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# INFINITELY MANY SOLUTIONS OF SUPERLINEAR FOURTH ORDER BOUNDARY VALUE PROBLEMS

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Abstract. We consider the boundary value problem

$$u^{(4)}(x) = g(u(x)) + p(x, u^{(0)}(x), \dots, u^{(3)}(x)), \quad x \in (0,1),$$
$$u(0) = u(1) = u^{(b)}(0) = u^{(b)}(1) = 0,$$

where:

(i) $g: \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\lim_{|\xi| \to \infty} g(\xi)/\xi = \infty$ ($g$ is *superlinear* as $|\xi| \to \infty$),
(ii) $p: [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and satisfies

$$|p(x, \xi_0, \xi_1, \xi_2, \xi_3)| \le C + \frac{1}{4}|\xi_0|, \quad x \in [0,1], \ (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4,$$

for some $C > 0$,
(iii) either $b = 1$ or $b = 2$.

We obtain solutions having specified nodal properties. In particular, the problem has infinitely many solutions.

## 1. Introduction

We consider the boundary value problem

$$u^{(4)}(x) = g(u(x)) + p(x, u^{(0)}(x), \dots, u^{(3)}(x)), \quad x \in (0,1), \tag{1.1}$$

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$$u(0) = u(1) = u^{(b)}(0) = u^{(b)}(1) = 0, \tag{1.2}$$

where:

(i) $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\lim_{|\xi|\to\infty} g(\xi)/\xi = \infty \tag{1.3}$$

($g$ is *superlinear* as $|\xi| \to \infty$),

(ii) $p\colon [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and satisfies

$$|p(x,\xi_0,\xi_1,\xi_2,\xi_3)| \le C + \frac{1}{4}|\xi_0|, \quad x \in [0,1],\ (\xi_0,\xi_1,\xi_2,\xi_3) \in \mathbb{R}^4, \tag{1.4}$$

for some $C > 0$,

(iii) either $b = 1$ or $b = 2$.

In order to state our results we first recall some standard notation to describe the nodal properties of solutions. For any integer $r \ge 0$, let $C^r[0,1]$ denote the standard Banach space of real valued, $r$-times continuously differentiable functions defined on $[0,1]$, with the norm $|u|_r = \sum_{i=0}^{r} |u^{(i)}|_0$, where $|\cdot|_0$ denotes the usual sup-norm on $C^0[0,1]$. Let

$$E = \{u \in C^3[0,1] : u \text{ satisfies (1.2)}\}, \quad X = E \cap C^4[0,1], \quad Y = C^0[0,1].$$

From now on $\nu$ will denote an element of $\{\pm\}$, that is, either $\nu = +$ or $\nu = -$. For each integer $k \ge 1$ and $\nu \in \{\pm\}$, let $S_{k,\nu}$ denote the set of $u \in E$ such that:

(i) $u$ has only simple zeros in $(0,1)$ and has exactly $k-1$ such zeros,

(ii) $\nu u^{(3-b)}(0) > 0$ and $u^{(3-b)}(1) \ne 0$ (with the obvious interpretation of $\nu u^{(3-b)}(0)$).

The sets $S_{k,\nu}$ are disjoint and open in $E$. A *solution* of (1.1)–(1.2) (and other boundary value problems below) is a function $u \in X$ satisfying (1.1), but when using nodal properties it is convenient to regard $u$ as an element of $E$. Our main result is the following theorem.

Theorem 1.1. *There exists an integer $k_0 \ge 1$ such that for all integers $k \ge k_0$ and each $\nu$ the problem* (1.1)–(1.2) *has at least one solution $u_{k,\nu} \in S_{k,\nu}$.*

Superlinear problems of similar form to (1.1)–(1.2) have been considered in many papers, particularly in the second and fourth order cases, with either periodic or separated boundary conditions, see for example [1]–[4], [8], [10] and the references therein. Specifically, the second order periodic problem is considered in [3], while [2], [4] consider the corresponding problem with separated boundary conditions, and results similar to Theorem 1.1 are obtained in each of these papers. The fourth order periodic problem is considered in [1], [8], while [10] considers a general $2m$th order problem with separated boundary conditions. The assumptions in [8] are closest to those adopted here, but only one solution is obtained there. The papers [1], [10] impose additional assumptions, similar to the condition

$$\lim_{|\xi_0|\to 0} (g(\xi_0) + p(x, \xi_0, \xi_1, \xi_2, \xi_3))/\xi_0 = 0,$$

and they obtain the result in Theorem 1.1, with the improvement that $k_0 = 1$. Such conditions are not imposed in the other papers cited above, so $k_0 = 1$ cannot be obtained in them. In fact, an example on p. 187 of [4] shows that the result of Theorem 1.1 is optimal for the second order problem, in the sense that the result need not be true with $k_0 = 1$, and an adaptation of this example also shows that Theorem 1.1 is optimal in this sense, see Remark 2.13 below.

Condition (1.4) can be regarded as a linear growth rate condition on $p$. In [3], in the second order case with $p$ a function of $u$ and $u'$, a more general linear growth rate condition is allowed. In Section 5 of [8], in the fourth order case, a boundedness condition of the form $|p(x, \xi_0, \xi_1, \xi_2, \xi_3)| \le q(x)$, where $q \in L^1(0, 1)$, is assumed.

We use a global bifurcation argument to prove the theorem instead of the continuation methods used in [2]–[4], [8] (a bifurcation method is also used in [10]). In [5] an $n$th order superlinear problem similar to the above is studied and a result similar to Theorem 1.1 is obtained. However, in [5] a single boundary condition is imposed at one end point and $n - 1$ conditions are imposed at the other end point, and the proof uses a shooting method which depends on this distribution of the boundary conditions.

## 2. Proof of Theorem 1.1

By redefining $g$ and $p$ suitably, if necessary, we may suppose that

$$g(\xi)\xi \ge 0, \quad \xi \in \mathbb{R}, \tag{2.1}$$

while retaining the growth conditions (1.3) and (1.4). For any $u \in X$ we define $e(u)\colon [0, 1] \to \mathbb{R}$ by $e(u)(x) = p(x, u^{(0)}(x), \dots, u^{(3)}(x))$, $x \in [0, 1]$. It follows from (1.4) that

$$|e(u)(x)| \le C + \frac{1}{4}|u(x)|, \quad x \in [0, 1]. \tag{2.2}$$

For any $s \in \mathbb{R}$ let $G(s) = \int_0^s g(\xi)\, d\xi \ge 0$, and for any $s \ge 0$ let

$$\gamma(s) = \max\{|g(\xi)| : |\xi| \le s\}, \quad \Gamma(s) = \max\{G(\xi) : |\xi| \le s\}.$$

We now consider the boundary value problem

$$u^{(4)} = \lambda u + \alpha(g(u) + e(u)), \quad u \in X, \tag{2.3}$$

where $\alpha \in [0, 1]$ is an arbitrary fixed number and $\lambda \in \mathbb{R}$. In the following lemmas $(\lambda, u) \in \mathbb{R} \times X$ will be an arbitrary solution of (2.3) while $R \ge 0$ will be an arbitrary number. Also, $\eta_1, \eta_2, \dots,$ will be constants and $\zeta_1, \zeta_2, \dots,$ will be continuous functions (from $[0,\infty)$ to $[0,\infty)$ unless stated otherwise) and these will depend only on $g$ and $p$, not on $(\lambda,u)$ or $\alpha$.

By (1.2) and Rolle's theorem, for any $u\in X$ each of the functions $u^{(j)}$, $j=0,\dots,3$, has a zero in $[0,1]$, so repeated application of the mean value theorem shows that

$$|u^{(j)}|_0 \le |u^{(j+1)}|_0, \quad j=0,\dots,3. \tag{2.4}$$

LEMMA 2.1. *There exists $\zeta_1$ such that if*

$$0\le\lambda\le R, \quad |u''|_0\le R, \tag{2.5}$$

*then $|u|_3\le\zeta_1(R)$.*

PROOF. By (2.2)–(2.5) $|u^{(4)}|_0\le(1/4)\zeta_1(R):=R^2+\gamma(R)+C+R$, so the result follows from (2.4). □

LEMMA 2.2. *For any $x_0,x_1\in[0,1]$,*

$$\begin{aligned}u''(x_1)^2+\lambda u(x_1)^2+2\alpha G(u(x_1))-2u'(x_1)u'''(x_1)&=u''(x_0)^2+\lambda u(x_0)^2\\&\quad+2\alpha G(u(x_0))-2u'(x_0)u'''(x_0)-2\alpha\int_{x_0}^{x_1}e(u)(\xi)u'(\xi)\,d\xi.\end{aligned}$$

PROOF. Multiply (2.3) by $u'$ and integrate. □

LEMMA 2.3.

(a) *There exists an increasing function $\zeta_2$ such that if $0\le\lambda\le R$, and $\sum_{j=0}^3|u^{(j)}(x_0)|\le R$, for some $x_0\in[0,1]$, then $|u''|_0\le\zeta_2(R)$.*
(b) *If $b=1$ then there exists a constant $\eta_1\ge0$ such that if $|u''|_0\ge\eta_1$ then $|u''(x_0)|\ge|u''|_0/2$ for $x_0\in\{0,1\}$.*

PROOF. (a) Choose $x_1\in[0,1]$ such that $|u''|_0=|u''(x_1)|$. If $x_1\in(0,1)$ then $u'''(x_1)=0$, while if $x_1\in\{0,1\}$ then $b=1$ (or $u\equiv0$) and $u'(x_1)=0$. In either case we obtain from (2.2), (2.4) and Lemma 2.2

$$|u''|_0^2\le R^2+R^3+2\Gamma(R)+2R^2+2\left(C+\frac14|u|_0\right)|u''|_0\le K(R)+2C|u''|_0+\frac12|u''|_0^2,$$

where $K(R)=(3+R)R^2+2\Gamma(R)$. Hence $|u''|_0\le\zeta_2(R):=\max\{1,2K(R)+4C\}$.

(b) Choosing $x_0\in\{0,1\}$ and $x_1$ as in the proof of part (a) now yields (using (1.2))

$$u''(x_0)^2\ge|u''|_0^2-2C|u''|_0-\frac12|u''|_0^2,$$

which proves the result. □

From now on it will be convenient to deal with the cases $b = 1$ and $b = 2$ separately (for instance, Lemma 2.4 below is clearly false when $b = 1$) so, until stated otherwise, we now suppose that $b = 2$. The modifications required to deal with the case $b = 1$ will be described below.

By (1.3) we can choose $\eta_2 \geq 1$ such that

$$|\xi| \geq \eta_2 \Rightarrow |g(\xi)| \geq C + \frac{1}{4}|\xi|. \tag{2.6}$$

We also define functions $\zeta_3$ and $\zeta_4$ (with $\zeta_4 \colon \mathbb{R} \to [\eta_2, \infty)$) by

$$\zeta_3(\xi) = \xi(\xi+\xi^2) + \gamma(\xi+\xi^2) + C + (\xi+\xi^2)/4, \quad \xi \geq 0,$$
$$\zeta_4(\xi) = \begin{cases} \eta_2 + \zeta_2(\xi + \xi^2 + \zeta_3(\xi)) + \eta_1 + 2(\xi^2 + \gamma(\xi) + C + \xi/4) & \text{for } \xi \geq \eta_2, \\ \zeta_4(\eta_2) & \text{for } \xi < \eta_2. \end{cases}$$

Clearly, these functions are increasing.

LEMMA 2.4. *If $R \geq \eta_2$, $0 \leq \lambda \leq R$ and $|u''|_0 \geq \zeta_4(R)$ then, for any $x_0 \in [0,1]$ with $|u(x_0)| \leq R$, we have $|u'(x_0)| \geq R^2$.*

PROOF. Suppose that for some $R \geq \eta_2$ there exists $x_0 \in (0,1)$ such that $|u(x_0)| \leq R$ and $|u'(x_0)| < R^2$. We will show that this is impossible if $|u''|_0 \geq \zeta_4(R)$. Suppose, for now, that $u''(x_0) \geq 0$ and $u'''(x_0) \geq 0$. We first show that $u(x) > -R - R^2$ for $x \in (x_0, 1]$. Suppose, on the contrary, that there exists $x \in (x_0, 1]$ such that $u(x) = -R - R^2$, and let $x_1 > x_0$ be the least such point, so that $u(x) \geq -R - R^2$ on $[x_0, x_1]$. Then, from Taylor's theorem, (2.1)–(2.3) and (2.6),

$$\begin{aligned} \frac{1}{2}u''(x_0) + \frac{1}{3!}u'''(x_0)(x_1 - x_0) &\leq -\frac{1}{4!}(\lambda u(y) + \alpha(g(u(y)) + e(u)(y))(x_1 - x_0)^2 \\ &\leq \frac{1}{4!}\zeta_3(R)(x_1 - x_0)^2, \end{aligned}$$

where $y \in [x_0, x_1]$. This implies that

$$u''(x_0) \leq \frac{1}{2}\zeta_3(R), \quad u'''(x_0) \leq \frac{1}{2}\zeta_3(R), \tag{2.7}$$

but, by Lemma 2.3, this is impossible if $|u''|_0 \geq \zeta_4(R)$, which proves that $u(x) \geq -R - R^2$ on $(x_0, 1]$. But now, from (1.2) and Taylor's theorem (as above),

$$0 = u''(1) \geq u''(x_0) + u'''(x_0)(1 - x_0) - \frac{1}{2}\zeta_3(R)(1 - x_0)^2,$$

so that (2.7) again holds and we again have a contradiction, which proves the result when $u''(x_0) \geq 0$ and $u'''(x_0) \geq 0$. A similar argument holds if $u''(x_0) \leq 0$ and $u'''(x_0) \leq 0$, while if $u''(x_0)$ and $u'''(x_0)$ have opposite signs then we consider $u$ on the interval $[0, x_0)$ and again use a similar argument.

Finally, if $x_0 \in \{0,1\}$ (so that $u(x_0) = 0$) and $|u(x_0)| < R^2$, then by continuity there exists $x_0' \in (0,1)$ such that $|u(x_0')| \le R$ and $|u'(x_0')| < R^2$, which contradicts the result just proved. □

We now consider the problem

$$u^{(4)} = \lambda u + \theta(|u''|_0/\zeta_4(\lambda))(g(u) + e(u)), \quad u \in X, \tag{2.8}$$

where $\theta\colon \mathbb{R} \to \mathbb{R}$ is an increasing, $C^\infty$ function with $\theta(s) = 0$, $s \le 1$ and $\theta(s) = 1$, $s \ge 2$ (we have replaced $\alpha$ in (2.3) with the function $\theta(|u''|_0/\zeta_4(\lambda))$). The nonlinear term in (2.8) is a continuous function of $(\lambda, u) \in \mathbb{R} \times X$ and is zero for $\lambda \in \mathbb{R}$, $|u''|_0 \le \zeta_4(\lambda)$, so (2.8) becomes a linear eigenvalue problem in this region and overall the problem can be regarded as a bifurcation (from $u = 0$) problem.

Regarding the linear problem, define the operator $L\colon X \to Y$ by $Lu = u^{(4)}$, $u \in X$. Corollary 2 and Theorems 1 and 3 of [7] show that the eigenvalue problem $Lu = \mu u$, has a set of eigenvalues $0 < \mu_1 < \mu_2 < \dots$ with $\lim_{k\to\infty} \mu_k = \infty$. Each eigenvalue $\mu_k$, $k \ge 1$, is simple (in the sense used in [9]; this follows from [7] and the formal self-adjointness of $L$ with respect to the $L^2(0,1)$ inner product) and has a corresponding eigenfunction $\phi_k \in S_{k,+}$. The next lemma now follows immediately.

Lemma 2.5. *The set of solutions $(\lambda, u)$ of (2.8) with $|u''|_0 \le \zeta_4(\lambda)$ is*

$$\{(\lambda, 0) : \lambda \in \mathbb{R}\} \cup \{(\lambda_k, t\phi_k) : k \ge 1,\ |t| \le \zeta_4(\lambda)/|\phi_k''|_0\}.$$

We also have the following global bifurcation result for (2.8).

Lemma 2.6. *For each $k \ge 1$ and $\nu$ there exists a connected set $\mathcal{C}_{k,\nu} \subset \mathbb{R} \times E$ of non-trivial solutions of (2.8) such that $\mathcal{C}_{k,\nu} \cup (\mu_k, 0)$ is closed and connected and:*

(i) *there exists a neighbourhood $N_k$ of $(\mu_k, 0)$ in $\mathbb{R} \times E$ such that $N_k \cap \mathcal{C}_{k,\nu} \subset \mathbb{R} \times S_{k,\nu}$,*
(ii) *either $\mathcal{C}_{k,\nu} \cap \mathcal{C}_{k',\nu'} \ne \emptyset$, for some $(k', \nu') \ne (k, \nu)$, or $\mathcal{C}_{k,\nu}$ meets infinity in $\mathbb{R} \times E$ (that is, there exists a sequence $(\lambda_n, u_n) \in \mathcal{C}_{k,\nu}$, $n = 1, 2, \dots$, such that $|\lambda_n| + |u_n|_3 \to \infty$).*

Proof. Since $L^{-1}\colon Y \to X$ exists and is bounded (see Corollary 3 of [7]), (2.8) can be rewritten in the form

$$u = \lambda L^{-1} u + \theta(|u''|_0/\zeta_4(\lambda)) L^{-1}(g(u) + e(u)), \tag{2.9}$$

and since $L^{-1}$ can be regarded as a compact operator from $Y$ to $E$, it is clear that finding a solution $(\lambda, u)$ of (2.8) in $\mathbb{R} \times X$ is equivalent to finding a solution of (2.9) in $\mathbb{R} \times E$. This problem is of the form considered in [9] (see also [6]) so the lemma follows from the results there. □

In the second order Sturm–Liouville problem considered in [9] nodal properties are preserved on the set $\mathcal{C}_{k,\nu}$ (that is, $\mathcal{C}_{k,\nu} \subset \mathbb{R} \times S_{k,\nu}$ with the appropriate definition of $S_{k,\nu}$) and this prevents the first alternative in part (ii) of the theorem occurring. For the problem (2.8) nodal properties need not be preserved so we must consider this alternative. However, we will rely on preservation of nodal properties for "large" solutions, encapsulated in the following result.

LEMMA 2.7. *If* $(\lambda, u)$ *is a solution of* (2.8) *with* $\lambda \geq 0$ *and* $|u''|_0 \geq \zeta_4(\lambda)$ *then* $u \in S_{k,\nu}$, *for some* $k \geq 1$ *and* $\nu$.

PROOF. If $u \notin S_{k,\nu}$ for any $k \geq 1$ and $\nu$ then $u$ must have a double zero, but this contradicts Lemma 2.4. □

In view of Lemmas 2.5 and 2.7, in the following lemmas we suppose that $(\lambda, u)$ is an arbitrary non-trivial solution of (2.8) with $\lambda \geq 0$ and $u \in S_{k,\nu}$, for some $k \geq 1$ and $\nu$.

LEMMA 2.8. *There exists an integer* $k_0 \geq 1$ (*depending only on* $\zeta_4(0)$) *such that if* $\lambda = 0$ *and* $\zeta_4(0) \leq |u''|_0 \leq 2\zeta_4(0)$ *then* $k < k_0$.

PROOF. Let $x_1$, $x_2$ be consecutive zeros of $u$. Then there exists $x_3 \in (x_1, x_2)$ such that $u'(x_3) = 0$, and hence, by Lemma 2.4 (with $R = \eta_2$), $|u(x_3)| \geq 1$. Hence, $|x_2 - x_1| \geq 2/|u'|_0$, and so, by (2.4), $k \leq |u''|_0/2 \leq \zeta_4(0)$. □

Now let

$$V_R(u) = \{x \in [0,1] : |u(x)| \geq R\}, \quad W_R(u) = \{x \in [0,1] : |u(x)| < R\}.$$

LEMMA 2.9. *Suppose that* $R \geq \eta_2$, $0 \leq \lambda \leq R$ *and* $|u''|_0 \geq \zeta_4(R)$. *Then* $W_R(u)$ *consists of exactly* $k+1$ *intervals, each of length less than* $2/R$, *and* $V_R(u)$ *consists of exactly* $k$ *intervals.*

PROOF. Lemma 2.4 implies that $|u'(x)| \geq R^2$ for all $x \in W_R(u)$, from which the result follows immediately. □

LEMMA 2.10. *There exists* $\zeta_5$, *satisfying* $\lim_{R\to\infty} \zeta_5(R) = 0$, *and* $\eta_3 \geq \eta_2$ *such that, for any* $R \geq \eta_3$, *if either*

(a) $0 \leq \lambda \leq R$ *and* $|u''|_0 = 2\zeta_4(R)$,

*or*

(b) $\lambda = R$ *and* $\zeta_4(R) \leq |u''|_0 \leq 2\zeta_4(R)$,

*then the length of each interval of* $V_R(u)$ *is less than* $\zeta_5(R)$.

PROOF. Define $H = H(R)$ by

$$H(R)^4 := \min\{R,\ \min\{g(\xi)/\xi : |\xi| \geq R\} - (C/R + 1/4)\},$$

and let $\zeta_5(R) := 2\pi/H(R)$. By (1.3), $\lim_{R\to\infty} H(R) = \infty$, so $\lim_{R\to\infty} \zeta_5(R) = 0$ and we may choose $\eta_3 \geq \eta_2$ sufficiently large that $H(R) > 0$ for all $R \geq \eta_3$.

Choose $x_0$, $x_2$ such that $u(x_0) = u(x_2) = R$ and $u > R$ on $(x_0, x_2)$, that is, $I := [x_0, x_2]$ is an interval of $V_R(u)$ (the case of intervals on which $u < 0$ is similar). By (2.8) and the construction of $H$, if either (a) or (b) holds then $u^{(4)}(x) \geq H^4 u(x) > 0$ for $x \in I$, and by Lemma 2.4, $u'(x_0) > 0$ and $u'(x_2) < 0$. Suppose now that $x_2 - x_0 > \zeta_5(R)$, that is $l := 2\pi/(x_2 - x_0) < H$. Defining $x_1 = (x_0 + x_2)/2$ and

$$v(x) = 1 + \cos l(x - x_1), \quad x \in I,$$

we have

$$\begin{gathered} v(x_0) = v(x_2) = v'(x_0) = v'(x_2) = v'''(x_0) = v'''(x_2) = 0, \\ v''(x_0) = v''(x_2) = l^2, \quad v^{(4)}(x) = l^4(v(x) - 1), \quad x \in I, \end{gathered}$$

and hence, from the above results,

$$\begin{aligned} 0 > l^2(u'(x_2) - u'(x_0)) &= \int_{x_0}^{x_2} \frac{d}{dx}(u'''v - uv''' - u''v' + u'v'')\,dx \\ = \int_{x_0}^{x_2} (u^{(4)}v - uv^{(4)})\,dx &\geq \int_{x_0}^{x_2} (H^4uv - l^4(v-1)u)\,dx > l^4 \int_{x_0}^{x_2} u\,dx > 0, \end{aligned}$$

and this contradiction shows that $x_2 - x_0 \leq \zeta_5(R)$, which proves the lemma (the final part of the proof is from p. 71 of [8]). □

Now choose an arbitrary integer $k \geq k_0$ and $\nu$, and choose $\Lambda > \max\{\eta_3, \mu_k\}$ such that

$$2(k+1)/\Lambda + k\zeta_5(\Lambda) < 1. \tag{2.10}$$

Let

$$\begin{aligned} B &= \{(\lambda, u) : 0 \leq \lambda \leq \Lambda,\ \zeta_4(\lambda) \leq |u''|_0 \leq 2\zeta_4(\Lambda)\}, \\ D_1 &= \{(\lambda, u) : 0 \leq \lambda \leq \Lambda,\ |u''|_0 = \zeta_4(\lambda)\}, \\ D_2 &= \{(0, u) : 2\zeta_4(0) \leq |u''|_0 \leq \zeta_4(\Lambda)\}. \end{aligned}$$

It follows from Lemma 2.5 that $\mathcal{C}_{k,\nu}$ "enters" $B$ through the set $D_1$, while from Lemma 2.7, $\mathcal{C}_{k,\nu} \cap B \subset \mathbb{R} \times S_{k,\nu}$. Thus, by Lemmas 2.1 and 2.6, $\mathcal{C}_{k,\nu}$ must "leave" $B$, and since $\mathcal{C}_{k,\nu}$ is connected it must intersect $\partial B$. However, Lemmas 2.8–2.10 (together with (2.10)) show that the only portion of $\partial B$ (other than $D_1$) which $\mathcal{C}_{k,\nu}$ can intersect is $D_2$. Thus there exists a point $(0, u_{k,\nu}) \in \mathcal{C}_{k,\nu} \cap D_2$, and clearly $u_{k,\nu}$ provides the desired solution of (1.1)–(1.2), which completes the proof of the theorem when $b = 2$.

We now suppose that $b = 1$ and describe the necessary modifications to the above argument to prove the result in this case. We assume from now on that

$\lambda \geq 0$ and $|u''|_0 \geq \zeta_4(\lambda) \geq \eta_1$ so that, in view of part (b) of Lemma 2.3, we can define numbers $z_0, z_1 \in (0,1)$ by the following conditions:

$$u'(z_0) = u'(z_1) = 0, \quad u'(x) \neq 0 \quad \text{for } x \in (0, z_0) \cup (z_1, 1).$$

As noted above, Lemma 2.4 is false if $b = 1$. However, the following modified result holds in this case.

LEMMA 2.11. *The result of Lemma* 2.4 *holds for* $x_0 \in [z_0, z_1]$.

PROOF. It follows from the boundary conditions (1.2) and the definition of the points $z_0$, $z_1$, that there exist points $s_0 \in (0, z_0)$, $s_1 \in (z_1, 1)$, such that $u''(s_0) = u''(s_1) = 0$. We now follow the proof of Lemma 2.4, except that in obtaining the second contradiction we use the point $s_1$ rather than 1. This proves the result in the case $u''(x_0) \geq 0$ and $u'''(x_0) \geq 0$. As in the proof of Lemma 2.4, the other cases are dealt with similarly. □

It is now clear that $u$ is strictly monotonic on the intervals $[0, z_0]$, $[z_1, 1]$ and Lemma 2.7 holds (using part (b) of Lemma 2.3). Furthermore, $|u(z_0)| > R$, $|u(z_1)| > R$, and if we define $y_0 \in [0, z_0)$, $y_1 \in (z_1, 1]$ by $|u(y_0)| = |u(y_1)| = R$ (so that $W_R^0(u) := [0, y_0)$, $W_R^1(u) := (y_1, 1]$, are the intervals of $W_R(u)$ containing 0 and 1 respectively), then $|u'(y_0)| \neq 0$, $|u'(y_1)| \neq 0$, so the only further result now needed to prove the theorem when $b = 1$ is the following.

LEMMA 2.12. *There exists* $\zeta_6$*, satisfying* $\lim_{R\to\infty} \zeta_6(R) = 0$*, such that if* $0 \leq \lambda \leq R$ *and* $|u''|_0 \geq \zeta_4(R)$ *then the length of each interval* $W_R^0(u)$*,* $W_R^1(u)$ *is less than* $\zeta_6(R)$.

PROOF. We consider $W_R^0(u)$ and suppose that $u \geq 0$ on $W_R^0(u)$ (the other cases are similar). Let

$$K = K(R) := R^2 + \gamma(R) + C + \frac{1}{4}R \leq \frac{1}{2}\zeta_4(R) \leq u''(0)$$

(by part (b) of Lemma 2.3). Then, using the definition of $\zeta_4$ and expressing the values of $u'(y_0)$ and $u(y_0)$, respectively, in terms of the Taylor expansion of $u$ about $x = 0$, yields the following results:

$$\begin{gathered} u''(0)y_0 + \frac{1}{2!}u'''(0)y_0^2 + \frac{1}{3!}Ky_0^3 \geq 0 \Rightarrow u'''(0)y_0 \geq -\frac{7}{3}u''(0), \\ R + \frac{1}{4!}K \geq \frac{1}{2}y_0^2(u''(0) + \frac{1}{3}u'''(0)y_0) \geq \frac{1}{9}y_0^2u''(0) \geq \frac{1}{18}y_0^2\zeta_4(R). \end{gathered}$$

Hence, $y_0 \leq \zeta_6(R) := \sqrt{18(R+K)/\zeta_4(R)}$, and it can readily be seen from the above definitions that $\lim_{R\to\infty} \zeta_6(R) = 0$, which proves the result. □

This completes the proof of the theorem for both $b = 1$ and $b = 2$.

REMARK 2.1. Consider the problem

$$u^{(4)} = u \max\{K^2, u^2\}, \quad u \in X.$$

Given any $k_0 \geq 1$, the method of proof of Lemma 2.9 can be used to show that if $K$ is sufficiently large then there is no solution of this problem with fewer than $k_0 - 1$ zeros, that is, there is no solution $u \in S_{k,\nu}$ for $k < k_0$. Thus Theorem 1.1 is optimal in the sense that the result need not hold for all $k \geq 1$.

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