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PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDED NONLINEARITIES

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Abstract. In this article we discuss the existence and non-existence of forced T-periodic solutions to ordinary differential equations of the form $u'' + g(u) = e(t)$. The results concern equations with bounded nonlinear terms g satisfying $g(s) > 0$ (or $g(s) < 0$) for all real numbers s, and $g(\pm\infty) = 0$. Variational and topological methods are employed.

1. Introduction

In this paper we study the existence and non-existence of T-periodic solutions to T-periodic nonlinear second order ordinary differential equations. We consider equations of the form (1.1) with periodic boundary conditions (1.2):

$$u'' + g(u) = e(t), \tag{1.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{1.2}$$

Notice that if the forcing function $e(t)$ is defined on the entire real line and is T-periodic then any solution to (1.1), (1.2) can be extended to the whole real line as a T-periodic solution of (1.1). Conversely, any T-periodic solution of (1.1) satisfies (1.2). We thus may refer to solutions of (1.1), (1.2) as T-periodic solutions of (1.1).

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Throughout this paper we assume that $g(u)$ is a bounded and continuous function. The problem (1.1), (1.2) with bounded or sublinear $g(u)$ has been the a subject of continuing interest. Most of this work relates to nonlinearities $g(u)$ such that either

(i) $g(u)u \geq 0$ for $|u|$ large, and analogous conditions for second order systems, or

(ii) $g(u)$ is periodic with zero mean, as in the pendulum equation when $g(u) = k\sin(u)$, or $g(u)$ is oscillatory.

In case (i) the problem and its extensions to systems and to problems at higher eigenvalues, and to elliptic boundary value problems "at resonance", have been the subject of a great deal of research since the papers Lazer ([3]), and Lazer and Leach ([4]). Both degree theoretic methods and critical point theory have been used. A recent paper on the periodic problem for conservative second order systems of the form (1.1) with a bounded nonlinearity was [11], in which critical point theory was applied; this work was related to the earlier work of Mawhin and Willem ([6]). In case (ii), the periodically forced pendulum equation and related problems for systems have been of continuing interest, and again both degree methods and critical point theory have been applied; see Mawhin and Willem ([5], [6]), Fonda and Zanolin ([2]), and Ortega ([7]). In this paper we are interested in a class of bounded nonlinearities g which do not belong to either of classes (i) or (ii). Instead we will include functions such as $g(u) = \exp(-u^2)$ and $g(u) = 1/(1+u^2)$.

We will make use of the following conditions.

(G1) Assume $g(s) > 0$ for all s and $g(\pm\infty) = 0$; let $m \geq 0$ be a number such that $|g(s)| \leq m$ for all $s \in \mathbb{R}$.

(G2) Let $G(s) = \int_0^s g(t)dt$, and assume there is a number $M \geq 0$ such that $|G(s)| \leq M$ for all $s \in \mathbb{R}$.

Any solution of our periodic problem (1.1), (1.2) would also be a critical point of the Lagrangian functional

$$\Psi(u) := \frac{1}{T}\int_0^T \left[\frac{1}{2}(u')^2 - G(u) + e\cdot u\right] ds$$

on the space H_T^1 of absolutely continuous functions on $[0,T]$ with $u(0) = u(T)$, and $u' \in L^2(0,T)$. However under our conditions (G1), (G2) and $\overline{e} > 0$, Ψ will not be bounded from below or above, nor does it have easily observed saddle behavior. To see that is not bounded from below or above, let $u = \overline{u}$ be constant. Then

$$\Psi(\overline{u}) = -G(\overline{u}) + \overline{e}\cdot\overline{u} \to \pm\infty \quad \text{as } \overline{u} \to \pm\infty.$$

Thus usual critical point methods cannot be applied in any readily apparent way. Instead we will restrict Ψ to the subspace $\widetilde{H}_T^1$ of T-periodic functions with mean value 0, and use a Lagrange multiplier. Let $\widetilde{u} \in \widetilde{H}^1_T$, and for each fixed $c \in \mathbb{R}$ let

$$\Phi_c(\widetilde{u}) := \frac{1}{T}\int_0^T \left[\frac{1}{2}(\widetilde{u}')^2 - G(\widetilde{u}+c) + \widetilde{e}\widetilde{u}\right] ds.$$

We will show that for each c, Φ_c has a minimum at some $\widetilde{w} \in \widetilde{H}^1_T$. It follows that

$$\frac{1}{T}\int_0^T [\widetilde{w}'\widetilde{v}' - g(\widetilde{w}+c)\widetilde{v} + \widetilde{e}\widetilde{v}]\, ds = 0 \quad \text{for all } \widetilde{v} \in \widetilde{H}^1_T$$

and thus there is a real number λ such that

$$\widetilde{w}'' + g(\widetilde{w}+c) = \widetilde{e} + \lambda.$$

That is, (1.1), (1.2) has a solution $u = \widetilde{w}+c$ for $e = \widetilde{e}+\overline{e}$ with $\overline{e} = \lambda$. We then use the method of sub- and supersolutions to prove the existence of periodic solutions for $0 < \overline{e} < \lambda$. All of this will now be developed in detail in the next section. For additional information regarding contemporary variational methods and results the reader is referred to the books of Mawhin and Willem ([5]), Rabinowitz ([8]), and Struwe ([10]). Regarding the method of sub- and supersolutions (also known as lower and upper solutions), a good recent survey is the paper of De Coster and Habets ([1]) or see the paper of Schmitt ([9]).

2. Theorems and proofs

For any function $w \in L^1((0,T),\mathbb{R})$ let

$$\overline{w} = \frac{1}{T}\int_0^T w(s)\, ds \quad \text{and} \quad \widetilde{w} = w - \overline{w}.$$

Let $g \in C(\mathbb{R},\mathbb{R})$ be a continuous function on $\mathbb{R}$, and $e \in C([0,T],\mathbb{R})$. As stated in the Introduction, we use the conditions:

(G1) Assume $g(s) > 0$ for all s and $g(\pm\infty) = 0$; let $|g|_\infty = \sup_{s\in\mathbb{R}} |g(s)|$.

(G2) Let $G(s) = \int_0^s g(t)\, dt$, and assume there is a number $M \geq 0$ such that $|G(s)| \leq M$ for all $s \in \mathbb{R}$.

Theorem 1. *Let $g \in C(\mathbb{R},\mathbb{R})$ satisfy* (G1) *and* (G2)*. Then for $e = \overline{e} + \widetilde{e} \in C([0,T],\mathbb{R})$ there is a number $\lambda^* = \lambda^*(\widetilde{e})$ satisfying $0 < \lambda^*(\widetilde{e}) \leq |g|_\infty$ such that the periodic problem* (1.1), (1.2) *has a solution if and only if $0 < \overline{e} \leq \lambda^*(\widetilde{e})$.*

Remark 1. If instead we assume

(G1') $g(s) < 0$ for all s and $g(\pm\infty) = 0$,

then there is a number $\lambda^*(\widetilde{e})$ satisfying $-|g|_\infty \leq \lambda^*(\widetilde{e}) < 0$ such that the conclusion holds provided $\lambda^*(\widetilde{e}) \leq \overline{e} < 0$.

The proof of Theorem 1 will be based upon a variational argument followed by an application of sub- and supersolutions.

Let $H^1_T = \{u \in C([0,T],\mathbb{R}) \mid u$ is absolutely continuous, $u(0) = u(T)$, and $u' \in L^2(0,T)\}$.

The norm in H^1_T may be taken to be $\|u\| = \|u\|_{L^2} + \|u'\|_{L^2}$, but an equivalent norm may be defined using the mean value $\overline{u} = \frac{1}{T}\int_0^T u(s)\,ds$. This norm is

$$\|u\|_1 = |\overline{u}| + \|u'\|_{L^2}$$

and this is the norm we shall use. H^1_T is a Hilbert space with inner product

$$(u,v)_1 = \overline{u}\overline{v} + \frac{1}{T}\int_0^T u'(s)v'(s)\,ds.$$

H^1_T may be written as the direct sum $H^1_T = \overline{H}^1_T + \widetilde{H}^1_T$ in the obvious notation; we identify $\overline{H}^1_T$ with $\mathbb{R}$.

Proof of Theorem 1. Suppose u is a solution to (1.1), (1.2). Integrating (1.1) over $[0,T]$ and using the periodicity conditions (1.2) we get

$$\frac{1}{T}\int_0^T g(u(s))\,ds = \overline{e}. \tag{2.1}$$

Since $0 < g(u(s)) \le |g|_\infty$ we have that $0 < \overline{e} \le |g|_\infty$ as a necessary condition.

$\widetilde{H}^1_T$ is a Hilbert space using the norm inherited from H^1_T. For each $c \in \mathbb{R}$ we define a C^1 functional Φ_c on $\widetilde{H}^1_T$ as follows: For $\widetilde{u} \in \widetilde{H}^1_T$

$$\Phi_c(\widetilde{u}) := \frac{1}{T}\int_0^T \frac{1}{2}(\widetilde{u}'(s))^2 - G(c+\widetilde{u}(s)) + \widetilde{e}(s)\widetilde{u}(s)\,ds.$$

We note that

$$\Phi_c(\widetilde{u}) = \frac{1}{2}\|\widetilde{u}\|_1^2 - \frac{1}{T}\int_0^T G(c+\widetilde{u}(s)) + \widetilde{e}(s)\widetilde{u}(s)\,ds.$$

Thus

$$\Phi_c(\widetilde{u}) \ge \frac{1}{2}\|\widetilde{u}\|_1^2 - k_1\|\widetilde{u}\|_1 - M$$

where k_1 is a constant which depends only on $\widetilde{e}$. Thus

$$\lim_{\|\widetilde{u}\|_1 \to \infty} \Phi_c(\widetilde{u}) = \infty$$

and Φ_c is coercive on $\widetilde{H}^1_T$. It is easily checked that it is also weakly lower semicontinuous on $\widetilde{H}^1_T$. Suppose $\{v_n\} \subset \widetilde{H}^1_T$ and $v_n \rightharpoonup v$ weakly in $\widetilde{H}^1_T$. Then $v_n \to v$ strongly in $L^2(0,T)$. By dominated convergence,

$$\lim_{n\to\infty} \frac{1}{T}\int_0^T G(c+v_n(s)) + \widetilde{e}(s)v_n(s)\,ds = \frac{1}{T}\int_0^T G(c+v(s)) + \widetilde{e}(s)v(s)\,ds.$$

Thus since the norm $\|\cdot\|_1$ is weakly lower semicontinuous,

$$\liminf_{n\to\infty} \Phi_c(v_n) \ge \Phi_c(v)$$

and Φ_c is weakly lower semicontinuous. Coercivity and weak lower semicontinuity on the Hilbert space $\widetilde{H}_T^1$ implies that Φ_c attains its minimum at some point $\widetilde{w}_c \in \widetilde{H}_T^1$, and $\Phi_c'(\widetilde{w}) = 0$. That is, for all $v \in \widetilde{H}_T^1$

$$(\widetilde{w}_c, v)_1 - \frac{1}{T}\int_0^T g(c + \widetilde{w}_c(s))v(s) + \widetilde{e}(s)v(s)\,ds = 0. \tag{2.2}$$

By regularity arguments (see [6]) $\widetilde{w}_c'$ is absolutely continuous, and by (2.2), there is a constant λ_c such that

$$\widetilde{w}_c''(t) + g(c + \widetilde{w}_c(t)) = \widetilde{e}(t) + \lambda_c$$

and

$$\widetilde{w}_c(0) = \widetilde{w}_c(T), \quad \widetilde{w}_c'(0) = \widetilde{w}_c'(T).$$

That is to say, (1.1), (1.2) with $\overline{e} = \lambda_c$ has the solution $u = c + \widetilde{w}_c$. Moreover, we see from the differential equation that $\widetilde{w}_c''(t)$ is continuous on $[0, T]$.

Now obviously $0 < \lambda_c \le |g|_\infty$. Let us define $\lambda^* = \lambda^*(\widetilde{e})$ by

$$\lambda^* = \sup_{c\in\mathbb{R}} \lambda_c \le |g|_\infty\,.$$

Note that certainly $0 < \lambda^*$. Now either

$$u'' + g(u) = \widetilde{e} + \lambda \tag{2.3}$$

with $\lambda = \lambda^*$ has a solution u satisfying periodicity conditions (1.2) or else there is a sequence $\{\lambda_n\}$ with $0 < \lambda_n < \lambda^*$ and $\lambda_n \nearrow \lambda^*$ such that (2.3) has with $\lambda = \lambda_n$ a solution u_n satisfying (1.2), for each $n \in \mathbb{N}$. In the latter case we see that there is a constant k_2 such that

$$|u_n''(t)| \le k_2 \quad \text{for all } t \in [0, T] \text{ and all } n \in \mathbb{N}.$$

Thus without loss of generality, writing $u_n = \overline{u}_n + \widetilde{u}_n$, we may conclude that $\{\widetilde{u}_n\}$ converges in $\widetilde{H}_T^1$ to some $\widetilde{U} \in \widetilde{H}_T^1$. We claim that the sequence of real numbers $\{\overline{u}_n\}$ must remain bounded; if it is not bounded; then without loss of generality we can assume $\overline{u}_n \to \infty$ as $n \to \infty$. Then $u_n(t) = \overline{u}_n + \widetilde{u}_n(t) \to \infty$ uniformly on $[0, T]$ and thus

$$g(\overline{u}_n + \widetilde{u}_n(t)) \to 0 \quad \text{as } n \to \infty.$$

We may conclude that the T-periodic function $\widetilde{U}$ satisfies

$$\widetilde{U}'' = \widetilde{e} + \lambda^*$$

which contradicts $\lambda^* > 0$. Thus the sequence $\{\overline{u}_n\}$ must remain bounded, and without loss of generality we may assume $\overline{u}_n \to c^*$. Now $u = \widetilde{U} + c^*$ is a solution of (2.3) with $\lambda = \lambda^*$ and satisfies the periodicity conditions.

Thus in any case there is a solution of (2.3), (1.2) with $\lambda = \lambda^*$. We will now show that there are also solutions of (2.3), (1.2) for all $0 < \lambda \le \lambda^*$.

Let $0 < \lambda < \lambda^*$. Let U be a solution of (2.3), (1.2) with $\lambda = \lambda^*$. Then $U \in C^2([0,T],\mathbb{R})$, U satisfies (1.2), and

$$U''(t) + g(U(t)) = \widetilde{e}(t) + \lambda^* > \widetilde{e}(t) + \lambda.$$

Thus U is a T-periodic subsolution of

$$u'' + g(u) = \widetilde{e}(t) + \lambda.$$

Let $c \in \mathbb{R}$ and let $W_c := U(t) + c$. Now $g(s) > 0$ for all s, and $W_0(t) = U(t)$ has compact range. Thus

$$\inf_{0 \le t \le T} g(U(t)) = m^* > 0.$$

However,

$$\lim_{c \to \infty} g(U(t) + c) = 0,$$

uniformly in $t \in [0,T]$. Thus there is a number $c_0 > 0$ so large that $W_{c_0}(t) = U(t) + c_0 > U(t)$ and

$$g(U(t) + c_0) < m^* \le g(U(t)) \quad \text{for } 0 \le t \le T.$$

Since $U(t) + c_0$ is continuous on $[0,T]$ we may actually conclude that there is a number $\varepsilon > 0$ such that

$$g(U(t) + c_0) < m^* - \varepsilon \quad \text{for } 0 \le t \le T.$$

Thus

$$W''_{c_0}(t) + g(W_{c_0}(t)) = U''(t) + g(U(t) + c_0)$$

and

$$\begin{aligned} U''(t) + g(U(t) + c_0) &< U''(t) + m^* - \varepsilon \\ &< U''(t) + m^* \le U''(t) + g(U(t)) = \widetilde{e}(t) + \lambda^*. \end{aligned}$$

Thus there is a number $\lambda_1 < \lambda^*$ such that for all $\lambda_1 \le \lambda < \lambda^*$ the function $W_{c_0}(t)$ satisfies

$$W''_{c_0}(t) + g(W_{c_0}(t)) < \widetilde{e}(t) + \lambda.$$

We have shown that there is a $\lambda_1 < \lambda^*$ such that for all $\lambda_1 < \lambda < \lambda^*$, the function $U(t)$ is a strict T-periodic subsolution, and $W_{c_0}(t)$ a strict T-periodic supersolution, of

$$u'' + g(u) = \widetilde{e}(t) + \lambda. \tag{2.4}$$

Moreover,

$$U(t) < W_{c_0}(t) = U(t) + c_0 \quad \text{for } 0 \le t \le T.$$

It now follows from results on sub- and supersolutions (see e.g. [1] or [9]) that (2.4) has for each $\lambda_1 < \lambda < \lambda^*$ a T-periodic solution $w_\lambda(t)$ satisfying

$$U(t) < w_\lambda(t) < U(t) + c_0.$$

Now let $\lambda_* = \inf\{\alpha : (2.4)$ has a T-periodic solution for all $\alpha < \lambda < \lambda^*\}$. From what we have shown, we know that

$$0 \le \lambda_* \le \lambda_1 < \lambda^* \le |g|_\infty .$$

It remains to show that $\lambda_* = 0$. If not, then $0 < \lambda_* \le \lambda_1$. We claim that (2.4) has a T-periodic solution for $\lambda = \lambda_*$. We know there is a T-periodic solution for all $\lambda_* < \lambda < \lambda^*$. Let $\{\lambda_n\}$ be a sequence of numbers with $\lambda_* < \lambda_n < \lambda^*$ and $\lambda_n \searrow \lambda_*$ as $n \to \infty$. Then for each $n \in \mathbb{N}$ there is a T-periodic solution $u_n = \overline{u}_n + \widetilde{u}_n$ of (2.4) with $\lambda = \lambda_n$. As in our previous argument showing that there is a T-periodic solution when $\lambda = \lambda^*$, we obtain a convergent (in $\widetilde{H}^1_T$) subsequence of the $\{\widetilde{u}_n\}$, which we may as well assume is our original sequence. Let $\widetilde{W} = \lim_{n\to\infty} \widetilde{u}_n(t)$. Also as before, if we suppose that $\{\overline{u}_n\}$ is an unbounded sequence, we may as well assume $\overline{u}_n \to \infty$. It follows that $u_n(t) = \overline{u}_n + \widetilde{u}_n(t) \to \infty$ uniformly on $[0,T]$, and thus $g(u_n(t)) \to 0$ as $n \to \infty$. It follows that $\widetilde{W}(t)$ is T-periodic and

$$\widetilde{W}''(t) = \widetilde{e}(t) + \lambda_*.$$

Thus if $0 < \lambda_*$ we again reach a contradiction. Thus $\lambda_* = 0$. We have thus shown that (2.4) has a T-periodic solution if and only if $0 < \lambda \le \lambda^* = \lambda^*(\widetilde{e})$. Obviously there is no T-periodic solution if $\lambda = 0$. This proves the theorem in the case that $g(s) > 0$ for all $s \in \mathbb{R}$. In the case that $g(s) < 0$ for all s, the proof is too much the same to write here. This completes our proof of the theorem.□

Remark 2. In most cases when $g(s) > 0$ we will have that $\lambda^* < |g|_\infty$.

Theorem 2. *Let $g \in C(\mathbb{R},\mathbb{R})$ and suppose* (G1) *and* (G2) *hold. Suppose additionally that there is no interval on which $g(s)$ is constant. Then the conclusions of Theorem* 1 *hold, and whenever $\widetilde{e}$ is not the zero function, we have $\lambda^* = \lambda^*(\widetilde{e}) < |g|_\infty$.*

Proof. Suppose $\widetilde{e} \ne 0$ and $\lambda^* = \lambda^*(\widetilde{e}) = |g|_\infty$. Then there is a T-periodic solution to the differential equation

$$u'' + g(u) = \widetilde{e} + |g|_\infty . \tag{2.5}$$

Integrating each side of the latter equation over $[0,T]$ and using the periodicity conditions shows that

$$\frac{1}{T}\int_0^T g(u(s))\,ds = |g|_\infty .$$

Hence

$$\frac{1}{T}\int_0^T [|g|_\infty - g(u(s))]\,ds = 0.$$

But $h(s) = |g|_\infty - g(u(s)) \ge 0$ and $h(s)$ is continuous for $0 \le s \le T$. Thus

$$g(u(s)) = |g|_\infty \quad \text{for } 0 \le s \le T.$$

But as $g(s)$ is not constant on any interval this implies that $u(s)$ =constant= $\overline{u}$ for $s \in [0,T]$. Thus $u'' = 0$ and since $u = \overline{u}$ is a solution of (2.5),

$$g(\overline{u}) = \widetilde{e}(t) + |g|_\infty .$$

But $g(\overline{u}) = |g|_\infty$, so we have $\widetilde{e}(t) = 0$ for all $0 \le t \le T$, contrary to hypothesis. Thus $\lambda^* < |g|_\infty$. $\square$

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