasons to study admissible pairs (and set-valued maps determined by them) follows from the famous Vietoris—Begle theorem (see e.g. [21])

[2]The notion of admissibility may be generalized to that of n-admissibility, $n\geq 1$, in the spirit of [12], [15]. For some applications, condition (ii) is not necessary.

which states that *if* $p : \Gamma \to X$ *is a Vietoris map, then the induced homomorphism* $p^* : H^*(X) \to H^*(\Gamma)$ *is an isomorphism.* This result however seems to be useless in our framework, where no standard (co)homological issues play a role. Therefore we state (a simplified form of) a cohomotopy version of the Vietoris theorem due to the second author (comp. [13], [14]).

Theorem 2.4. *Let* $Y' \subset Y$ *be path-connected* ANR*s* (*absolute neighbourhood retracts*) *such that the pair* (Y, Y') *is* 1*-connected and* Y, Y' *have homotopy types of compact polyhedra. If* $p : (\Gamma, \Gamma') \to (X, X')$, *where* $\Gamma' \subset \Gamma$, $X' \subset X$ *are closed and* $\Gamma' = p^{-1}(X')$, *is a Vietoris map and* $\dim X < \infty$, *then the transformation*

$$p^{\#} : [X, X'; Y, Y'] \to [\Gamma, \Gamma'; Y, Y'],$$

induced by p *is bijective.*

Corollary 2.5. *If* $\dim X < \infty$ *and* $p : (\Gamma, \Gamma') \to (X, X')$ *is as in Theorem* 2.4, *then for any* $n \geq 0$, $z \in \mathbb{R}^{n+1}$ *and* $\varepsilon > 0$, *the transformations*

$$\begin{aligned} p^{\#} : \pi^n(X, X') &\to \pi^n(\Gamma, \Gamma'), \\ p^{\#} : [X, X'; \mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus B^{n+1}(z, \varepsilon)] &\to [\Gamma, \Gamma'; \mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus B^{n+1}(z, \varepsilon)], \\ p^{\#} : [X, X'; D^{n+1}, S^n] &\to [\Gamma, \Gamma'; D^{n+1}, S^n], \end{aligned}$$

induced by p *are bijective.*

Proof. The case $n \geq 1$ follows from Theorem 2.4 and if $n = 0$, then the assertion is trivial. □

Definition 2.6. We say that two admissible pairs $X \xleftarrow{p_0} \Gamma_0 \xrightarrow{q_0} Y$ and $X \xleftarrow{p_1} \Gamma_1 \xrightarrow{q_1} Y$ are homotopic if there exists an admissible pair $X \times [0,1] \xleftarrow{R} \Gamma \xrightarrow{S} Y$ such that $\Gamma_0, \Gamma_1 \subset \Gamma$ (up to the homeomorphisms $j_k : \Gamma_k \to \Gamma$, $k = 0, 1$ embedding Γ_k as a subset in Γ) and the following diagram is commutative

$$\begin{tikzcd} X \arrow[d, "i_0"'] & \Gamma_0 \arrow[l, "p_0"'] \arrow[d, "j_0"'] \arrow[dr, "q_0"] & \\ X \times I & \Gamma \arrow[l, "R"'] \arrow[r, "S"] & Y \\ X \arrow[u, "i_1"] & \Gamma_1 \arrow[l, "p_1"] \arrow[u, "j_1"] \arrow[ur, "q_1"'] & \end{tikzcd}$$

where $i_k(x) = (x, k)$, $k = 0, 1$, $x \in X$. We write also $(R, S) : (p_0, q_0) \simeq (p_1, q_1)$.

Example 2.7. (i) If $X \xleftarrow{p_0} \Gamma_0 \xrightarrow{q_0} Y$, $X \xleftarrow{p_1} \Gamma_1 \xrightarrow{q_1} Y$ and $\Gamma = \Gamma_0 = \Gamma_1$, there is a Vietoris map $R : \Gamma \times [0,1] \to X \times [0,1]$ such that $R(\omega, k) = (p_k(\omega), k)$, $k = 0, 1$, and the maps q_0, q_1 are homotopic (i.e. there is $S : q_0 \simeq q_1 : \Gamma \times [0,1] \to Y$), then $(R, S) : (p_0, q_0) \simeq (p_1, q_1)$. In particular if $p_0 = p_1$, then defining $R(\omega, t) = (p_0(\omega), t)$, $y \in \Gamma$, $t \in [0,1]$, we get the same result.

(ii) If $p : (\Gamma, \Gamma') \to (X, X')$ is a Vietoris map ($p^{-1}(X') = \Gamma'$) and $q : (\Gamma, \Gamma') \to (Y, Y')$, where Y, Y' satisfy assumptions of Theorem 2.4, then there is a unique (up to a homotopy) map $f : (X, X') \to (Y, Y')$ such that $h : f \circ p \simeq q$. Then admissible pairs (p, q) and $(p, f \circ p)$ are homotopic.

Proposition 2.8. *Let $p : (\Gamma, \Gamma') \to (X, X')$ be a Vietoris map and $q : (\Gamma, \Gamma') \to (Y, Y')$, where Y is a space and Y' is closed. Then, for any $n \geq 0$, the formula*

$$(p, q)^{\#} := (p^{\#})^{-1} \circ q^{\#}$$

correctly defines the transformation

$$(p, q)^{\#} : \pi^n(Y, Y') \to \pi^n(X, X').$$

If $(p_0, q_0) \simeq (p_1, q_1)$, then $(p_0, q_0)^{\#} = (p_1, q_1)^{\#}$.

Proof. The correctness follows from Corollary 2.5, the last assertion from Definition 2.6 and the fact that $i_0^{\#} = i_1^{\#}$. □

3. Finite dimensional case

Let U be an open bounded subset of $\mathbb{R}^m$ and $\operatorname{cl} U \xleftarrow{p} \Gamma \xrightarrow{q} \mathbb{R}^n$, $m \geq n \geq 1$ be an admissible pair (by Remark 2.3, $\dim \Gamma < \infty$) and suppose that $0 \notin q(p^{-1}(x))$ for $x \in \operatorname{bd} U$. It implies that there is $\varepsilon > 0$ such that $q(p^{-1}(\operatorname{bd} U)) \subset \mathbb{R}^n \setminus B^n(0, \varepsilon)$.

Consider the following sequence of maps:

$$\begin{aligned}(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\varepsilon)) \xleftarrow{q} (p^{-1}(\operatorname{cl} U), p^{-1}(\operatorname{bd} U)) \xrightarrow{p} (\operatorname{cl} U, \operatorname{bd} U)\\ \xrightarrow{i_1} (\mathbb{R}^m, \mathbb{R}^m \setminus U) \xleftarrow{i_2} (\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0, r)),\end{aligned}$$

where $r > 0$ is such that $U \subset B^m(0, r)$ and i_1, i_2 are inclusions. By the excision property (2.1), $i_1^{\#}$ is a bijection. Hence, in view of Proposition 2.8, we have defined the transformation

$$\begin{aligned}(1)\quad \mathcal{K} := i_2^{\#} \circ (i_1^{\#})^{-1} \circ (p, q)^{\#} : \pi^n(S^n) = \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0, \varepsilon))\\ \to \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0, r)) = \pi^n(S^m).\end{aligned}$$

Definition 3.1. By the *generalized degree* of the pair (p, q) on U we understand the element

$$\deg((p, q), U, 0) := \mathcal{K}(\mathbf{1}) \in \pi^n(S^m).$$

(Recall that $\mathbf{1}$ denotes the generator of $\pi^n(S^n) \cong \mathbb{Z}$.)

It is clear that this definition does not depend on the choice of ε and r.

Remark 3.2. It is not difficult to check that if $n = m$, then $\deg((p, q), U, 0) \in \pi^n(S^n)$ is nothing else but the ordinary degree of the pair as constructed in e.g. [10].

Theorem 3.3. *The generalized degree has the following properties:*

(i) (Existence) *If* $\deg((p,q),U,0) \neq 0 \in \pi^n(S^m)$, *then there is* $x \in U$ *such that* $0 \in q(p^{-1}(x))$ *(and* $p(\omega) = q(\omega)$ *for some* $\omega \in p^{-1}(U)$*).*

(ii) (Localization) *If* U_1 *is open and* $0 \notin q(p^{-1}(\operatorname{cl} U \setminus U_1))$, *then* $\deg((p,q), U_1, 0)$ *is defined and equal to* $\deg((p,q),U,0)$.

(iii) (Homotopy Invariance) *If pairs*

$$\operatorname{cl} U \xleftarrow{p_0} \Gamma_0 \xrightarrow{q_0} \mathbb{R}^n \quad \text{and} \quad \operatorname{cl} U \xleftarrow{p_1} \Gamma_1 \xrightarrow{q_1} \mathbb{R}^n$$

are homotopic, i.e. $(R,S) : (p_0,q_0) \simeq (p_1,q_1)$ *and* $0 \notin S(R^{-1}(x,t))$ *for* $x \in \operatorname{bd} U$, $t \in [0,1]$, *then* $\deg((p_0,q_0),U,0) = \deg((p_1,q_1),U,0)$.

(iv) (Additivity) *Assume that* $m < 2n-1$. *If* U_1, U_2 *are open subsets of* U *such that* $U_1 \cap U_2 = \emptyset$ *and* $0 \notin q(p^{-1}(x))$ *for* $x \in U \setminus (U_1 \cup U_2)$, *then*

$$\deg((p,q),U,0) = \deg((p,q),U_1,0) + \deg((p,q),U_2,0).$$

Proof. (i) Assume to the contrary that $0 \notin q(p^{-1}(U))$. Then

$$q(p^{-1}(\operatorname{cl} U), p^{-1}(\operatorname{bd} U)) \to (\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\varepsilon)),$$

factorizes through $(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n \setminus B^n(0,\varepsilon))$ and, consequently, the transformation $(p,q)^{\#}$ is trivial, hence so is $\mathcal{K}$ and $\deg((p,q),U,0) = 0 \in \pi^n(S^m)$.

(ii) Changing ε if necessary we may assume that $B = B^n(0,\varepsilon) \cap q(p^{-1}(\operatorname{cl} U \setminus U_1)) = \emptyset$. Consider the diagram:

$$\begin{tikzcd}
 & \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \arrow[dl, "(p,q)^{\#}"'] \arrow[d, "(\overline{p},\overline{q})^{\#}"'] \arrow[dr, "(p_1,q_1)^{\#}"] & \\
\pi^n(\operatorname{cl} U, \operatorname{bd} U) & \pi^n(\operatorname{cl} U, U \setminus U_1) \arrow[l] \arrow[r] & \pi^n(\operatorname{cl} U_1, \operatorname{bd} U_1) \\
\pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus U) \arrow[u] \arrow[r, phantom, "="] \arrow[dr] & \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus U) \arrow[u] \arrow[d] & \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus U_1) \arrow[u] \arrow[l] \arrow[dl] \\
 & \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus C) &
\end{tikzcd}$$

where $C = B^m(0,r)$, r is such that $U \subset C$, $\overline{p} := p_{|(p^{-1}(\operatorname{cl} U), p^{-1}(\operatorname{cl} U \setminus U_1))}$, $\overline{q} := q_{|(p^{-1}(\operatorname{cl} U), p^{-1}(\operatorname{cl} U \setminus U_1))}$, $p_1 := p_{|(p^{-1}(\operatorname{cl} U_1), p^{-1}(\operatorname{bd} U_1))}$, $q_1 := q_{|(p^{-1}(\operatorname{cl} U_1), p^{-1}(\operatorname{bd} U_1))}$ and all unmarked arrows are induced by inclusions. This diagram is commutative. Its first "column" corresponds to $\mathcal{K}$ while the third one to $\deg((p,q),U_1,0)$. Hence the assertion.

(iii) Changing ε we may assume that $B^n(0,\varepsilon) \cap S(R^{-1}(\operatorname{bd} U \times [0,1])) = \emptyset$. The assertion follows from the second part of Proposition 2.8.

(iv) Without loss of generality we may assume that

$$q : (p^{-1}(\operatorname{cl} U), p^{-1}(\operatorname{bd} U)) \to (\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,1)).$$

Moreover, by (ii), let $U = U_1 \cup U_2$.

Define for $i = 1, 2$ two maps $q_i : (\Gamma, \Gamma') \to (\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,1))$ as follows:

$$q_i(w) = \begin{cases} q(w) & \text{for } w \in p^{-1}(U_i), \\ s_0 & \text{for } w \in p^{-1}(\operatorname{cl} U \setminus U_i), \end{cases}$$

where $s_0 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Of course in view of our assumptions they both are continuous.

One can immediately show that $\deg((p, q_i), U, 0) = \deg((p, q), U_i, 0)$.

Let $v \in \mathbf{1}$ (comp. Definition 3.1). Obviously, to prove the property, it is sufficient to check that

$$[v \circ q] = [v \circ q_1] + [v \circ q_2],$$

where the addition "$+$" in $\pi^n(\Gamma, \Gamma')$ is defined following [11] (it is in fact the same addition as the one defined by the bijection $p^{\#}$, so we do not need any assumption concerning a dimension of Γ). We omit this not difficult, technical verification. □

4. Infinite dimensional case

Let E, E' be Banach spaces and let $F : E \to E'$ be a *Fredholm operator* (i.e. bounded linear and such that its kernel $\operatorname{Ker}(F)$ and cokernel $\operatorname{Coker}(F) := E'/\operatorname{Im}(F)$, where $\operatorname{Im}(F)$ is the image of F, are finite dimensional) with index

$$i(F) := \dim \operatorname{Ker}(F) - \dim \operatorname{Coker}(F) = k \geq 0.^3$$

Since both $\operatorname{Ker}(F)$ and $\operatorname{Im}(F)$ are direct summands in E and E', respectively, we may consider continuous linear projections $P : E \to E$ and $Q : E' \to E'$, such that $\operatorname{Ker} F = \operatorname{Im}(P)$ and $\operatorname{Ker} Q = \operatorname{Im}(F)$. Clearly E, E' split into (topological) direct sums

$$\operatorname{Ker}(P) \oplus \operatorname{Ker}(F) = E, \quad \operatorname{Im}(Q) \oplus \operatorname{Im}(F) = E'.$$

Moreover, since $\operatorname{Im}(F)$ is a closed subspace of E', $F|_{\operatorname{Ker} P}$ is a linear homeomorphism onto $\operatorname{Im}(F)$. Note also that F is proper when restricted to a closed bounded set.

Let $X \subset E$ be open and consider an admissible pair $X \xleftarrow{p} \Gamma \xrightarrow{q} E'$ such that q is *locally compact* (i.e. each point $\omega \in \Gamma$ has a neighbourhood V_ω such that $\operatorname{cl} q(V_\omega)$ is compact) and the set $C := \{x \in X \mid F(x) \in q(p^{-1}(x))\}$ is compact. The collection of such pairs will be denoted by $\mathcal{D}_c(X, F)$.

[3] Observe that if $F : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then F is Fredholm and $i(F) = m - n$.

It follows that the set $A := \{\omega \in \Gamma \mid F \circ p(\omega) = q(\omega)\}$ being closed and contained in $p^{-1}(C)$ is compact, too. Therefore, in view of the local compactness of q, there is an open set $V \supset A$ such that $\operatorname{cl} q(V)$ is compact.

Choose a bounded open set $U \subset E$, such that

$$C \subset U \subset \operatorname{cl} U \subset X \quad \text{and} \quad p^{-1}(U) \subset V.$$

There is $\varepsilon_0 > 0$ such that

$$\{y \in E' \mid \exists_{x \in \operatorname{bd} U}\ y \in F(x) - q(p^{-1}(x))\} \cap B^{E'}(0, 2\varepsilon_0) = \emptyset.$$

Take $\varepsilon \in (0, \varepsilon_0]$ and let $l_\varepsilon : \operatorname{cl} q(p^{-1}(U)) \to E'$ be a Schauder projection of the compact set $K := \operatorname{cl} q(p^{-1}(U))$ into a finite dimensional subspace Z of E', such that $||l_\varepsilon(y) - y||_{E'} < \varepsilon$ for $y \in \operatorname{cl} q(p^{-1}(U))$. Denote by L' the finite dimensional subspace of $\operatorname{Im}(F)$ such that $Z \subset L = L' \oplus \operatorname{Im}(Q)$. Put $T := F^{-1}(L)$, $U_L = U \cap T$. It is clear that the closure $\operatorname{cl} U_L$ (in T) is contained in $\operatorname{cl} U \cap T$ and its boundary $\operatorname{bd} U_L$ (relative T) in $\operatorname{bd} U \cap T$. Further let $p_L = p|_{p^{-1}(\operatorname{cl} U_L)}$, $q_L = l_\varepsilon \circ q|_{p^{-1}(\operatorname{cl} U_L)}$ and $G = F|_T : T \to L$. Observe, that p_L is a Vietoris map and G is a Fredholm operator of index

$$i(G) = \dim T - \dim L = k.$$

Enlarging L' if necessary we may assume that $\dim L := n \geq k + 2$. Putting $\dim T = m = n + k$ we arrive in a situation discussed in Section 2.

Definition 4.1. By the generalized index of a pair $(p, q) \in \mathcal{D}_c(X, F)$ we understand the element

$$\operatorname{Ind}_F((p, q), X) = \deg((p_L, G \circ p_L - q_L), U_L, 0) \in \Pi_k.$$

By definition $\deg((p_L, G \circ p_L - q_L), U_L, 0)$ belongs to $\pi^n(S^m)$ but since $m < 2n - 1$ we know that $\pi^n(S^m) \cong \Pi_k$.

Let us now prove, that this definition is correct, i.e. does not depend on the choice of U, $\varepsilon \in (0, \varepsilon_0]$, l_ε and L'. First assume that U, ε and l_ε are fixed and let $N' = L' \oplus Y$, where $Y \subset \operatorname{Im}(F)$ and $\dim Y = 1$. Put $N := N' \oplus \operatorname{Im}(Q)$, $M := F^{-1}(N)$ and $U_N := U \cap M$, $p_N := p|_{p^{-1}(\operatorname{cl} U_N)}$, $q_N := l_\varepsilon \circ q|_{p^{-1}(\operatorname{cl} U_N)}$ and $G' = F|_M$. Introduce an orientation in N and observe that L cuts N into two closed halfspaces denoted, according to the orientation, by N^+ and N^-, respectively. Then $N^+ \cup N^- = N$, $N^+ \cap N^- = L$,

$$L \setminus B^{E'}(0, \varepsilon) = (N^+ \setminus B^{E'}(0, \varepsilon)) \cap (N^- \setminus B^{E'}(0, \varepsilon))$$

and

$$N \setminus B^{E'}(0, \varepsilon) = (N^+ \setminus B^{E'}(0, \varepsilon)) \cup (N^- \setminus B^{E'}(0, \varepsilon)).$$

Hence the Mayer–Vietoris operator

$$\Delta_1 : \pi^n(L, L \setminus B^{E'}(0,\varepsilon)) \to \pi^{n+1}(N, N \setminus B^{E'}(0,\varepsilon))$$

is well defined. It is easy to check that Δ_1 is a bijection.

In a similar manner $M = M^+ \cup M^-$, where $M^\pm = F^{-1}(N^\pm)$, $T = M^+ \cap M^-$;

$$T \setminus B^E(0,r) = (M^+ \setminus B^E(0,r)) \cap (M^- \setminus B^E(0,r))$$

and

$$M \setminus B^E(0,r) = (M^+ \setminus B^E(0,r)) \cup (M^- \setminus B^E(0,r)),$$

where $r > 0$ is such that $U \subset B^E(0,r)$. Hence again we have defined

$$\Delta_2 : \pi^n(T, T \setminus B^E(0,r)) \to \pi^{n+1}(M, M \setminus B^E(0,r))$$

being a bijection. Further $U_N^+ \cup U_N^- = U_N$, where $U_N^\pm = U \cap M^\pm$, $U_N^+ \cap U_N^- = U_L$, $\operatorname{bd} U_N = (\operatorname{bd} U_N^+ \setminus N^-) \cup (\operatorname{bd} U_N^- \setminus N^+)$ and $\operatorname{bd} U_L = (\operatorname{bd} U_N^+ \setminus N^-) \cap (\operatorname{bd} U_N^- \setminus N^+)$. The Mayer–Vietoris map

$$\Delta : \pi^n(\operatorname{cl} U_L, \operatorname{bd} U_L) \to \pi^{n+1}(\operatorname{cl} U_N, \operatorname{bd} U_N)$$

is defined. In order to show that $\deg((p_L, G \circ p_L - q_L), U_L, 0) = \deg((p_N, G \circ p_N - q_N), U_N, 0)$ we have to show that the following diagram is commutative (recall Definition 3.1)

$$\begin{tikzcd}
\pi^n(L, L \setminus B^{E'}) \arrow[r, "(p_l, G\circ p_L - q_L)^\#"] \arrow[d, "\Delta_1"'] \arrow[dr, phantom, "(1)"] & \pi^n(\operatorname{cl} U_L, \operatorname{bd} U_L) \arrow[d, "\Delta"] \arrow[dr, phantom, "(2)"] & \pi^n(T, T \setminus U_L) \arrow[l, "i_1^\#"'] \arrow[r, "i_2^\#"] & \pi^n(T, T \setminus B^E) \arrow[d, "\Delta_2"] \\
\pi^{n+1}(N, N \setminus B^{E'}) \arrow[r, "(p_N, G'\circ p_N - q_N)^\#"] & \pi^{n+1}(\operatorname{cl} U_N, \operatorname{bd} U_N) & \pi^{n+1}(M, M \setminus U_N) \arrow[l, "\overline{i_1}^\#"'] \arrow[r, "\overline{i_2}^\#"] & \pi^{n+1}(M, M \setminus B^E)
\end{tikzcd}$$

where $B^{E'} := B^{E'}(0,\varepsilon)$, $B^E := B^E(0,r)$, $i_1^\#$, $i_2^\#$, $\overline{i_1}^\#$ and $\overline{i_2}^\#$ are induced by the inclusions.

The commutativity of "box" (2) is easy to check. To prove it for "box" (1) observe that by Theorem 2.4 and Corollary 2.5, there are unique maps $f_L : (\operatorname{cl} U_L, \operatorname{bd} U_L) \to (L, L \setminus B^{E'})$ and $f_N : (\operatorname{cl} U_N, \operatorname{bd} U_N) \to (N, N \setminus B^{E'})$ such that $f_L \circ p_L \simeq G \circ p_L - q_L$ and $f_N \circ p_N \simeq G' \circ p_N - q_N$. Hence

$$f_L^\# = (p_L, G \circ p_L - q_L)^\# : \pi^n(L, L \setminus B^{E'}) \to \pi^n(\operatorname{cl} U_L, \operatorname{bd} U_L)$$

and

$$f_N^\# = (p_N, G' \circ p_N - q_N)^\# : \pi^{n+1}(N, N \setminus B^{E'}) \to \pi^{n+1}(\operatorname{cl} U_N, \operatorname{bd} U_N).$$

It, therefore, remains to show that $\Delta\circ f_L^{\#}=f_N^{\#}\circ\Delta_1$. To that end define a new map $f_N'\colon(\mathrm{cl}\,U_N,\mathrm{bd}\,U_N)\to(N,N\setminus B^{E'})$ as follows. Observe that $M=T\oplus Y'$ where $Y'=(F|_{\mathrm{Ker}\,P})^{-1}(Y)$. If $x=x_1+x_2$, $x_1\in T$, $x_2\in Y'$, then put $f_N'(x)=f_L(x_1)+F(x_2)$. Let

$$h_L\colon(p^{-1}(\mathrm{cl}\,U_L),p^{-1}(\mathrm{bd}\,U_L))\times[0,1]\to(L,L\setminus B^{E'})$$

be a homotopy joining $f_L\circ p_L$ to $G\circ p_L-q_L$ and consider the map

$$h\colon p^{-1}(\mathrm{cl}\,U_N)\times\{0,1\}\cup p^{-1}(\mathrm{cl}\,U_L)\times[0,1]\to N$$

given by the formula

$$h(\omega,t)=\begin{cases} f_N'\circ p_N(\omega) & \text{for } \omega\in p^{-1}(\mathrm{cl}\,U_N),\ t=0,\\ G'\circ p_N(\omega)-q_N(\omega) & \text{for } \omega\in p^{-1}(\mathrm{cl}\,U_N),\ t=1,\\ h_L(\omega,t) & \text{for } \omega\in p^{-1}(\mathrm{cl}\,U_L),\ t\in[0,1].\end{cases}$$

Note that if $\omega\in p^{-1}(\mathrm{bd}\,U_N)$ and $t=0,1$ or $\omega\in p^{-1}(\mathrm{bd}\,U_L)$ and $t\in[0,1]$, then $h(\omega,t)\notin B^{E'}$. Moreover if $\omega\in p^{-1}(\mathrm{cl}\,U_N^{\pm})$, $t=0,1$, then $h(\omega,t)\in N^{\pm}$ (for $q_N(\omega)\in L$). Hence there exists an extension h_N of h onto : $p^{-1}(\mathrm{cl}\,U_N)\times[0,1]$ such that $h_N\colon(p^{-1}(\mathrm{cl}\,U_N),p^{-1}(\mathrm{bd}\,U_N))\times[0,1]\to(N,N\setminus B^{E'})$ yielding a homotopy joining $f_N'\circ p_N$ to $G'\circ p_N-q_N$. In view of the homotopy uniqueness of f_N we gather that $f_N\simeq f_N'$. Now, having the definition of Δ and Δ_1 in mind we easily see that indeed $\Delta\circ f_L^{\#}=f_N'^{\#}\circ\Delta_1=f_N^{\#}\circ\Delta_1$.

If L' is replaced by a space $L''=L'\oplus Y'$ where $Y'\subset\mathrm{Im}\,(F)$ and $1<\dim Y'<\infty$, the iterating above procedure we see indeed that our definition does not depend upon the choice of L'.

Now we shall show an independence of our definition of ε and l_ε. Suppose $\varepsilon'\in(0,\varepsilon_0]$ and let $l_{\varepsilon'}\colon K\to Z'\subset E'$ be another Schauder projection, such that $||l_{\varepsilon'}(y)-y||_{E'}<\varepsilon'$ on K. Take a finite dimensional subspace L' in $\mathrm{Im}\,(Q)$ such that $L=L'\oplus\mathrm{Im}\,(Q)$ contains both Z and Z'. If $q_L'=l_{\varepsilon'}\circ(q|_{p^{-1}(\mathrm{cl}\,U_L)})$, then $S(w,t)=(1-t)q_L(\omega)+tq_L'(\omega)$ for $\omega\in p^{-1}(\mathrm{cl}\,U_L)$ provides a homotopy between q_L and q_L' such that $G\circ p_L(\omega)\neq S(\omega,t)$ for $\omega\in p^{-1}(\mathrm{bd}\,U_L)$ and $t\in[0,1]$. Hence in view of Example 2.7(i), pairs $(p_L,G\circ p_L-q_L)$ and $(p_L,G\circ p_L-q_L)$ are homotopic and the homotopy invariance of deg shows that $\deg((p_L,G\circ p_L-q_L),U_L,0)=\deg((p_L,G\circ p_L-q_L'),U_L,0)$.

An independence of U follows as a consequence of the localization property of deg.

Definition 4.2. Given admissible pairs (p_0,q_0), (p_1,q_1) we say that they are *homotopic in* $\mathcal{D}_c(X,F)$ if there is $(R,S)\colon(p_0,q_0)\simeq(p_1,q_1)$ such that S is locally compact and the set $\{x\in X\mid f(x)\in S(R^{-1}(x,t))$ for some $t\in[0,1]\}$ is compact.

It is clear that pairs homotopic in $\mathcal{D}_c(X,F)$ belong to $\mathcal{D}_c(X,F)$, too.

Theorem 4.3. *The generalized coincidence index of pairs from* $\mathcal{D}_c(X,F)$ *has the following properties (like earlier* $C = \{x \in X | F(x) \in q(p^{-1}(x))\}$*):*

(i) (Existence) *If* $\mathrm{Ind}_F((p,q),X) \neq 0$, *then there is* $x \in X$ *such that* $F(x) \in q(p^{-1}(x))$.

(ii) (Localization) *If* $X' \subset X$ *is open and* $C \subset X'$, *then* $= \mathrm{Ind}_F((p,q),X')$ *is defined and equal to* $\mathrm{Ind}_F((p,q),X)$.

(iii) (Homotopy Invariance) *If* (p_0,q_0), (p_1,q_1) *are homotopic in* $\mathcal{D}_c(X,F)$, *then* $\mathrm{Ind}_F((p_0,q_0),X) = \mathrm{Ind}_F((p_1,q_1),X)$.

(iv) (Additivity) *If* X_1, X_2 *are open disjoint subsets of* X *such that* $C \subset X_1 \cup X_2$, *then*

$$\mathrm{Ind}_F((p,q),X) = \mathrm{Ind}_F((p,q),X_1) + \mathrm{Ind}_F((p,q),X_2).$$

(v) (Restriction) *If* $q(p^{-1}(X)) \subset Y$, *where* Y *is a closed subspace of* E', *then* $\mathrm{Ind}_F((p,q),X) = \mathrm{Ind}_G((p'q'),X \cap T)$, *where* $T := F^{-1}(Y \oplus \mathrm{Im}\,(Q))$, $p' = p|_{p^{-1}(\mathrm{cl}\,X \cap T)}$, $q' = q_{p^{-1}(\mathrm{cl}\,X \cap T)}$ *and* $G = F|_T$.

Proof. (i) By definition $\mathrm{Ind}_F((p,q),X) = \deg((p_{L_n}, G_n \circ p_{L_n} - q_{L_n}), U_{L_n})$, where L_n, l_n, G_n and $U_{L_n} = U \cap F^{-1}(L_n)$ correspond to a choice of $\varepsilon_n \in (0,\varepsilon_0]$ such that $\lim_{n\to\infty} \varepsilon_n = 0$. By the existence property of deg, there is a sequence $x_n \in U_{L_n}$ such that $F(x_n) \in q_{L_n}(p_{L_n}^{-1}(x_n)) = l_n \circ q(p^{-1}(x_n))$. Hence there is a sequence $\omega_n \in p^{-1}(x_n)$ such that $||F(x_n) - q(\omega_n)|| < \varepsilon_n \to 0$. But since q is compact in $p^{-1}(U)$, after passing to a subsequence if necessary, $q(\omega_n) \to z \in E'$. Since F restricted to $\mathrm{cl}\,U$ is proper, we gather again, without loss of generality, that $x_n \to x \in X$. The upper semicontinuity of $q(p^{-1}(\,\cdot\,))$ implies that $F(x) \in q(p^{-1}(x))$.

Properties (ii)–(iv) follow directly from the definition and respective properties of deg.

Property (v) follows easily from the definition since $(p',q') \in \mathcal{D}_c(X \cap T, G)$, $i(G) = k$ and an "admissible" space L from Definition 4.1 may be chosen in $Y \oplus \mathrm{Im}\,(Q)$. □

5. F-fundamentally restrictible maps

Let E, E', X and F be as in Section 4. Let $\varphi : X \multimap E'$ be a set-valued map.

Definition 5.1. A closed convex set $K \subset E'$ is called F-*fundamental* for φ, provided

(i) $\varphi(F^{-1}(K) \cap X) \subset K$, and

(ii) if for $x \in X$, $F(x) \in \overline{\mathrm{conv}}\,(\varphi(x) \cup K)$, then $F(x) \in K$.

It is clear that for any φ some F-fundamental set exists.

Observe that if $E = E'$ and $F = \mathrm{id}_E$ is the identity on E, then K is nothing else but a fundamental set for φ in the sense of e.g. [4] (see also references therein).

Some properties of F-fundamental sets are summarized in the following result (comp. [6]).

Proposition 5.2.

(i) *If K is an F-fundamental set for φ, then $\{x \in X \mid F(x) \in \varphi(x)\} \subset F^{-1}(K)$.*

(ii) *If K_1, K_2 are F-fundamental sets for φ, then so is $K = K_1 \cap K_2$.*

(iii) *If $P \subset K$ and K is an F-fundamental set for φ, then so is $K' = \overline{\mathrm{conv}}\,(\varphi(F^{-1}(K) \cap X) \cup P)$.*

(iv) *If K is the intersection of all F-fundamental sets for φ, then $K = \overline{\mathrm{conv}}\,(\varphi(F^{-1}(K) \cap X))$.*

(v) *For any $A \subset E'$, there exists an F-fundamental set K such that $K = \overline{\mathrm{conv}}\,(\varphi(F^{-1}(K) \cap X) \cup A)$.*

Definition 5.3. We say that φ is an *F-fundamentally restrictible map* if there exists a *compact* F-fundamental set for φ.

Let us collect some important examples of F-fundamentally restrictible set-valued maps.

Example 5.4. Let $F : E \to E'$ be an arbitrary Fredholm operator.

(a) If $\varphi : X \to E'$ is compact (i.e. $\mathrm{cl}\,\varphi(X)$ is compact), then $K = \overline{\mathrm{conv}}\,(\varphi(X))$ is a compact F- fundamental set for φ; hence φ is F-fundamentally restrictible.

(b) Let μ be a measure of noncompactness in E' having usual properties (see e.g. [1]) and let φ be F-condensing in the sense that, for any bounded set $A \subset X$, if $\mu(\varphi(A)) \geq \mu(F(A))$, then A is compact. If φ is bounded, then one shows that an F-fundamental set K, satisfying $K = \overline{\mathrm{conv}}\,(\varphi(F^{-1}(K) \cap X) \cup \{y\})$ for some $y \in E'$ (see Proposition 5.2) is compact; hence φ is F-fundamentally restrictible.

(c) If φ is an F-set contraction (i.e. there exists $k \in (0,1)$, such that for any bounded $A \subset X$, $\mu(\varphi(A)) \leq k\mu(F(A))$), then φ is F-condensing and therefore F-fundamentally restrictible.

Some other examples one can find in [6] and in [7].

Definition 5.5.

(i) Let $X \xleftarrow{p} \Gamma \xleftarrow{q} E'$ be an admissible pair. A closed convex set K is called *F-fundamental* for (p,q) if it is so for the set-valued map $\varphi(x) = q(p^{-1}(x))$.

(ii) The pair (p,q) is *F-fundamentally restrictible* if so is the above map φ and if K, K' are two compact disjoint F-fundamental sets for (p,q), then there exists a finite number of compact F-fundamental sets $K_1, \dots , K_n$ for φ such that $K \cap K_1 \neq \emptyset$, $K_n \cap K' \neq \emptyset$ and $K_i \cap K_{i+1} \neq \emptyset$ for all $1 \leq i \leq n-1$.

Remark 5.6. Observe that if *a priori* we knew that $\{x \in X \mid F(x) \in q(p^{-1}(x))\} \neq \emptyset$, then any two F-fundamental sets intersect. Moreover, any admissible pair (p, q) determining a set-valued map belonging to any of classes discussed in Example 5.4 (and some others, too) satisfies this condition (see [6]).

Now we are going to define a generalized index of coincidence between F and an F-fundamentally restrictible pair $X \xleftarrow{p} \Gamma \xrightarrow{q} E'$. Suppose that $C := \{x \in X \mid F(x) \in q(p^{-1}(x))\}$ is bounded and closed. The class of such pairs will be denoted by $\mathcal{D}(X, F)$. Therefore there is an open bounded set U such that $C \subset U \subset \operatorname{cl} U$. Let K_0 be *any* compact F-fundamental set for (p, q). In view of Proposition 5.2(i), C is contained in $F^{-1}(K_0) \cap \operatorname{cl} U$. Since $F|_{\operatorname{cl} U}$ is proper, we gather that C being obviously closed is also compact. Now let consider a map

$$q_{|p^{-1}(F^{-1}(K_0)\cap X)} : p^{-1}(F^{-1}(K_0) \cap X) \to E,$$

According to Definition 5.1, the range of this map is contained in K_0. Hence it has a compact extension $\overline{q} : \Gamma \to K_0$.[4]

It is clear that $\{x \in X \mid F(x) \in \overline{q}(p^{-1}(x))\} = C$. Hence $(p, \overline{q}) \in \mathcal{D}_c(X, F)$ and we are in a position to define an index via methods from Section 3.

Definition 5.7. By the *generalized index of* $(p, q) \in \mathcal{D}(X, F)$ we understand the element

$$\operatorname{Ind}_F((p, q), X) := \operatorname{Ind}_F((p, \overline{q}), X) \in \Pi_k.$$

Let us show that this definition is correct, i.e. does not depend on the choice of a compact F-fundamental set K_0 and an extension $\overline{q}$ of $q_{|p^{-1}(F^{-1}(K_0)\cap X)}$. Assume that K_1 is another compact F-fundamental set for (p, q) and let $\overline{q_1} : \Gamma \to K_1$ be a compact extension of $q_{|p^{-1}(F^{-1}(K_1)\cap X)}$. In view of Definition 5.5 we can assume without loss of generality that $K_2 = K_1 \cap K_0 \neq \emptyset$. By Proposition 5.2(ii), K_2 is a compact F-fundamental set for (p, q), too. Let $r_i : E' \to K_i, i = 0, 1, 2$ be retractions and consider a map $S : \Gamma \times [0, 1]$ given by

$$S(w, t) = \begin{cases} (1 - 4t)\overline{q}(w) + 4t(r_0 \circ q)(w) & \text{for } t \in [0, 1/4], \\ (2 - 4t)(r_0 \circ q)(w) + (4t - 1)(r_2 \circ q)(w) & \text{for } t \in (1/4, 1/2], \\ (3 - 4t)(r_2 \circ q)(w) + (4t - 2)(r_1 \circ q)(w) & \text{for } t \in (1/2, 3/4], \\ (4 - 4t)(r_1 \circ q)(w) + (4t - 3)\overline{q_1}(w) & \text{for } t \in (3/4, 1]. \end{cases}$$

and $w \in \Gamma$. It is clear that $S(\Gamma \times [0, 1]) \subset K_0 \cup K_1$ and therefore S is compact. Moreover, $S(\,\cdot\,, 0) = \overline{q}$, $S(\,\cdot\,, 1) = \overline{q_1}$. Hence, introducing $R : \Gamma \times [0, 1] \to X \times [0, 1]$

[4] For instance one can take any retraction $r : E' \to K_0$ and define $\overline{q} := r \circ q$.

by $R(w,t)=(p(w),t)$ for $w\in\Gamma$, $t\in[0,1]$, we see that $(R,S):(p,\overline{q})\simeq(p,\overline{q_1})$. Assume that $F(x)\in S(R^{-1}(x,t))$ for some $x\in X$ and $t\in[0,1]$. It is easy to check, using properties (i) and (ii) of Definition 5.1, that $x\in C$; hence $\{x\in X|F(x)\in S(R^{-1}(x,t))$ for some $t\in[0,1]\}$ is compact and, by the homotopy invariance (see Theorem 4.3(iii)), $\mathrm{Ind}_F((p,\overline{q}),X)=\mathrm{Ind}_F((p,\overline{q_1}),X)$.

Definition 5.8. Given F-fundamentally restrictible pairs (p_0,q_0), (p_1,q_1) we say that they are (F,K)*–homotopic* (written $(p_0,q_0)\simeq_K(p_1,q_1)$) if there is a homotopy $(R,S):(p_0,q_0)\simeq(p_1,q_1)$ such that the set $\{x\in X\mid F(x)\in S(R^{-1}(x,t))$ for some $t\in[0,1]\}$ is bounded and closed in E and K is a compact F-fundamental set for any map $X\ni x\mapsto S(R^{-1}(x,t))$ where $t\in[0,1]$.

At first glance the above definition of homotopic pairs is enough for our next considerations (comp. Theorem 5.10), but in applications we need more general one (which e.g. guarantees equivalence relation in $\mathcal{D}(X,F)$).

Definition 5.9. Two F-fundamentally restrictible pairs (p_0,q_0), (p_1,q_1) are *homotopic in* $\mathcal{D}(X,F)$ if there is a finite number of compact convex sets $K_1,\dots,K_n$ and F-fundamentally restrictible pairs $(r_1,s_1),\dots,(r_{n-1},s_{n-1})$ such that

$$(p_0,q_0)\simeq_{K_1}(r_1,s_1)\simeq_{K_2}\dots\simeq_{K_{n-1}}(r_{n-1},s_{n-1})\simeq_{K_n}(p_1,q_1).$$

It is clear that F-fundamentally restrictible pairs (p_0,q_0), (p_1,q_1) homotopic in $\mathcal{D}(X,F)$ belong to $\mathcal{D}(X,F)$, too.

Theorem 5.10. *The generalized index* Ind_F *on* $\mathcal{D}(X,F)$ *has all properties from Theorem* 4.3 (*with some obvious adjustments*).

Proof. Properties (i), (ii) and (iv) follow from the very definition and respective properties of $\mathrm{Ind}_F((p,\overline{q}),X)$ (we sustain here the notation from the paragraph preceding Definition 5.7).

As concerns (iii), without loss of generality we can prove this property only for pairs being (F,K)-homotopic, where K is a compact convex subset of E'. Suppose that $(R,S):(p_0,q_0)\simeq_K(p_1,q_1)$. By Definition 5.8, $S(R^{-1}((F^{-1}(K)\cap X)\times[0,1]))\subset K$; take an arbitrary extension $\overline{S}:\Gamma\to K$ of $S|_{R^{-1}((F^{-1}(K)\cap X)\times[0,1])}$. It is then clear that $\overline{q_l}=\overline{S}\circ j_l$, $l=0,1$ (see Definition 5.8) is a compact extension of $q_l|_{p_l^{-1}(F^{-1}(K)\cap X)}$ onto Γ_l ($l=0,1$). Moreover $(R,\overline{S}):(p_0,\overline{q_0})\simeq(p_1,\overline{q_1})$ in $\mathcal{D}_c(X,F)$. Hence

$$\mathrm{Ind}_F((p_0,q_0),X)=\mathrm{Ind}_F((p_0,\overline{q_0}),X)=\mathrm{Ind}_F((p_1,\overline{q}_1),X)=\mathrm{Ind}_F((p_1,q_1),X).$$

(v) Under the assumptions and notations of (v) from Theorem 4.3, first observe that, if K is a compact F-fundamental set for (p,q), then $K\cap(Y\oplus\mathrm{Im}\,(Q))$ is G-fundamental for (p',q'); therefore $(p',q')\in\mathcal{D}(X\cap T,G)$. Let $\overline{q}:\Gamma\to$

$K \cap Y$ be a compact extension of $q|_{p^{-1}(F^{-1}(K)\cap X)}$. Then by definition and Theorem 4.3(v),

$$\mathrm{Ind}_F((p,q),X) = \mathrm{Ind}_F((p,\overline{q}),X) = \mathrm{Ind}_G((p',\overline{q}'),X),$$

where $\overline{q}' = \overline{q}_{|p^{-1}(X\cap T)}$. But one easily sees that $\overline{q}'$ is a compact extension of $q'|_{p'^{-1}(G^{-1}(K\cap Y\oplus \mathrm{Im}\,(Q))\cap X\cap T)}$ and, hence,

$$\mathrm{Ind}_G((p',q'),X) = \mathrm{Ind}_G((p',\overline{q}'),X). \qquad \square$$

6. Final remarks

The idea how to define the coincidence index from Section 3 follows from [15]. Here however we presented a direct and more natural approach avoiding tedious and technical applications of the so-called "infinite dimensional" cohomotopy. Observe moreover that in Sections 3, 4, the standing assumption that p is a Vietoris map may be slightly generalized. Namely one may assume that p has the following property: for any finite-dimensional subspace $Y' \subset E'$, $p_{|p^{-1}(X\cap F^{-1}(Y))}$ is a Vietoris map. Of course it does not change anything as concerns the fibres of p but relaxes a bit condition (ii) of Definition 2.2.

If one wants to get rid of *any* assumptions concerning the dimensionality of the preimages of p, then one has to admit a different geometric assumption concerning fibres. Recall that a proper surjection $p : \Gamma \to X$ is *cell-like* if, for any $x \in X$, the fibre $p^{-1}(x)$ is a *cell-like set*, i.e. for any embedding of $p^{-1}(x)$ into an ANR, it is contractible in each of its neighbourhoods (see [16], [2]). It is clear that the fibres of a cell-like map are acyclic and, for instance, a proper surjection with contractible fibres is cell-like.

A result, implicitly contained in [5] (comp. [14], [15]) and similar to that of Theorem 2.4 says that: *given a cell-like map $p : \Gamma \to X$, closed sets $\Gamma' \subset \Gamma$, $X' \subset X$ such that $p^{-1}(X') = \Gamma'$ and a pair (Y, Y') of ANR, the transformation $p^{\#}[X, X'; Y, Y'] \to [\Gamma, \Gamma'; Y, Y']$, induced by p, is bijective provided* $\dim X < \infty$.

We see that it complements Theorem 2.4 in the case when $\dim \Gamma = \infty$.

All results of the paper may be put into the context of pairs (p, q), where p is a cell-like mapping. This is important from the view-point of applications, where we often meet set-valued maps of such form.

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Dorota Gabor and Wojciech Kryszewski
Faculty of Mathematics and Computer Science
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, POLAND

E-mail address: dgabor@mat.uni.torun.pl, wkrysz@mat.uni.torun.pl