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STRUCTURE OF STEADY STATES FOR STREATER'S ENERGY-TRANSPORT MODELS OF GRAVITATING PARTICLES

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Dedicated to Professor Andrzej Granas

Abstract. Energy-transport models introduced by R. F. Streater describe the evolution of the density and temperature of a cloud of self-gravitating particles. We study the existence of steady states with prescribed mass and energy for these models.

1. Introduction and equations

The systems of evolution partial differential equations introduced recently by R. F. Streater, cf. [26]–[29] for further extensions, generalize the classical Smoluchowski equation [25] proposed as a description of the dynamics of particles subject to an external potential and Brownian diffusion. We extended Streater's models in the paper [9] to the case of self-interacting particles which mathematically corresponds to the coupling to a mean field Poisson equation.

The Streater's models in [9] extend also classical drift-diffusion systems of parabolic-elliptic equations for self-interacting charged particles (Nernst–Planck and Debye–Hückel models in electrolytes, semiconductors, plasma physics, cf. e.g. [7] and [4]), and those for gravitationally attracting particles (cf. e.g. [31] and [11]). The above mentioned systems are isothermal, i.e. they do not take into account the evolution of the temperature.

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As it can be expected for those models, solutions for electrically interacting particles are global in time and tend to steady states ([4], [7]), while for attracting particles finite time blow-up of solutions may occur. This phenomenon is sometimes referred to as a gravitational collapse ([10], [3], [11]).

From a PDE theorist point of view, Streater's systems are of nonclassical type. They consist of (formally) parabolic equations governing the evolution of the density $u \geq 0$ (or rather densities $u_1, \ldots, u_N$ of different particle species) of a cloud of particles. These equations take into account the Brownian diffusion of particles and their collective motion caused by the gradient of the (mean field) potential generated by themselves. Another equation governs the evolution of the temperature ϑ and it involves thermal diffusion, heat production and convection terms. That equation, representing the balance of heat, is not of parabolic type.

These models preserve the charge or mass, the energy (so that they satisfy the first law of thermodynamics), and they are compatible with the second law of thermodynamics.

One of the major mathematical difficulties in an analysis of the evolution of Streater's models is connected with the lack of good a priori estimates for the temperature ϑ which may lead to possible degeneracies and/or singularities in the equation (where the temperature enters via $1/\vartheta$ factor).

Related energy-transport models in solid state physics considered in e.g. [13], [16] and [17], also display such difficulties.

Steady states and large time asymptotics of solutions of the evolution problem have been considered recently in [5] in the case of repulsive electric interactions. In particular, the existence and uniqueness of steady states has been established for certain external potentials φ_0 and domains Ω, the convergence to steady states in bounded domains has been proved, and the intermediate asymptotics in the case of the whole space $\mathbb{R}^d$ was shown for radially symmetric solutions of the evolution problem.

The analysis of the gravitational interaction of particles has been begun in the paper [6] where a special boundary condition (the homogeneous Dirichlet) has been assumed for the potential φ. That condition simplified the analysis of the structure of steady states but was not so physically relevant as the "free" condition considered in the present paper. Actually, this "free" condition is more natural and it may lead to new interesting qualitative phenomena in Streater's systems like gravitational collapse in the evolution problem (cf. [3], [10], [8] and [11] for the case of isothermal models).

Some aspects of the analysis of the existence of steady states in the present paper are different from those in the case of the Dirichlet condition when the geometry and topology of the domain play an important role, and the structure of the set of steady states depends on the shape of the domain in quite a sensitive way.

Note that the existence of solutions of the initial-boundary value problem (and the Cauchy problem) is only partially known, even for systems without self-interaction, cf. [27]. A proof of a global existence result is open and seems very difficult, cf. comments in [9] and [5]. A rather delicate analysis in [13], [16] does not extend to the present case.

Although the present paper can be viewed as a companion paper of [6] (e.g. we will use here in our computations some results concerning radially symmetric solutions from [6]), the methods used are in many aspects different.

Our study of the steady state problem for Streater's system involves the Poisson–Boltzmann–Emden equation arising in the statistical mechanics of gravitating particles, cf. e.g. [1], and in Onsager's approach to turbulence for the Euler equation in two dimensions (see [18], and for three-dimensional extensions – [20]). Classical results in [14], [15] as well as (a modification of) the variational approach to the two-dimensional problem in [12] will be applied here.

An alternative approach to the existence of solutions in two- and higher dimensional domains is based on the topological Leray–Schauder principle, see also [10], [19] and [22].

The method of moments will be useful in the present paper as a substitute of the Rellich–Pohozaev identity in the analysis of steady states with the Dirichlet boundary condition.

Notation. $|v|_p$ denotes the $L^p(\Omega)$ norm of the function v; various inessential constants will be denoted by C, even if they may vary from line to line.

We consider the system of equations for the (single) density $u \geq 0$ of a cloud of the (identical) gravitationally attracting particles, the temperature $\vartheta > 0$ and the potential φ generated by u

$$
\begin{aligned}
u_t &= \nabla \cdot \bigg(\kappa \bigg(\nabla u + \frac{u}{\vartheta} \nabla(\varphi + \varphi_0) \bigg) \bigg), \\
(u\vartheta)_t &= \nabla \cdot (\lambda \nabla \vartheta) + \nabla \cdot (\kappa(\vartheta \nabla u + u \nabla(\varphi + \varphi_0))) \\
&\quad + \nabla(\varphi + \varphi_0) \cdot \bigg(\kappa \bigg(\nabla u + \frac{u}{\vartheta} \nabla(\varphi + \varphi_0) \bigg) \bigg), \\
\Delta \varphi &= u.
\end{aligned} \tag{1}
$$

The coefficients κ, λ are nonnegative functions of x, u, ϑ, φ, and they can vanish only at $\vartheta = 0$. Note that the case $\kappa(\vartheta) = \vartheta$ was postulated by M. Smoluchowski in the seminal paper ([25]). The relation $\kappa(\vartheta) = o(\vartheta)$ as $\vartheta \to 0$ might be assumed for the consistency with the third law of thermodynamics.

The above system is studied either in a bounded connected domain $\Omega \subset \mathbb{R}^d$ or in the whole space $\mathbb{R}^d$. In the first case, the equations are supplemented by the boundary conditions

$$\frac{\partial u}{\partial \nu} + \frac{u}{\vartheta}\frac{\partial}{\partial \nu}(\varphi + \varphi_0) = 0, \quad \frac{\partial \vartheta}{\partial \nu} = 0, \tag{2}$$

where $\partial/\partial\nu$ denotes the exterior normal derivative on the boundary $\partial\Omega$. These conditions express that there are neither mass flux nor heat flux across the boundary $\partial\Omega$ of the domain. The potential φ satisfies the "free" condition

$$\varphi = E_d * u_\Omega, \tag{3}$$

where $u_\Omega(x) = u(x)$ for $x \in \Omega$, $u_\Omega(x) = 0$ for $x \notin \Omega$, and E_d is the fundamental solution of the Laplacian in $\mathbb{R}^d$: $E_2(z) = (1/2\pi)\log|z|$, $E_d(z) = -((d-2)\sigma_d)^{-1}|z|^{2-d}$, σ_d is the area of the unit sphere in $\mathbb{R}^d$. The paper [6] has dealt with the Dirichlet condition $\varphi = 0$ on the boundary $\partial\Omega$ which is mathematically somewhat simpler, but the free condition is physically more relevant. The latter is, however, not classical in the PDE theory.

It is reasonable to assume that the given function φ_0 satisfies the integrability condition $\exp(-\varphi_0/\vartheta_1) \in L^1(\Omega)$ for some $\vartheta_1 > 0$, i.e. the external potential φ_0 is *confining* at the temperature $\vartheta_1 > 0$. In the sequel, we will mainly consider the case $\varphi_0 \equiv 0$, referring to [11] for an analysis and interpretation of some cases with singular potentials φ_0. "Small" potentials φ_0 are not confining in unbounded domains like $\Omega = \mathbb{R}^d$ which implies the nonexistence of nontrivial steady states, cf. [5] for a similar situation with electric interactions.

The system (1) with the boundary conditions (2) and (3) is supplemented by the initial data prescribed at $t = 0$: $u(x,0) = u_0(x)$, $\vartheta(x,0) = \vartheta_0(x)$.

The boundary conditions (2) guarantee that the total mass $M = \int_\Omega u\,dx$ and the total energy

$$E = \int_\Omega u\left(\vartheta + \varphi_0 + \frac{1}{2}\varphi\right)dx \tag{4}$$

are preserved, at least for classical solutions.

For sufficiently smooth solutions of (1)–(3), the (neg)entropy

$$W = \int_\Omega u \log\left(\frac{u}{\vartheta}\right)dx \tag{5}$$

is decreasing, and the following production of entropy formula holds

$$\frac{dW}{dt} = -\int_\Omega \lambda\frac{|\nabla\vartheta|^2}{\vartheta^2}\,dx - \int_\Omega \kappa u\left|\frac{\nabla u}{u} + \frac{1}{\vartheta}\nabla(\varphi+\varphi_0)\right|^2 dx. \tag{6}$$

The entropy relation (6) implies that for steady states $\langle u, \vartheta, \varphi\rangle$ one has $\vartheta = \text{const}$, and the flux $\nabla u + (u/\vartheta)\nabla(\varphi+\varphi_0)$ vanishes a.e. in Ω, so that $\nabla(ue^{(\varphi+\varphi_0)/\vartheta}) = 0$. Thus u has the Boltzmann-distributed form $u = \gamma e^{-(\varphi+\varphi_0)/\vartheta}$, where $\gamma = M(\int_\Omega e^{-(\varphi+\varphi_0)/\vartheta}\,dx)^{-1}$, because $\int_\Omega u\,dx = M$, irrespective of the coefficients κ, λ.

Note that if the second boundary condition in (2) is replaced by the Dirichlet condition $\vartheta = \vartheta_1$, with some $\vartheta_1 > 0$, the total energy is no longer conserved. Nevertheless, all the stationary solutions still have the above Boltzmann form. To see this, let us calculate the time derivatives of the energy E in (4) and the entropy W in (5) with no mass flux condition but without making precise the boundary conditions for φ and ϑ

$$\frac{dE}{dt} = \frac{1}{2}\int_{\partial\Omega}\left(\varphi_t\frac{\partial\varphi}{\partial\nu} - \varphi\frac{\partial\varphi_t}{\partial\nu}\right)d\sigma + \int_{\partial\Omega}\lambda\frac{\partial\vartheta}{\partial\nu}\,d\sigma$$

and

$$\frac{dW}{dt} = -\int_\Omega \lambda\frac{|\nabla\vartheta|^2}{\vartheta^2}\,dx - \int_\Omega \kappa u\left|\frac{\nabla u}{u} + \frac{1}{\vartheta}\nabla\varphi\right|^2 dx - \int_{\partial\Omega}\frac{\lambda}{\vartheta}\frac{\partial\vartheta}{\partial\nu}\,d\sigma.$$

If the walls of the reservoir Ω are kept at the constant temperature ϑ_1, then $dE/dt = dW/dt = 0$ for the stationary solution $\langle u, \vartheta, \varphi\rangle$ implies $\int_{\partial\Omega}\lambda(\partial\vartheta/\partial\nu)\,d\sigma = 0$, the flux satisfies $\nabla u + (u/\vartheta)\nabla\varphi = 0$ a.e., $|\nabla\vartheta|^2/\vartheta^2 = 0$ a.e. hence $\vartheta = \vartheta_1$ in Ω.

Now we study stationary solutions of the system (1), with $\varphi_0 \equiv 0$, i.e. the (rescaled) Poisson–Boltzmann–Emden equation

$$\Delta\varphi = M\frac{e^{-\varphi/\vartheta}}{\int_\Omega e^{-\varphi/\vartheta}\,dx} \quad \text{in } \Omega, \tag{7}$$

with the condition (3) for φ, so that $u = \Delta\varphi$ holds.

The problem (3), (7) has a physical meaning as the steady state equation for the gravitational Streater's model only in dimensions $d \le 3$. In the sequel, we shall consider its scaled version (10) below on domains of arbitrary dimension d. The reason for this is that the properties of solutions do not change much for $d \ge 3$, while they are very different for $d = 2$ and $d = 3$.

Scaling the potential satisfying (7) with the condition (3) as $\varphi = \vartheta\psi$, the energy (4) becomes

$$E = M\vartheta + \frac{1}{2}\vartheta^2\int_\Omega \psi\Delta\psi\,dx. \tag{8}$$

The problem of finding a solution of (7) with prescribed energy E and mass $0 < M < \infty$ is therefore equivalent to looking for a solution of the equation

$$\frac{E}{\vartheta^2} = \left(\frac{E}{M^2}\right)m^2 = m + \frac{1}{2}\int_\Omega \psi\Delta\psi\,dx \equiv \mathcal{E}(m,\psi). \tag{9}$$

Here $m = M/\vartheta$ and ψ solves the Poisson–Boltzmann–Emden equation

$$\psi = \frac{m}{\int_\Omega e^{-\psi}\,dx}E_d * e^{-\psi} \quad \text{in } \Omega, \tag{10}$$

in the range of admissible $m > 0$. In general, this is a union of intervals contained in an interval with the endpoints 0 and $m_\Omega \le \infty$, but $m_\Omega < \infty$ for, e.g. star-shaped domains Ω or even $m_\Omega = 0$ for $\Omega = \mathbb{R}^d$, $d \ge 3$.

We restrict our attention to solutions in $L^\infty(\Omega) \cap H^1(\Omega)$, for which the energy is finite, cf. a comment after Remark 3.3.

The relation $E/M^2 = \mathcal{E}(m,\psi)/m^2$ is to be satisfied for a solution ψ of (10). In cases when this problem has multiple solutions, it is useful to define the quantity

$$\mathcal{F}(m) = \inf_{\psi - \text{a solution of } (10)} \mathcal{E}(m,\psi), \tag{11}$$

where (10) is the Poisson–Boltzmann–Emden equation with the fixed parameter m. Note that here the meanings of $\mathcal{E}$, $\mathcal{F}$ are different from those in [6], where (because of the Dirichlet condition imposed on φ and ψ), e.g. $\mathcal{E} = m - (1/2)\int_\Omega |\nabla\psi|^2\,dx$.

2. The two-dimensional case

In this section we will study the Poisson–Boltzmann–Emden equation (7) with the free condition (3) subject to the energy constraint (8). A variational principle analogous to the *Microcanonical Variational Principle* in [12] will be our main tool in this analysis.

Let us consider the entropy functional

$$S(\rho) = -\int_\Omega \rho \log \rho\,dx \tag{12}$$

on the set $D(1,e_p)$ of densities of nonnegative measures $\rho \ge 0$, $\int_\Omega \rho\,dx = 1$, under the constraint of the fixed potential energy

$$e_p = -\frac{1}{2}\int\int_{\Omega\times\Omega} E_d(x-y)\rho(x)\rho(y)\,dx\,dy, \tag{13}$$

where $d = 2$ so that $-E_2(x-y) = -(1/2\pi)\log|x-y|$ is a symmetric positive definite kernel, $e_p > 0$.

Proposition 2.1. *If $\Omega \subset \mathbb{R}^2$ is a bounded domain, then the maximization problem*

$$S(e_p) \equiv \sup_{\rho \in D(1,e_p)} S(\rho) \tag{14}$$

has a unique solution $\rho = \rho(e_p)$ such that $S(\rho(e_p)) = S(e_p)$, for each $e_p > 0$, and its potential $\varphi = E_2 * \rho$ satisfies (7) with $M = 1$ and some $\vartheta > 0$. Moreover, $S(e_p)$ is a continuous function of e_p.

Remark 2.2. The temperature ϑ is obtained as a Lagrange multiplier for the variational problem (14) so that "true" mass in (10) is $m = 1/\vartheta$. As we remarked already, $m \le m_\Omega$, and $m_\Omega < \infty$ may occur, e.g. in star-shaped domains.

Remark 2.3. Formally, the Euler–Lagrange equation for the problem (14) is exactly (7) with $M = 1$, but its derivation needs that $\rho > 0$ a.e. in Ω. This may be proved as in [12, Part II], for related problem where E_2 is replaced by the Green function G_Ω of the domain Ω.

Proof. The idea is the same as in Proposition 2.1 in Part II of [12]. Since $\rho \log \rho \ge -1/e$, $S(e_p)$ is finite. Given a maximizing sequence ρ_n for $S(e_p)$ with $\rho_n \in D(1, e_p)$, there exists a weak limit ρ in the sense of the weak convergence of measures. Since $-S(\rho_n) = \int_\Omega \rho_n \log \rho_n \, dx$ is bounded, by the de la Vallée-Poussin criterion, the limit $\rho \in L^1(\Omega)$. Moreover, $S(\rho) \ge S(e_p)$ by the upper semicontinuity of the entropy S follows, and it remains to prove that

$$-\frac{1}{2} \int \int_{\Omega\times\Omega} E_2(x-y)\rho(x)\rho(y)\, dx\, dy = e_p.$$

To check this let us split the integral

$$-\int \int_{\Omega\times\Omega} E_2(x-y)\rho(x)\rho(y)\, dx\, dy = I(\varepsilon) + I^c(\varepsilon)$$

into the integrals $I(\varepsilon)$ over the set $\{(x,y) \in \Omega \times \Omega : |x-y| < \varepsilon\}$ and $I^c(\varepsilon)$ over the complement of this set where $|x-y| \ge \varepsilon$. Since $\lim_{n\to\infty} I^c(\varepsilon) = -\int \int_{|x-y|\ge\varepsilon} E_2(x-y)\rho(x)\rho(y)\, dx\, dy \to 2e_p$ as $\varepsilon \searrow 0$ (no singularities under the integral appear, and $\int \int_{\Omega\times\Omega} E_2(x-y)\rho(x)\rho(y)\, dx\, dy$ is absolutely convergent), we need only to show that $\lim_{\varepsilon\searrow 0} I(\varepsilon) = 0$ uniformly with respect to n. Indeed, for $\varepsilon \le 1$,

$$\begin{aligned} 0 \le I(\varepsilon) &\le -\frac{1}{2\pi} \int \int_{|x-y|<\varepsilon} \log|x-y| \rho_n(x)\rho_n(y)\, dx\, dy \\ &\le -\frac{1}{2\pi} \int \int_{|x-y|<\varepsilon} |x-y|^{-1} \log|x-y|\, dx\, dy \\ &\quad + \frac{1}{2\pi} \int \int_{|x-y|<\varepsilon} \log(\rho_n(x)\rho_n(y))\rho_n(x)\rho_n(y)\, dx\, dy, \end{aligned}$$

where the last inequality is obtained by splitting the integration range into the set where $\rho_n(x)\rho_n(y) < |x-y|^{-1}$ and its complement. Thus, we have

$$\begin{aligned} I(\varepsilon) \le &\frac{1}{2\pi} \int \int_{|x-y|<\varepsilon} |x-y|^{-1} |\log|x-y||\, dx\, dy \\ &+ 2|s(\rho_n)| \sup_{x\in\Omega} \int_{|x-y|<\varepsilon} \rho_n(y)\, dy, \end{aligned}$$

which tends to 0 together with $\varepsilon \searrow 0$, uniformly in n, because of the $L^1(\Omega)$-bound on $\rho_n \log \rho_n$.

The remainder of the proof of Proposition 2.1 follows the lines of that of Proposition 2.3 in Part II of [12], where the authors proved that ρ cannot vanish on a set of positive measure. The idea is to show that one might improve the value of $S(\rho)$ by adding mass on the set where ρ vanishes (and, of course, normalizing mass and the potential energy). The uniqueness of the maximizer follows from the strict convexity of S. It is also of importance here that the energy $\mathcal{E}$ varies continuously along the branch of maximizers of (14). This can be proved similarly as was in Propositions 2.2, 2.4 in [12, Part II]. □

Remark 2.4. Observe that the scaling $\varrho = \mu\rho$, $\mu > 0$, permits us to extend the above result to the maximization problem of the quantity

$$-\int_\Omega \varrho \log \varrho \, dx = \mu S(\rho) - \mu \log \mu$$

on the set

$$D(\mu, \mu^2 e_p) = \bigg\{ \varrho : \varrho \geq 0, \int_\Omega \varrho \, dx = \mu, \\ -\frac{1}{2} \int\int_{\Omega\times\Omega} E_2(x-y)\varrho(x)\varrho(y)\, dx\, dy = \mu^2 e_p \bigg\}.$$

Remark 2.5. Note also that $\int_\Omega e^{-\varphi/\vartheta}\, dx < \infty$ for $\varphi = E_2 * \rho$ and $\vartheta > 0$ in bounded two-dimensional domains, but a *uniform* (with respect to ρ) bound for this integral can be given whenever $1/\vartheta < 4\pi$, cf. the proof of Theorem 1(i) in [10], and a discussion of the Moser–Trudinger inequality (11) in [6].

Note that the Microcanonical Variational Principle is of little use in higher dimensions ($d \geq 3$) because the limit ρ of a maximizing sequence $\{\rho_n\}$, in general, does not satisfy the energy constraint (13). The proof of Proposition 2.1 cannot be extended to $d \geq 3$ because of the singularity of the fundamental solution $E_d(x-y) \sim |x-y|^{2-d}$, much stronger than that of E_2.

Next we recall from [10, Theorem 1(i)] a result on the unique solvability of the Poisson–Boltzmann–Emden equation (10) for small $m > 0$.

Proposition 2.6. *The equation* (10) *in each bounded planar domain* Ω *has a unique solution* ψ *in* $H^1(\Omega)$ *for all sufficiently small* $m > 0$. *These solutions form a continuous branch of solutions* $\langle m, \psi\rangle$ *of* (10) *in* $[0,\infty)\times L^\infty(\Omega)$ *emanating from* $\langle 0, 0\rangle$.

Proof. The idea of the proof is to get an a priori bound on the quantity $\int_\Omega \exp(s|\psi|)\, dx$ for some $s > 1$, then a bound for $|\psi|_\infty$, and finally conclude with the use of the Leray–Schauder principle applied to the fixed point problem for the (compact) operator $\mathcal{T}(\psi) = m(\int_\Omega e^{-\psi}\, dx)^{-1} E_2 * e^{-\psi}$ considered in $L^\infty(\Omega)$. This is achieved for $m \in (0, 4\pi)$, and for small $m > 0$ the operator $\mathcal{T}(\psi)$ turns out to be a contraction mapping, so the uniqueness of solutions follows.

Let us give some details of the above mentioned a priori bounds.

For $m \in (0,4\pi)$ fix $\beta \in (m,4\pi)$ and consider $0 \le u \in L^1(\Omega)$, $0 < |u|_1 < \beta$, and $\psi = E_2 * u$. Using the Jensen inequality one has, for $s \in (1,4\pi/\beta)$,

$$\begin{aligned}\int_\Omega \exp(s|\psi|)\,dx &\le \int_\Omega \exp\left(s\beta|u|_1^{-1}\int_\Omega \frac{1}{2\pi}|\log|x-y||u(y)\,dy\right)dx\\ &\le \int_\Omega |u|_1^{-1}\left(\int_\Omega u(y)|x-y|^{-s\beta/(2\pi)}\,dy\right)dx\\ &\le \int_\Omega |u|_1^{-1}u(y)\left(\int_\Omega |x-y|^{-s\beta/(2\pi)}\,dx\right)dy\\ &\le \sup_{y\in\Omega}\int_\Omega |x-y|^{-s\beta/(2\pi)}\,dx \le C(\beta) < \infty.\end{aligned}$$

The constant $C(\beta)$ is bounded, for instance, by $2\pi(2-s\beta/(2\pi))^{-1}(2\delta)^{2-s\beta/(2\pi)} < \infty$, where $\delta = \operatorname{diam}\Omega$, because Ω is contained in a disc of the radius 2δ. Then we have

$$\begin{aligned}|\Omega| &\le \left(\int_\Omega e^{-\psi}\,dx\right)^{s/(s+1)}\left(\int_\Omega e^{s\psi}\,dx\right)^{1/(s+1)}\\ &\le \left(\int_\Omega e^{-\psi}\,dx\right)^{s/(s+1)}|\exp(|\psi|)|_s^{s/(s+1)},\end{aligned}$$

which implies

$$\left(\int_\Omega e^{-\psi}\,dx\right)^{-1} \le |\Omega|^{-1-1/s}|\exp(|\psi|)|_s.$$

Finally, we obtain with $1/s + 1/s' = 1$

$$\left|m\left(\int_\Omega e^{-\psi}\,dx\right)^{-1}E_2 * e^{-\psi}\right|_\infty \le C(\Omega,\beta)m\sup_{x\in\Omega}\left(\int_\Omega |\log|x-y||^{s'}\,dy\right)^{1/s'} < \infty$$

which is the required (uniform in $\{0 \le u \in L^1(\Omega) : |u|_1 \le \beta\}$) $L^\infty(\Omega)$-bound. The Leray–Schauder theorem concludes the proof. □

Corollary 2.7. *In any bounded planar domain Ω $\mathcal{E}(m,\psi) \ge m - Cm^2$ with some $C = C(\Omega)$ as $m \searrow 0$.*

Proof. Indeed, $\int_\Omega \psi\Delta\psi\,dx = \mathcal{O}(m^2)$ since $\int_\Omega \Delta\psi\,dx = m$, and $|\psi|_\infty = \mathcal{O}(m)$ as $m \searrow 0$ as was proved before. □

We are ready now to prove the main theorem of this section

Theorem 2.8. *If $\Omega \subset \mathbb{R}^2$ is a bounded domain, then the Poisson–Boltzmann–Emden equation* (7) *for arbitrary $M > 0$ and $E \in \mathbb{R}$ has a solution satisfying the energy relation* (8)*.*

Proof. From Proposition 2.6 we have a unique continuous branch of solutions ψ_m for small $m > 0$, so that $\mathcal{F}(m) = \mathcal{E}(m,\psi_m)$. Corollary 2.7 implies that

$\mathcal{E}(m,\psi_m)$ is strictly positive for small $m>0$. Moreover, from Proposition 2.1, and scaling properties of solutions of the Microcanonical Variational Principle in Remark 2.4, it follows that

$$\inf_{m>0}\mathcal{E}(m,\psi_m)=-\infty$$

for the solutions ψ_m of (10) corresponding to the MVP. Indeed, since solutions of (10) are unique for small $m>0$, the values of $m=1/\vartheta$ cannot accumulate at 0 when $e_p\to\infty$.

Observe that $\mathcal{E}$ is continuous along the branch of solutions ψ_m of (10) consisting of the maximizers of MVP. Moreover, the values of $\mathcal{E}$ fill up a half-line $(-\infty,\mathcal{E}_0]$ with some $\mathcal{E}_0>0$. These properties of $\mathcal{E}(m,\psi_m)$ put together imply the solvability of the equation (8) for every $E/M^2\in\mathbb{R}$. □

It is worth noting here that for the problem in smooth star-shaped domains with the Dirichlet boundary condition for the potential φ considered in [6], it follows from [12, Part II, p. 251], that solutions of (10) with $\mathcal{E}(m,\psi)\to-\infty$ correspond to $m\to 8\pi$.

It may happen that all the variational solutions correspond (as seen in [12]) to the solutions of (10) with m bounded by a constant depending on Ω. However, in general, there exist solutions of (10) which are not necessarily maximizers of (14) for larger m, e.g. for all $m>0$ in annuli. The situation is different in star-shaped domains: similarly as in [6] for the Poisson–Boltzmann–Emden equation with the Dirichlet boundary condition on star-shaped domains, we have the following nonexistence result

Proposition 2.9. *If Ω is a bounded star-shaped domain in $\mathbb{R}^d$, $d\geq 2$, then the equation* (10) *has no $L^\infty(\Omega)\cap H^1(\Omega)$ solution for $m>m_\Omega$ with some $m_\Omega<\infty$.*

Proof. We may suppose without loss of generality that Ω is star-shaped with respect to the origin $0\in\mathbb{R}^d$. The equation (10), as the steady state problem for $v_t=\nabla\cdot(\kappa(\nabla v+v\nabla\psi))$ with $v=\Delta\psi$, is equivalent to

$$\nabla v+v\nabla\psi=0,\quad v=\Delta\psi, \tag{15}$$

in the class of solutions ψ of (10) in $L^\infty(\Omega)\cap H^1(\Omega)$. Multiplying the first equation in (15) by x, we get after an integration by parts $\int_\Omega\nabla v\cdot x\,dx+\int_\Omega v\nabla\psi\cdot x\,dx=0$. Then, another integration by parts and the symmetrization of the second (double) integral leads to

$$\begin{aligned}-dm&+\int_{\partial\Omega}v\ x\cdot\nu\,d\sigma\\&+\frac{1}{2}\int\int_{\Omega\times\Omega}v(x)v(y)\frac{1}{\sigma_d|x-y|^d}\{(x-y)\cdot x+(y-x)\cdot y\}\,dx\,dy=0.\end{aligned}$$

Since $x \cdot \nu \geq 0$ on $\partial\Omega$ for any star-shaped domain Ω, we arrive at the inequality

$$-2dm + \int\int_{\Omega\times\Omega} v(x)v(y)\sigma_d^{-1}|x-y|^{2-d}\,dx\,dy \leq 0, \tag{16}$$

or

$$m^2\sigma_d^{-1}\delta^{2-d} \leq 2dm,$$

where $\delta = \operatorname{diam}\Omega$. In particular, this gives $m \leq 8\pi$ in bounded star-shaped planar domains (irrespective of their size), and $m \leq 2d\sigma_d\delta^{d-2} \equiv C(\Omega)$ in d-dimensional ($d \geq 3$) domains Ω, so that $m_\Omega < \infty$ in these cases. □

The above proof may be extended to a class of domains which are not star-shaped but rather "dumbbell-like", cf. [3, remark after Theorem 1].

In the last part of this section we present some computations for radially symmetric solutions which will sharpen the estimates for the energy $\mathcal{E}(m,\psi)$ in radially symmetric planar domains.

EXAMPLE 1. Consider the ball $B = \{x \in \mathbb{R}^d : r = |x| < A\}$. It is clear that the radially symmetric solution ψ of (10) is of the form $\psi = \psi_D + \psi(A)$, where ψ_D is the solution satisfying the Dirichlet condition on ∂B, $\psi_D(A) = 0$. Hence we have

$$\int_B \psi\Delta\psi\,dx = -\int_B |\nabla\psi|^2\,dx + \int_B \psi(A)\Delta\psi\,dx = -\int_B |\nabla\psi|^2\,dx + m\psi(A), \tag{17}$$

and it suffices to estimate the value of $\psi(A)$.

If $d = 2$ then

$$\psi(A) = \int_B \Delta\psi(y)\frac{1}{2\pi}\log|x-y|\,dy \leq \frac{m}{2\pi}\log(2A),$$

where $|x| = A$. Thus we obtain

$$\mathcal{E}(m,\psi) \leq m - \frac{1}{2}\int_B |\nabla\psi_D|^2\,dx + \frac{m^2}{4\pi}\log(2A)$$

and $\liminf_{m\nearrow 8\pi}\mathcal{E}(m,\psi) = \lim_{m\nearrow 8\pi}\mathcal{E}(m,\psi) = -\infty$, because $(\partial/\partial r)\psi_D(r) = 4Ar((Ar)^2 + 8\pi/m - 1)^{-1}$, and

$$\begin{aligned} A^{-2}\int_B |\nabla\psi_D|^2\,dx &= 32\pi\int_0^1 r^3\left(r^2 + \frac{8\pi}{m} - 1\right)^{-2} dr \\ &= -2m - 16\pi\log\left(1 - \frac{m}{8\pi}\right) \to \infty \end{aligned} \tag{18}$$

when $0 < m \to m_B = 8\pi$, as in Example 1 in Section 4 of [6].

The result on the solvability of the problem (7)–(8) for $\Omega = \mathbb{R}^2$ is qualitatively the same as in Theorem 2.8:

Example 2. Let $\Omega = \mathbb{R}^2$. It is easy to check that there exists a family of radially symmetric solutions with $\partial\psi(r)/\partial r = 4r(k+r^2)^{-1}$, where the parameter $k > 0$ is an arbitrary positive number. All these solutions have their mass equal to $m = 8\pi$.

Integrating we obtain $\psi(r) = 2\log(k + r^2) - 2\log k + \psi(0)$, so that

$$(19)\quad \begin{aligned}\int_{\mathbb{R}^2} \psi\Delta\psi\,dx &= 2\pi\int_0^\infty \Delta\psi(r)\psi(r)r\,dr \\ &= 2\pi\int_0^\infty \frac{8kr}{(k+r^2)^2}(2\log(k+r^2) - 2\log k + \psi(0)) \\ &= 16\pi(\log k + 1) - 16\pi\log k + 8\pi\psi(0) = 16\pi + 8\pi\psi(0).\end{aligned}$$

Finally, we have

$$\begin{aligned}\psi(0) &= 2\pi\int_0^\infty \Delta\psi(r)\frac{1}{2\pi}r\log r\,dr = \frac{8}{k}\int_0^\infty \frac{r\log r}{(1+(r/\sqrt{k})^2)^2}\,dr \\ &= 8\biggl(\biggl(\int_0^1 + \int_1^\infty\biggr)\frac{z\log z}{(1+z^2)^2}\,dz + \int_0^\infty \frac{1}{2}\frac{z\log k}{(1+z^2)^2}\,dz\biggr) = 2\log k,\end{aligned}$$

and therefore $\mathcal{E}$ assumes all the real values from $-\infty$ to ∞.

It is worth noting that in the case of $\Omega = \mathbb{R}^2$ it is not reasonable to consider the energy defined as in [6] because $\int_{\mathbb{R}^2} |\nabla\psi|^2\,dx = \infty$ for each nontrivial ψ.

Our last example in this section is an analysis of radially symmetric solutions in planar annuli where the problem of existence of steady states with the prescribed energy and mass has qualitatively a different character than in preceding cases. Note that, besides radial solutions, there exist also nonradial solutions, cf. [24], [30] and Corollary 2.10 below.

Example 3. Let $\Omega = \{x \in \mathbb{R}^d : 0 < a < r = |x| < A < \infty\}$ be the annulus with the inner radius a and the outer radius A. For radially symmetric solutions of (10) the potential ψ is a radial function, ψ increases on $[0, \infty)$, and $\psi(x) = \psi(0)$ for $|x| < a$. Thus, for $d = 2$ we have

$$m\psi(0) = \int_\Omega \psi(0)\Delta\psi\,dx \le \int_\Omega \psi\Delta\psi\,dx \le \int_\Omega \psi(A)\Delta\psi\,dx = m\psi(A),$$

and it is sufficient to note that

$$\begin{aligned}\psi(0) &= \int_\Omega \Delta\psi(y)\frac{1}{2\pi}\log|y|\,dy \ge \frac{m}{2\pi}\log a, \\ \psi(A) &= \int_\Omega \Delta\psi(y)\frac{1}{2\pi}\log|x-y|\,dy \le \frac{m}{2\pi}\log(2A), \quad |x| = A.\end{aligned}$$

Indeed, $m + (m^2/4\pi)\log a \le \mathcal{E}(m,\psi) \le m + (m^2/4\pi)\log(2A)$, and there exists $\ell_0 \in \mathbb{R}$ such that for $E/M^2 \le \ell_0$ there are no radially symmetric steady states of (7) satisfying (8) in the annulus. It is not difficult to see that there exists a number $\ell_1 \in \mathbb{R}$ ($\ell_1 \geq \ell_0$) such that if $E/M^2 \geq \ell_1$, then there is a radially symmetric steady state.

It is a bit more difficult to prove that $\ell_0 = \ell_1$ for annuli in two dimensions, which follows from the existence of a single connected branch of radial solutions. We refer the reader to Proposition 5.8 in [6] for details of that reasoning in the case of the Dirichlet condition for ψ.

Comparing the results on radial solutions in Example 3 with Theorem 2.8 we arrive at the following corollary expressing the phenomenon of symmetry breaking for solutions of the Poisson–Boltzmann–Emden equation in two-dimensional annuli. This was already observed for solutions with the prescribed Dirichlet homogeneous condition in [23], cf. also [24], [30], by delicate and seemingly more difficult arguments compared to ours.

Corollary 2.10. *There exist solutions of* (10) *which are not radially symmetric.*

Proof. As was established in Example 3 for arbitrary radially symmetric solutions one has $\mathcal{E}(m,\psi)m^{-2} \geq m^{-1} - C(\Omega)$ with a constant $C(\Omega)$ depending on the geometry of the annulus only. Therefore, the bound $E/M^2 \geq -C(\Omega)$ holds. On the other hand, as in the proof of Theorem 2.8, we obtain variational solutions with $M = 1$ and $e_p > 0$ arbitrarily large, so that $E = E/M^2 = \vartheta - e_p$ could be negative and arbitrarily large. Note that $m = 1/\vartheta$ cannot tend to 0 as $e_p \to \infty$ by the uniqueness of solutions for small m obtained in Proposition 2.6. Therefore, there exist solutions of the MVP which are not radially symmetric.□

The same argument applies to the case of the Poisson–Boltzmann–Emden equation with the Dirichlet boundary condition. We need to take into account Proposition 2.1 in Part II of [12] and Proposition 5.8 in [6].

3. The higher dimensional problem

Our aim in this section is to analyze the existence of solutions of (8) in d-dimensional domains Ω with $d \geq 3$. The ultimate goal is to show that if $E/M^2 \leq \ell_0$ for some $\ell_0 \in \mathbb{R}$, then there are no solutions, while if $E/M^2 \geq \ell_1$ for some $\ell_1 \geq \ell_0$, then there exist solutions, i.e. the situation is similar to that for radially symmetric solutions in two-dimensional annuli.

Recall that Proposition 2.9 states that there are no solutions of (10) with sufficiently large m (i.e. for $m > m_\Omega$ with some $m_\Omega \leq 2d\sigma_d(\operatorname{diam}\Omega)^{d-2}$) in bounded star-shaped domains Ω.

Let us begin with some particular examples, continuing Example 1 from the preceding section.

Example 1_d. Recall that, based on computations in [14], [15], one can prove that $\inf_{m\in[0,m_B)}\mathcal{F}(m) > -\infty$ in the d-dimensional balls, $d\geq 3$. The radially symmetric solutions form a single connected branch which can be parametrized, and if $3\leq d\leq 9$, the solutions of (10) for given m are not, in general (i.e. for moderate values of m), unique. Note also that $m_B < 2d\sigma_d$ if $3\leq d\leq 9$, and $m_B = 2\sigma_d$, $\psi_{m_B}\notin L^\infty$ if $d\geq 10$. Moreover, the endpoint of the branch of bounded solutions corresponds to an unbounded solution with $\Delta\psi(r) = 2(d-2)r^{-2}$ whose energy is, however, finite: $\mathcal{E}(2\sigma_d,\psi) > -\infty$. Here, again $\ell_1 = \ell_0$ but this constant could be different from that in [6] for the problem with the Dirichlet condition.

Let us start with a simple observation that $\partial\psi(r)/\partial r = m\sigma_d^{-1}r^{1-d}$ for every $r\geq A$. Moreover, since $\lim_{r\to\infty}\psi(r) = 0$, we have $\psi(A) = \psi(r) - \int_A^r m\sigma_d^{-1}\rho^{1-d}\,d\rho = -\int_A^\infty m\sigma_d^{-1}\rho^{1-d}\,d\rho = -m(\sigma_d(d-2))^{-1}A^{2-d}$. This leads to

$$\int_B \psi\Delta\psi\,dx = \int_B \Delta\psi(\psi_D+\psi(A))\,dx = -\int_B |\nabla\psi_D|^2\,dx - m^2(\sigma_d(d-2))^{-1}A^{2-d},$$

where ψ_D denotes a solution of (10) with the Dirichlet condition (see (17)), and $\inf_{m\in[0,m_B)}\mathcal{F}(m) > -\infty$. Finally, the claimed existence of $\ell_0=\ell_1$ follows.

Continuing Example 2 we note that

Example 2_d. There is no radial solution of (10) with finite mass $m>0$ in the whole space $\mathbb{R}^d$ (and other radial solutions have their energy equal to $-\infty$).

This follows from another formulation of the problem (10) (in the, so called, *integrated densities*) suggested by [14] and [15]. Since this is not of primary interest here, we skip the details referring to the equations (18)–(20) in [6] which easily lead to the above mentioned nonexistence result.

An extension of the analysis performed in Example 3 (and in [6], Proposition 5.8) reveals a much richer structure of radially symmetric solutions of (10) on higher dimensional annuli.

Example 3_d. There exists $\ell_0\in\mathbb{R}$ and $\ell_1\geq\ell_0$ such that for $E/M^2\leq\ell_0$, the problem (7)–(8) does not have solutions, and there is a solution for each $E/M^2\geq\ell_1$. In general, solutions with given $m>0$ are unique on "thin" annuli (i.e. those with A/a close to 1), and they are not unique on "thick" annuli. Moreover, the radially symmetric solutions are unique for sufficiently small $m>0$.

For the proof of former statement, we can easily see that

$$m - m^2(2\sigma_d(d-2))^{-1}a^{2-d} \leq \mathcal{E}(m,\psi) \leq m - m^2(2\sigma_d(d-2))^{-1}(2A)^{2-d},$$

because

$$\int_\Omega \psi\Delta\psi\,dx \ge m\psi(0) = -m\int_\Omega \Delta\psi(y)(\sigma_d(d-2))^{-1}|y|^{2-d}\,dy$$
$$\ge -m^2(\sigma_d(d-2))^{-1}a^{2-d}.$$

The second bound is valid in arbitrary bounded domain Ω with $\operatorname{diam}\Omega = 2A$ because

$$\int_\Omega \psi\Delta\psi\,dx = -\int\int_{\Omega\times\Omega} \Delta\psi(x)\Delta\psi(y)(\sigma_d(d-2))^{-1}|x-y|^{2-d}\,dx\,dy \tag{20}$$
$$\le -m^2(\sigma_d(d-2))^{-1}(\operatorname{diam}\Omega)^{2-d},$$

and this holds for every (not necessarily stationary) configuration of particles, so that

$$\frac{E}{M^2} \le \frac{\vartheta}{M} - C(\Omega)$$

for some $C(\Omega) > 0$.

A more precise estimate can be given along the lines of the proof of Proposition 5.8 in [6]. The a priori estimates there permit us to prove existence of steady states on annuli using the topological Leray–Schauder principle.

Our next result concerns general star-shaped domains Ω in $\mathbb{R}^d$ where, as was proved in Proposition 2.9, $m_\Omega < \infty$.

Proposition 3.1. *For each $L^\infty(\Omega)\cap H^1(\Omega)$ solution of the problem* (7) *with the condition* (3) *in a star-shaped domain $\Omega \subset \mathbb{R}^d$ the inequality*

$$\frac{E}{M^2} \ge -\frac{2}{d-2}\frac{\vartheta}{M}$$

holds.

Proof. We employ again (as in the proof of Proposition 2.9) the method of moments. Since the stationary density has the Boltzmann form, the equation for steady states of (1) becomes

$$\nabla u + \frac{u}{\vartheta}\nabla\varphi = 0.$$

Multiplying this by x and integrating by parts, we get after the symmetrization, similarly to Proposition 2.9

$$\int_\Omega \nabla u\cdot x\,dx + \frac{1}{\vartheta}\int_\Omega u\nabla\varphi\cdot x\,dx = 0,$$

$$-dM\vartheta + \vartheta\int_{\partial\Omega} u\,x\cdot\nu\,d\sigma$$
$$+\frac{1}{2}\int\int_{\Omega\times\Omega} u(x)u(y)\frac{1}{\sigma_d|x-y|^d}\{(x-y)\cdot x + (y-x)\cdot y\}\,dx\,dy = 0,$$

and

$$-2dM\vartheta + I \equiv -2dM\vartheta + \int\int_{\Omega\times\Omega} u(x)u(y)\sigma_d^{-1}|x-y|^{2-d}\,dx\,dy \le 0.$$

Thus we arrive at the inequality $I \le 2dM\vartheta$ which leads to

$$\begin{aligned} E &= M\vartheta - \frac{1}{2}\int\int_{\Omega\times\Omega} u(x)u(y)(\sigma_d(d-2))^{-1}|x-y|^{2-d}\,dx\,dy \\ &= M\vartheta - (2(d-2))^{-1}I \ge M\vartheta - d(d-2)^{-1}M\vartheta \\ &= -2(d-2)^{-1}M\vartheta. \end{aligned}$$

This means $E/M^2 \ge -2(d-2)^{-1}\vartheta/M$, and returning to the energy relation (8): $(E/M^2)m^2 \ge -2m/(d-2)$. □

Theorem 3.2. *If Ω is a bounded domain in $\mathbb{R}^d$, $d \ge 3$, then there exists a constant $\ell_1 \in \mathbb{R}$ such that for any $E/M^2 > \ell_1$, there is a nontrivial bounded solution of the problem* (3), (7)–(8).

Proof. There exist solutions of (10) with small $m > 0$, cf. [10, Theorem 1]. They belong to a continuous branch $\langle \mu, \psi\rangle \in [0,\infty) \times L^\infty(\Omega)$, $\psi = \mu E_d * e^{-\psi}$ which emanates from $\langle 0, 0\rangle$ with $\mu = m(\int_\Omega e^{-\psi}\,dx)^{-1} \le m|\Omega|^{-1}$ because $\psi < 0$ in Ω by negativity of E_d. This is a consequence of the Leray–Schauder topological principle applied to (a subspace of) $L^\infty(\Omega)$. Thus, the function ψ is uniformly small in $L^\infty(\Omega)$ with respect to small $m \ge 0$: $|\psi|_\infty = \mathcal{O}(m)$ as $m \searrow 0$, and $\mathcal{E}(m,\psi) = m + (1/2)\int_\Omega \psi\Delta\psi\,dx = m - \mathcal{O}(m^2)$. □

Remark 3.3. The nonexistence for $E/M^2 \le \ell_0$ in *star-shaped* domains would follow from the result of Proposition 3.1 and from a lower bound for the energy of the form $\mathcal{F}(m) \ge \alpha m$ with some $\alpha \in \mathbb{R}$. This, in turn, would be a consequence of the uniqueness of solutions of (10) with small $m > 0$. Indeed, then $\mathcal{E}(m,\psi) \ge m - Cm^2$ for small $m > 0$. However, we can prove the latter only for densities $\Delta\psi$ in an appropriate Morrey space containing $L^{d/2}(\Omega)$, see Theorem 1(ii) in [10], and the remark after the proof of Lemma 5.3 in [6].

The uniqueness of solutions for small $m > 0$ for the problem in d-dimensional star-shaped domains with the Dirichlet condition imposed on ψ is a consequence of a result of X. Cabré and P. Majer cited in Theorem 1.4 in [21]. Note also that there exist irregular solutions of the Poisson–Boltzmann–Emden equation in d-dimensional domains, $d \ge 3$, which have density singularities like $c|x|^{-2}$, but here c is a sufficiently large constant, cf. a discussion in [2], [21], and the proof of Theorem 1(ii) in [10]. For instance, $2(d-2)|x|^{-2}$ is the density of such an unbounded solution in the ball in $\mathbb{R}^d$, see Example 1_d.

We believe that the examples of the structure of steady states described for balls, radially symmetric solutions in annuli and in star-shaped domains are qualitatively generic for all bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 3$. We formulate a conjecture, analogous to that in Section 5.4 of [6].

Conjecture. For every bounded domain in $\mathbb{R}^d$, $d \geq 3$, there exist $\ell_0 \in \mathbb{R}$ such that the problem (3), (7)–(8) has a solution with given $M > 0$ and $E \in \mathbb{R}$ provided $E/M^2 > \ell_0$, and it does not have solution if $E/M^2 < \ell_0$.

Evidently, if $\Omega \subset \mathbb{R}^d$ is such that (10) has solutions for all $m > 0$, i.e. $m_\Omega = \infty$, the analysis of the local behavior of the function $\mathcal{F}(m) \sim m - Cm^2$ in a vicinity of $\langle 0, 0\rangle$ and a global lower bound for $\mathcal{F}(m) \geq m - Cm^2$, will imply the conjectured result, provided the set of the bounded solutions is a single connected branch.

It would be also of interest to check for which domains $\inf_{m\in(0,m_\Omega)} \mathcal{F}(m) < 0$ because in such a case $\ell_0 < 0$ holds. Of course, $\mathcal{E}(m,\psi) = m + (1/2)\int_\Omega \psi\Delta\psi \leq m - C(\Omega)m^2$ as was proved in (20), Example 3_d, shows the above for domains with m_Ω sufficiently large or even $m_\Omega = \infty$.

4. Concluding remarks

It is well known that for the isothermal evolution problem a solution may cease to exist if the initial data u_0 is large enough, i.e. $\int_\Omega u_0\,dx > 8\pi$ in the two-dimensional case or u_0 is of *high concentration* if $d \geq 3$. The latter result implies that there exist initial data of arbitrarily small mass which lead to a finite time blow-up of solutions. Such a blow-up is accompanied by an unlimited growth of each $L^p(\Omega)$-norm of solution, $p > 1$, and even $\lim_{t\nearrow t_0}\int_\Omega u\log u\,dx = \infty$ for some $t_0 < \infty$. We refer the reader to [3] for the precise announcements and proofs.

On the other hand, it was proved in [9] that in two-dimensional domains there is no such a blow-up of solutions of the gravitational Streater's model in the sense of the above mentioned result, i.e. in particular, $\int_\Omega u\log u\,dx$ remains bounded.

Based on our results concerning the existence of steady states in Streater's models, one can conjecture that if $d \geq 3$, the domain is star-shaped and E/M^2 is initially very negative: $E/M^2 \ll -1$, then the solution cannot be global in time. Of course, such initial data do exist, for instance, the Gaussian densities u_0 (restricted to the domain Ω) satisfy this condition provided their variance is small enough, i.e. they are highly concentrated. This conjecture is supported by the observation that in higher dimensional domains there is no steady state to which $u(t)$ could accumulate as $t \to \infty$. Therefore, one expects that either the density $u(t)$ of the solution blows up in a finite time (a gravitational collapse occurs), or the temperature $\vartheta(t)$ becomes infinite (a thermal runaway takes place). Unfortunately, we cannot prove this assertion in the general case.

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