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SOME RESULTS FOR JUMPING NONLINEARITIES

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Abstract. We discuss the calculation of critical groups for jumping nonlinearities as the resonance set is crossed. In addition, we produce a counterexample showing that even "generically" the resonance set is more complicated than previously thought.

In this paper, we establish two main results. Firstly we obtain several new formulae for critical groups of jumping nonlinearities (in the sense of [3]). In particular, we improve considerably some results of ours in [4] and in particular answer a question in [21] (and generalize the main result of [21]). In a much more general case, we construct an exact sequence in cohomology for the change in critical groups as we cross the resonance set. Secondly, we construct a counterexample showing that even geometrically the resonance set for jumping nonlinearities is more complicated than previously thought (despite claims to the contrary in the literature). Here the resonance set A_0 is the set of $(\mu,\nu)\in\mathbb{R}^2$ for which

$$\begin{aligned} -\Delta u &= \mu u^+ + \nu u^- && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

has a non-trivial solution. This set has been extensively studied (cp. [4], [5], [14], [17], [26]) but is still poorly understood.

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Note that critical group computations are of interest for a number of reasons. They are of use when one considers whether the equation

$$(2) \qquad -\Delta u = g(u) - f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

has a non-trivial solution for every $f \in L^2(\Omega)$. Here $g: R \to R$ is continuous such that $y^{-1}g(y) \to \mu(\nu)$ as $y \to \infty(-\infty)$. Indeed, if allow more general sublinear perturbations of $\mu u^+ + \nu u^-$, a result in Section 3 of [8] shows that the critical groups of the mapping $u \to -\Delta u - \mu u^+ - \nu u^-$ in a certain sense determine the answer to this problem. Secondly, the critical groups can be used to give results on the multiplicity of solutions of (2) for certain f (cp. [10]). Thirdly, the critical groups are of importance in deciding if there are non-trivial solutions where $g(0) = 0$ and $f \equiv 0$. Here we also need to make assumptions on the behaviour of g near zero. Finally, as we will show in [11], these critical group computations are important in Conley index calculations for competing species systems with diffusion and large interactions. Note that in all these computations, we usually need to assume $(\mu, \nu) \notin A_0$ so that the structure of A_0 is of considerable interest.

1. The main index calculation

Let $\lambda_1 < \lambda_2 < \ldots$ denote the distinct eigenvalues of $-\Delta$ with Dirichlet boundary conditions on Ω, let

$$A_0 = \{(\mu, \nu) \in \mathbb{R}^2 : -\Delta u = u\mu^+ + \nu u^- \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \text{ has a non-trivial solution}\}$$

and let

$$f_{\mu,\nu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \mu(u^+)^2 - \nu(u^-)^2.$$

It is well known that A_0 is closed. (Basic results on A_0 can be found in [4]).

Then, if $(\mu, \nu) \notin A_0$, the critical groups of zero, $c_j(f_{\mu,\nu}, 0)$, are defined to be the relative cohomology of $T \equiv \{u \in B_1 : f_{\mu,\nu}(u) \le 0\}$ relative to $T \setminus \{0\}$ (for a suitable coefficient group). Here B_1 is the unit ball in $\dot{W}^{1,2}(\Omega)$ (see [3]). Note that we are using here that, if $(\mu, \nu) \notin A_0$, zero is the only critical point of $f_{\mu,\nu}$ and $f_{\mu,\nu}(0) = 0$. This is also the cohomology of the Conley index of the isolated critical point zero of $-\nabla f_{\mu,\nu}$ for the natural parabolic (or gradient flow cp. Rybakowski [24]). Then, by homotopy invariance, these critical groups are independent of (μ, ν) for (μ, ν) in a component of $\mathbb{R}^2 \setminus A_0$. Thus, if (μ, ν) lies in a component of $\mathbb{R}^2 \setminus A_0$ intersecting the diagonal $\mu = \nu$, the critical groups are easy to calculate since the critical groups are then those of a linear map. Note that the set A_0 is still poorly understood which complicates the study of the critical groups.

It is also easy to show that $f_{\mu,\nu}$ and $f_{\nu,\mu}$ have the same critical groups. Thus we may assume $\nu \ge \mu$.

Suppose now that λ_i is one of the distinct eigenvalues $-\Delta$ (with boundary condition) where $i > 1$. It has been proved by several people (see [16] or [26]) that in the "box" $\widetilde{B} = \{(\mu,\nu) : \lambda_{i-1} < \mu,\ \nu < \lambda_{i+1},\ \nu \geq \mu\}$, there are two decreasing continuous curves C_1, C_2 in $\widetilde{B}$ both parametrized by $\mu, \nu = h_j(\mu)$, $j = 1, 2$, both passing through (λ_i, λ_i) and both having in their closure points with $\mu = \lambda_{i-1}$ or $\nu = \lambda_{i+1}$, such that $C_1 \cup C_2 \subseteq A_0$, C_2 is above C_1 (not necessarily strictly) and such that $(\mu,\nu) \notin A_0$ if $(\mu,\nu) \in \widetilde{B}$ and $\nu < h_1(\mu)$ or $\nu > h_2(\mu)$. If $\nu < h_1(\mu)$ or $\nu > h_2(\mu)$ it is easy to see that (μ,ν) lies in a component of $\mathbb{R}^2 \setminus A_0$ intersecting the diagonal and hence the critical groups are easy to calculate. Note also that the structure of A_0 in the domain between C_1 and C_2 may be quite complicated (see [1] or [4] or [17]). Our main result is the following. Here k_i is the sum of the multiplicities of the eigenvalues less than λ_i.

THEOREM 1. *Assume that $\lambda_{i-1} < \mu < \lambda_i$ such that $h_1(\mu)$ is defined and $\varepsilon > 0$ such that $(\mu,\nu) \notin A_0$ if $h_1(\mu) < \nu < h_1(\mu) + \varepsilon$. Then, for such ν, $c_{j+k_i}(f_{\mu,\nu}, 0) = \widetilde{H}^{j-1}(\widetilde{S})$ for $j \geq 1$, where $\widetilde{S} = \{u \in \dot{W}^{1,2}(u) : \|u\|_2 = 1,\ -\Delta u = \mu u^+ + h_1(\mu)u^-\}$ and $c_j(f_{\mu,\nu}, 0) = 0$ if $j \leq k_i$.*

REMARKS. Note that we need to use Alexander-Spanier cohomology (see [19]) in the calculation of the cohomology of S because we are unsure S is a "nice" space. We proved an equivalent theorem in [4] under a non-degeneracy hypothesis on the second derivatives of $f_{\mu,\nu}$ on S. (This extra hypothesis has the advantage that it ensures that $(\mu,\nu) \notin A_0$ if $h_1(\mu) < \nu < h_i(\mu) + \varepsilon$.) In [4], we also found that $c_{k_i}(f_{\mu,\nu}, 0) = 0$ for $\nu > h_1(\mu)$ and $(\mu,\nu) \in A_0$ even if $\nu \geq \lambda_{i+1}$. Note that $c_{k_i}(f_{\mu,\nu}, 0) = G$ (where G is the coefficient group) if $\mu \leq \nu < h_1(\mu)$. We could also use duality theorems to obtain results for homology. We could also use a similar proof (plus the Alexander duality theorem in cohomology) to obtain an analogous formula for the critical groups for $\nu = h_2(\mu) - t$ where t is small and positive. (The result is for homology in this case.) This was already proved in [21]. As they note, it seems difficult to extend their method to our case. As in [21], we could easily extend our method to cover the cases of decreasing reasonably smooth curves $\mu = k_1(s), \nu = k_2(s)$ approaching $(\mu, h_1(\mu))$ in $\mathbb{R}^2 \setminus A_0$ with little change in the proof . It seems this is not a great improvement in applications. We could use prove similar results for other boundary conditions.

Let us now give the proof (aside from two technical lemmas). Let N_i denote the eigenspace corresponding to the eigenvalue λ_i and let Z_i be its orthogonal complement in $\dot{W}^{1,2}(\Omega)$. We write $u = n + z$ where $n \in N_i$, $z \in Z_i$ and let P be the corresponding orthogonal projection onto N_i. Then, as in [4], one can easily see from the contraction mapping theorem that if $\lambda_{i-1} < \mu, \nu < \lambda_i$, the equation

$$(I - P)(-\Delta(w + n) - \mu(w + n)^+ - \nu(w + n)^-) = 0$$

has a unique solution $w = S_{\mu,\nu}(n)$ in Z_i where $S_{\mu,\nu}$ is positive homogeneous in n and continuous in (μ,ν,n). Let

$$F_{\mu,\nu}(n) \equiv P(-\Delta(S_{\mu,\nu}(n)+n) - \mu(S_{\mu,\nu}(n)+n)^+ - \nu(S_{\mu,\nu}(n)+n)^-) \tag{3}$$

and $\widetilde{f}_{\mu,\nu}(n) = f_{\mu,\nu}(n + S_{\mu,\nu}(n))$. As in [4], we see that $F_{\mu,\nu}$ is the gradient of $\widetilde{f}_{\mu,\nu}$ and that

$$c_j(f_{\mu,\nu},0) = c_{j-k_i}(\widetilde{f}_{\mu,\nu},0) = \widetilde{H}^{j-k_i}(\{n \in N_i : \widetilde{f}_{\mu,\nu}(n) < 0,\ \|n\| = 1\})$$

(and is zero if $j \le k_i$). The last inequality assumes 0 is not a local minimum of $\widetilde{f}_{\mu,\nu}$. On the other hand, it is easy to see that $\widetilde{S}$ is homeomorphic to $S_i = \{n \in N_i : \|n\| = 1,\ F_{\mu,h_1(\mu)}(n) = 0\}$ and hence we see that it suffices to prove our result for $F_{\mu,\nu}$. Let $M_{\mu,\nu} = \{n \in N_i : \|n\| = 1,\ \widetilde{f}_{\mu,\nu}(n) \le 0\}$.

We will prove two technical lemmas

LEMMA 1. *$\widetilde{f}_{\mu,\nu}(n)$ is strictly decreasing in ν if $n \neq 0$.*

LEMMA 2. *If $h_i(\mu) < \nu_1 < \nu_2 < h_1(\mu) + \varepsilon$, the natural inclusion of M_{μ,ν_1} into M_{μ,ν_2} is a homotopy equivalence and M_{μ,ν_1} is homotopy equivalent to its interior.*

REMARK. Note that $M_{\mu,\nu_1} \subseteq M_{\mu,\nu_2}$.

Assuming these lemmas, we complete the proof of Theorem 1. Note that it is proved in [4] that if $\nu < h_1(\mu), 0$ is the unique global minimum of $\widetilde{f}_{\mu,\nu}$ and hence by continuity $\widetilde{f}_{\mu,h_1(\mu)}(n) \ge 0$ on N_i. Since $\widetilde{f}_{\mu,\nu}(n) = 0$ if n is a critical point of $\widetilde{f}_{\mu,\nu}$ (by the homogeneity), we see that $\widetilde{f}_{\mu,h_1(\mu)} \ge 0$ on the unit sphere N_i^1 of N_i and the zeros are the solutions of $F_{\mu,h_1(\mu)}(n) = 0$ on N_i^1 (that is S_i). Suppose ν_i are decreasing in $(h_1(\mu), h_1(\mu)+\varepsilon)$ and converge to $h_1(\mu)$. It is easy to see that $\bigcap M_{\mu,\nu_i} = M_{\mu,h_1(\mu)}$. (Here we are using Lemma 1.) Since the M_{μ,ν_i} are compact and since the natural inclusion induce isomorphisms of the cohomology of the M_{μ,ν_i}, we see from theorems on inverse limits in Alexander Spanier cohomology (cp. Massey [19, p. 238]) that M_{μ,ν_j} (and hence its interior which is $\{n \in N_i^1 : \widetilde{f}_{\mu,\nu_i}(n) < 0\}$) has the same cohomology as $\widetilde{S}_i$ and our result follows. (A similar argument appears in [12].) Hence it suffices to prove the two lemmas.

PROOF OF LEMMA 1. The main difficulty is to prove that if $\|n\| = 1$

$$\frac{\partial}{\partial \nu}\widetilde{f}_{\mu,\nu}(n) = -\frac{1}{2}\|(n + S_{\mu,\nu}(n))^-\|_2^2. \tag{4}$$

If this is true, we see that $\partial \widetilde{f}_{\mu,\nu}(n)/\partial\nu \le 0$ and either our claim follows or $n + S_{\mu,\nu}(n) \ge 0$ almost everywhere on Ω. If $n + S_{\mu,\nu}(n) \ge 0$ on Ω, our equation for $S_{\mu,\nu}$ becomes

$$(I-P)(-\Delta(n + S_{\mu,\nu}(n)) - \mu(n + S_{\mu,\nu}(n)) = 0.$$

Since $I-P$ commutes with $-\Delta$ and the identity map, this equations reduces to finding a solution $w = S_{\mu,\nu}(n) \in Z_i$ of $-\Delta w - \mu w = 0$ (with the boundary condition). Clearly $w = 0$ is a solution. Since the solution is unique, $S_{\mu,\nu}(n) = 0$. Then $n + S_{\mu,\nu}(n) = n$ which is non-zero. This contradicts our claim that $n = n + S_{\mu,\nu}(n)$ is non-negative on Ω since n does not vanish identically and n is orthogonal to ϕ_1.

It remains to prove (4). As usual, with jumping nonlinearities we have difficulty with smoothness, in particular, we are unsure that $S_{\mu,\nu}$ is differentiable in ν. Since all the terms in (3) are Lipschitz in μ, ν, n, it is easy to see that $S_{\mu,\nu}(n)$ is Lipschitz in ν uniformly for n with $\|n\| = 1$. Then

$$\begin{aligned} A &\equiv \widetilde{f}_{\mu,\nu_1}(n) - \widetilde{f}_{\mu,\nu}(n) = f_{\mu,\nu_1}(n + S_{\mu,\nu_1}(n)) - f_{\mu,\nu}(n + S_{\mu,\nu}(n)) \\ &= -\frac{1}{2}(\nu_1 - \nu)\|(n + S_{\mu,\nu_1}(n))^-\|_2^2 + f_{\mu,\nu}(n + S_{\mu,\nu_1}(n)) - f_{\mu,\nu}(n + S_{\mu,\nu}(n)) \end{aligned}$$

(by the formula for f). Now f is C^1 in u as is easily seen. Hence we see that

$$\begin{aligned} A = &-\frac{1}{2}(\nu_1 - \nu)\|(n + S_{\mu,\nu_1}(n))^-\|_2^2 \\ &+ f'_{\mu,\nu}(n + S_{\mu,\nu}(n))(S_{\mu,\nu_1}(n) - S_{\mu,\nu}(n)) + o(\|S_{\mu,\nu_1}(n) - S_{\mu,\nu}(n)\|_2)). \end{aligned}$$

Now, since $S_{\mu,\nu}(n)$ is Lipschitz in ν, the last term is $o|\nu_1 - \nu|$. Moreover, the second term is zero. This follows because $S_{\mu,\nu_1}(n) - S_{\mu,\nu}(n) \in Z_i$ and the equation solved by $S_{\mu,\nu}(n)$ can be rewritten as $f'_{\mu,\nu}(n + S_{\mu,\nu}(n))w = 0$ if $w \in Z_i$.

Hence we see that

$$-(\nu - \nu_1)^{-1}(\widetilde{f}_{\mu,\nu}(n) - \widetilde{f}_{\mu,\nu_1}(n)) = \frac{1}{2}\|(n + S_{\mu,\nu_1}(n))^-\|_2^2 + o(1)$$

and the differentiability in ν and the formula for the derivative follow easily. This completes the proof of Lemma 1. □

PROOF OF LEMMA 2. Suppose $h_1(\mu) < \nu < \nu_1 < h_1(\mu) + \varepsilon$. Since $\nabla\widetilde{f}_{\mu,\nu} = F_{\mu,\nu}$, we see that $\nabla\widetilde{f}_{\mu,\nu}(n)$ is non trivial if $\|n\| = 1$ and $(\mu,\nu) \in \mathbb{R}^2 \setminus A_0$. In particular, if in addition, we assume that $\widetilde{f}_{\mu,\nu}(n) = 0$, then some tangential component of $\nabla\widetilde{f}_{\mu,\nu}(n)$ is non-zero (since the equation $\widetilde{f}_{\mu,\nu}(tn) = t^2\widetilde{f}_{\mu,\nu}(n)$ for $t \geq 0$ ensures that the radial component is zero). Now it is easy to see that $\widetilde{f}_{\mu,\nu}$ is C^1 in n and hence by the implicit function theorem, $M_{\mu,\nu} = \{n \in N_i^1 : \widetilde{f}_{\mu,\nu}(n) \leq 0\}$ is a C^1-manifold with boundary (possibly not a connected manifold) with interior $\{n \in N_i^1 : \widetilde{f}_{\mu,\nu}(n) < 0\}$.

Now, by the proof of Lemma 1, $\partial\widetilde{f}_{\mu,\nu}(n)/\partial\nu$ exists and is continuous in μ, ν, n. Now as in the construction of tubular neighbourhoods (see [2]), we can find a C^1 vector field z defined on the boundary of $M_{\mu,\nu}$ which is close to the normal to $\partial M_{\mu,\nu}$ in N_i^1 (and hence $\partial\widetilde{f}_{\mu,\nu}(n)/\partial z \neq 0$ on the boundary $\partial M_{\mu,\nu}$). Since points near $\partial M_{\mu,\nu}$ can be then uniquely written in the form $n + tz(n)$ where $n \in \partial M_{\mu,\nu}$ and t is small (as in [2]), we see by deforming along these lines that int $M_{\mu,\nu}$ has the homotopy type of $M_{\mu,\nu}$. Moreover, if ν_1 is near ν, we see by applying the implicit function theorem to $\widetilde{f}_{\mu,\nu_1}(n) = 0$ that $\partial M_{\mu,\nu_1}$ can be written in the form $\{n + t(n,\nu_1)z(n) : n \in \partial M_{\mu,\nu}\}$ where t is continuous and $t(n,\nu) = 0$. Hence we can easily deform along $n + tz(n)$ to see that if ν_1, ν_2 are close to ν and $\nu_1 < \nu_2$ the natural inclusion of M_{μ,ν_1} into M_{μ,ν_2} is a homotopy equivalence as required. This completes the proof of Lemma 2. $\square$

REMARK. One can use compactness to show that $\|(n + S_{\mu,h_1(\mu)}(n))^-\|_2$ and $\|(n+S_{\mu,h_1\mu}(n))^+\|_2$ have positive lower bounds on N_i^1, therefore one can prove a similar result even for directions close to the positive quadrant. This idea could also be used to improve slightly the main result in [21]. Note that the proof could be simplified a little for directions in the interior of the positive quadrant.

2. A partial generalization

In this short section, we generalize the result of Section 1 to cases where we cross other curves in A_0. The idea here is to obtain an exact sequence relating the cohomology index for $(\mu,\nu+\varepsilon)$ and $(\mu,\nu-\varepsilon)$. Note that one can use the proof of Theorem 1 to relate the cohomology index for $(\mu,\nu+\varepsilon)$ to the homology of $\{n \in N_i^1 : \widetilde{f}_{\mu,\nu}(n) \le 0\}$ but in the general case this does not seem a very convenient object to compute with.

More formally, we assume that $(\mu,\nu) \in A_0$ and there is an $\varepsilon > 0$ such that $(\mu,\nu+t) \neq A_0$ if $0 < |t| < \varepsilon$ (though we could generalize this to directions in the first quadrant or even slightly outside the first quadrant). Choose λ_i, λ_j so $\lambda_i < \mu,\nu < \lambda_j$. Then the equation (1) can be reduced by a Liapounov–Schmidt reduction to a finite dimensional problem $\widetilde{F}_{\mu,\nu}(w) = 0$ where $w \in Y$. Here Y is the subspace spanned by the eigenvectors corresponding to eigenvalues in (λ_i,λ_j). As before, $\widetilde{F}_{\mu,\nu}$ is the gradient of $\widehat{f}_{\mu,\nu}$. We assume that $\widehat{f}_{\mu,\nu}|_{Y^1}$ has zero as an isolated critical value where Y^1 is the unit sphere in Y. We suspect that this assumption is independent of the reduction. Note that it always holds under a regularity assumption like that in Section 1 of [4].

THEOREM 2. *Under the above assumptions, there exist maps c_n, γ_n, k_n so the following sequence is exact*

$$\xrightarrow{i_q} \widetilde{H}^q(\Sigma h(-\nabla\widehat{f}_{\mu,\nu}|_{Y^1}, S)) \xrightarrow{j_q} c_q(\widehat{f}_{\mu,\nu+\varepsilon}, 0)$$
$$\xrightarrow{k_q} c_q(\widehat{f}_{\mu,\nu-\varepsilon}, 0) \xrightarrow{i_{q+1}} \widetilde{H}^{q+1}(\Sigma h(-\nabla\widehat{f}_{\mu,\nu}|_{Y^1}, S)) \xrightarrow{j_{q+1}}$$

Here Σ denotes a suspension, the gradient is the gradient on Y^1 and S is a neighbourhood of the critical points of $\widehat{f}_{\mu,\nu}|_{Y^1}$ corresponding to the critical value zero (*and this homotopy index is for the map on Y^1*).

REMARK. We will discuss after the proof further properties of the maps k_q etc. For example, the k_q are effectively induced by an inclusion. Theorem 2 makes it rather difficult for the critical groups of $\widehat{f}_{\mu,\nu}$ (and thus of $f_{\mu,\nu}$) not to be different for ν replaced by $\nu \pm \varepsilon$ unless $h(-\nabla \widehat{f}_{\mu,\nu}|_{Y_1}, S)$ has trivial cohomology. Recall that the critical groups of $\widehat{f}_{\mu,\nu}$ and $f_{\mu,\nu}$ are the same up to a shift in indices.

Proof. Note that Lemmas 1 and 2 of Section 1 continue to hold for this case (with the same proof). By the same argument as there, we see up to isomorphism, $h(-\widetilde{F}_{\mu,\nu+\varepsilon}, 0)$ has the same cohomology except (for a shift of 1) as that of $\{w \in Y^1 : \widehat{f}_{\mu,\nu}(w) \leq 0\}$. Here we need to see Alexander–Spanier cohomology.

We use a similar argument but with Steenrod homology to prove that (up the same shift of 1) $h(-\nabla \widehat{f}_{\mu,\nu-\varepsilon}, 0)$ has the same Steenrod homology as $\widetilde{Q} = \{w \in Y^1 : \widehat{f}_{\mu,\nu}(w) < 0\}$. We need to explain this a little more. Much as before, if ε_i decrease to zero $Q_i = \{w \in Y^1 : \widehat{f}_{\mu,\nu-\varepsilon_i}(w) \leq 0\}$ is compact, increasing in i and their union is $\widetilde{Q}_i$. Hence the homology of $\widetilde{Q}$ is the direct limit of the homology of $\widetilde{Q}_i$ (for Steenrod homology). Here we use Lemma 9.1 in [19]. We then argue as before to deduce that $\widetilde{Q}_i$ and $\widetilde{Q}$ have the same Steenrod homology for large i. Moreover, we can deform $\widetilde{Q}$ along gradient to show it has the homotopy type of a "nice" compact set. Hence, since the homology is finitely generated, we can use duality between homology and cohomology for compact spaces (cp. Massey [19, Corollary 4.18]) to see that $\widetilde{Q}_i$ and $\widetilde{Q}$ have the same Alexander–Steenrod cohomology for large i for Z coefficients (and hence for all coefficients).

On the other hand, if zero as an isolated critical point of $\widehat{f}_{\mu,\nu}|_{Y^1}$, we can argue as in [7] and [8] to deduce that the homotopy index of $-\nabla \widehat{f}_{\mu,\nu}$ on S is usually obtained as follows. We choose $\varepsilon > 0$ such that zero is the only critical value of $\widehat{f}|_{Y^1}$ in $[-\varepsilon, \varepsilon]$. We can then argue as in the proof of Proposition 3.5 in [12] to deduce that the cohomology of the homotopy index of $-\nabla \widehat{f}$ on S is the relative Alexander–Spanier cohomology of $\{\widehat{f} \leq 0\} \cap \mathcal{U}, \{\widehat{f} \leq 0, x$ is not a critical point $\widehat{f}|_{Y^1}$ with $\widehat{f}(x) = 0\} \cap \mathcal{U}$ where $\mathcal{U}$ is a suitable neighbourhood of the critical set corresponding to the critical point zero in Y^1. By excision we can take $\mathcal{U} = Y^1$. Note that $\{x \in Y^1 : \widehat{f} \leq 0$, x is not a critical point with $f(x) = 0\}$ has the same cohomology as $\{x \in Y^1 : \widehat{f} < 0\}$ since both can be deformed by the flow onto $\widehat{f} \leq -\varepsilon/2$. Hence we see that the cohomology of the homotopy index of $-\nabla \widehat{f}$ on S is simply the relative cohomology of $\{x \in Y^1 : \widehat{f} \leq 0\}$ relative to $\{x \in Y^1 : \widehat{f} < 0\}$. Thus our theorem is really the cohomology sequence for the pair $(\widehat{f} \leq 0,\ \widehat{f} < 0)$. (Note that we suspend the homotopy index on Y^1 to get the dimensions to be correct). The proof need slight modifications for the case where $\widehat{f} < 0$ is empty (that is 0 is a local minimum of $\widehat{f}_{\mu,\nu}$) but it is easy to see the result still holds. $\square$

Remarks. There are many special cases which are much simpler. If the critical groups are trivial for $\nu - \varepsilon$, the exact sequence implies the critical groups for $\nu+\varepsilon$ of $\widetilde{f}_{\mu,\nu+\varepsilon}$ are simply those of a suspension of the map $-\nabla\widehat{f}$ on S. A similar result holds if the critical groups are trivial for $\nu+\varepsilon$. With a little care it can be shown that Theorem 1 (and the Perera–Schechter result) are a special case of this theorem (but under a slightly additional assumption on $\widehat{f}|_{Y^1}$.)

If the homotopy index $-\nabla\widehat{f}|_{Y^1}$ on S is trivial our result implies that the critical groups are the same for $\nu+\varepsilon$ and $\nu-\mu$. Conversely, it in general seems rather difficult (but maybe not impossible) for the critical groups to be the same for $\nu\pm\varepsilon$ if the homotopy index of $-\nabla\widehat{f}|_{Y^1}$ on S is non-trivial. We need to be in the situation where $\{x\in Y^1:\widehat{f}(x)\le 0\}$ and $\{x\in Y^1:\widehat{f}(x)<0\}$ have the same cohomology but the natural inclusion of these spaces does not induce in isomorphism of cohomology. (For example this can only occur if the reduced cohomology of the spaces is not-trivial at two adjacent levels j, $j+1$.) Note here that the mapping k_q is up to isomorphism the map of $\widetilde{H}^{q-1}\{\widetilde{f}\le 0\}$ of $\widetilde{H}^{q-1}\{\widehat{f}<0\}$ induced by inclusion. We now consider the calculation of $\widetilde{H}(h(-\nabla\widehat{f},S))$. If $\widetilde{S}_i$ are the components of S (assumed to be only finitely many), it is easy to see from Mayer-Vietoris theorems as in [19]) that $\widetilde{H}(h(-\nabla\widehat{f},S))=\bigoplus_i\widetilde{H}(h(-\nabla\widehat{f},\widetilde{S}_i))$. Thus the calculation reduces to components. If $\widetilde{S}_i$ is a non-degenerate C^1-manifold of dimension j_i and $\widetilde{k}_i$ unstable directions in Y^1 and either $\widetilde{S}_i$ is a point (or is simply connected) or the coefficients are Z_2 or $\widetilde{k}_i=0$ or $n-j_i-1$, then the Thom isomorphism theorem ensures (cp. Spanier [27, p. 259]), that $\widetilde{H}^j(h(-\nabla\widehat{f},\widetilde{S}_i))=H^{j-\widetilde{k}_i}(\widetilde{S}_i)$. (This is much easier to prove if $\widetilde{S}_i$ is a point and, in general we need Z_2 coefficient because of orientability problems. If $\widetilde{k}_i=n-j_i-1$, we also need to replace cohomology by homology in the right hand side of our formula.) Thus we can calculate the cohomology of the homotopy index in a number of cases. Moreover, if $\widetilde{S}_i$ consists of isolated non-degenerate points (or more generally $\widetilde{S}_i$ are non-degenerate manifold caused by a symmetry group which acts transitively on $\widetilde{S}_i$) one can argue as in the proof of Theorem 2 in [4] to prove that $(\mu,\nu+t)\notin A_0$ if $0<|t|<\varepsilon$. Hence in this case, one of the assumptions in Theorem 2 automatically holds. We need to be careful in the interpretation of the theorem if Y is one dimensional.

COROLLARY 1. *Under the assumptions of Theorem* 2*,*

$$\operatorname{degree}(-\widetilde{F}_{\mu,\nu+\varepsilon},0)-\operatorname{degree}(-\widetilde{F}_{\mu,\nu-\varepsilon},0)=\operatorname{degree}(-\nabla\widehat{f}|_{Y^1},S).$$

PROOF. This follows from the proof of Theorem 2 since (by [24, p. 205]) $\chi(\widehat{f}\le 0)-\chi(\widehat{f}<0)=\chi(\widehat{f}\le 0/\widehat{f}<0)$ and since deg $(-\nabla\widetilde{f}_{\mu,\nu\pm\varepsilon},0)=\chi(\widehat{f}_{\mu,\nu\pm\varepsilon}|_{Y^1}\le 0)-1$ (for example, see [7, p. 14]). Here χ denotes the Euler characteristic. □

REMARK. As usual, we could reformulate this in terms of degrees on $L^2(\Omega)$. Note that one has to be careful in the interpretation of the degree on Y^1 if Y is one-dimensional. This case is uninteresting because the result is well known in this case.

Lastly for this section we want to point out that the three solution result in [11] holds under much weaker assumptions on the linearity. It suffices to assume that the nonlinearity $f(y)$ is globally Lipschitz and $y^{-1}f(y) \to \mu(\nu)$ as $y \to \infty$ ($y \to -\infty$). The proof of this uses weak convergence and a modification of ideas of Katriel (see [15]) to prove that the global reduction result in [11] is still valid for large t and the proof continues as before. We will publish this elsewhere because the ideas are also useful for improving the applications of results on bifurcation from a simple eigenvalue and bifurcation from infinity for partial differential equations. Note that there are a number of results on at least 3 solutions in [18] with no Lipschitz condition on f at all but the 3 solution result in [11] covers some cases of (μ, ν) which are not covered by their result.

3. A counter-example

As before, we consider the problem

$$\begin{aligned} -\Delta u &= \mu u^+ + \nu u^- && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5}$$

Let $T = \{(\mu, \nu, u) : (\mu, \nu, u) \text{ solves (5)}, \|u\|_{1,2} = 1\}$. We say that $(\widetilde{\mu}, \widetilde{\nu}, \widetilde{u}) \in T$ is regular if $-\Delta - (\widetilde{\mu}\widetilde{\chi}_{\widetilde{u}>0} + \widetilde{\nu}\chi_{\widetilde{u}<0})I$ (plus the boundary condition) has only a one-dimensional kernel in $\dot{W}^{1,2}(\Omega)$. (Note that the kernel is non-trivial because it contains u.) Here $\widetilde{\chi}_D$ denotes the characteristic function of the set D. The interest in regularity is that it implies that if $\widetilde{\mu}, \widetilde{\nu} > \lambda_1$ then near $(\widetilde{\mu}, \widetilde{\nu}, \widetilde{u})$, T can be written as $(\mu, \nu(\mu), u(\mu))$ where $\nu(\widetilde{\mu}) = \widetilde{\nu}, u(\widetilde{\mu}) = \widetilde{u}$ and ν and u are continuous in μ and $\nu(\mu)$ is decreasing in μ (cp. [22] or [23] or [25]). Alternatively it could be parametrized locally by ν. Thus, if every point of T off the diagonal is regular, T has a simple structure. Note that the set A_0 defined in Section 1 is the image of T under the natural projection on the first two coordinates. We say that T is regular if every point of T off the diagonal is regular. There have been several attempts ([22], [25]) to prove that, for generic Ω, T is regular. We give a counterexample showing that this is false. In fact, we prove somewhat more. Our example show that for an open set of Ω in a natural topology, the local representation of T described above must fail at some point of T. (Thus near this point, T is no longer a continuous curve or the curve must "bend back".) Hence the structure of T is not so simple.

Our construction is based on a simple idea. We find an example where $\lambda_{i-2} < \lambda_{i-1} < \lambda_i < \lambda_{i+1}$ where λ_{i-1} and λ_i are simple and a closed semicircle $\widehat{T}$ contained in the interior of the "box" $\{\lambda_{i-2} < \mu, \nu < \lambda_{i+1}, \nu \geq \mu\}$ such that the "ends" of $\widehat{T}$ are on the diagonal and (λ_j, λ_j) for $j = i-1$, i are inside $\widehat{T}$ and such that $\widehat{T} \cap \widehat{P}T$ consists of 6 points each of which is regular. Here $\widehat{P}$ is the natural projection of $\mathbb{R}^2 \times \dot{W}^{1,2}(\Omega)$ onto $\mathbb{R}^2$. Then there must be a point of $\widehat{P}T$ inside $\widehat{T}$ not on the diagonal where regularity fails. To see this, one simply notes that near a regular point T is a simple curve parametrized by μ (where ν is decreasing in μ) and hence if regularity always hold this curve must continue till it hits the diagonal where $\mu = \nu$. (Note that it is easy to see there are no compactness troubles.) Thus if regularity holds off the diagonal, we have six distinct curves in T meeting the diagonal. They can only meet the diagonal at $(\lambda_{i-1}, \lambda_{i-1})$ or (λ_i, λ_i) (because if $(\mu, \mu) \in A_0, \mu$ is an eigenvalue of $-\Delta$). On the other hand near a simple eigenvalue, it is well known that T consists of 2 non-intersecting curves (cp. [14]). Hence we have a contradiction and our claim follows. Finally, to complete our example, it is easy to see that our assumptions persist under smooth perturbations of Ω and hence we have the required example. (The only non-obvious point is that regular solutions persist under smooth perturbations. We will sketch a proof of this a little later.) Note that our argument would not be affected if there were other non-regular points on $\widehat{T}$.

What we actually do is to construct our example is to construct an example as above except that instead of having two simple eigenvalues λ_{i-1}, λ_i we have a double eigenvalue between λ_{i-2} and λ_{i+1}. We then use a smooth perturbation of Ω and a theorem of Uhlenbeck [28] (or Micheletti [20]) to split the eigenvalue. As before, the rest of the structure is unchanged in this perturbation.

The example we now require is in fact the one in Section 3 of [4] though we need to make extra calculations to check that our construction is valid. We will produce an example of a double eigenvalue $\widehat{\mu}$ such that the function $\|u^+\|_2^2$ on $\{u : -\Delta u = \widehat{\mu} u \in \Omega,\ u = 0 \text{ on } \partial\Omega,\ \|u\|_2 = 1\}$ has 6 critical points each of which is non-degenerate in the sense of [4]. Then, by the theorem in Section 2 of [4], T near $(\widehat{\mu}, \widehat{\mu})$ will consist of 6 non-intersecting curves. We need to check that near $(\widehat{\mu}, \widehat{\mu})$ each of the points on these curves is regular. If we prove this, we can then simply make the semicircle $\widehat{T}$ a small semicircle centre $(\widehat{\mu}, \widehat{\mu})$ and we will have the required example.

Thus our example reduces to some rather tedious checking (which is complicated by limited smoothness of our maps). However, we will do this which will give our example.

Firstly, we need to check that if we have a regular point of T and we perturb the domain, there is a point of T for the perturbed Ω nearby. (Standard linear operator theory ensures it will still be regular.) To do this we need to consider the known construction of the curve near $(\widetilde{\mu}, \widetilde{\nu}, \widetilde{u})$. Let $W = \{u \in \dot{W}^{1,2}(\Omega) \cap W^{2,2}(\Omega) : \|u\|_2 = 1\}$. Then the curve is obtained by solving the equation $(u, \nu) \in W \times R \to -\Delta u - \mu u^+ - \nu u^- = 0$ for (u, ν) as a function of μ near $(\widetilde{u}, \widetilde{\nu})$ and μ near $\widetilde{\mu}$. The key point is to use the contraction mapping theorem and note that the map $(\widehat{u},\tau) \to -\Delta\widehat{u} - \widetilde{\mu}\widetilde{\chi}_{\widetilde{u}>0}\widehat{u} - \widetilde{\nu}\widetilde{\chi}_{\widetilde{u}<0}\widehat{u} - \tau(\widetilde{u})^-$ is a bijection as a map of $Y \times R$ to $L^2(\Omega)$ where Y is the tangent space to W at $\widetilde{u}$. Since $(-\Delta - (\widetilde{\mu}\widetilde{\chi}_{\widetilde{u}>0} - \widetilde{\nu}\widetilde{\chi}_{\widetilde{u}<0})I)^{-1}$ will change continuously under domain perturbation (cp. [9]), it is rather easy to check the contraction theory argument will still hold for the perturbed domain (and indeed the neighbourhoods have size locally independent of Ω). A similar but more complicated argument appears in Step 2 of the proof of Theorem 3 in [9]. This completes the proof of this claim.

Secondly, we need to know that if $\widehat{\mu}$ is a multiple eigenvalue of $-\Delta - \widehat{\mu}I$ (plus the boundary condition on Ω) and if u_0 is a non-degenerate critical point of $\int_\Omega (u^+)^2$ on the unit sphere $\widetilde{N}$ in $N_{\widehat{\mu}} = \{u \in W^{2,2}(\Omega) : -\Delta u = \widehat{\mu}u,\ u = 0$ on $\partial\Omega\}$ then the branch of solutions obtained are regular solutions. This is not difficult but we need to consider the construction of the branch more carefully. Firstly note, as ever, we need only prove the regularity for the reduced operator $F_{\mu,\nu}$ where we make a Liapounov–Schmidt reduction to the space $N_{\widehat{\mu}}$. In other words, we need to prove the kernel is one-dimensional. The construction of the solutions is as follows. Let $k = \|u_0^+\|_2^2$ and note that $0 < k < 1$. We look for solutions $u = u_0 + o(1)$, $\mu = \widehat{\mu} + ta$, $\nu = \widehat{\mu} + tb$ where t is positive and small, $a^2 + b^2 = 1$ and $a\|u_0^+\|_2^2 + b\|u_0^-\|_2^2$ is close to zero. In this case, it is shown in [4] that the equation $F_{\mu,\nu}(n) = 0$ becomes

$$-tP(a(n + S_t n)^+ + b(n + S_t n)^-) = 0$$

where $S_t w$ satisfies a Lipchitz condition of $o(t)$ in w and P is the orthogonal projection onto $N_{\widehat{\mu}}$.

To check the kernel we can remove the factor t. Note that our operator is differentiable because at a solution $n + S_t n$ will only vanish on a set of measure zero. Hence we can use a similar but easier argument to the proof of Theorem 2 in [4] to check that for small t the derivative in n is $P((a_0\widetilde{\chi}_{u_0>0} + b_0\widetilde{\chi}_{u_0<0})I) + o(1)$. Here $a_0^2 + b_0^2 = 1$, $a_0\|u_0^+\|_2^2 + b_0\|u_0^-\|_2^2 = 0$.

Hence we only have to prove that the map $n \to P((a_0\widetilde{\chi}_{u_0>0} + b_0\widetilde{\chi}_{u_0<0})n)$ has only a one dimensional kernel. This is exactly what is meant by non-degeneracy and hence we have proved our claim.

It remains to check that for our example in [4] the second eigenvalue $\widehat{\mu}$ has the property that the map $\|u^+\|_2^2$ on the unit sphere in $N_{\widehat{\mu}}$ has exactly 6 critical points and they are non-degenerate. We showed in [4] that the eigenspace $N_{\widehat{\mu}}$ is close to the subspace $\widehat{N}$ spanned by $\Phi_1(x) = (\phi_1(x) - \phi_3(x))/\sqrt{2}$ and $\Phi_2(x) = (\phi_1(x) - 2\phi_2(x) + \phi_3(x))/\sqrt{6}$. Here we mean close in $L^p(\widetilde{B})$ for $\widetilde{B}$ a large ball for all p with $p < \infty$. Note that the normalization for Φ_2 is in error in [4]. We explain our notation here. Ω is close to $B_1 \cup B_2 \cup B_3$ where B_i are disjoint symmetrically placed balls of radius 1. Moreover, $\phi_i(x)$ is the normalized positive first eigenfunction on B_i (and defined to be zero elsewhere). Moreover, our function $\widetilde{F} = \|u^+\|_2^2$ on $N_{\widehat{\mu}}$ is close to the corresponding function $\widehat{F} = \|u^+\|_2^2$ on $\widehat{N}$. (There are some technical issues here on closeness which we return to at the end of the proof.) We will prove that $\widehat{F}$ has exactly 6 critical points on the unit sphere $\widehat{N}^1$ in $\widehat{N}$ and each of these is non-degenerate. The result will follow from this and some simple perturbation arguments. Because of the rotational symmetry of our problem (rotations though an angle of $2\pi/3$) it will suffice to prove the non-degeneracy at 2 points on separate orbits. This is all a rather tedious calculation. Note that $\widehat{N}^1$ can be written as $\{\sin s\Phi_1(x) + \cos s\Phi_2(x) : 0 \le s \le 2\pi\}$. Note also that $\widehat{F}$ is the sum of three terms, one from each ball. Denote these terms by $\widehat{F}_i$ for $i = 1, 2, 3$. We split our computation into different ranges of s noting that by the rotational symmetry we need only consider an interval of s of length $2\pi/3$. For $-\pi/2 \le s \le \pi/2$, it is easy to see that $\widehat{\Phi}(s, x) = \sin$ s $\Phi_1(x) + \cos s\Phi_2(x) \le 0$ on B_2 and hence $\widehat{F}_2 \equiv 0$ if $-\pi/2 \le s \le \pi/2$. On the other hand $\widehat{\Phi}(s, x) = ((1/\sqrt{2}) \sin s + (1/\sqrt{6}) \cos s)\phi_1(x)$ on B_1. Hence we easily see that $\widehat{F}_1 = (((1/\sqrt{2}) \sin s + (1/\sqrt{6}) \cos s)^+)^2$. Similarly $\widehat{F}_3 = ((-(1/\sqrt{2}) \sin s + (1/\sqrt{6}) \cos s)^+)^2$.

Hence we see that on $[-\pi/6, \pi/6]$

$$\begin{aligned}\widehat{F}(s) &= \left(\frac{1}{\sqrt{2}} \sin s + \frac{1}{\sqrt{6}} \cos s\right)^2 + \left(-\frac{1}{\sqrt{2}} \sin s + \frac{1}{\sqrt{6}} \cos s\right)^2 \\ &= \sin^2 s + \frac{1}{3} \cos^2 s = 1 - \frac{2}{3} \cos^2 s.\end{aligned}$$

In fact that $s = 0$ is a critical point comes from a reflection symmetry. It is an easy computation that $s = 0$ is a non-degenerate critical point and is the only critical point on $[-\pi/6, \pi/6]$. On $[\pi/6, \pi/2], (-1/\sqrt{2}) \sin s + (1/\sqrt{6}) \cos s < 0$ and hence $\widehat{F}_3 = 0$. Hence on $[\pi/6, \pi/2]$, $\widehat{F}(s) = ((1/\sqrt{2}) \sin s + (1/\sqrt{6}) \cos s)^2$ which has critical points where $(1/\sqrt{2}) \sin s + (1/\sqrt{6}) \cos s = 0$ or $(1/\sqrt{2}) \cos s - (1/\sqrt{6}) \sin s = 0$ that is $\tan s = \sqrt{3}$ or $-1/\sqrt{3}$. The second is clearly impossible for $s \in [\pi/6, \pi/2]$ while the first give a unique solution of $\pi/3$. Hence we have exactly 2 critical points on $[-\pi/6, \pi/2]$. An easy computation shows that $\widehat{F}''(\pi/3) \ne 0$. From this and our symmetry our claim on critical points follows. However, we need a little more. An easy check shows that at all our critical points $\widehat{\Phi}(s)$ is non-zero on all of $B_1 \cup B_2 \cup B_3$. We will need this for our perturbation process. In fact, because of the symmetries (which are retained on Ω) we need only check this for the two critical points above. For $s = 0$, this is obvious while it is obviously true for $s = \pi/3$ on B_1, since $F_1'(\pi/3) > 0$ and it obviously true on B_2. On B_3, $\widehat{\Phi}(\pi/3) = (-(1/\sqrt{2}) \sin(\pi/3) + (1/\sqrt{6}) \cos(\pi/3))\phi_3(x)$ which is non-zero on B_3 which proves our claim.

Our map of interest is really the map $\widetilde{F} = \int_\Omega (u^+)^2$ on N_μ. Then $\widetilde{F}'(u)h = 2\int_\Omega u^+ h$ and $\widetilde{F}''(u)(h, k) = 2\int_\Omega \widetilde{\chi}_{u>0} hk$ where the second derivative is only known to exist if $\{x \in \Omega : u(x) \ne 0\}$ has measure zero. Now it is easy to see that critical points of $\widetilde{F}$ can only occur near those of $\widehat{F}$. Conversely, it is easy to see that there are critical points of $\widetilde{F}$ near those of $\widehat{F}$. (This was implicitly proved in [4]). Our argument below shows that any critical point of $\widetilde{F}$ near a local minimum of $\widehat{F}$ is a local minimum of $\widetilde{F}$ (and a similar result holds for local maximum). Hence the local uniqueness of the critical points of $\widetilde{F}$ follows (and hence $\widetilde{F}$ has exactly 6 critical points on the sphere).

At the critical points of $\widehat{F}$ corresponding to $s_1 = 0$ or $s_2 = \pi/3$, our earlier results shows that $\widehat{\Phi}(s_i)$ is non-zero almost everywhere on $B_1 \cup B_2 \cup B_3$ and hence $\widehat{F}''(\widehat{\Phi}(s_i))$ exist for $i = 1, 2$ and

$$\widehat{F}''(\widehat{\Phi}(s_i))(h,k) = 2\int_{B_1\cup B_2\cup B_3} \widetilde{\chi}_{\widehat{\Phi}(s_i)>0} hk.$$

Similarly, if s is near s_i such that $\Phi(s)$ is a critical points of $\widetilde{F}$, we know from general theory (as in [4]) that $\Phi(s)$ only vanishes on a set of Ω of zero measure. Thus $\widetilde{F}''(\Phi(s))$ exists and

$$\widetilde{F}''(\Phi(s))(\widetilde{h},\widetilde{k}) = 2\int_\Omega \widetilde{\chi}_{\Phi(s)>0}\widetilde{h}\widetilde{k}.$$

Now, if $\|\widetilde{h}\|_2 = \|\widetilde{k}\|_2 = 1$ where $\widetilde{h}, \widetilde{k} \in N_{\widehat{\mu}}$, then as in [4] standard elliptic theory ensures that we have a bound for $\|\widetilde{h}\|_\infty + \|\widetilde{k}\|_\infty$ which is independent of $\widehat{\mu}$ and Ω (provided we have a bound for $\widehat{\mu}$). Hence we can use Holder's inequality to show that, given $\varepsilon > 0$ there is a $\delta > 0$ independent of Ω such that $|2\int_{\mathcal{U}} \widetilde{\chi}_{\Phi(s)>0}\widetilde{h}\widetilde{k}| < \varepsilon$ if $m(\mathcal{U}) < \delta$, $h, k \in N_{\widehat{\mu}}$, $\|h\|_2 = \|k\|_2 = 1$. Similar properties hold for $\widehat{F}''(\widehat{\Phi}(s_i))$.

Hence we easily see that if s is near s_i $\widetilde{F}''(\Phi(s))(\widetilde{h},\widetilde{k})$ is uniformly close to $\widehat{F}''(\widehat{\Phi}(s_i))(h,k)$ where $\|\widetilde{h}\|_2 = \|\widetilde{k}\|_2 = \|h\|_2 = \|k\|_2 = 1$, $\widetilde{h}, \widetilde{k} \in N_{\widehat{\mu}}$, $h, k \in \widehat{N}$ and h and $\widetilde{h}$ and k and $\widetilde{k}$ are corresponding elements of $N_{\widehat{\mu}}$ and $\widehat{N}$. Note that we have used here the $\widehat{\Phi}(s_i)$ is non-zero almost everywhere on $B_1 \cup B_2 \cup B_3$. Since $\widehat{F}|_{\widehat{N}^1}$ has only non-degenerate critical points, it follows that $\widetilde{F}|_{N^1_{\widehat{\mu}}}$ has only non-degenerate critical points (which proves our non-degeneracy condition for $\widetilde{F}$). Moreover, if $\Phi(s)$ is a critical points of $\widetilde{F}$ with s near s_i, $\Phi(s)$ is a local minimum $\widetilde{F}|_{N^1_{\widehat{\mu}}}$ if $i = 1$ and is a local maximum if $i = 2$. This completes our construction.

Remarks. (1) If one examines our construction, one sees that in the region of A_0 where we show regularity fails, we see that the dimension of the kernel of $-\Delta - (\mu\widetilde{\chi}_{u>0} + \nu\widetilde{\chi}_{u<0})I$ (with the boundary condition) is at most 2-dimensional. Thus, in this case, one might hope the behaviour of T is still fairly simple. (Locally our problem is equivalent to a bifurcation problem with a one-dimensional kernel.)

(2) Our construction is quite flexible. It need not be based on balls. It can easily be modified to obtain examples on star shaped domains and for a number of other boundary conditions. (If the boundary conditions are Neumann, we

need to choose our example so the "joining strips" in the diagram in [4] are short while to obtain star shaped examples we need to arrange so there are no joining strips.) Note that the symmetry is important in our construction.

(3) It is possible to prove that in our example "generically" T consists of C^1-curves and 4 of the curves move in to the diagonal while the remaining 2 curves join up and "cancel" in some form of saddle node type bifurcation (for this region of μ, ν). This is by a rather tedious bifurcation theory argument. As usual the lack of smoothness makes the proof much more tedious. Note that this result will continue to hold for nearby domains.

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