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FREE BOUNDARY PROBLEM FOR A VISCOUS HEAT-CONDUCTING FLOW WITH SURFACE TENSION

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Abstract. In the paper the equations describing the motion of a drop of a viscous heat-conducting capillary fluid bounded by a free surface are examined. Assuming that the viscosity coefficients, the coefficient of heat-conductivity, the pressue and the specific heat at constant volume of the fluid depend on its density and temperature we prove the existence of a global in time solution which is close to a constant state for any moment of time.

1. Introduction

In this paper we study the motion of a drop of a viscous compressible heat-conducting capillary fluid, the viscosity and the heat-conductivity coefficients of which depend on its density and temperature.

Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain of the drop at time t. Let $v = v(x,t)$ be the velocity of the fluid, $\rho = \rho(x,t)$ the density, $\theta = \theta(x,t)$ the temperature, $f = f(x,t)$ the external force field per unit mass, $r = r(x,t)$ the heat sources per unit mass, $\bar\theta = \bar\theta(x,t)$ the heat flow per unit surface, $p = p(\rho,\theta)$ the pressure, $c_v = c_v(\rho,\theta)$ the specific heat at constant pressure, $\mu = \mu(\rho,\theta)$ and $\nu = \nu(\rho,\theta)$ the viscosity coefficients, $\varkappa = \varkappa(\rho,\theta)$ the coefficient of heat-conductivity, σ the

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constant coefficient of the surface tension, p_0 the external (constant) pressure. Then the motion of the drop is described by the following system of equations:

$$
(1.1)\quad
\begin{aligned}
&\rho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = \rho f && \text{in } \widetilde{\Omega}^T,\\
&\rho_t + \operatorname{div}(\rho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\rho c_v(\theta_t + v\cdot\nabla\theta) - \operatorname{div}(\varkappa\nabla\theta) - \theta p_\theta \operatorname{div} v \\
&\qquad - \frac{\mu}{2}\sum_{i,j=1}^{3}(v_{ix_j}+v_{jx_i})^2 - (\nu-\mu)(\operatorname{div} v)^2 = \rho r && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\overline{n} - \sigma H\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\frac{\varphi_t}{|\nabla\varphi|} && \text{on } \widetilde{S}^T,\\
&\varkappa(\rho,\theta)\frac{\partial\theta}{\partial n} = \overline{\theta} && \text{on } \widetilde{S}^T,\\
&\rho|_{t=0} = \rho_0,\ v|_{t=0} = v_0,\ \theta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned}
$$

where $\widetilde{\Omega}^T = \bigcup_{t\in(0,T)}\Omega_t\times\{t\}$, $\widetilde{S}^T = \bigcup_{t\in(0,T)} S^T\times\{t\}$, $S_t = \partial\Omega_t$, $\varphi(x,t) = 0$ describes S_t (at least locally), $\overline{n}$ is the unit outward vector normal to the boundary, i.e. $\overline{n} = \nabla\varphi/|\nabla\varphi|$; $\Omega = \Omega_t|_{t=0} = \Omega_0$. By $\mathbb{T} = \mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T} = \{T_{ij}\}_{i,j=1,2,3} = \{\mu(\rho,\theta)S_{ij}(v) + (\nu(\rho,\theta)-\mu(\rho,\theta))\delta_{ij}\operatorname{div} v - p(\rho,\theta)\delta_{ij}\}_{i,j=1,2,3},$$

where $\mathbb{S}(v) = \{S_{ij}(v)\}_{i,j=1,2,3} = \{v_{ix_j} + v_{jx_i}\}_{i,j=1,2,3}$ is the velocity deformation tensor.

Moreover, H is the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\overline{n} = \Delta_{S_t}(t)x, \quad x = (x_1,x_2,x_3),$$

where $\Delta_{S_t}(t)$ is the Laplace–Beltrami operator on S_t. Let S_t be given locally by $x = x(s_1,s_2,t)$, $(s_1,s_2)\in U\subset\mathbb{R}^2$, where U is an open set. Then

$$(1.2)\qquad \Delta_{S_t}(t) = g^{-1/2}\frac{\partial}{\partial s_\alpha}\left(g^{1/2}g^{\alpha\beta}\frac{\partial}{\partial s_\beta}\right), \quad \alpha,\beta = 1,2,$$

where the summation convention over repeated indices is assumed, $g = \det\{g_{\alpha\beta}\}_{\alpha,\beta=1,2}$, $g_{\alpha\beta} = \partial x/\partial s_\alpha\cdot\partial x/\partial s_\beta$, $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$.

Finally, thermodynamic considerations imply $c_v > 0$, $\varkappa > 0$, $\nu > \mu/3 > 0$, $\sigma > 0$.

From $(1.1)_{2,5}$ it follows that the total mass of the fluid in Ω_t is conserved, i.e.

$$(1.3)\qquad \int_{\Omega_t}\rho\,dx = \int_\Omega \rho_0\,d\xi = M.$$

In this paper we prove the existence of global-in-time solutions to (1.1) which are close to an equilibrium solution.

Assume that $p_\rho > 0$, $p_\theta > 0$ for $\rho, \theta \in \mathbb{R}^1_+$ and consider the equation

$$p\bigg(\frac{M}{(4/3)\pi R_e^3}, \theta_e\bigg) = p_0 + \frac{2\sigma}{R_e}. \tag{1.4}$$

We assume that there exist $R_e > 0$ and $\theta_e > 0$ satisfying (1.4). Then we have the following definition.

Definition 1.1. Let $f = r = \overline{\theta} = 0$. By an equilibrium (constant) state we mean a solution $(v, \theta, \rho, \Omega_t)$ of problem (1.1) such that $v = 0$, $\theta = \theta_e$, $\rho = \rho_e$, $\Omega_t = \Omega_e$ for $t \ge 0$, where $\rho_e = M/((4/3)\pi R_e^3)$, Ω_e is a ball of radius R_e, $R_e > 0$ and $\theta_e > 0$ satisfy equation (1.4).

In order to prove the existence of solutions to problem (1.1) we have to introduce the Lagrangian coordinates which are initial data to the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \Omega. \tag{1.5}$$

Integrating (1.5) we obtain the following relation between the Eulearian x and the Lagrangian coordinates:

$$x = \xi + \int_0^t u(\xi, t')\, dt' \equiv X_u(\xi, t),$$

where $u(\xi,t) = v(X_u(\xi,t),t)$. Moreover, we have $\Omega_t = \{x \in \mathbb{R}^3 : x = X_u(\xi,t),\ \xi \in \Omega\}$, $S_t = \{x \in \mathbb{R}^3 : x = X_u(\xi,t),\ \xi \in S\}$.

Using the Lagrangian coordinates we write problem (1.1) in the form

$$\begin{aligned}
&\eta u_t - \mathrm{div}_u \mathbb{T}_u(u,p) = \eta g && \text{in } \Omega^T = \Omega \times (0,T),\\
&\eta_t + \eta \mathrm{div}_u u = 0 && \text{in } \Omega^T,\\
&\eta c_v \vartheta_t - \mathrm{div}_u(\varkappa \nabla_u \vartheta) + \vartheta p_\vartheta \mathrm{div}_u u &&\\
&\qquad - \frac{\mu}{2}\sum_{i,j=1}^3 (\xi_{x_i} \cdot \nabla_\xi u_j + \xi_{x_j} \cdot \nabla_\xi u_i)^2 &&\\
&\qquad - (\nu - \mu)(\mathrm{div}_u u)^2 = \eta k && \text{in } \Omega^T,\\
&\mathbb{T}_u(u,p)\overline{n}_u - \sigma H \overline{n}_u = -p_0 \overline{n}_u && \text{on } S^T = S \times (0,T),\\
&\varkappa(\eta, \vartheta)\overline{n}_u \cdot \nabla \vartheta = \overline{\vartheta} && \text{on } S^T,\\
&\eta|_{t=0} = \rho_0,\ u|_{t=0} = v_0,\ \vartheta|_{t=0} = \theta_0 && \text{in } \Omega,
\end{aligned} \tag{1.6}$$

where $\eta(\xi,t) = \rho(X_u(\xi,t),t)$, $\vartheta(\xi,t) = \theta(X_u(\xi,t),t)$, $g(\xi,t) = f(X_u(\xi,t),t)$, $k(\xi,t) = r(X_u(\xi,t),t)$, $\overline{\vartheta}(\xi,t) = \overline{\theta}(X_u(\xi,t),t)$, $\overline{n}_u(\xi,t) = \overline{n}(X_u(\xi,t),t)$, $\nabla_u = \xi_{ix}\partial_{\xi_i} = \{\xi_{ix_j}\partial_{\xi_i}\}_{j=1,2,3}$,

$$\begin{aligned}
\mathbb{T}_u(u,p) = &- pI + \mathbb{D}_u(u) = \{-p(\eta,\vartheta)\delta_{ij} + \mu(\eta,\vartheta)(\partial_{x_i}\xi_k\partial_{\xi_k}u_j\\
&+ \partial_{x_j}\xi_k\partial_{\xi_k}u_i) + (\nu(\eta,\vartheta) - \mu(\eta,\vartheta))\delta_{ij}\mathrm{div}_u u\}_{i,j=1,2,3},
\end{aligned}$$

$I = \{\delta_{ij}\}_{i,j=1,2,3}$, is the unit matrix,

$$\operatorname{div}_u u = \nabla_u \cdot u = \partial_{x_i}\xi_k \partial_{\xi_k} u_i, \quad \operatorname{div}_u \mathbb{T}_u(u,p) = \{\partial_{x_j}\xi_k \partial_{\xi_k} T_{uij}(u,p)\}_{i=1,2,3}$$

($\partial_{x_i}\xi_k$ are the elements of the matrix ξ_x which is inverse to $x_\xi = I + \int_0^t u_\xi(\xi,t')\,dt'$) and the summation over repeated indices is assumed.

This paper consists of five sections. In Section 2 notation and auxiliary results are introduced. In Section 3 we prove the local existence theorem for problem (1.1) (see Theorem 3.3). In Section 4 we derive some estimates which are necessary for the global existence (see Lemmas 4.1–4.2). Finally, Section 5 is devoted to the global existence theorem for problem (1.1) (see Theorems 5.1–5.2).

In this paper we prove the global existence of a solution

$$\begin{aligned}(u,\vartheta,\eta) \in{}& W_2^{2+\alpha,2+\alpha/2}(\Omega^T) \times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\\ &\times C([0,T]; W_2^{1+\alpha}(\Omega)) \cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T) \quad (\alpha \in (3/4,1))\end{aligned}$$

to problem (1.1), which is close to the equilibrium state (see Definition 1.1). To obtain this result we use methods similar to those of [8], [7], [4], [15], [12] and [13].

Papers [15], [12] and [13] are concerned with problem (1.1) in the case when ν, μ and $\varkappa$ do not depend on ρ and θ. In [15] the local existence theorem for problem (1.1) is proved, while papers [12] and [13] are devoted to the global existence of a solution to this problem. Moreover, the regularity of solutions obtained in papers [15] and [13] is the same as in this paper.

In papers [8] and [7] the motion of compressible barotropic viscous capillary fluids bounded by a free surface is considered, while in [4] the authors study the global motion of a compressible barotropic viscous fluid with boundary slip condition.

Finally, papers [11], [9] and [10] are concerned with free boundary problems for equations describing the motion of viscous incompressible capillary fluids with the surface tension dependent on the temperature.

2. Notation and auxiliary results

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary S. The Sobolev–Slobodetskiĭ space with the norm

$$\begin{aligned}\|u\|_{W_2^{k+\alpha,k/2+\alpha/2}(\Omega^T)} = \Big[& \sum_{|\beta|+2i\le k} \|D_x^\beta \partial_t^i u\|^2_{L_2(\Omega^T)}\\ &+ \sum_{|\beta|=k} \int_0^T \int_\Omega\int_\Omega \frac{|D_x^\beta u(x,t) - D_{x'}^\beta u(x't)|^2}{|x-x'|^{3+2\alpha}}\,dx\,dx'\,dt\\ &+ \int_\Omega \int_0^T\int_0^T \frac{\partial_t^{[k/2]} u(x,t) - \partial_{t'}^{[k/2]} u(x,t')|^2}{|t-t'|^{1+\alpha+k-2[k/2]}}\,dx\,dt\,dt'\Big]^{1/2},\end{aligned}$$

where $D_x^\beta = \partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\partial_{x_3}^{\beta_3}$, $\beta = (\beta_1,\beta_2,\beta_3)$ is multi-index, $|\beta| = \beta_1+\beta_2+\beta_3$, we denote by $W_2^{k+\alpha,k/2+\alpha/2}(\Omega^T)$, $k\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,1)$.

Similarly we can define the norms in space $W_2^{k+\alpha}(\Omega)$ and $W_2^{k+\alpha,k/2+\alpha/2}(S^T)$. Moreover, we use the notation:

$$\begin{aligned}
\|u\|_{W_2^{k+\alpha,k/2+\alpha/2}(\Omega^T)} &= \|u\|_{k+\alpha,\Omega^T},\\
\|u\|_{W_2^{k+\alpha}(\Omega)} &= \|u\|_{k+\alpha,Q},\quad Q\in\{\Omega,S,S^1\}\ (S^1 \text{ is the unit sphere}),\\
\|u\|_{L_p(Q)} &= |u|_{p,Q},\quad p\in[1,\infty],\ Q\in\{\Omega,S\},\\
\|u\|_{L_2(Q)} &= \|u\|_{0,Q},\quad Q\in\{\Omega,S,\Omega^T,S^T\},\\
\|u\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} &= \bigg[\|u\|^2_{2+\alpha,\Omega^T} + T^{-\alpha}\bigg(\|u_t\|^2_{0,\Omega^T} + \sum_{|\beta|=2}\|\partial_x^\beta u\|^2_{0,\Omega^T}\bigg)\\
&\qquad + \sup_{t\le T}\|u(\,\cdot\,,t)\|^2_{1+\alpha,\Omega}\bigg]^{1/2},\\
\|u\|_{Q^T}^{(\alpha,\alpha/2)} &= (\|u\|^2_{\alpha,Q^T} + T^{-\alpha}\|u\|^2_{0,Q^T})^{1/2},\quad Q\in\{\Omega,S\},\\
[u]_{\alpha,\Omega^T,x} &= \bigg(\int_0^T dt\int_\Omega\int_\Omega \frac{|u(x,t)-u(x',t)|^2}{|x-x'|^{3+2\alpha}}\,dx\,dx'\bigg)^{1/2},\\
[u]_{\alpha,\Omega^T,t} &= \bigg(\int_\Omega dx\int_0^T\int_0^T \frac{|u(x,t)-u(x,t')|^2}{|t-t'|^{1+2\alpha}}\,dt\,dt'\bigg)^{1/2}.
\end{aligned}$$

Next, by $B_p^l(\mathbb{R}^n)$, $p\in[1,\infty]$, $l\in\mathbb{R}^1_+$, $l\notin\mathbb{Z}$, we denote the isotropic Besov spaces. By the Golovkin theorem from [3] the norms of spaces $B_p^l(\mathbb{R}^n)$ and the Sobolev–Slobodetskiĭ spaces $W_p^l(\mathbb{R}^n)$ ($l\notin\mathbb{Z}$) are equivalent.

In the paper we use the following imbedding for the Besov spaces (see [1, Section 18]):

$$D_x^\sigma B_p^l(\mathbb{R}^n)\subset B_q^\varrho(\mathbb{R}^n)\quad\text{for } \frac{n}{p}-\frac{n}{q}+|\sigma|+\varrho\le l, \tag{2.1}$$

and the following interpolation inequality

$$\|D_x^\sigma u\|_{B_q^\varrho(\mathbb{R}^n)} \le \varepsilon^{1-\varkappa}\|u\|_{B_p^l(\mathbb{R}^n)} + c\varepsilon^{-\varkappa}\|u\|_{L_p(\mathbb{R}^n)}, \tag{2.2}$$

where $\varkappa = (n/p-n/q+|\sigma|+\varrho)/l$ (which holds for $n/p-n/q+|\sigma|+\varrho<l$).

In the above notation $B_p^l(\mathbb{R}^n)$ with $l\in\mathbb{Z}_+$ is the Sobolev space.

All above remarks can be applied for spaces of functions defined on a bounded domain $\Omega\subset\mathbb{R}^n$, and by using a partition of unity we can also define spaces of traces on the boundary of Ω and formulate corresponding trace theorems.

By $C_B^k(Q)$ ($Q\subset\mathbb{R}^n$ is a domain) we denote the space of functions $u\in C^k(Q)$ such that $D^\sigma u$ is bounded on Q with the norm

$$\|u\|_{C_B^1(Q)} = \max_{0\le|\sigma|\le k}\sup_{x\in Q}|D^\sigma u(x)|.$$

Now, we introduce a partition of unity $(\{\widetilde{\Omega}_l\},\{\zeta_l\})$, $\Omega=\bigcup_{l\in\mathcal{M}\cup\mathcal{N}}\widetilde{\Omega}_l$, $\sum_{l\in\mathcal{M}\cup\mathcal{N}}\zeta_l(x)=1$ for $x\in\Omega$ (where $\widetilde{\Omega}_l$ with $l\in\mathcal{M}$ are interior subdomains and $\widetilde{\Omega}_l$ with $l\in\mathcal{N}$ are boundary subdomains), which is used in Section 4 to derive one of inequalities necessary for the global existence. Let $\widetilde{\Omega}_l$ be one of the $\widetilde{\Omega}_{l,s}$ and $\zeta_l(\xi)$ be the corresponding function. If $\widetilde{\Omega}_l$ is an interior subdomain then let $\widetilde{\omega}_l\subset\widetilde{\Omega}_l$ and $\zeta_l(\xi)=1$ for $\xi\in\widetilde{\omega}_l$. Otherwise we assume that $\overline{\widetilde{\Omega}}_l\cap S\neq\phi$, $\overline{\widetilde{\omega}}_l\cap S\neq\phi$, $\overline{\widetilde{\omega}}_l\subset\overline{\widetilde{\Omega}}_l$. Let $\beta\in\overline{\widetilde{\omega}}_l\cap S\subset\overline{\widetilde{\Omega}}_l\cap S$, $\widetilde{S}_l\equiv S\cap\widetilde{\Omega}_l$. Introduce local coordinates connected with $\{\xi\}$ by

$$y_i=\alpha_{ij}(\xi_j-\beta_j),\quad \alpha_{3i}=n_i(\beta),\quad i=1,2,3, \tag{2.3}$$

where α_{ij} is a constant orthogonal matrix such that $\widetilde{S}_l$ is determined by

$$y_3=F(y_1,y_2),\quad F\in W_2^{5/2+\alpha},$$

and

$$\widetilde{\Omega}_l=\{y:|y_i|<d,\ i=1,2,\ F(y')<y_3<F(y')+d,\ y'=(y_1,y_2)\}. \tag{2.4}$$

Next, we introduce functions u', ϑ' and η' by

$$u'_{ij}(y)=\alpha_{ij}u_j(\xi)|_{\xi=\xi(y)},\quad \vartheta'(y)=\vartheta(\xi)|_{\xi=\xi(y)},\quad \eta'(y)=\eta(\xi)|_{\xi=\xi(y)},$$

where $\xi=\xi(y)$ is the inverse transformation to (2.3). Further we introduce new variables by

$$z_i=y_i,\quad i=1,2,\quad z_3=y_3-\widetilde{F}(y),\quad y\in\widetilde{\Omega}_l,$$

which will be denoted by $z=\Phi(y)$, where $\widetilde{F}$ is an extension of F to $\widetilde{\Omega}_l$. Let

$$\widehat{\Omega}_l=\Phi(\widetilde{\Omega}_l)=\{z:|z_k|<d,\ k=1,2,\ 0<z_3<d\} \tag{2.5}$$

and $\widehat{S}_l=\Phi(\widetilde{S}_l)$. Define

$$\widehat{f}(z)=f'(y)|_{y=\Phi^{-1}(z)},\quad f\in\{u,\vartheta,\eta\}.$$

Introduce $\widehat{\nabla}_k=\xi_{lx_k}z_{i\xi_l}\nabla_{z_i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi)=\Phi(\psi(\xi))$ and $y=\psi(\xi)$ is defined by (2.3). Introduce also the notation

$$\begin{aligned}
\widetilde{u}_l(\xi)&=u(\xi)\zeta_l(\xi), & \widetilde{\vartheta}_{\sigma l}(\xi)&=\vartheta_\sigma(\xi)\zeta_l(\xi),\\
\widetilde{\eta}_{\sigma l}(\xi)&=\eta_\sigma(\xi)\zeta_l(\xi), & \xi&\in\widetilde{\Omega}_l,\ \widetilde{\Omega}_l\cap S=\phi,\\
\widetilde{u}_l(z)&=\widehat{u}(z)\widehat{\zeta}_l(z), & \widetilde{\vartheta}_{\sigma l}(z)&=\widehat{\vartheta}_\sigma(z)\widehat{\zeta}_l(z),\\
\widetilde{\eta}_{\sigma l}(z)&=\widehat{\eta}_\sigma(z)\widehat{\zeta}_l(z), & z&\in\widehat{\Omega}_l=\Phi(\widetilde{\Omega}_l),\ \overline{\widetilde{\Omega}}_l\cap S\neq\phi,
\end{aligned}$$

where $\widehat{\zeta}_l(z)=\zeta_l(\xi)|_{\xi=\chi^{-1}(z)}$, $\eta_\sigma=\eta-\varrho_e$, $\vartheta_\sigma=\vartheta-\theta_e$; ϱ_e, θ_e are introduced by Definition 1.1.

Then problem (1.6) takes the following form in an interior subdomain:

$$
\begin{aligned}
&\eta\widetilde{u}_{it}-\nabla_j T_{ij}(\widetilde{u},\widetilde{p}_\sigma)=-\nabla_{u_j}B_{uij}(u,\zeta)-T_{uij}(u,p_\sigma)\nabla_{u_j}\zeta\\
&\qquad-(\nabla_j T_{ij}(\widetilde{u},\widetilde{p}_\sigma)-\nabla_{u_j}T_{uij}(\widetilde{u},\widetilde{p}_\sigma))\equiv k_{1i},\quad i=1,2,3,\\
&\widetilde{\eta}_{\sigma t}+\varrho_e\nabla\cdot\widetilde{u}=\varrho_e u\cdot\nabla_u\zeta-\eta_\sigma\nabla_u\cdot u\zeta+\varrho_e(\nabla\cdot\widetilde{u}-\nabla_u\cdot\widetilde{u})\equiv k_2,\\
&\eta c_v\widetilde{\vartheta}_{\sigma t}-\nabla\cdot(\varkappa\nabla\widetilde{\vartheta}_\sigma)+\theta_e p_\vartheta(\varrho_e,\theta_e)u\cdot\nabla_u\zeta\\
&\quad=\eta\widetilde{k}+\bigg[\frac{\mu}{2}\sum_{i,j=1}^{3}(\xi_{kx_i}\partial_{\xi_k}u_j+\xi_{kx_j}\partial_{\xi_k}u_i)^2+(\nu-\mu)(\nabla_u\cdot u)\bigg]\zeta\\
&\qquad+\theta_e p_\vartheta(\varrho_e,\theta_e)u\cdot\nabla_u\zeta+(\theta_e p_\vartheta(\varrho_e,\theta_e)-\vartheta p_\vartheta(\varrho,\theta))\nabla_u\cdot u\zeta\\
&\qquad+\theta_e p_\vartheta(\varrho_e,\theta_e)(\nabla\cdot\widetilde{u}-\nabla_u\cdot\widetilde{u})-\varkappa(\eta,\vartheta)(\nabla_u^2\zeta\vartheta_\sigma+2\nabla_u\zeta\cdot\nabla_u\vartheta_\sigma)\\
&\qquad-\nabla_u\varkappa\cdot\nabla_u\zeta\vartheta_\sigma-\zeta\nabla_u\varkappa\cdot\nabla_u\vartheta_\sigma-[\nabla\cdot(\varkappa\nabla\vartheta_\sigma)-\nabla_u\cdot(\varkappa\nabla_u\vartheta_\sigma)]\\
&\quad\equiv\eta\widetilde{k}+k_3,
\end{aligned}
\tag{2.6}
$$

where $p_\sigma = p - 2\sigma/R_e - p_0$ (R_e is introduced in Definition 1.1), $\widetilde{p}_\sigma = p_\sigma\zeta$, $\mathbb{B}_u(u,\zeta)=\{B_{uij}(u,\zeta)\}_{i,j=1,2,3}=\{\mu(u_i\nabla_{u_j}\zeta+u_j\nabla_{u_i}\zeta)+(\nu-\mu)\delta_{ij}u\nabla_u\zeta\}_{i,j=1,2,3}$, $\nabla_{u_j}=\xi_{ix_j}\partial_{\xi_i}$ and in a boundary subdomain

$$
\begin{aligned}
&\widehat{\eta}\widetilde{u}_{it}-\nabla_j T_{ij}(\widetilde{u},\widetilde{p}_\sigma)=-\widehat{\nabla}_j\widehat{B}_{ij}(\widehat{u},\widehat{\zeta})-\widehat{T}_{ij}(\widehat{u},\widehat{p}_\sigma)\widehat{\nabla}_j\widehat{\zeta}\\
&\qquad-(\nabla_j T_{ij}(\widetilde{u},\widetilde{p}_\sigma)-\widehat{\nabla}_j\widehat{T}_{ij}(\widetilde{u},\widetilde{p}_\sigma))\equiv k_{4i},\quad i=1,2,3,\\
&\widetilde{\eta}_{\sigma t}+\varrho_e\nabla\cdot\widetilde{u}=\varrho_e\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}-\widehat{\eta}_\sigma\widehat{\nabla}\cdot\widehat{u}\widehat{\zeta}+\varrho_e(\nabla\cdot\widetilde{u}-\widehat{\nabla}\cdot\widetilde{u})\equiv k_5,\\
&\widehat{\eta}\widehat{c}_v\widetilde{\vartheta}_{\sigma t}-\nabla\cdot(\widehat{\varkappa}\nabla\widetilde{\vartheta}_\sigma)+\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)\nabla\cdot\widetilde{u}\\
&=\widehat{\eta}\widehat{k}+\bigg[\frac{\widehat{\mu}}{2}\sum_{i,j=1}^{3}(\widehat{\nabla}_i\widehat{u}_j+\widehat{\nabla}_j u_i)^2+(\widehat{\nu}-\widehat{\mu})(\widehat{\nabla}\cdot\widehat{u})^2\bigg]\widehat{\zeta}\\
&\quad+\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta}+(\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)-\widehat{\vartheta}p_{\widehat{\vartheta}}(\widehat{\eta},\widehat{\vartheta}))\widehat{\nabla}\cdot u\widehat{\zeta}\\
&\quad+\theta_e p_{\widehat{\vartheta}}(\varrho_e,\theta_e)(\nabla\cdot\widetilde{u}-\widehat{\nabla}\cdot\widetilde{u})-\widehat{\varkappa}(2\widehat{\nabla}^2\widehat{\zeta}\widehat{\vartheta}_\sigma+2\widehat{\nabla}\widehat{\zeta}\cdot\widehat{\nabla}\widehat{\vartheta}_\sigma)\\
&\quad-\widehat{\nabla}\widehat{\varkappa}\cdot\widehat{\nabla}\widehat{\zeta}\widehat{\vartheta}_\sigma-\widehat{\zeta}\widehat{\nabla}\widehat{\varkappa}\cdot\widehat{\nabla}\widehat{\vartheta}_\sigma-[\nabla\cdot(\widehat{\varkappa}\nabla\widetilde{\vartheta}_\sigma)-\widehat{\nabla}\cdot(\widehat{\varkappa}\widehat{\nabla}\vartheta_\sigma)]\equiv\widehat{\eta}\widetilde{k}+k_6,\\
&\mathbb{T}(\widetilde{u},\widetilde{p}_\sigma)\widehat{n}'=\sigma\Delta_{\widehat{S}}\widehat{\xi}\cdot\widehat{\zeta}+\sigma\Delta_{\widehat{S}}\int_0^t\widetilde{u}\,dt'+\mathbb{T}(\widetilde{u},\widetilde{p}_\sigma)\widehat{n}'-\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma)\widehat{n}\\
&\qquad+\widehat{\mathbb{B}}(\widehat{u},\widehat{\zeta})\widehat{n}-\sigma\bigg(2\widehat{\nabla}\int_0^t\widehat{u}\,dt'\widehat{\nabla}\widehat{\zeta}+\int_0^t\widehat{u}\,dt'\widehat{\nabla}^2\widehat{\zeta}\bigg)+\frac{2\sigma}{R_e}\widehat{n}\widehat{\zeta}\equiv\frac{2\sigma}{R_e}\widehat{n}\widehat{\zeta}+k_7,\\
&\widehat{\varkappa}\widehat{n}'\cdot\nabla\widetilde{\vartheta}_\sigma=\widetilde{\widetilde{\vartheta}}+\widehat{\varkappa}\widehat{n}\cdot\widehat{\nabla}\widehat{\zeta}\widehat{\vartheta}_\sigma+\widehat{\varkappa}(\widehat{n}'\cdot\nabla\widetilde{\vartheta}_\sigma-\widehat{n}\cdot\widehat{\nabla}\widetilde{\vartheta}_\sigma)\equiv\widetilde{\widetilde{\vartheta}}+k_8,
\end{aligned}
\tag{2.7}
$$

where $\widehat{p}_\sigma = p_\sigma(\widehat{\eta},\widehat{\vartheta})$, $\widehat{c}_v = c_v(\widehat{\eta},\widehat{\vartheta})$, $\widehat{\varkappa}=\varkappa(\widehat{\eta},\widehat{\vartheta})$, $\widehat{\mu}=\mu(\widehat{\eta},\widehat{\vartheta})$, $\widehat{\nu}=\nu(\widehat{\eta},\widehat{\vartheta})$, $\widehat{\mathbb{T}}$ and $\widehat{\mathbb{B}}$ indicate that operator ∇_u is replaced by $\widehat{\nabla}$; $\widehat{n}'=(0,0,1)$, $\widehat{n}$ is the vector $\overline{n}_u$ written in z coordinates and

$$\Delta_{\widehat{S}}(t)=\frac{1}{\sqrt{g_{\widehat{u}}}}\frac{\partial}{\partial z_\gamma}\bigg(\sqrt{g_{\widehat{u}}}g_{\widehat{u}}^{\gamma\delta}\frac{\partial}{\partial z_\delta}\bigg),$$

$\{g_{\widehat{u}}^{\gamma\delta}\}$ is the inverse matrix to $\{g_{\widehat{u}\gamma\delta}\}$,

$$g_{\widehat{u}\gamma\delta} = \frac{\partial x}{\partial z_\gamma} \cdot \frac{\partial x}{\partial z_\delta}, \quad x = \widehat{\xi} + \int_0^t \widehat{u}(z,t')\,dt', \quad g_{\widehat{u}} = \det\{g_{\widehat{u}\gamma\delta}\}.$$

In particulary $g_{\widehat{u}}(0) = 1+F_{z_1}^2+F_{z_2}^2$, $g_{\widehat{u}}^{11}(0) = g_{\widehat{u}}^{-1}(0)(1+F_{z_2}^2)$, $g_{\widehat{u}}^{22}(0) = g_{\widehat{u}}^{-1}(0)(1+F_{z_1}^2)$, $g_{\widehat{u}}^{12}(0) = g_{\widehat{u}}^{21}(0) = -g_{\widehat{u}}^{-1}(0)F_{z_1}F_{z_2}$ and we assume that d from (2.4) and (2.5) is so small that

$$|F_{z_i}| \le 1/4, \quad i = 1,2.$$

Moreover, we can write

$$p_\sigma(\eta,\vartheta) = p_1\eta_\sigma + p_2\vartheta_\sigma,$$

where $p_1(\eta,\vartheta) = \int_0^1 p_\eta(\varrho_e+s(\eta-\varrho_e),\vartheta)\,ds$ and $p_2(\vartheta) = \int_0^1 p_\vartheta(\varrho_e,\theta_e+s(\vartheta-\theta_e))\,ds$.

3. Local existence

First, consider the following auxiliary problem

$$\begin{aligned}
&\eta u_t - \mathrm{div}_w \mathbb{D}_w(u) = F && \text{in } \Omega^T,\\
&\mu(\eta,\gamma)\Pi_0\Pi_w\mathbb{S}_w(u)\overline{n}_w = \Pi_0 G_1 && \text{on } S^T,\\
&\overline{n}_0\cdot\mathbb{D}_w(u)\overline{n}_w - \sigma\overline{n}_0\cdot\Delta_w(t)\int_0^t u\,dt' = G_2 && \text{on } S^T,\\
&u|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\mathbb{D}_w(u) &= \mu(\eta,\gamma)\mathbb{S}_w(u) + (\nu(\eta,\gamma)-\mu(\eta,\gamma))\mathrm{div}_w u I,\\
\mathbb{S}_w(u) &= \{\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i\}_{i,j=1,2,3},\\
\mathrm{div}_w\mathbb{D}_w(u) &= \{\partial_{x_j}\xi_k\partial_{\xi_k}D_{wij}(u)\}_{i=1,2,3},
\end{aligned}$$

$\partial_{x_i}\xi_k$ $(i,k=1,2,3)$ are elements of matrix ξ_x which is inverse to $x_\xi = I + \int_0^t w_\xi(\xi,t')\,dt'$, $\overline{n}_w = \overline{n}(X_w(\xi,t),t)$, $\Delta_w(t)$ is given by (1.2) with $x = \xi + \int_0^t w(\xi,t')\,dt'$, $\Pi_w g = g - \overline{n}_w(\overline{n}_w\cdot g)$, $G_2 = G_2^{(1)} + \sigma\int_0^t G_2^{(2)}\,dt'$.

We prove.

Lemma 3.1. *Let $S \in W_2^{3/2+\alpha}$, $\eta \in C([0,T];W_2^{1+\alpha}(\Omega)) \cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, $1/\eta \in L_\infty(\Omega^T)$, $\gamma \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $w \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\alpha \in (3/4,1)$, $\nu \in C^3(\mathbb{R}^2)$, $\mu \in C^3(\mathbb{R}^2)$ and assume that for some $a > 0$*

$$T^a\Big[\|w\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} + \|\gamma\|_{\Omega^T}^{(\alpha+2,\alpha/2+1} + \sup_{0\le t\le T}\|\eta\|_{1+\alpha,\Omega} + \|\eta\|_{1+\alpha,\Omega^T} + \Big(\int_0^T \|\eta_t\|_{1+\alpha,\Omega}^2\,dt\Big)^{1/2}\Big] \le \delta, \tag{3.2}$$

where δ is a sufficiently small constant. Let

$$
\begin{gathered}
F \in W_2^{\alpha,\alpha/2}(\Omega^T), \qquad G_1 \in W_2^{1/2+\alpha,1/4+\alpha}(S^T), \\
G_2^{(1)} \in W_2^{1/2+\alpha,1/4+\alpha/2}(S^T), \quad G_2^{(2)} \in W_2^{\alpha-1/2,\alpha/2-1/4}(S^T), \quad v_0 \in W_2^{1+\alpha}(\Omega)
\end{gathered}
$$

and let the following compatibility conditions be satisfied:

$$
\begin{aligned}
&\mu(\eta|_{t=0},\gamma|_{t=0})\Pi_0\mathbb{S}(v_0)\overline{n}_0|_S = \Pi_0 G_1|_{t=0}, \\
&\overline{n}_0\cdot[\mu(\eta|_{t=0},\gamma|_{t=0})\mathbb{S}(v_0) + (\nu(\eta|_{t=0},\gamma|_{t=0}) \\
&\qquad -\mu(\eta|_{t=0},\gamma|_{t=0}))\operatorname{div} v_0 I]\overline{n}_0|_S = G_2^{(1)}|_{t=0},
\end{aligned}
$$

where $\mathbb{S}(v_0) = \{v_{0i\xi_j} + v_{0j\xi_i}\}_{i,j=1,2,3}$. Then there exists a unique solution $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ of problem (3.1) and

$$
\begin{aligned}
(3.3)\quad \|u\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} \le \overline{\phi}_1\bigg(T, \left|\frac{1}{\eta}\right|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}\bigg) &\cdot (\|F\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|G_1\|_{1/2+\alpha,S^T} + \|G_2^{(1)}\|_{1/2+\alpha,S^T} \\
&+ \|G_2^{(2)}\|_{S^T}^{(\alpha-1/2,\alpha/2-1/4)} + \|v_0\|_{1+\alpha,\Omega}),
\end{aligned}
$$

where $\overline{\phi}_1$ is a positive continuous nondecreasing function of its arguments.

Proof. The existence can be proved by using the method of successive approximations, and the uniqueness of a solution follows from estimate (3.3). Thus, we shall only derive (3.3). First, consider the problem

$$
(3.4)\qquad
\begin{aligned}
&\eta u_t - \mu\nabla_w^2 u - \nu\nabla_w\nabla_w\cdot u = F && \text{in } \Omega^T, \\
&\mu\Pi_0\Pi_w\mathbb{S}_w(u)\overline{n}_w = \Pi_0 G_1 && \text{on } S^T, \\
&\overline{n}_0\cdot\mathbb{D}_{0w}(u)\overline{n}_w - \sigma\overline{n}_0\cdot\Delta_w(t)\int_0^t u\,dt' = G_2 && \text{on } S^T, \\
&u|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
$$

where μ, ν are positive constants, $\mathbb{D}_{0w}(u) = \mu\mathbb{S}_w(u) + (\nu-\mu)\operatorname{div}_w uI$, the data S, η, w, F, G_1, G_2 satisfy the assumptions of the lemma and moreover

$$
\begin{aligned}
&\mu\Pi_0\mathbb{S}(v_0)\overline{n}_0|_S = \Pi_0 G_1|_{t=0}, \\
&\overline{n}_0\cdot[\mu\mathbb{S}(v_0) + (\nu-\mu)\operatorname{div} v_0 I]\overline{n}_0|_S = G_2^{(1)}|_{t=0}.
\end{aligned}
$$

Then in view of (3.2), using the methods of [7] (see also [2]) we obtain the unique existence of a solution $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ of (3.4) satisfying the estimate

$$
\begin{aligned}
(3.5)\quad \|u\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} \le \overline{\phi}_2\bigg(T, \left|\frac{1}{\eta}\right|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}\bigg)&(\|F\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|G_2^{(1)}\|_{1/2+\alpha,S^T} \\
&+ \|G_2^{(2)}\|_{S^T}^{(\alpha-1/2,\alpha/2-1/4)} + \|v_0\|_{1+\alpha,\Omega}).
\end{aligned}
$$

In order to prove estimate (3.3) introduce a partition of unity $(\{\zeta_k(\xi)\},\{\Omega_k\})$ (see [5], [6]) such that $\operatorname{supp}\zeta_k \subset \Omega_k$, $k = 1,\dots,N$, $\sum_{k=1}^N \zeta_k(\xi) = 1$ for $\xi \in \Omega$,

$\zeta_k \ge 0$, $0 \le n_0 \le \sum_{k=1}^N \zeta_k^2(\xi) \le N_0$ and $|D_\xi^\alpha \zeta_k(\xi)| \le c\lambda^{-|\alpha|}$, where $\lambda =$ max diam Ω_k. Let $u_k = u\zeta_k$, $v_{0k} = v_0\zeta_k$, $F_k = F\zeta_k$, $G_{1k} = G_1\zeta_k$, $G_{2k} = G_2\zeta_k$, $Q_k = \Omega_k \times (0,T)$, $\sum_k = \partial\Omega_k \times (0,T)$. Then (3.1) yields the system of problems:

$$
\begin{aligned}
&\eta u_{kt} - \mu_k \nabla_w^2 u_k - \nu_k \nabla_w \nabla_w \cdot u_k \\
&\qquad = F_k + \operatorname{div}_w(\mathbb{D}_w(u_k) - \mathbb{D}_{kw}(u_k)) + R_1 \equiv H_1 \qquad \text{in } Q_k,\\
&\mu_k \Pi_0 \Pi_w \mathbb{S}_w(u_k)\overline{n}_w \\
&\qquad = \Pi_0 G_{1k} + (\mu_k - \mu)\Pi_0\Pi_w \mathbb{S}_w(u_k)\overline{n}_w + R_2 \equiv H_2 \quad \text{on } \textstyle\sum_k,\\
&\overline{n}_0 \cdot \mathbb{D}_{kw}(u_k)\overline{n}_w - \sigma \overline{n}_0 \cdot \Delta_w(t)\int_0^t u_k\,dt' \\
&\qquad = G_{2k} + \overline{n}_0 \cdot (\mathbb{D}_{kw}(u_k) - \mathbb{D}_w(u_k))\overline{n}_w + R_3 \\
&\qquad \equiv H_3^{(1)} + \int_0^t H_3^{(2)}\,dt' \qquad \text{on } \textstyle\sum_k,\\
&u_k|_{t=0} = v_{0k} \qquad \text{in } \Omega_k,
\end{aligned}
\tag{3.6}
$$

where R_i $(i = 1,2,3)$ contain terms with lower order derivatives,

$$
\mu_k = \mu(\eta(\xi_k,t_k),\gamma(\xi_k,t_k)), \quad \nu_k = \nu(\eta(\xi_k,t_k),\gamma(\xi_k,t_k))
$$

and $(\xi_k,t_k) \in Q_k$. Since each problem (3.6) has the form (3.4), estimate (3.5) holds for u_k, i.e.

$$
\begin{aligned}
\|u_k\|_{Q_k}^{(\alpha+2,\alpha/2+1)} \le{}& \overline{\phi}_2\bigg(T, \bigg|\frac{1}{\eta}\bigg|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}\bigg) \\
&\cdot (\|H_1\|_{Q_k}^{(\alpha,\alpha/2)} + \|H_2\|_{1/2+\alpha,\sum_k} + \|H_3^{(1)}\|_{1/2+\alpha,\sum_k} \\
&+ \|H_3^{(2)}\|_{\sum_k}^{(\alpha-1/2,\alpha/2-1/4)} + \|v_{0k}\|_{1+\alpha,\Omega}).
\end{aligned}
\tag{3.7}
$$

Now, we estimate the terms on the right-hand side of (3.7). First, consider

$$
\begin{aligned}
I_1 &= \operatorname{div}_w(\mathbb{D}_w(u_k) - \mathbb{D}_{kw}(u_k)) \\
&= (\mu(\eta,\gamma) - \mu_k)\nabla_w^2 u_k + (\nu(\eta,\gamma) - \nu_k)\nabla_w\nabla_w \cdot u_k + R_4 \equiv I_2 + I_3 + R_4,
\end{aligned}
$$

where R_4 contains terms with lower order derivatives.

In order to estimate I_2 we rewrite it in the form

$$
\begin{aligned}
I_2 ={}& [\mu_\eta(c_1,c_2)(\eta(\xi,t) - \eta(\xi_k,t)) + \mu_\eta(c_1,c_2)(\eta(\xi_k,t) - \eta(\xi_k,t_k)) \\
&+ \mu_\gamma(c_1,c_2)(\gamma(\xi,t) - \gamma(\xi_k,t)) + \mu_\gamma(c_1,c_2)(\gamma(\xi_k,t) - \gamma(\xi_k,t_k))]\nabla_w^2 u_k \\
\equiv{}& \sum_{i=1}^4 K_i,
\end{aligned}
$$

where $c_1 = \eta_k + s(\eta - \eta_k)$, $c_2 = \gamma_k + s(\gamma - \gamma_k)$, $0 < s < 1$.

First, we estimate

$$
[K_1]_{\alpha,Q_k,\xi}^2 \le c\int_{\Omega_k}\int_{\Omega_k}\int_0^T |\xi - \xi_k|^{2\beta}\frac{|\eta(\xi,t) - \eta(\xi_k,t)|^2}{|\xi - \xi_k|^{2\beta}}
\tag{3.8}
$$

$$
\begin{aligned}
&\cdot \frac{|\partial_\xi^2 u_k(\xi,t) - \partial_{\xi'}^2 u_k(\xi',t)|^2}{|\xi-\xi'|^{3+2\alpha}}\, d\xi\, d\xi'\, dt' \\
&+ c\int_{\Omega_k}\int_{\Omega_k}\int_0^T \frac{|\eta(\xi,t)-\eta(\xi',t)|^2 |\partial_\xi^2 u_k|^2}{|\xi-\xi'|^{3+2\alpha}}\, d\xi\, d\xi'\, dt' \\
&+ c\int_{\Omega_k}\int_{\Omega_k}\int_0^T |\xi-\xi_k|^{2\beta} \frac{|\eta(\xi,t)-\eta(\xi_k,t)|^2}{|\xi-\xi_k|^{2\beta}} \\
&\cdot \frac{|\int_0^t (w_\xi - w_{\xi'})\, dt'|^2}{|\xi-\xi'|^{3+2\alpha}} |\partial_\xi^2 u_k|^2\, d\xi\, d\xi'\, dt \\
\le\, & c\lambda^{2\beta} \sup_{0\le t\le T} \sup_{\xi,\xi'\in\Omega_k} \frac{|\eta(\xi,t)-\eta(\xi',t)|^2}{|\xi-\xi'|^{2\beta}} \|u_k\|^2_{2+\alpha,Q_k} \\
&+ c\sup_{0\le t\le T}\left(\int_{\Omega_k}\int_{\Omega_k} \frac{|\eta(\xi,t)-\eta(\xi',t)|^4}{|\xi-\xi'|^{3+4(\frac14+\alpha)}}\, d\xi\, d\xi'\right)^{1/2} \\
&\cdot \int_0^T \left(\int_{\Omega_k}\int_{\Omega_k} \frac{|\partial_\xi^2 u_k|^4}{|\xi-\xi'|^2}\, d\xi\, d\xi'\right)^{1/2} dt \\
&+ c\lambda^{2\beta} T \sup_{0\le t\le T} \sup_{\xi,\xi'\in\Omega_k} \frac{|\eta(\xi,t)-\eta(\xi' t)|^2}{|\xi-\xi'|^{2\beta}} \\
&\cdot \int_0^T \left(\int_{\Omega_k}\int_{\Omega_k} \frac{|w_\xi - w_{\xi'}|^4}{|\xi-\xi'|^{3+4(\frac14+\alpha)}}\, d\xi\, d\xi'\right)^{1/2} dt \\
&\cdot \int_0^T \left(\int_{\Omega_k}\int_{\Omega_k} \frac{|\partial_\xi^2 u_k|^4}{|\xi-\xi'|^2}\, d\xi\, d\xi'\right)^{1/2} dt \\
\le\, & c\lambda^{2\beta} \sup_{0\le t\le T} \|\eta\|^2_{C^\beta(\overline{\Omega})} (1 + T\|w\|^2_{2+\alpha,\Omega^T}) \|u_k\|^2_{2+\alpha,Q_k} \\
&+ c\sup_{0\le t\le T} \|\eta\|^2_{1+\alpha,\Omega} \\
&\cdot \left(\varepsilon \int_0^T \|u_k\|^2_{2+\alpha,\Omega_k} dt + c(\varepsilon)\int_0^T \|u_k\|^2_{0,\Omega_k}\, dt\right),
\end{aligned}
$$

where $0<\beta<\alpha-1/2$ and we have used the imbedding $W_2^{1+\alpha}(\Omega)\subset W_4^{1/4+\alpha}(\Omega)$ and the interpolation inequality

$$
\int_0^T |\partial_\xi^2 u_k|^2_{4,\Omega_k}\, dt \le \varepsilon \int_0^T \|u_k\|^2_{2+\alpha,\Omega_k}\, dt + c(\varepsilon)\int_0^T |u_k|^2_{2,\Omega_k}\, dt,
$$

which hold for $\alpha > 3/4$.

Using the imbedding $W_2^{1+\alpha}(\Omega)\subset C^\beta(\overline{\Omega})$ (which holds for $0<\beta<\alpha-1/2$) estimate (3.8) yields

$$
\begin{aligned}
[K_1]^2_{\alpha,Q_k,\xi} \le\, & c(\lambda^{2\beta} + \lambda^{2\beta}\|w\|^2_{2+\alpha.\Omega^T} + \varepsilon) \sup_{0\le t\le T} \|\eta\|^2_{1+\alpha,\Omega} \|u_k\|^2_{2+\alpha,Q_k} \\
&+ c(\varepsilon) T \sup_{0\le t\le T} \|\eta\|^2_{1+\alpha,\Omega} \sup_{0\le t\le T} \|u_k\|^2_{0,\Omega_k}.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
[K_1]^2_{\alpha/2,Q_k,t} \le{}& c\int_{\Omega_k}\int_0^T\int_0^T |\xi-\xi_k|^{2\beta}\frac{|\eta(\xi,t)-\eta(\xi_k,t)|^2}{|\xi-\xi_k|^{2\beta}}\\
&\cdot\frac{|\partial_\xi^2 u_k(\xi,t)-\partial_\xi^2 u_k(\xi,t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt'\\
&+c\int_{\Omega_k}\int_0^T\int_0^T\frac{|\eta(\xi,t)-\eta(\xi,t')|^2|\partial_\xi^2 u_k|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt'\\
&+c\int_{\Omega_k}\int_0^T\int_0^T|\eta(\xi,t)-\eta(\xi_k,t)|^2\frac{|\int_{t'}^t w_\xi\,d\tau|^2}{|t-t'|^{1+\alpha}}|\partial_\xi^2 u_k|^2\,dt\,dt'\\
\le{}& c\lambda^{2\beta}\Big(\sup_{0\le t\le T}\|\eta\|^2_{C^\beta(\overline{\Omega})}\|u_k\|^2_{2+\alpha,Q_k}\\
&+T^{1-\alpha}\int_0^T|w_\xi|^2_{4,\Omega_k}\,dt\int_0^T|\partial_\xi^2 u_k|^2\,dt\Big)\\
&+c\int_0^T\int_0^T\frac{|\int_{t'}^t\eta_t\,d\tau|^2_{4,\Omega_k}|\partial_\xi^2 u_k|^2_{4,\Omega_k}}{|t-t'|^{1+\alpha}}\,dt\,dt'\\
\le{}& c[\lambda^{2\beta}(1+T^{1-\alpha}\|w\|^2_{2+\alpha,\Omega^T})\sup_{0\le t\le T}\|\eta\|^2_{1+\alpha,\Omega}\\
&+T^{1-\alpha}\|\eta_t\|^2_{\alpha,\Omega^T}]\|u_k\|^2_{2+\alpha,Q_k}.
\end{aligned}
$$

Now, we estimate K_2. First, we obtain

$$
\begin{aligned}
[K_2]^2_{\alpha,Q_k,\xi} \le{}& c\int_{\Omega_k}\int_{\Omega_k}\int_0^T|\eta(\xi_k,t)-\eta(\xi_k,t_k)|^2\\
&\cdot\bigg[\frac{|\int_0^t(w_\xi-w_{\xi'})\,dt'|^2}{|\xi-\xi'|^{3+2\alpha}}|\partial_\xi^2 u_k|^2\\
&+\frac{|\partial_\xi^2 u_k(\xi,t)-\partial_{\xi'}^2 u_k(\xi',t)|^2}{|\xi-\xi'|^{3+2\alpha}}\bigg]\,d\xi\,d\xi'\,dt\\
\le{}& c\int_0^T|\eta(\xi,t)-\eta(\xi,t_k)|^2_{\infty,\Omega_k}\\
&\cdot\bigg[\int_{\Omega_k}\int_{\Omega_k}\frac{|\partial_\xi^2 u_k(\xi,t)-\partial_{\xi'}^2 u_k(\xi',t)|^2}{|\xi-\xi'|^{3+2\alpha}}\,d\xi\,d\xi'\,dt\\
&+T\int_0^T\bigg(\int_{\Omega_k}\int_{\Omega_k}\frac{|w_\xi-w_{\xi'}|^4}{|\xi-\xi'|^{3+4(1/4+\alpha)}}\,d\xi\,d\xi'\bigg)^{1/2}dt\\
&\cdot\bigg(\int_{\Omega_k}\int_{\Omega_k}\frac{|\partial_\xi^2 u_k|^4}{|\xi-\xi'|^2}\,d\xi\,d\xi'\bigg)^{1/2}\bigg]\\
\le{}& c\int_0^T(\varepsilon\|\eta(\xi,t)-\eta(\xi,t_k)\|^2_{1+\alpha,\Omega_k}\\
&+c(\varepsilon)|\eta(\xi,t)-\eta(\xi,t_k)|^2_{2,\Omega_k})(1+T\|w\|^2_{2+\alpha,\Omega^T})\|u_k\|^2_{2+\alpha,\Omega_k}\,dt
\end{aligned}
$$

$$\le c(\varepsilon \sup_{0\le t\le T} \|\eta\|^2_{1+\alpha,\Omega} + c(\varepsilon)T\|\eta_t\|^2_{0,\Omega^T})(1+T\|w\|^2_{2+\alpha,\Omega^T})\|u_k\|^2_{2+\alpha,Q_k}.$$

Next, we get

$$\begin{aligned}
[K_2]^2_{\alpha/2,Q_k,t} \le{}& c\int_{\Omega_k}\int_0^T\int_0^T |\eta(\xi_k,t)-\eta(\xi_k,t_k)|^2 \\
&\cdot\left[\frac{|\int_{t'}^t w_\xi\, d\tau|^2|\partial_\xi^2 u_k|^2}{|t-t'|^{1+\alpha}} + \frac{|\partial_\xi^2 u_k(\xi,t)-\partial_\xi^2 u_k(\xi,t')|^2}{|t-t'|^{1+\alpha}}\right] d\xi\, dt\, dt' \\
&+ c\int_{\Omega_k}\int_0^T\int_0^T \frac{|\eta(\xi_k,t)-\eta(\xi_k,t')|^2}{|t-t'|^{1+\alpha}}|\partial_\xi^2 u_k|^2\, d\xi\, dt\, dt' \\
\le{}& c\int_0^T\int_0^T (\varepsilon\|\eta(\xi,t)-\eta(\xi,t_k)\|^2_{1+\alpha,\Omega_k} + c(\varepsilon)|\eta(\xi,t)-\eta(\xi,t_k)|^2_{2,\Omega_k}) \\
&\cdot\bigg(T^{1-\alpha}\|w\|^2_{2+\alpha,\Omega^T}|\partial_\xi^2 u_k|^2_{2,\Omega_k} \\
&+ \frac{|\partial_\xi^2 u_k(\xi,t)-\partial_\xi^2 u_k(\xi,t')|^2_{2,\Omega_k}}{|t-t'|^{1+\alpha}}\bigg) d\xi\, dt\, dt' \\
&+ c\int_{\Omega_k}\int_0^T\int_0^T \frac{|\int_{t'}^t \eta_t(\xi_k,\tau)d\tau|^2}{|t-t'|^{1+\alpha}}|\partial_\xi^2 u_k|^2\, d\xi\, dt\, dt' \\
\le{}& c\bigg[(\varepsilon \sup_{0\le t\le t}\|\eta\|^2_{1+\alpha,\Omega} + c(\varepsilon)T\|\eta_t\|^2_{0,\Omega^T})(1+T^{1-\alpha}\|w\|^2_{2+\alpha,\Omega^T}) \\
&+ T^{1-\alpha}\int_0^T \|\eta_t\|^2_{\alpha,\Omega} dt]\|u_k\|^2_{2+\alpha,Q_k}\bigg].
\end{aligned}$$

In the above estimates of K_1 and K_2, c and $c(\varepsilon)$ are positive continuous increasing functions of $|\eta|^2_{\infty,\Omega^T}$ and $|\gamma|^2_{\infty,\Omega^T}$.

The terms K_3, K_4 and I_3 can be estimated in the same way as K_1 and K_2. The terms R_1 and R_4 are estimated by using the imbedding (2.1) and the interpolation inequality (2.2) as follows:

$$R_1 + R_4 \le (\varepsilon + c(\varepsilon)T^{a_1})(Xb_1(X)+1)(\|u_k\|^{(\alpha+2,\alpha/2+1)}_{Q_k} + \|u\|^{(\alpha+2,\alpha/2+1)}_{\Omega^T}),$$

where

$$\begin{aligned}
X ={}& \|w\|^{(\alpha+2,\alpha/2+1)}_{\Omega^T} + \|\gamma\|^{(\alpha+2,\alpha/2+1)}_{\Omega^T} + \sup_{0\le t\le T}\|\eta\|_{1+\alpha,\Omega} + \|\eta\|_{1+\alpha,\Omega^T} \\
&+ \left(\int_0^T \|\eta_t\|^2_{1+\alpha,\Omega}\, dt\right)^{1/2},
\end{aligned}$$

b_1 is a continuous positive increasing function, $a_1 > 0$ is a constant.

The boundary terms in (3.6) can be estimated in the same way. Thus, summarizing the above considerations and assuming that λ, ε, T and δ from (3.2) are sufficiently small we get

$$(3.9)\quad \|u_k\|_{Q_k}^{(\alpha+2,\alpha/2+1)} \le \overline{\phi}_2\bigg(T, \bigg|\frac{1}{\eta}\bigg|_{\infty,\Omega^T}, |\eta|_{\infty,\Omega^T}\bigg) \cdot [(\varepsilon + c(\varepsilon)T^{a_2})(Xb_2(X)+1)\|u\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} + \|F_k\|_{Q_k}^{(\alpha,\alpha/2)} + \|G_{1k}\|_{1/2+\alpha,\sum_k} + \|G_{2k}^{(1)}\|_{1/2+\alpha,\sum_k} + \|G_{2k}^{(2)}\|_{\sum_k}^{(\alpha-1/2,\alpha/2-1/4)} + \|v_{0k}\|_{1+\alpha,\Omega}],$$

where b_2 is a continuous positive increasing function and $a_2 > 0$ is a constant.

Now, summing (3.9) over all sets of the partition of unity and assuming that ε is sufficiently small we obtain (3.3) and the assertion of the lemma. □

In the same way the following lemma can be proved.

LEMMA 3.2. *Let* $S \in W_2^{3/2+\alpha}$, $\eta \in C([0,T]; W_2^{1+\alpha}(\Omega)) \cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$, $1/\eta \in L_\infty(\Omega^T)$, $\gamma \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $w \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$, $\alpha \in (3/4,1)$, $c_v \in C^2(\mathbb{R}^2)$, $\varkappa \in C^3(\mathbb{R}^2)$ *and assume* (3.2). *Let* $K \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $\overline{\vartheta} \in W_2^{1/2+\alpha,1/4+\alpha}(S^T)$, $\theta_0 \in W_2^{1+\alpha}(\Omega)$ *and let the compatibility condition holds:*

$$\varkappa(\eta|_{t=0}, \gamma|_{t=0})\overline{n}_0 \cdot \nabla_w \theta_0|_S = \overline{\vartheta}|_{t=0}.$$

Then there exists a unique solution $\vartheta \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *of the problem*

$$\begin{aligned} &\eta c_v(\eta,\gamma)\vartheta_t - \operatorname{div}_w(\varkappa(\eta,\gamma)\nabla_w\vartheta) = K && \text{in } \Omega^T,\\ &\varkappa(\eta,\gamma)\overline{n}_0 \cdot \nabla_w \vartheta = \overline{\vartheta} && \text{on } S^T,\\ &\vartheta|_{t=0} = \theta_o && \text{in } \Omega. \end{aligned}$$

Moreover, the following estimate is satisfied

$$\|\vartheta\|_{\Omega^T}^{(\alpha+2,\alpha/2+1)} \le \overline{\phi}_3\bigg(T, \bigg|\frac{1}{\eta c_v(\eta,\gamma)}\bigg|_{\infty,\Omega^T}, |\eta c_v(\eta,\gamma)|_{\infty,\Omega^T}\bigg) \cdot (\|K\|_{\Omega^T}^{(\alpha,\alpha/2)} + \|\overline{\vartheta}\|_{1/2+\alpha,S^T} + \|\vartheta_0\|_{1+\alpha,\Omega}),$$

where $\overline{\phi}_3$ *is a positive continuous nondecreasing function of its arguments.*

Now, using Lemmas 3.1–3.2, Lemma 6.1 of [2] and the method of successive approximations (see [15]) we obtain the following theorem.

THEOREM 3.3. *Let* $S \in W_2^{5/2+\alpha}$, $\varrho_0 \in W_2^{1+\alpha}(\Omega)$, $\theta_0 \in W_2^{1+\alpha}(\Omega)$, $\vartheta_0 \in W_2^{1+\alpha}(\Omega)$, $\alpha \in (3/4,1)$, $c_v \in C^2(\mathbb{R}^2)$, $c_v > 0$, $p \in C^3(\mathbb{R}^2)$, $\nu \in C^3(\mathbb{R}^2)$, $\mu \in C^3(\mathbb{R}^2)$, $\varkappa \in C^3(\mathbb{R}^2)$, $\varkappa > 0$, $\nu > \mu/3 > 0$; $f, r \in C_B^2(\mathbb{R}^3 \times \mathbb{R}_+^1)$, $\overline{\theta} \in C_B^3(\mathbb{R}^3 \times \mathbb{R}_+^1)$

and let the following compatibility conditions be satisfied:

$$
\begin{aligned}
&\mu(\varrho_0,\theta_0)\Pi_0\mathbb{S}(v_0)\overline{n}_0 = 0 && \text{on } S,\\
&\overline{n}_0\cdot[\mu(\varrho_0,\theta_0)\mathbb{S}(v_0) + (\nu(\varrho_0,\theta_0)-\mu(\varrho_0,\theta_0))\cdot \operatorname{div} v_0 I]\overline{n}_0 &&\\
&\qquad = \overline{n}_0\cdot(p(\varrho_0,\theta_0)-p_0)\overline{n}_0 + \sigma H(\,\cdot\,,0) && \text{on } S,\\
&\varkappa(\varrho_0,\theta_0)\overline{n}_0\cdot\nabla_\xi\theta_0 = \overline{\theta}(\xi,0) && \text{on } S.
\end{aligned}
$$

Then there exists $T > 0$ such that there exists a unique solution $(u,\vartheta,\eta) \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times C([0,T];W_2^{1+\alpha}(\Omega))\cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)$ of problem (1.6) *and the following estimates hold for $t\le T$*

$$\|u\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)} + \|\vartheta\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)} \le A$$

and

$$
\begin{aligned}
(\|u\|_{\Omega^t}^{(\alpha+2,\alpha/2)} &+ \|\vartheta_\sigma\|_{\Omega^t}^{(\alpha+2,\alpha/2+1)} + \|\eta_\sigma\|_{1+\alpha,\Omega^t} + \sup_{0\le\tau\le t}\|\eta_\sigma\|_{1+\alpha,\Omega})^2\\
&\le \overline{\phi}(T,A)\bigg[\|v_0\|_{1+\alpha,\Omega}^2 + \|\varrho_{\sigma0}\|_{1+\alpha,\Omega}^2 + \|\theta_{\sigma0}\|_{1+\alpha,\Omega}^2\\
&\qquad + \|k\|_{\alpha,\Omega^t}^2 + \|\overline{\vartheta}\|_{\alpha+1/2,S^t}^2 + \bigg\|H(\,\cdot\,,0)+\frac{2}{Re}\bigg\|_{\alpha+1/2,S^t}^2\bigg],
\end{aligned}
$$

where A is a constant, $\overline{\phi}$ is a positive continuous nondecreasing function of its arguments, $\eta_\sigma = \eta-\varrho_e$, $\vartheta_\sigma = \vartheta-\theta_e$, $\varrho_{\sigma0} = \varrho_0-\varrho_e$, $\theta_{\sigma0} = \theta_0-\theta_e$.

4. Some estimates for the global existence

In this section we derive some estimates for the local solution which are crucial in the proof of the global existence. We use the same methods as in the case of coefficients μ, ν and $\varkappa$ independent of ϱ and θ.

First, in the same way as Theorem 3.5 from [16] we prove the following lemma.

Lemma 4.1. *Let*

$$
\begin{aligned}
(u,\vartheta,\eta) \in\, & W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\times W_2^{2+\alpha,1+\alpha/2}(\Omega^T)\\
&\times W_2^{1+\alpha,1/2+\alpha/2}(\Omega^T)\cap C([0,T];W_2^{1+\alpha}(\Omega)),
\end{aligned}
$$

$\alpha\in(3/4,1)$, be a local solution to problem (1.1) *and let the assumptions of Theorem* 3.3 *be satisfied and let $f=0$. Then for any $0<t_0<T$ and $\lambda>0$, $u\in C([t_0+\lambda,T];W_2^{2+\alpha}(\Omega))$ and the estimate holds*

$$
\begin{aligned}
\sup_{t_0+\lambda\le t\le T}&\|u\|_{2+\alpha,\Omega}^2 + \sup_{t_0+\lambda\le t\le T}\|\vartheta_\sigma\|_{2+\alpha,\Omega}^2\\
&\le c(K)(\|u\|_{2+\alpha,\Omega^T}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega^T}^2 + \|r\|_{C_B^2(\mathbb{R}^3\times\mathbb{R}_+^1)}^2 + \|\overline{\theta}\|_{C_B^3(\mathbb{R}^3\times\mathbb{R}_+^1)}^2),
\end{aligned}
$$

where

$$K = \|u\|^2_{2+\alpha,\Omega^T} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega^T} + \sup_{0\le t\le T}\|u\|^2_{1+\alpha,\Omega} + \sup_{0\le t\le T}\|\vartheta_\sigma\|^2_{1+\alpha,\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\Omega^T} + \sup_{0\le t\le T}\|\eta_\sigma\|^2_{1+\alpha,\Omega},$$

$c(K)$ *is a positive nondecreasing continuous function of* K *depending also on* T.

In order to obtain the other estimates we assume the following condition:

$$\begin{cases} \Omega_t \text{ is diffeomarphic to a ball and} \\ S_t \text{ can be described by } |x| = r = R(\omega,t), \quad \omega\in S^1, \end{cases} \tag{4.1}$$

where $t\le T$ (T is the time of the local existence, S^1 is the unit sphere).

Moreover, we assume:

$$\varrho_1 < \varrho(x,t) < \varrho_2, \quad \theta_1 < \theta(x,t) < \theta_2 \tag{4.2}$$

for all $x\in\overline{\Omega}_t$ and $t\in[0,T]$.

By $\Delta^s(h)f(\xi)$ we denote the s-th difference of f such that

$$\Delta^s(h)f(\xi) = \sum_{k=0}^s c_s^k(-1)^{s-k}f(\xi+kh), \quad c_s^k = \binom{s}{k}$$

and by $\Delta_t^k(h)f(t)$ we denote

$$\Delta_t^k(h)f(t) = \sum_{j=0}^k c_k^j(-1)^{k-i}f(t+jh).$$

Introduce the functions:

$$\begin{aligned}
\phi_1(t,\widetilde{\Omega}_i) &= \int_{\widetilde{\Omega}_i}\bigg(\eta\widetilde{u}_i^2 + \frac{p_1}{\varrho_e}\widetilde{\eta}_{\sigma i}^2 + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\widetilde{\vartheta}_{\sigma i}^2\bigg)d\xi \\
&+ \int_{\mathbb{R}^3}dh\int_{\widetilde{\Omega}_i}d\xi\bigg(\frac{\eta|\Delta^2(h)\widetilde{u}_i|^2}{|h|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(h)\widetilde{\eta}_{\sigma i}|^2}{\varrho_e|h|^{3+2(1+\alpha)}} + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(h)\widetilde{\vartheta}_{\sigma i}|^2}{|h|^{3+2(1+\alpha)}}\bigg) \\
&+ \int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_{\widetilde{\Omega}_i}\bigg(\frac{\mu(\eta,\vartheta)}{4}|\mathbb{S}(\Delta_{t'}\widetilde{u}_i)|^2 + \frac{\nu(\eta,\vartheta)-\mu(\eta,\vartheta)}{2}|\operatorname{div}\Delta_{t'}\widetilde{u}|^2\bigg)\,d\xi\bigg|_{t'=t-h_0} \\
&+ \frac{1}{2}\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_{\widetilde{\Omega}_i}(\varkappa(\eta,\vartheta)|\nabla(\Delta_{t'}\widetilde{\vartheta}_{\sigma i})|^2)\,d\xi|_{t'=t-h_0} \quad \text{for } i\in\mathcal{M}
\end{aligned}$$

and

$$\begin{aligned}
\phi_2(t,\widehat{\Omega}_i) &= \int_{\mathbb{R}^2}d\tau\int_{\widehat{\Omega}_i}dz\bigg(\frac{\widehat{\eta}|\Delta^2(\tau)\widetilde{u}_i|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{p_1}{\varrho_e}\frac{|\Delta^2(\tau)\widetilde{\eta}_{\sigma i}|^2}{|\tau|^{2+2(1+\alpha)}} \\
&+ \frac{p_2\widehat{\eta}c_v(\widehat{\eta},\widehat{\vartheta})}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(\tau)\widetilde{\vartheta}_{\sigma i}|^2}{|\tau|^{2+2(1+\alpha)}} + \frac{|\Delta(\tau)\widehat{\nabla}_3\widetilde{\eta}_{\sigma i}|^2}{|\tau|^{2+2\alpha}} + \frac{\widehat{\eta}|\Delta(\tau)\widetilde{u}_{iz}|^2}{|\tau|^{2+2\alpha}} \\
&+ \frac{p_1}{\varrho_e}\frac{|\Delta(\tau)\widetilde{\eta}_{\sigma i}|^2}{|\tau|^{2+2\alpha}}\bigg) + \int_{\mathbb{R}^1_+}dn\int_{\widehat{\Omega}_i}dz\frac{|\Delta_3(n)\widehat{\nabla}_3\widetilde{\eta}_{\sigma i}|^2}{|n|^{1+2\alpha}}
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&+ \frac{\sigma}{4} \int_{\mathbb{R}^2} d\tau \int_{\widehat{S}_i} d\tau_{\widehat{S}} \frac{(\sum_{\gamma=1}^2 \int_0^t \Delta(\tau) \widetilde{u}_{iz_\gamma z_\gamma} dt')^2}{|\tau|^{2+2(1+\alpha)}} \\
&+ \frac{\sigma}{4} \int_{\mathbb{R}^2} d\tau \int_{\widehat{S}_i} dz_{\widehat{S}} \frac{(\sum_{\gamma=1}^2 \int_0^t \Delta^2(\tau) \widetilde{u}_{iz_\gamma} dt')^2}{|\tau|^{2+1(1+\alpha)}} \\
&+ \int_{\widehat{\Omega}_i} \left(\widehat{\eta} \widetilde{u}_i^2 + \frac{p_1}{\varrho_e} \widetilde{\eta}_{\sigma i} + \frac{p_2 \widehat{\eta} c_v(\widehat{\eta}, \widehat{\vartheta})}{\theta_e p_\vartheta(\varrho_e, \theta_e)} \widetilde{\vartheta}_{\sigma i}^2 \right) dz \\
&+ \sigma \int_{\mathbb{R}^2} dz \int_{\widehat{S}_i} dz_{\widehat{S}} \Bigg[\Bigg(4\sqrt{\widehat{c}} \frac{\Delta(\tau)((H(\,\cdot\,,0) + \frac{2}{R_e}) \widehat{n}_0 \widehat{\zeta}_i)}{|\tau|^{1+\alpha}} \\
&+ \frac{1}{4\sqrt{\widehat{c}}} \frac{\int_0^t \Delta^3(\tau) \widetilde{u}_i dt}{|\tau|^{1+2+\alpha}} \Bigg)^2 + \Bigg(\frac{4\Delta(\tau)((H(\,\cdot\,,0) + \frac{2}{R_e}) \widehat{n}_0 \widehat{\zeta}_i}{|\tau|^{1+\alpha}} \\
&+ \frac{1}{4} \frac{\sum_{\gamma=1}^2 \int_0^t \Delta(\tau) \widetilde{u}_{iz_\gamma z_\gamma} dt'}{|\tau|^{1+\alpha}} \Bigg)^2 \Bigg] \\
&+ \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widehat{\Omega}_i} \Bigg(\frac{\mu(\widehat{\eta}, \widehat{\vartheta})}{4} |\mathbb{S}(\Delta_t \widetilde{u}_i)|^2 \\
&+ \frac{\nu(\widehat{\eta}, \widehat{\vartheta}) - \mu(\widehat{\eta}, \widehat{\vartheta})}{2} |\operatorname{div} \Delta_{t'} \widetilde{u}|^2 \Bigg) dz|_{t'=t-h_0} \\
&+ \frac{1}{2} \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widehat{\Omega}_i} (\varkappa(\widehat{\eta}, \widehat{\vartheta}) |\nabla(\Delta_{t'} \widetilde{\vartheta}_{\sigma i}|^2) \, dz|_{t'=t-t_0} \quad \text{for } i \in \mathcal{N},
\end{aligned}
$$

where in (4.3) $\tau = (\tau_1, \tau_2)$, $\Delta^s(\tau) f(z) = \sum_{k=0}^s \binom{s}{k} f(z' + k\tau, z_3)$, $z' = (z_1, z_2)$, $\Delta^s(n) f(z) = \sum_{k=0}^s \binom{s}{k} f(z', z_3 + kn)$, $\{g_{\widehat{u}}^{\gamma\delta}\}$ is the inverse matrix to $\{g_{\widehat{u}\gamma\delta}\}$ and $g_{\widehat{u}\gamma\delta} = \partial x / \partial z_\gamma \cdot \partial x / \partial z_\delta$, $x = \widehat{\xi} + \int_0^t \widehat{u}(z, t') \, dt'$, $\widehat{c} > 0$ is a constant such that

$$
\int_{\mathbb{R}^2} d\tau \int_{\widehat{S}_i} dz_{\widehat{S}} \frac{(\int_0^t \Delta^3(\tau) \widetilde{u}_i \, dt')^2}{|\tau|^{2+2(1+\alpha)}} \le \widehat{c} \int_{\mathbb{R}^2} d\tau \int_{\widehat{S}_i} dz_{\widehat{S}} \frac{\sum_{\gamma=1}^2 (\int_0^t \Delta^2(\tau) \widetilde{u}_{iz_\gamma} \, dt')^2}{|\tau|^{2+2(1+\alpha)}}
$$

(see [1]).

The following lemma holds.

Lemma 4.2. *Let the assumptions* (4.1) *and* (4.2) *be satisfied. Assume that there exists a local solution to problem* (1.1). *Let* $f = 0$, $k \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $\overline{\vartheta} \in W_2^{\alpha+1/2,\alpha/2+1/4}(S^T)$, *where* $\alpha \in (3/4, 1)$, *T is the time of the local existence. Then*

$$
\begin{aligned}
(4.4) \quad \phi(t, \Omega) &+ \overline{c}(\|u\|_{2+\alpha,\Omega\times(t_*,t)}^2 + \|\vartheta_\sigma\|_{2+\alpha,\Omega\times(t_*,t)}^2 + \|\eta_\sigma\|_{1+\alpha,\Omega\times(t_*,t)}^2) \\
&\le c(t, Z_3)[\|u\|_{0,\Omega\times(t_*,t)}^2 + \|\vartheta_\sigma\|_{0,\Omega\times(t_*,t)}^2 + \|\eta_\sigma\|_{0,\Omega\times(t_*,t)}^2 \\
&\quad + \|k\|_{\alpha,\Omega\times(t_*,t)}^2 + \|\overline{\vartheta}\|_{\alpha+1/2,S\times(t_*,t)}^2 \\
&\quad + \varepsilon_1 Z_4 + Z_1(Z_1 + Z_2 + \|u\|_{2+\alpha,\Omega^t}^2) + (Z_1 + Z_4) Z_5] + \phi(t_*, \Omega),
\end{aligned}
$$

where $0 \le t_* < t \le T$, $\phi(t, \Omega) = \sum_{i\in\mathcal{M}} \phi_1(t, \widetilde{\Omega}_i) + \sum_{i\in\mathcal{N}} \phi_2(t, \widehat{\Omega}_i)$, $c = c(t, Z_3)$ *is a positive continuous function nondecreasing with respect to its arguments,* $\overline{c} \le 1$

is a positive constant, $\varepsilon_1 \in (0,1)$ *is a sufficiently small constant and*

$$\begin{aligned}
&Z_1 = \|u\|^2_{2+\alpha,\Omega\times(t_*,t)} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega\times(t_*,t)} + \|\eta_\sigma\|^2_{1+\alpha,\Omega\times(t_*,t)},\\
&Z_2 = \sup_{t_*\le t\le t}\|u\|^2_{1+\alpha,\Omega} + \sup_{t_*\le t\le t}\|\vartheta_\sigma\|^2_{1+\alpha,\Omega} + \sup_{t_*\le t\le t}\|\eta_\sigma\|^2_{1+\alpha,\Omega},\\
&Z_3 = [\|u\|^{(\alpha+2,\alpha/2+1)}_{\Omega^t} + \|\vartheta_\sigma\|^{(\alpha+2,\alpha/2+1)}_{\Omega^t}]^2 + \|\eta_\sigma\|^2_{1+\alpha,\Omega^t} + \sup_{0\le t'\le t}\|\eta_\sigma\|^2_{1+\alpha,\Omega},\\
&Z_4 = (t-t_*)\Big\|H(\,\cdot\,,0)+\frac{2}{R_e}\Big\|^2_{\alpha,S^1} + \int_{t_*}^t \|R(\,\cdot\,,t')-R(\,\cdot\,,0)\|^2_{2+\alpha,S^1}\,dt'\\
&Z_5 = \Big\|\int_0^t u\,dt'\Big\|^2_{2+\alpha,S} = \sum_{i\in\mathcal N}\Big\|\int_0^t \widetilde u_i\,dt'\Big\|^2_{2+\alpha,\widehat S_l}.
\end{aligned}$$

Proof. The way of the proof is similar as in ([8], [4], [17]). First, we obtain the estimate in an interior subdomain $\widetilde\Omega = B_{2\lambda}(\xi_0) = \{\xi\in\mathbb R^3 : |\xi-\xi_0| < 2\lambda\}$ for

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\bigg(\frac{\eta|\Delta^2(z)\widetilde u|^2}{|z|^{3+2(1+\alpha)}} + \frac{p_1|\Delta^2(z)\widetilde\eta_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}} + \frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\widetilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}}\bigg)\\
&\quad + c_1\int_{\mathbb R^3}dz\frac{\|\Delta^2(z)\widetilde u\|^2_{1,B_{2\lambda}(\xi_0)}}{|z|^{3+2(1+\alpha)}} + c_2\int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{p_2|\nabla(\Delta^2(z)\widetilde\vartheta_\sigma|^2}{|z|^{3+2(1+\alpha)}},
\end{aligned}$$

where $3 = 1$ in $B_{\lambda+\varepsilon_0}(\xi_0)$ for some $-\lambda < \varepsilon_0 < \lambda$. In comparison to the case of u, ν and $\varkappa$ independent of ϱ and θ (see [17]) we have to estimate the following additional terms:

$$\begin{aligned}
I_1 &= \int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta^2(z)\mu(\eta,\vartheta)(\partial_{\xi_i}u_j+\partial_{\xi_j}u_i)\Delta^2(z)\widetilde u_{i\xi_j}}{|z|^{3+2(1+\alpha)}}\\
&\quad + \int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta^2(z)(\nu(\eta,\vartheta)-\mu(\eta,\vartheta))\delta_{ij}\operatorname{div}u\Delta^2(z)\widetilde u_{i\xi_j}}{|z|^{3+2(1+\alpha)}},\\
I_2 &= 2\int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta(z)\mu(\eta,\vartheta)(\Delta(z)\partial_{\xi_i}u_j+\Delta(z)\partial_{\xi_j}u_i)\Delta^2(z)\widetilde u_{i\xi_j}}{|z|^{3+2(1+\alpha)}}\\
&\quad + 2\int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta(z)(\nu(\eta,\vartheta)-\mu(\eta,\vartheta))\delta_{ij}\Delta(z)\operatorname{div}u\Delta^2(z)\widetilde u_{i\xi_j}}{|z|^{3+2(1+\alpha)}},\\
I_3 &= \int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta^2(z)k_{1i}\Delta^2(z)\widetilde u_i}{|z|^{3+2(1+\alpha)}},\\
I_4 &= \int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta^2(z)\varkappa(\eta,\vartheta)\widetilde\vartheta_{\sigma\xi}\cdot\Delta^2(\tau)\widetilde\vartheta_{\sigma\xi}}{|z|^{3+2(1+\alpha)}},\\
I_5 &= 2\int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta(z)\varkappa(\eta,\vartheta)\Delta(z)\widetilde\vartheta_{\sigma\xi}\cdot\Delta^2(z)\widetilde\vartheta_{\sigma\xi}}{|z|^{3+2(1+\alpha)}},\\
I_6 &= \int_{\mathbb R^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{\Delta^2(z)k_3\Delta^2(z)\widetilde\vartheta_\sigma}{|z|^{3+2(1+\alpha)}}.
\end{aligned}$$

We get

$$
\begin{aligned}
|I_1| &\le c\int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{(|\Delta^2(z)\eta_\sigma| + |\Delta^2(z)\vartheta_\sigma|)|u_\xi||\Delta^2(z)\widetilde{u}_\xi|}{|z|^{3+2(1+\alpha)}} \\
&\le \varepsilon \int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^2(z)\widetilde{u}_\xi|^2}{|z|^{3+2(1+\alpha)}} + c(\varepsilon)|u_\xi|^2_{\infty,\Omega}(\|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\Omega}) \\
&\le \varepsilon\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}} + c(\varepsilon)\|u\|^2_{2+\alpha,\Omega}(\|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\Omega})
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &\le \varepsilon \int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^2(z)\widetilde{u}_\xi|^2}{|z|^{3+2(1+\alpha)}} \\
&\quad + c(\varepsilon)\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{(|\Delta(z)\eta_\sigma|^2 + |\Delta(z)\vartheta_\sigma|^2)|\Delta(z)u_\xi|^2}{|z|^{3+2(1+\alpha)}} \\
&\le \varepsilon\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}} + c(\varepsilon)\bigg[\bigg(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\eta_\sigma|^{2p_1}}{|z|^{3+p_1(1+\alpha)}}\bigg)^{1/p_1} \\
&\quad + \bigg(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)\vartheta_\sigma|^{2p_1}}{|z|^{3+p_1(1+\alpha)}}\bigg)^{1/p_1}\bigg] \\
&\quad \cdot \bigg(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta(z)u_\xi|^{2p_2}}{|z|^{3+p_2(1+\alpha)}}\bigg)^{1/p_2} \\
&\le \varepsilon\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}} + c(\varepsilon)(\|\vartheta_\sigma\|^2_{1+\alpha,\Omega} + \|\eta_\sigma\|^2_{1+\alpha,\Omega})\|u\|^2_{2+\alpha,\Omega},
\end{aligned}
$$

where $1/p_1 + 1/p_2 = 1$ and we have used the imbeddings $B_2^{1+\alpha}(\Omega) \subset B_{2p_1}^{(1+\alpha)/2}(\Omega)$ and $B_2^{1+\alpha}(\Omega) \subset B_{2p_2}^{(1+\alpha)/2}(\Omega)$ which hold for $\alpha \ge 1/2$ and p_1, p_2 satisfying $3/2 - 3/2p_1 + (1+\alpha)/2 \le 1+\alpha$ and $3/2 - 3/2p_2 + (1+\alpha)/2 \le 1+\alpha$.

I_3 and I_4 can be estimated by using similar imbeddings as in the cases of I_1 and I_2. I_5 consists of several terms, most of which can be estimated in the same way as I_i $(i = 1,\ldots,4)$. Therefore, we estimate only one term of I_5 which has the qualitative form

$$
I_5^1 = \int_{\mathbb{R}^3} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{(\Delta^2(z)\eta_{\sigma\xi} + \Delta^2(z)\vartheta_{\sigma\xi})\cdot\vartheta_{\sigma\xi}\Delta^2(z)\widetilde{\vartheta}_\sigma}{|z|^{3+2(1+\alpha)}}.
$$

We have

$$
\begin{aligned}
|I_5^1| &\le c|\vartheta_{\sigma\xi}|_{\infty,\Omega}\bigg(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^2(z)\eta_{\sigma\xi}|^{p_1} + |\Delta^2(z)\vartheta_{\sigma\xi}|^{p_1}}{|z|^{3+p_1(1+\alpha)/2}}\bigg)^{1/p_1} \\
&\quad \cdot \bigg(\int_{B_{2\lambda}(0)} dz \int_{B_{2\lambda}(\xi_0)} d\xi \frac{|\Delta^2(z)\widetilde{\vartheta}_\sigma|^{p_2}}{|z|^{3+p_2(1+\alpha)}}\bigg)^{1/p_2} \\
&\le \varepsilon\|\widetilde{\vartheta}_\sigma\|^2_{2+\alpha,\Omega} + c(\varepsilon)\|\vartheta_\sigma\|^2_{2+\alpha,\Omega}(\|\eta_\sigma\|^2_{1+\alpha,\Omega} + \|\vartheta_\sigma\|^2_{1+\alpha,\Omega}),
\end{aligned}
$$

where $1/p_1+1/p_2=1$ and we have used the imbeddings $B_2^\alpha(\Omega)\subset B_{p_1}^{(1+\alpha)/2}(\Omega)$ and $B_2^{2+\alpha}(\Omega)\subset B_{p_2}^{1+\alpha}(\Omega)$ which hold for p_1 and p_2 satisfying $3/2-3/p_1+(1+\alpha)/2\le\alpha$ and $3/2-3/p_2+1+\alpha\le 2+\alpha$.

The other terms are estimated in [17] and [4]. Hence we obtain the following inequality

$$
\begin{aligned}
(4.5)\quad & \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\bigg(\frac{\eta|\Delta^2(z)\widetilde{u}|^2}{|z|^{3+2(1+\alpha)}}+\frac{p_1|\Delta^2(z)\widetilde{\eta}_\sigma|^2}{\varrho_e|z|^{3+2(1+\alpha)}}\\
&\quad+\frac{p_2\eta c_v(\eta,\vartheta)}{\theta_e p_\vartheta(\varrho_e,\theta_e)}\frac{|\Delta^2(z)\widetilde{\vartheta}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\bigg)+c_1\int_{\mathbb{R}^3}dz\frac{\|\Delta^2(z)\widetilde{u}\|^2_{1,B_{2\lambda}(\xi_0)}}{|z|^{3+2(1+\alpha)}}\\
&\quad+c_2\int_{\mathbb{R}^3}dz\int_{B_{2\lambda}(\xi_0)}d\xi\frac{p_2|\nabla(\Delta^2(z)\widetilde{\vartheta}_\sigma|^2}{|z|^{3+2(1+\alpha)}}\\
&\le\varepsilon(\|u\|^2_{2+\alpha,\widetilde{\Omega}}+\|\eta_\sigma\|^2_{1+\alpha,\widetilde{\Omega}}+\|\vartheta_\sigma\|^2_{2+\alpha,\widetilde{\Omega}})\\
&\quad+\psi_1\Big(\frac{1}{\varepsilon},a\Big)\Big[X_1X_2(1+X_2)+X_1(1+X_2^2)\int_0^t\|u\|^2_{2+\alpha,\Omega}\,dt'\\
&\quad+\|\eta_\sigma\|^2_{1+\alpha,\Omega}\|\widetilde{u}_t\|^2_{\alpha,\widetilde{\Omega}}\Big]+\psi_2\Big(\frac{1}{\varepsilon},a\Big)(\|u\|^2_{1+\alpha,\widetilde{\Omega}}+\|\widetilde{\eta}_\sigma\|^2_{\alpha,\widetilde{\Omega}}\\
&\quad+\|\widetilde{\vartheta}_\sigma\|^2_{\alpha,\widetilde{\Omega}}+\|\vartheta_\sigma\|^2_{1+\alpha,\widetilde{\Omega}})+\psi_3\Big(\frac{1}{\varepsilon},a\Big)\|\widetilde{k}\|^2_{\alpha,\widetilde{\Omega}},
\end{aligned}
$$

where $X_1=\|u\|^2_{2+\alpha,\Omega}+\|\eta_\sigma\|^2_{1+\alpha,\Omega}+\|\vartheta_\sigma\|^2_{2+\alpha,\Omega}$, $X_2=\|u\|^2_{1+\alpha,\Omega}+\|\eta_\sigma\|^2_{1+\alpha,\Omega}+\|\vartheta_\sigma\|^2_{1+\alpha,\Omega}$, $a=T^{1/2}(\int_0^T\|u\|^2_{2+\alpha,\Omega}\,dt')^{1/2}$, ψ_i $(i=1,2,3)$ are positive continuous functions, $\varepsilon>0$ is a sufficiently small constant.

Now, we want to estimate $\int_{\widetilde{\Omega}}\|\widetilde{u}_{t'}\|^2_{\alpha/2,(t_*,t)}\,d\xi$. To do this apply $\Delta_t(h)$ to $(2.6)_1$. We get

$$
\begin{aligned}
&\eta\Delta_t\widetilde{u}_{it}-\nabla_jT_{ij}(\Delta_t\widetilde{u}_t,\Delta_t\widetilde{p}_\sigma)\\
&\qquad=\eta\Delta_t\widetilde{u}_{it}-\Delta_t(\eta\widetilde{u}_{it})+\Delta_tk_{1i}+[\Delta_t\nabla_jT_{ij}(\widetilde{u},\widetilde{u}_\sigma)-\nabla_jT_{ij}(\Delta_t\widetilde{u},\Delta_t\widetilde{p}_\sigma)].
\end{aligned}
$$

It suffices to estimate only the third and the fourth terms on the right-hand side (multiplied by $\Delta_{t'}\widetilde{u}_{t'}$) because the others are the same as in the case of constant μ and ν. Moreover, we choose only these terms of Δ_tk_{1i} and of $[\Delta_t\nabla_jT_{ij}(\widetilde{u},\widetilde{p}_\sigma)-\nabla_jT_{ij}(\Delta_t\widetilde{u},\Delta_t\widetilde{p}_\sigma)]$ which have the qualitative forms $\int_0^{t'}u_\xi\,d\tau(\Delta_t\eta_\sigma+\Delta_t\vartheta_\sigma)\widetilde{u}_{\xi\xi}$ and $\int_0^t u_\xi\,d\tau(\Delta_t\eta_{\sigma\xi}+\Delta_t\vartheta_{\sigma\xi})\widetilde{u}_\xi$ because the other terms can be estimated similarly. We have

$$
\begin{aligned}
(4.6)\quad&\int_0^{h_0}\frac{dh}{h^{1+\alpha}}\int_{t_*}^{t-h_0}dt'\int_{\widetilde{\Omega}}\bigg|\int_0^{t'}u_\xi\,d\tau\bigg|(|\Delta_{t'}\eta_\sigma|+|\Delta_{t'}\vartheta_\sigma|)|\widetilde{u}_{\xi\xi}||\Delta_{t'}\widetilde{u}_{t'}|\,d\xi\\
&\le\varepsilon\|\widetilde{u}\|^2_{2+\alpha,\widetilde{\Omega}\times(t_*,t)}\\
&\quad+c(\varepsilon)\int_0^{h_0}dh\int_{t_*}^{t-h_0}dt'\int_{\widetilde{\Omega}}\frac{|\int_0^{t'}u_\xi\,d\tau|^2|\int_{t'}^{t'+h}u_\xi\,d\tau|^2|\widetilde{u}_{\xi\xi}|^2}{h^{1+\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
&+ c(\varepsilon)\int_0^{h_0} dh \int_{t_*}^{t-h_0} dt' \int_{\widetilde\Omega} \frac{|\int_0^{t'} u_\xi\, d\tau|^2 |\int_{t'}^{t'+h} \vartheta_{\sigma\tau}\, d\tau|^2 |\widetilde u_{\xi\xi}|^2}{h^{1+\alpha}} \\
&\le \varepsilon \|\widetilde u\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)} + c(\varepsilon) t^{2-\alpha}\|u\|^2_{2+\alpha,\widetilde\Omega^t} \\
&\quad \cdot \left(\|u\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)} \|\widetilde u\|^2_{2,\widetilde\Omega\times(t_*,t)} + \int_{t_*}^t |\vartheta_{\sigma\tau}|^2_{4,\widetilde\Omega}\, d\tau \int_{t_*}^t |\widetilde u_{\xi\xi}|^2_{4,\widetilde\Omega}\, d\tau \right) \\
&\le \varepsilon \|\widetilde u\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)} + c(\varepsilon) t^{2-\alpha}\|u\|^2_{2+\alpha,\widetilde\Omega^t} \|\widetilde u\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)} \\
&\quad \cdot (\|u\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)} + \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)})
\end{aligned}
$$

and

$$
\begin{aligned}
(4.7)\quad &\int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{t_*}^{t-h_0} dt' \int_{\widetilde\Omega} \left| \int_0^{t'} u_\xi\, d\tau \right| (|\Delta_{t'}\eta_{\sigma\xi}| + |\Delta_{t'}\vartheta_{\sigma\xi}|) |\widetilde u_\xi| |\Delta_{t'}\widetilde u_{t'}|\, d\xi \\
&\le c\int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_0^{t-h_0} dt' \left| \int_0^{t'} u_\xi\, d\tau \right|_{\infty,\widetilde\Omega} \left(\left| \int_{t'}^{t'+h} u_{\xi\xi}\, d\tau \right|_{4,\widetilde\Omega} \right. \\
&\quad \left. + |\Delta_{t'}\vartheta_{\sigma\xi}|_{4,\widetilde\Omega} \right) |\widetilde u_\xi|_{4,\widetilde\Omega} |\Delta_{t'}\widetilde u_{t'}|_{2,\widetilde\Omega} \\
&\le \varepsilon \|\widetilde u\|^2_{2+\alpha,\widetilde\Omega^t} + c(\varepsilon) t^{2-\alpha} \|u\|^2_{2+\alpha,\widetilde\Omega^t} (\|u\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)} \\
&\quad + \|\vartheta_\sigma\|^2_{2+\alpha,\widetilde\Omega\times(t,t^*)}) \sup_{t_*\le t'\le t} \|\widetilde u\|^2_{1+\alpha,\widetilde\Omega}.
\end{aligned}
$$

Taking into account (4.6), (4.7) and calculations from [17] and [4] we obtain the inequality

$$
\begin{aligned}
(4.8)\quad &\int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{t_*}^{t-h_0} dt' \int_{\widetilde\Omega} \eta |\Delta_{t'}\widetilde u_{t'}|^2\, d\xi + \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widetilde\Omega} \left(\frac{\mu(\eta,\vartheta)}{4} |\mathbb{S}(\Delta_{t'}\widetilde u)|^2 \right. \\
&\quad \left. + \frac{\nu(\eta,\vartheta) - \mu(\eta,\vartheta)}{2} |\operatorname{div} \Delta_{t'}\widetilde u|^2 \right) d\xi|_{t'=t-h_0} \\
&\le \int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{\widetilde\Omega} \left(\frac{\mu(\eta,\vartheta)}{4} |\mathbb{S}(\Delta_{t'}\widetilde u)|^2 \right. \\
&\quad \left. + \frac{\nu(\eta,\vartheta) - \mu(\eta,\vartheta)}{2} |\operatorname{div} \Delta_{t'}\widetilde u|^2 \right) d\xi|_{t'=t_*} \\
&\quad + \varepsilon(\|u\|^2_{2+\alpha,\Omega\times(t_*,t)} + \|\vartheta_\sigma\|^2_{2+\alpha,\Omega\times(t_*,t)} + \|\eta_\sigma\|^2_{1+\alpha,\Omega\times(t_*,t)}) \\
&\quad + d\|\widetilde\vartheta_\sigma\|^2_{2+\alpha,\widetilde\Omega\times(t_*,t)} + c(\varepsilon,t,Z_3)(\|u\|^2_{0,\Omega\times(t_*,t)} + \|\vartheta_\sigma\|^2_{0,\Omega\times(t_*,t)} \\
&\quad + \|\eta_\sigma\|^2_{0,\Omega\times(t_*,t)} + \|k\|^2_{0,\Omega\times(t_*,t)} + Z_1 Z_2),
\end{aligned}
$$

where $c(\varepsilon, t, Z_3)$ is a positive continuous function increasing with respect to t and Z_3.

An estimate for $\int_{\widetilde\Omega} \|\widetilde\vartheta_{\sigma t'}\|^2_{\alpha/2,(t_*,t)}\, d\xi$ can be derived by using similar calculations.

Now, we have to obtain estimates for boundary subdomains. We concentrate ourselves only on deriving the estimate for $\int_{\widehat\Omega} \|\widetilde u_t\|^2_{\alpha/2,(t_*,t)}\, dz$. As before we apply

$\Delta_t(h)$ to equation $(2.7)_1$ and to boundary condition $(2.7)_3$. We estimate only the term which has the qualitative form $(\Delta_t\eta_\sigma + \Delta_t\vartheta_\sigma)\int_0^t \widehat{u}_z\, d\tau\widetilde{u}_z$. We get

$$\begin{aligned}\int_0^{h_0} \frac{dh}{h^{1+\alpha}} \int_{t_*}^{t-h_0} dt' \int_{\widehat{S}} dz_{\widehat{S}}(|\Delta_t\widehat{\eta}_\sigma| + |\Delta_t\widehat{\vartheta}_\sigma|)\Big|\int_0^t \widehat{u}_z d\tau\Big||\widetilde{u}_z||\Delta_{t'}\widetilde{u}_{t'}| \\ \le \varepsilon\|\widetilde{u}\|^2_{2+\alpha,\widehat{\Omega}\times(t_*,t)} \\ + c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\int_0^{h_0} dh \int_{t_*}^{t-h_0} dt' \int_{\widehat{S}} dz_{\widehat{S}} \frac{|\int_{t'}^{t'+h}\widehat{u}_z\, d\tau|^2|\widetilde{u}_z|^2}{h^{3/2+\alpha}} \\ + c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\int_0^{h_0} dh \int_{t_*}^{t-h_0} dt' \int_{\widehat{S}} dz_{\widehat{S}} \frac{|\Delta_t\widehat{\vartheta}_\sigma|^2|\widetilde{u}_z|^2}{h^{3/2+\alpha}} \\ \equiv \varepsilon\|\widetilde{u}\|^2_{2+\alpha,\widehat{\Omega}\times(t_*,t)} + L_1 + L_2.\end{aligned}$$

First, we have

$$\begin{aligned}L_1 &\le c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\int_0^{h_0} dh \int_{t_*}^{t-h_0} dt' \frac{\sup_{t_*\le t'\le t}|\widehat{u}_z|^2_{2,\widehat{S}}|\widetilde{u}_z|^2_{\infty,\widehat{S}}}{h^{1/2-\alpha}} \\ &\le c(\varepsilon)t^{5/2-\alpha}\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\|\widetilde{u}\|^2_{2+\alpha,\widehat{\Omega}\times(t_*,t)}\sup_{t_*\le\tau\le t}\|\widehat{u}\|^2_{1+\alpha,\widehat{\Omega}}.\end{aligned}$$

Next, we get

$$\begin{aligned}L_2 &\le c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\int_0^{h_0} dh \int_{t_*}^{t-h_0} dt' \frac{|\Delta_t\widehat{\vartheta}_\sigma|^2_{8,\widehat{S}}|\widetilde{u}_z|^2_{8/3,\widehat{S}}}{h^{3/2+\alpha}} \\ &\le c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\sup_{t_*\le t'\le t}|\widetilde{u}_z|^2_{8/3,\widehat{S}}\int_0^{h_0} dh \int_{t_*}^{t-h_0} dt' \frac{|\Delta_t\widehat{\vartheta}_\sigma|^2_{8,\widehat{S}}}{h^{1+2(1/4+\alpha/2)}} \\ &\le c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\sup_{t_*\le t\le t}\|\widetilde{u}_z\|^2_{\alpha-1/2,\widehat{S}}\int_0^{h_0} dh \int_{t_*}^{t-h_0} dt' \frac{\|\Delta_t\widehat{\vartheta}_\sigma\|^2_{1,\widehat{S}}}{h^{1+2(1/4+\alpha/2)}} \\ &\le c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\sup_{t_*\le t\le t}\|\widetilde{u}\|^2_{1+\alpha,\widehat{\Omega}}(\|\widehat{\vartheta}_{\sigma z}\|^2_{1/2+\alpha,\widehat{S}^t} + \|\widehat{\vartheta}_\sigma\|^2_{1/2+\alpha,\widehat{S}^t}) \\ &\le c(\varepsilon)t\|\widehat{u}\|^2_{2+\alpha,\widehat{\Omega}^t}\sup_{t_*\le t\le t}\|\widetilde{u}\|^2_{1+\alpha,\widehat{\Omega}}\|\widehat{\vartheta}_\sigma\|^2_{2+\alpha,\widehat{\Omega}^t}.\end{aligned}$$

The remaining terms are estimated similarly, so we obtain for $\widehat{\Omega}$ the analogous estimate to (4.8). The other estimates for boundary subdomains are derived in the same way as (4.5), (4.8) and the corresponding estimates from [17] and [4]. Thus, inequality (4.4) holds. □

5. Global existence

First, we introduce some notation. Assume

$$p_0 \ne 0, \quad \overline{\theta} \ge 0 \tag{5.1}$$

and define functions γ_i $(i = 1,2,3)$ by (see [12]–[14]):

$$\begin{aligned}
\cosh\gamma_1 &\equiv \frac{\nu_0}{\mu_0^3} - 1 \quad \text{for } \nu_0 \in I_1 \equiv (2\mu_0^3, \infty),\\
\cos\gamma_2 &\equiv \frac{\nu_0}{\mu_0^3} - 1 \quad \text{for } \nu_0 \in I_2 \equiv (\mu_0, 2\mu_0^3],\\
\cos\gamma_3 &\equiv 1 - \frac{\nu_0}{\mu_0^3} \quad \text{for } \nu_0 \in I_2 \equiv (0, \mu_0^3],
\end{aligned}$$

where $\mu_0 = \widetilde{c}\sigma(\beta - 1/3)/3p_0\beta$, $\nu_0 = d(\beta-1)/2p_0\beta$, $\widetilde{c} = (36\pi)^{1/3}$, $\beta > 1$ is a constant,

$$d = \int_\Omega \varrho_0(v_0^2/2 + e(\varrho_0,\theta_0))\,d\xi + p_0|\Omega| + \sigma|S| + \sup_{0\le t\le T}\int_0^t dt' \int_{S_{t'}} \overline{\theta}(s,t')\,ds,$$

e is the internal energy of fluid per unit mass.

Next, assume

$$e_1 < e(\varrho,\theta) < e_2 \quad \text{for } \varrho \in (\varrho_1,\varrho_2),\ \theta \in (\theta_1,\theta_2), \tag{5.2}$$

where $\varrho_1 = \varrho_e - l$, $\varrho_2 = \varrho_e + l$, $\theta_1 = \theta_e - l$, $\theta_2 = \theta_e + l$ and $l > 0$ is a constant such that $\varrho_e - l > 0$ and $\theta_e - l > 0$.

Now, we can introduce functions (see [12]–[14]):

$$\begin{aligned}
\Phi_1(\mu_0,\gamma_1,p_0,\beta,e_1,\varrho_2,M) &= \frac{p_0\mu_0^{3\beta}}{\beta-1}\left(2\cosh\frac{\gamma_1}{3} - 1\right)^{3(\beta-1)}\\
&\quad\cdot\left[2(\cosh\gamma_1 + 1) - \frac{\beta-1}{\beta-1/3}\left(2\cosh\frac{\gamma_1}{3} - 1\right)^2\right] - \frac{e_1}{\varrho_2^\alpha}M^\beta,\\
\Phi_2(\mu_0,\gamma_2,p_0,\beta,e_1,\varrho_2,M) &= \frac{p_0\mu_0^{3\beta}}{\beta-1}\left(2\cos\frac{\gamma_2}{3} - 1\right)^{3(\beta-1)}\\
&\quad\cdot\left[2(\cos\gamma_2 + 1) - \frac{\beta-1}{\beta-1/3}\left(2\cos\frac{\gamma_2}{3} - 1\right)^2\right] - \frac{e_1}{\varrho_2^\alpha}M^\beta,\\
\Phi_3(\mu_0,\gamma_3,p_0,\beta,e_1,\varrho_2,M) &= \frac{p_0\mu_0^{3\beta}}{\beta-1}\left[2\cos\left(\frac{\pi}{3} - \frac{\gamma_3}{3}\right) - 1\right]^{3(\beta-1)}\\
&\quad\cdot\left\{2(1-\cos\gamma_3) - \frac{\beta-1}{\beta-1/3}\left[2\cos\left(\frac{\pi}{3} - \frac{\gamma_3}{3}\right) - 1\right]^2\right\} - \frac{e_1}{\varrho_2^\alpha}M^\beta,
\end{aligned}$$

where $\alpha = \beta - 1$. Moreover, denote:

$$\begin{aligned}
Q_1 &= \mu_0^3\left(2\cosh\frac{\gamma_1}{3} - 1\right)^3,\\
Q_2 &= \mu_0^3\left(2\cos\frac{\gamma_2}{3} - 1\right)^3,\\
Q_3 &= \mu_0^3\left[2\cos\left(\frac{\pi}{3} - \frac{\gamma_3}{3}\right) - 1\right]^3.
\end{aligned}$$

Before formulating the global existence theorem for problem (1.1) we present assumptions which are necessary to prove this theorem.

We assume:

$$(5.3)\qquad \begin{cases} (v_0,\theta_0,\varrho_0)\in W_2^{1+\alpha}(\Omega)\times W_2^{1+\alpha}(\Omega)\times W_2^{1+\alpha}(\Omega),\\ \|v_0\|^2_{1+\alpha,\Omega}+\|\varrho_{\sigma 0}\|^2_{1+\alpha,\Omega}+\|\theta_{\sigma 0}\|^2_{1+\alpha,\Omega}\le \alpha_1, \quad \text{where } \alpha\in(3/4,1), \end{cases}$$

$$(5.4)\qquad \|r\|^2_{C_B^2(\mathbb{R}^3\times(0,\infty))}+\|\overline{\theta}\|^2_{C_B^3(\mathbb{R}^3\times(0,\infty))}\le\overline{\delta},$$

$$(5.5)\qquad \int_\Omega \varrho_0\,d\xi=M,\quad \int_\Omega\varrho_0\xi\,d\xi=0,\quad \int_\Omega \varrho_0 v_0\,d\xi=0,$$

$$(5.6)\qquad \begin{cases} \Omega \text{ is diffeomorphic to a ball},\\ S \text{ is described by } |\xi|=\widetilde{R}(\omega),\ \omega\in S^1\ (S^1 \text{ is the unit sphere}),\\ \|\widetilde{R}(\omega)-Re\|^2_{1,S^1}\le\alpha_2, \end{cases}$$

$$(5.7)\qquad S\in W_2^{5/2+\alpha}\quad\text{and}\quad \left\|H(\,\cdot\,,0)+\frac{2}{Re}\right\|^2_{\alpha+1/2,S^1}\le\alpha_3,$$

$$(5.8)\qquad \left\|H(\,\cdot\,,0)+\frac{2}{Re}\right\|^2_{\alpha,S^1}\le\alpha_4,$$

$$(5.9)\qquad \begin{aligned} &c_v\in C^2(\mathbb{R}^2),\quad p\in C^3(\mathbb{R}^2),\quad \mu,\nu\in C^3(\mathbb{R}^3),\quad \varkappa\in C^3(\mathbb{R}^3),\\ &e\in C^1(\mathbb{R}^1_+\times\mathbb{R}^1_+), \end{aligned}$$

$$(5.10)\qquad \begin{aligned} &\nu>\frac{1}{3}\mu>0,\quad \varkappa>0,\quad c_v=e_\theta>0,\\ &p_\varrho>0\ \ p_\theta>0\ \text{ for }\ \varrho,\theta>0, \end{aligned}$$

$$(5.11)\qquad \begin{cases} \text{parameters } \mu_0,\ \nu_0,\ p_0,\ \beta, e_1,\ \varrho_2,\ M \text{ satisfy one of the relations } (A_i)\\ \text{where } (A_i):\nu_0\in I_i,\ 0<\Phi_i(\mu_0,\gamma_i,p_o,\beta,e_1,\varrho_2,M)\le\delta_0,\\ |Q_i-|\Omega_e||\le\delta_1,\ i\in\{1,2,3\}. \end{cases}$$

In the above assumptions α_i $(i=1,\dots,4)$, $\overline{\delta}$, δ_0, δ_1 are positive constants.

It is proved in [12] (see also [13], [14]) that in the case when $\nu_0\in I_i$ and $0<\Phi_i(\mu_0,\gamma_i,p_0,\beta,e_1,\varrho_2,M)\le\delta_0$ for $i\in\{1,2,3\}$ we have

$$(5.12)\qquad ||\Omega_t|-Q_i|\le C\delta\quad\text{for } t\in[0,T],$$

where $\delta^2=c\delta_0$, $C>0$ is a constant, T is the time of the local existence.

Hence we can add to the previous assumptions (5.1)–(5.11) the following assumption:

$$\begin{aligned}(5.13)\quad \int_\Omega \varrho_0 \frac{v_0^2}{2}\,d\xi + \int_\Omega \varrho_0(e(\varrho_0,\theta_0)-e_1)\,d\xi + p_0(|\Omega|-Q_i+\delta_2)\\ + \sigma[|S| - \widetilde{c}(Q_i-\delta_2)^{2/3}] + \sup_t \int_0^t dt' \int_{S_{t'}} \overline{\theta}(x,t')\,ds \le \delta_3,\end{aligned}$$

where $\delta_2 \in (0,1/2)$ is a constant so small that $Q_i - \delta_2 > 0$ for $i = 1,2,3$ and δ from $C\delta \le \delta_2$.

Now, we formulate

Theorem 5.1. *Let assumptions* (5.1)–(5.11), (5.13) *and the assumptions of Lemma* 4.1 *be satisfied. Let $e_\varrho > 0$ for $\varrho,\theta > 0$. Then for sufficiently small constants α_i $(i = 1,\dots,4)$, $\overline{\delta}$ and δ_i $(i = 0,\dots,3)$ there exists a global solution to problem* (1.1) *such that*

$$\begin{aligned}(u,\vartheta_\sigma,\eta_\sigma) \in W_2^{2+\alpha,1+\alpha/2}(\Omega_{kT}\times(kT,t)) \times W_2^{2+\alpha,1+\alpha/2}(\Omega_{kT}\times(kT,t))\\ \times C([kT,t];W_2^{1+\alpha}(\Omega_{kT})) \cap W_2^{1+\alpha,1/2+\alpha/2}(\Omega_{kT}\times(kT,t)),\end{aligned}$$

and

$$\|u(t)\|^2_{1+\alpha,\Omega_{kT}} + \|\eta_\sigma(t)\|^2_{1+\alpha,\Omega_{kT}} + \|\vartheta_\sigma(t)\|^2_{1+\alpha,\Omega_{kT}} \le \widetilde{c}\alpha_1$$

for $kT \le t \le (k+1)T$, $k \in \mathbb{N}\cup\{0\}$, where u, η_σ, ϑ_σ denote v, ϱ_σ, θ_σ written in the Lagrangian coordinates $\xi \in \Omega_{kT}$, T is the time of the local existence and $\widetilde{c} > 0$ is a constant. Moreover, $S_t \in W_2^{5/2+\alpha}$ for $t \in \mathbb{R}^1_+$.

We prove Theorem 5.1 in the same way as Theorem 5.1 of [13] by using Theorem 3.3, Lemmas 4.1, 4.2 and Lemmas 3.2, 4.2–4.7 of [13].

In the case $e_\varrho = 0$ similarly as in [13] we obtain the following theorem.

Theorem 5.2. *Let assumptions* (5.1)–(5.11) *and the assumptions of Lemma* 4.1 *be satisfied. Let $e_\varrho = 0$, $p = a\varrho e$, where $a > 0$ is a constant. Moreover, let*

$$\begin{aligned}\int_\Omega \varrho_0 \frac{v_0^2}{2}\,d\xi + \int_\Omega (\varrho_0 e(\varrho_0,\theta_0) - \varrho_1 e_1)\,d\xi + p_0(|\Omega|-Q_i+\delta_2)\\ + \sigma[|S| - \widetilde{c}(Q_i-\delta_2)^{2/3}] + \sup_t \int_0^t dt' \int_{S_{t'}} \overline{\theta}(s,t')dt' \le \delta_3,\end{aligned}$$

where $\widetilde{c} = (36\pi)^{1/3}$, $t \le T$, $\delta_2 \in (0,1/2)$ is a constant so small that $Q_i - \delta_2 > 0$ for $i = 1,2,3$ and δ from (5.12) *is so small that $C\delta \le \delta_2$. Then the assertion of Theorem* 5.1 *holds.*

The proof of Theorem 5.2 is the same as the proof of Theorem 5.1.

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