Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 19, 2002, 109–121

EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR A NONLOCAL BOUNDARY VALUE PROBLEM

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Abstract. Sufficient conditions are given for the existence of multiple positive solutions of a boundary value problem of the form $x''(t)+q(t)f(x(t)) = 0$, $t \in [0,1]$, $x(0) = 0$ and $x(1) = \int_\alpha^\beta x(s)dg(s)$, where $0 < \alpha < \beta < 1$. A weaker boundary value problem is used to get information on the corresponding integral operator. Then the results follow by applying the Krasnosel'skiĭ fixed point theorem on a suitable cone.

1. Introduction

We deal with the existence of multiple positive solutions of a second order ordinary differential equa tion of the form

$$x''(t) + q(t)f(x(t)) = 0, \quad t \in [0,1], \tag{1.1}$$

which satisfy the conditions

$$x(0) = 0, \tag{1.2}$$

$$x(1) = \int_\alpha^\beta x(s)\,dg(s), \tag{1.3}$$

where $0 < \alpha < \beta < 1$ and $g\colon [\alpha,\beta] \to \mathbb{R}$ is an increasing function.

Nonlocal boundary value problems of the form (1.1)–(1.3) constitute a natural extension of two-point, three-point and multi-point boundary value problems,

2000 *Mathematics Subject Classification.* 34B18.

Key words and phrases. Nonlocal boundary value problems, multiple positive solutions, Krasnosel'skiĭ's fixed point theorem.

studied extensively in the litereture. We refer to Bitsadze ([3]) and Bitsadze and Samarskiĭ ([4]) from the early sixties, followed by a great number of authors, see, e.g. [11]–[13], [15]–[17], [21]–[23]. On the other hand the problem of the existence of positive solutions for various types of boundary value problems is recently the subject of several papers (see e.g. [1], [2], [5]–[10], [14]–[18], [23], [24]). All these works concern problems with restrictions on the slope of the solutions (see e.g. [14]–[17]) and problems with restrictions on the solutions themselves. And as the first class is concerned the things seem to be simple, because some rather mild conditions may guarrantee the existence of a fixed point of the corresponding integral operator, which is positive. The situation becomes interesting in the case which is discussed in this paper: The integral condition (1.3) concerning values of the solution does not lead to a positive integral operator and an application of the Krasnosel'skiĭ fixed point theorem on cones is not directly applicable. To overcome this problem we consider a new representation of the operator by using the (seemingly weaker) boundary value problem of the form

$$\begin{aligned} u''(t)+q(t)f(x(t)) &= 0, \qquad\qquad t\in[0,1],\\ u(0) &= 0,\\ u(1) &= u(\xi)g(\beta), \end{aligned}$$

where $x\in C([0,1],\mathbb{R})$ and $\xi\in[\alpha,\beta]$ are given. Then we find it more convenient to apply the well known fixed point theorem due to Krasnosel'skiĭ [19], which states as follows:

Theorem 1.1. *Let $\mathcal{B}$ a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$, with $0\in\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$, and let*

$$A\colon \mathbb{K}\cap(\Omega_2\setminus\overline{\Omega_1})\to\mathbb{K}$$

be a completely continuous operator such that either

$$\begin{aligned} &\|Au\|\le\|u\|,\quad u\in\mathbb{K}\cap\partial\Omega_1\quad\text{and}\quad \|Au\|\ge\|u\|,\quad u\in\mathbb{K}\cap\partial\Omega_2\\ \text{or}\quad &\|Au\|\ge\|u\|,\quad u\in\mathbb{K}\cap\partial\Omega_1\quad\text{and}\quad \|Au\|\le\|u\|,\quad u\in\mathbb{K}\cap\partial\Omega_2. \end{aligned}$$

Then A has a fixed point in $\mathbb{K}\cap(\Omega_2\setminus\overline{\Omega_1})$.

One of the most advantage of this theorem is that it can help to estimate the maximum values of the solutions. Moreover it can provide information on the least number of the solutions of such problems by applying it repeatedly. The latter works in the same way as the classical Bolzano–Weierstrass Theorem may give information on the number of zeros of continuous functions on intervals of the real line.

In this paper we apply Theorem 1.1 and obtain existence results for one, two and three positive solutions of the boundary value problem (1.1)–(1.3). Our motivations are the problems examined mainly in [15]–[17], [24] and especially in [23]. We among others extend the results given in [23].

The paper is organized as follows: Section 1 contains the basic preliminaries and some very useful lemmas. These lemmas imply several corollaries. The main results are given in Section 3, where two applications are also presented.

2. Preliminaries and some lemmas

In the sequel we shall denote by $\mathbb{R}$ the real line and by I the interval $[0,1]$. Then $C(I)$ will denote the space of all continuous functions $x\colon I \to \mathbb{R}$. Let $C_0(I)$ be the space of all continuous functions $x\colon I \to \mathbb{R}$, with $x(0) = 0$. The spaces $C(I)$ and $C_0(I)$ become Banach spaces when they are furnished with the usual sup-norm $\|\cdot\|$.

For the function g we make the following assumption:

(A1) $g\colon [\alpha,\beta] \to \mathbb{R}$ is an increasing function such that $\beta(g(\beta) - g(\alpha)) < 1$.

It is clear that without loss of generality we can (and shall) assume that $g(\alpha) = 0$.

Lemma 2.1. *If $x \in C_0(I)$ is a concave function satisfying condition* (1.3) *and g is a function satisdying* (A1), *then we have*

(i) $x(t) \geq 0$, $t \in I$ *and*
(ii) $x(t) \geq \mu\|x\|$, $t \in [\alpha,1]$, *where*

$$\mu := \min\{\gamma, 1-\beta, (\beta-\alpha)\gamma g(\beta)\} \quad \textit{and} \quad \gamma := \min\{\alpha, 1-\beta, (1-\beta)/(1-\alpha)\}.$$

(*Notice that* $0 < \mu < 1$.)

Proof. (i) If $x(1) \geq 0$, then, by the concavity of x and the fact that $x(0) = 0$, we have $x(t) \geq 0$, $t \in I$.

Assume that $x(1) < 0$. From (1.3), (A1) and the mean value theorem, it follows that there is $\xi_x \in [\alpha,\beta]$ such that $x(1) = x(\xi_x)g(\beta)$ (notice that $g(\alpha) = 0$). Moreover, since $g(\beta) > 0$ and $x(1) < 0$, we have $x(\xi_x) < 0$. This and $g(\beta)\beta < 1$ lead to

$$x(1) = g(\beta)x(\xi_x) > \frac{1}{\beta}x(\xi_x) \geq \frac{1}{\xi_x}x(\xi_x),$$

which contradicts the concavity of x.

(ii) First we shall prove that, if x is a concave function in $C_0(I)$, then

$$x(s) \geq \gamma\|x\|, \quad s \in [\alpha,\beta].$$

Indeed let $t_0 \in I$ be such that $\|x\| = x(t_0)$. We distinguish three cases: (Notice that $x \geq 0$ by the argument we proved in (i).)

(1) $\beta \le t_0$. Then $s \le t_0$, for every $s \in [\alpha, \beta]$ and, since x is a concave function, we have $sx(t_0) \le t_0 x(s)$. This implies $\alpha\|x\| \le x(s)$ and hence

$$\gamma\|x\| \le x(s).$$

(2) $\alpha \le t_0 \le \beta$. If $s \in [\alpha, t_0]$ then again, following case (1), we obtain $\alpha\|x\| \le x(s)$.

Let $s \in (t_0, \beta]$. Then observe that

$$\frac{1-s}{1-t_0} \le \frac{x(s)-x(1)}{x(t_0)-x(1)},$$

because of the concavity of the function x. Thus we have

$$\begin{aligned}(1-s)x(t_0) &\le (1-t_0)x(s) + (1-s-1+t_0)x(1)\\ &= (1-t_0)x(s) + (t_0-s)x(1) \le (1-t_0)x(s),\end{aligned}$$

since $t_0 \le s$. This implies that

$$(1-\beta)x(t_0) \le (1-\alpha)x(s) \quad \text{or} \quad \frac{1-\beta}{1-\alpha}x(t_0) \le x(s)$$

and finally, $\gamma\|x\| \le x(s)$.

(3) $t_0 < \alpha$. Then $t_0 < s$ for every $s \in [\alpha, \beta]$ and, following the same arguments as in case (2) above, we obtain

$$(1-s)x(t_0) \le (1-t_0)x(s) \quad \text{or} \quad (1-\beta)\|x\| \le x(s)$$

and so $\gamma\|x\| \le x(s)$.

Now, in order to show that $x(s) \ge \mu\|x\|$, $s \in [\alpha, 1]$, we distinguish the cases $x(\beta) < x(1)$ and $x(1) \le x(\beta)$.

If $x(\beta) < x(1)$, then by the concavity, for every $s \ge \beta$, we have $x(\beta) \le x(s)$. Therefore, by the above first part of our proof, for all $s \in [\alpha, 1]$ it holds

$$\begin{aligned}x(s) &\ge \min\{\min\{x(s) : s \in [\alpha, \beta]\}, \min\{x(s) : s \in [\beta, 1]\}\}\\ &\ge \min\{\gamma\|x\|, x(\beta)\} = \gamma\|x\|.\end{aligned}$$

If $x(1) \le x(\beta)$ then again, by the concavity, we have $x(s) \ge x(1)$, for every $s \in [\beta, 1]$. Therefore, from (1.3), for any such s we have

$$x(s) \ge \int_\alpha^\beta x(r)\,dg(r) \ge \gamma\|x\|(\beta-\alpha)g(\beta).$$

Hence in any case it holds $x(s) \ge \mu\|x\|$, $s \in [\alpha, 1]$ and the proof is complete. □

To proceed we need give our basic assumptions on the functions f and q. We shall do that by presenting also the appropriate remarks and giving the lemmas which are needed in the sequel. Assume that

(A2) $f\colon \mathbb{R} \to \mathbb{R}$, $q\colon [0,1] \to [0, \infty)$ are continuous functions, with $f(x) \ge 0$, when $x \ge 0$ and q is not identically zero on $[\beta, 1]$.

It is easy to see that the problem (1.1)–(1.3) is equivalent to the operator equation

$$x = Ax, \quad x \in C_0(I),$$

where A is the completely continuous operator defined by

$$\begin{aligned}(2.1)\quad Ax(t) :=& \delta t \int_0^1 (1-s)q(s)f(x(s))\,ds\\ &- \delta t \int_\alpha^\beta \int_0^r (r-s)q(s)f(x(s))\,ds\,dg(r) - \int_0^t (t-s)q(s)f(x(s))\,ds,\end{aligned}$$

where we have set

$$\delta := \frac{1}{1-\int_\alpha^\beta s\,dg(s)} (>0).$$

Now define the set $\mathbb{K} := \{x \in C_0(I) : x \text{ is concave and (1.3) holds}\}$, and observe that it is a cone in $C_0(I)$. By the first argument of Lemma 2.1 the cone $\mathbb{K}$ consists of nonnegative functions.

Lemma 2.2. *If the functions f, q, g satisfy assumptions* (A1), (A2), *then it holds*

$$A\mathbb{K} \subset \mathbb{K}.$$

Proof. Fix a $x \in \mathbb{K}$. Then we have $f(x(t)) \geq 0$ for all t. It can also be easily seen that

$$(Ax)''(t) = -q(t)f(x(t)), \quad t \in I,$$

and $(Ax)'' \leq 0$. This implies that Ax is a concave function. Moreover, it is clear that Ax satisfies the boundary conditions (1.2), (1.3), which proves the result.□

Now we assume:

(A3) There exist $\nu > 0$ with

$$\delta\nu \int_0^1 (1-s)q(s)ds \leq 1$$

and $N > 0$ such that

$$f(u) \leq \nu N \quad \text{for all } u \in [0, N]$$

or, equivalently,

$$f_s(N) := \sup_{0\leq u\leq N} f(u) \leq \nu N.$$

Lemma 2.3. *For all $x \in \mathbb{K}$, with $\|x\| = N$, we have $\|Ax\| \leq \|x\|$.*

Proof. Indeed, if $\|x\| = N$ then $0 \leq x(s) \leq N$ for every $s \in I$. (Keep in mind Lemma 2.1(1).) Then, by (A3) and (2.1), for every $t \in I$, we have

$$Ax(t) \leq \delta t \int_0^1 (1-s)q(s)f(x(s))\,ds \leq \delta t\nu\|x\| \int_0^1 (1-s)q(s)\,ds \leq \|x\|. \qquad \square$$

Next we set

$$f_i(w) := \inf\{f(z) : z \in [\mu w, w]\},$$

where μ is the positive number defined in Lemma 2.1. We suppose that:

(A4) There exist $\lambda > 0$, with

$$1 - \beta g(\beta) \leq \lambda\alpha \int_\beta^1 (1-s)q(s)\,ds \tag{2.2}$$

and $M > 0$ such that

$$f_i(M) \geq \lambda M. \tag{2.3}$$

Lemma 2.4. *For all $x \in \mathbb{K}$, with $\|x\| = M$, we have $\|Ax\| \geq \|x\|$.*

Proof. Fix $x \in \mathbb{K}$ with $\|x\| = M$. Then, from Lemma 2.1, we have

$$\mu\|x\| \leq x(t) \leq \|x\|, \quad t \in [\alpha, 1]$$

and therefore it holds

$$f(x(s)) \geq f_i(\|x\|), \quad s \in [\beta, 1]. \tag{2.4}$$

Now observe that the function $u(t) := Ax(t)$, $t \in I$ is the unique solution of the boundary value problem

$$u'' + q(t)f(x(t)) = 0, \quad t \in I, \quad u(0) = 0, \quad u(1) = \int_\alpha^\beta u(s)\,dg(s).$$

We let

$$E(u) := \left\{\xi \in [\alpha, \beta] : \int_\alpha^\beta u(s)\,dg(s) = u(\xi)\int_\alpha^\beta dg(s) = u(\xi)g(\beta)\right\}$$

be the set of all mean values of u with respect to the (Borel) measure generated by the function g. Obviously $E(u)$ is a compact set. Consider the point

$$\xi_u := \min E(u).$$

It is clear that u solves the boundary value problem

$$y'' + q(t)f(x(t)) = 0, \quad t \in I, \quad y(0) = 0, \quad y(1) = y(\xi_u)g(\beta)$$

and so, u is the function given by the closed formula

$$\begin{aligned} u(t) = {} & \zeta_u t \int_0^1 (1-s)q(s)f(x(s))\,ds \\ & - \zeta_u t g(\beta)\int_0^{\xi_u} (\xi_u - s)q(s)f(x(s))\,ds - \int_0^t (t-s)q(s)f(x(s))\,ds \end{aligned}$$

for $t \in I$, where

$$\zeta_u := \frac{1}{1 - \xi_u g(\beta)}.$$

Notice that $\alpha \le \xi_u \le \beta$ and, in view of (A1), $\zeta_u > 0$. Then we have

$$\begin{aligned}
(Ax)(\xi_u) &= u(\xi_u) \\
&= \zeta_u \xi_u \int_0^1 (1-s)q(s)f(x(s))\,ds \\
&\quad - \zeta_u \xi_u g(\beta) \int_0^{\xi_u} (\xi_u - s)q(s)f(x(s))\,ds - \int_0^{\xi_u} (\xi_u - s)q(s)f(x(s))\,ds \\
&= \xi_u \zeta_u \int_0^1 (1-s)q(s)f(x(s))\,ds - \zeta_u \int_0^{\xi_u} (\xi_u - s)q(s)f(x(s))\,ds \\
&= \xi_u \zeta_u \int_0^1 q(s)f(x(s))\,ds - \xi_u \zeta_u \int_0^1 sq(s)f(x(s))\,ds \\
&\quad - \xi_u \zeta_u \int_0^{\xi_u} q(s)f(x(s))\,ds + \zeta_u \int_0^{\xi_u} sq(s)f(x(s))\,ds \\
&= \xi_u \zeta_u \int_{\xi_u}^1 (1-s)q(s)f(x(s))\,ds + \zeta_u(1-\xi_u) \int_0^{\xi_u} sq(s)f(x(s))\,ds.
\end{aligned}$$

Taking into account (2.4), (2.3) and (2.2) we finally obtain that

$$\begin{aligned}
(Ax)(\xi_u) &\ge \zeta_u \alpha \int_\beta^1 (1-s)q(s)\lambda\|x\|\,ds \\
&= \lambda \zeta_u \alpha \|x\| \int_\beta^1 (1-s)q(s)\,ds \ge \zeta_u(1-\beta g(\beta))\|x\| \ge \|x\|.
\end{aligned}$$

Clearly this argument implies the result. □

Now we assume that the quantities

$$T_0 := \lim_{u\to 0} \frac{f(u)}{u} \quad \text{and} \quad T_\infty := \lim_{u\to\infty} \frac{f(u)}{u}$$

exist. The previous lemmas imply the following corollaries:

COROLLARY 2.5. *If $T_0 = 0$, then there exists $m_0 > 0$ such that for every $m \in (0, m_0]$ and for every $x \in \mathbb{K}$, with $\|x\| = m$, we have $\|Ax\| \le \|x\|$.*

PROOF. Let $\varepsilon > 0$ be such that

$$\delta\varepsilon \int_0^1 (1-s)q(s)\,ds \le 1.$$

Then, since $T_0 = 0$, there exists $m_0 > 0$ such that for every $u \in (0, m_0]$ we have $f(u) \le \varepsilon u$. Let now $m \in (0, m_0]$ be fixed. For every $u \in (0, m]$ we have $f(u) \le \varepsilon u \le \varepsilon m$. Thus assumption (A3) is valid with $\nu := \varepsilon$ and $N := m$. So, Lemma 2.3 applies. □

Corollary 2.6. *If $T_\infty = \infty$ then there exists $H_0 > 0$ such that, for every $H \geq H_0$ and for every $x \in \mathbb{K}$, with $\|x\| = H$, we have $\|Ax\| \geq \|x\|$.*

Proof. Choose $R > 0$ so that

$$1 - \beta g(\beta) \leq R\alpha \int_\beta^1 (1-s)q(s)\,ds.$$

Since $T_\infty = \infty$, there is a $H_1 > 0$ such that for every $u \geq H_1$ we have $f(u) \geq Ru/\mu$. Set $H_0 := H_1/\mu$. Then for any $H \geq H_0$ we have $\mu H \geq H_1$. So, if $u \in [\mu H, H]$ then $u \geq H_1$. Hence

$$f(u) \geq \frac{1}{\mu} Ru \geq RH.$$

Therefore

$$f_i(H) \geq RH.$$

Hence assumption (A4) is valid with $\lambda := R$ and $M := H$, and Lemma 2.4 applies. □

Corollary 2.7. *If $T_0 = \infty$ then there exists $h_0 > 0$ such that, for every $h \in (0, h_0)$ and for every $x \in \mathbb{K}$, with $\|x\| = m$, we have $\|Ax\| \geq \|x\|$.*

Proof. Choose $R > 0$ so that

$$1 - \beta g(\beta) \leq R\alpha \int_\beta^1 (1-s)q(s)\,ds.$$

Since $T_0 = \infty$, there is a $h_1 > 0$ such that, for every $u \in (0, h_1]$, we have $f(u) \geq Ru\mu$. Set $h_0 := \mu h_1$. Then for any $h \in (0, h_0]$ and $u \in [\mu h, h]$ we have $0 < u \leq h \leq h_0 = \mu h_1 < h_1$ and thus

$$f(u) \geq \frac{1}{\mu} Ru \geq Rh.$$

Therefore

$$f_i(h) \geq Rh.$$

So, assumption (A4) is valid with $\lambda := R$ and $M := h$, and Lemma 2.4 applies. □

Corollary 2.8. *If $T_\infty = 0$, then there exists $D_0 > 0$ large as we want such that, for every $x \in \mathbb{K}$, with $\|x\| = D_0$, we have $\|Ax\| \leq \|x\|$.*

Proof. Let $\varepsilon > 0$ be such that

$$\delta\varepsilon \int_0^1 (1-s)q(s)\,ds \leq 1.$$

We distinguish two cases:

(1) Assume first that f is bounded. Then there is a $b > 0$ such that $f(u) \leq b$, for all $u \geq 0$. We set $D_0 := b/\varepsilon$. Then, for all $v \leq D_0$, it holds $f(v) \leq b = \varepsilon D_0$. Hence $f_s(D_0) \leq \varepsilon D_0$.

(2) If f is not bounded then, since $T_\infty = 0$, there is D_0 so large as we want such that $f_s(D_0) = f(D_0) \le \varepsilon D_0$.

In any case, assumption (A3) holds with $\nu := \varepsilon$ and $N := D_0$. Finally, Lemma 2.3 applies. $\square$

3. Main results

Now it is time to state and prove our main results.

Theorem 3.1. *Consider the functions f, q, g satisfying the assumptions* (A1), (A2) *and moreover, one of the following statements:*

(i) (A3) *and* (A4),
(ii) (A3) *and* $T_0 = \infty$,
(iii) (A3) *and* $T_\infty = \infty$,
(iv) (A4) *and* $T_0 = 0$,
(v) (A4) *and* $T_\infty = 0$,
(vi) $T_0 = 0$ *and* $T_\infty = \infty$,
(vii) $T_0 = \infty$ *and* $T_\infty = 0$.

Then the boundary value problem (1.1)–(1.3) *admits at least one positive solution.*

Proof. The result of the theorem is easily obtained if we apply Theorem 1.1 to the completely continuous operator A on the cone $\mathbb{K}$ and use Lemmas 2.3 and 2.4 if (i) holds, Lemma 2.3 and Corollary 2.7 if (ii) holds, Lemma 2.3 and Corollary 2.6 if (iii) holds, Lemma 2.4 and Corollary 2.5 if (iv) holds, Lemma 2.4 and Corollary 2.8 if (v) holds, Corollaries 2.5 and 2.6 if (vi) holds, and Corollaries 2.7 and 2.8 if (vii) holds. In all cases we keep in mind Lemma 2.2. $\square$

Remark 1. In any case the application of Theorem 1.1 provides information on the norm of the fixed points, namely of the maximum value of the corresponding solution of the problem. For instance, in case (i) the norm of the solution lies in the interval $(\min\{M, N\}, \max\{M, N\})$.

Theorem 3.2. *Assume that the functions f, q, g satisfy the assumptions* (A1)–(A4). *Moreover, let one of the following statements holds:*

(i) $M < N$ *and* $T_0 = 0$,
(ii) $M < N$ *and* $T_\infty = \infty$,
(iii) $N < M$ *and* $T_0 = \infty$,
(iv) $N < M$ *and* $T_\infty = 0$.

Then the boundary value problem (1.1)–(1.3) *admits at least two positive solutions.* (*In any case the remark of Theorem* 3.1 *keeps in force.*)

Proof. As in the proof of Theorem 3.1, we apply (twice) Theorem 1.1 on the completely continuous operator A on the cone $\mathbb{K}$ and use Lemmas 2.3, 2.4 in connection with Corollary 2.5 if (i) holds, Corollary 2.6 if (ii) holds, Corollary 2.7 if (iii) holds and Corollary 2.8 if (iv) holds. Again, keep in mind Lemma 2.2. □

Theorem 3.3. *Consider the functions f, q, g satisfying the assumptions* (A1)–(A4)*, and moreover, one of the following:*

(i) $M < N$, $T_0 = 0$ *and* $T_\infty = \infty$,
(ii) $N < M$, $T_0 = \infty$ *and* $T_\infty = 0$.

Then the boundary value problem (1.1)–(1.3) *admits at least three positive solutions.* (*In any case the remark of Theorem* 3.1 *keeps in force.*)

Proof. The result follows, as in the previous theorems. We use now Lemmas 2.3, 2.4 in connection with Corollaries 2.5, 2.6 if (i) holds and Corollaries 2.7, 2.8, if (ii) holds. □

Remark 2. Theorem 1 in [23] deals with the cases (vi) and (vii) of Theorem 3.1 above. Moreover, the three-point boundary condition used in [23] is a special form of the general boundary condition (1.3). This shows that Theorem 1 of [23] is a special case of our Theorem 3.1. This claim is also shown in the following example.

Example 1. Consider the boundary value problem

$$x''(t) + \frac{1}{2}(x(t) - 2)^2 e^{x(t)-1} = 0, \quad t \in I, \tag{3.1}$$

$$x(0) = 0 \quad \text{and} \quad x(1) = 2x(0.25), \tag{3.2}$$

which is a very special case of the problem (1.1)–(1.3). Here we have

$$g(s) = \begin{cases} 0 & \text{if } 1/8 \leq s < 1/4, \\ 2 & \text{if } s = 1/4. \end{cases}$$

In this problem we have $f(u) = (u - 2)^2 e^{u-1}$, $u \in [0, \infty)$. To set it in our situation we let $q(t) = 1/2$, $\alpha := 1/8$, $\beta := 1/4$, $\delta := 2$, $\gamma := 1/8$ and $\mu := 1/32$. It is obvious that the function f does not satisfy the assumption (i) or (ii) of Theorem 1.1 in [23]. So this theorem does not imply any existence result for the boundary value problem (3.1)–(3.2).

Consider a N such that $0 < N < 2$. Then, taking into account that $f(u)$ decreases for $u < 2$, it is easy to see that $f_s(N) = 4/e$. Moreover, since $\int_0^1 q(s)(1 - s)\,ds = (1/2)\int_0^1 (1 - s)\,ds = 1/4$, we must have that $\nu \leq 2$. Choose $\nu = 2$. Thus for $N := 2/e$ we have $f_s(N) = 4/e \leq \nu N$. Hence assumption (A3) is satisfied.

On the other hand, since $\int_\beta^1 q(s)(1 - s)\,ds = (1/2)\int_{1/4}^1 (1 - s)\,ds = 9/64$, we can get $\lambda := 256/9$. So, for instance, if $M := 224$ then it holds $(\mu M - 2)^2 e^{\mu M - 1} \geq \lambda M$. Since the function $f(u)$ increases in the interval $[2, \infty)$, we have $f_i(M) = f(\mu M) = (\mu M - 2)^2 e^{\mu M - 1} \geq \lambda M$. Hence assumption (A4) is satisfied. Moreover, if we choose $h := 0.05$ then it holds $(h-2)^2 e^{h-1} \geq \lambda h$ and hence assumption (A4) is also satisfied. Finally we observe that $T_0 = \infty$. Therefore, since assumptions (A1), (A2) are obviously satisfied by Theorem 3.2 (case (iii)), it follows that the boundary value problem (3.1)–(3.2) admits at least two positive solutions x_1, x_2 such that

$$0.05 < \|x_1\| < \frac{2}{e} < \|x_2\| < 224.$$

The lower bound for $\|x_1\|$ can be found via the arguments in the proof of Corollary 2.7.

EXAMPLE 2. Consider the continuous function

$$f(u) := \begin{cases} 4u^2 & \text{if } u \leq 15.2, \\ 924.16 & \text{if } 15.2 \leq u \leq 1848.32, \\ \dfrac{25}{92416}u^2 & \text{if } 1848.32 \leq u, \end{cases}$$

and formulate the boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in I, \tag{3.3}$$

$$x(0) = 0 \quad \text{and} \quad x(1) = x(0.1) + 0.9x(0.5). \tag{3.4}$$

Here we set

$$g(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq 0.1, \\ 1 & \text{if } 0.1 < s < 0.5, \\ 1.9 & \text{if } 0.5 \leq s \leq 1. \end{cases}$$

For the function f we observe that it holds

$$T_0 = 0 \quad \text{and} \quad T_\infty = \infty.$$

Also we have

$$\delta := \frac{1}{1 - 0.1 - 0.9 \times 0.5} = \frac{100}{45}, \quad \gamma := 0.1, \quad \mu := 0.076.$$

Choose

$$\nu := 0.5, \quad \lambda := 4,$$

as well as

$$M := 200, \quad N := 1848.32, \quad m_0 := 0.125 \quad \text{and} \quad H_0 := 2560000.$$

Then observe that assumptions of case (i) of Theorem 3.3 are satisfied and hence we conclude that problem (3.3)–(3.4) admits at least three positive solutions x_1, x_2, x_3. Using the arguments in the proofs of Corollaries 2.5 and 2.6 we can immediately see that these solutions satisfy

$$0.125 < \|x_1\| < 200 < \|x_2\| < 1848.32 < \|x_3\| < 2560000.$$

Acknowledgments. The authors would like to express their thanks to the referee for his/her comments on the last example of the paper.

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Manuscript received November 28, 2001

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