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THE PASCAL THEOREM AND SOME ITS GENERALIZATIONS

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Dedicated to Professor Andrzej Granas

Abstract. We present two generalizations of the famous Pascal theorem to the case of algebraic curves of degree 3.

1. Introduction

The 350 years old theorem of B. Pascal [9] says, that if a hexagon is inscribed in a conic, then the opposite sides of the hexagon meet in three colinear points. The dual version of this result is called the Brianchon theorem and says that if a conic is inscribed in a hexagon, then the diagonals of the hexagon intersect at one point. These theorems remain true in some degenerate cases, e.g. when the hexagon degenerates to a pentagon. There exist essentially two proofs of the Pascal theorem, one uses projective geometry methods and the cross-ratio invariant (see Section 2), while the other one relies on the Cayley–Bacharach theorem (see Section 3). It seems that such a beautiful results should have generalizations. For example, the projective proof of the Pascal theorem uses the fact that a conic is a (projective) rational curve. There exist rational curves of higher degrees, e.g. a cubic with one point of self-intersection. There are, however, only few works in this direction. Probably the most interesting is the

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paper [4] by D. Eisenbud, M. Green and J. Harris devoted to generalization of the Cayley–Bacharach theorem to higher dimensions.

In this paper we prove two generalizations of the Pascal theorem (Theorem 4.4 and 5.1 below) which are in the same style as Pascal's result, i.e. that some points, obtained as results of intersections of algebraic curves lie on a straight line. Theorem 4.4 deals with a general cubic intersected by three lines in nine points. One constructs a conic through five of them and two lines through the remaining four. One obtains three additional points that turn out to lie on straight line. The proof of this result is a standard application of the Cayley–Bacharach theorem. Theorem 5.1 is more subtle. It deals with a rational cubic (i.e. a cubic with a double point) with 8 generic points. One constructs two pairs of conics, each of them through four of these points and the double point. The two quartics defined in this way (each is a sum of two conics) define 4 additional points in their intersection. It turns out that these 4 points lie on a straight line. The proof is analytic and uses the notion of multi-dimensional residuum, applied in a non-trivial case. (In fact, we did not expected such result; it has surprised us a little). We prove also a generalization of the Brianchon theorem (Theorem 5.3), the dual version of Theorem 5.1. It is restricted to simply connected cubics, with cusp singularity and one inflection point. For a configuration of 8 lines tangent to such cubic one constructs two pairs of conics, each tangent to 4 of the lines and to the line tangent at the inflection point. One obtains 4 additional lines tangent to the both corresponding quartics. These lines turn out intersect at one point.

Now we know, how to generalize Theorem 4.4 to curves of higher degrees. Probably there exists also a generalization of Theorem 5.1 to rational curves of higher degrees. It seems that, using Theorem 5.1 or some kind of its inverse, one could provide a geometrical construction of 12 different rational cubics through 8 points in $\mathbb{C}P^2$ in general position (see [7]). The subject seems to be highly interesting. We intend to continue investigations in future papers.

The plan of the article is following: in Section 2 we present the classical Pascal's proof of Pascal theorem. In Section 3 we introduce the notion of multi-dimensional residuum and prove the Cayley–Bacharach theorem. In Section 4 we present the analytic proof of Pascal theorem, of its inverse and of Theorem 4.4. Section 5 contains the proof of Theorems 5.1 and 5.3.

2. The cross-ratio and conics

The *cross-ratio* of a quadruple of different points $a_1, \ldots, a_4 \in \mathbb{C}P^1$ is defined as

$$\operatorname{cr}(a_1, \ldots, a_4) = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_4)(a_2 - a_3)}, \tag{2.1}$$

if all points lie on the affine part $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$, and

$$\mathrm{cr}(a_1, a_2, a_3, \infty) = \frac{a_1 - a_2}{a_2 - a_3}. \tag{2.2}$$

It is, of course, the limit of (2.1).

LEMMA 2.1. *The cross-ratio is invariant under the action of* $\mathrm{PSL}(2, \mathbb{C})$ – *the group of automorphisms of* $\mathbb{C}P^1$.

PROOF. If $\sigma\colon z \to (\alpha z + \beta)/(\gamma z + \delta)$, $\alpha\delta - \beta\gamma = 1$ is an automorphism of $\mathbb{C}P^1$, then

$$\sigma(a) - \sigma(b) = \frac{a - b}{(\gamma a + \delta)(\gamma b + \delta)}. \tag{2.3}$$ □

The following proposition will be used in the geometrical proof of the Pascal theorem.

Let A and B be two different projective lines on a projective plane $\mathbb{C}P^2$, let us denote by o their unique intersection point. Choose points a_1, a_2, a_3 on A, and b_1, b_2, b_3 on B. Define the lines A_1, A_2 and A_3, where A_j passes through a_j and b_j (see Figure 1).

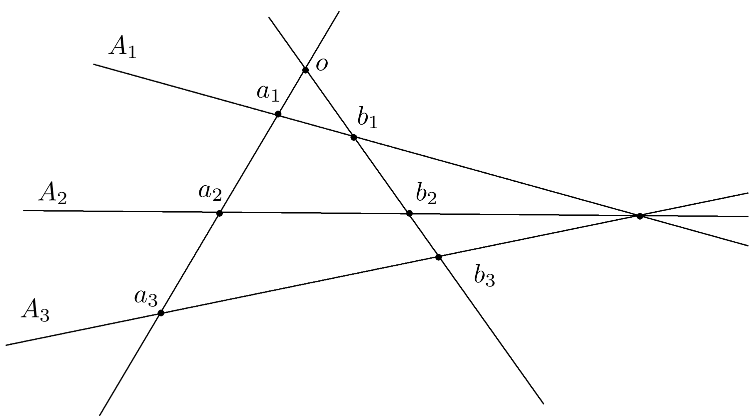

FIGURE 1

PROPOSITION 2.2. *The lines A_1, A_2 and A_3 intersect at one point if and only if the cross-ratios* $\mathrm{cr}(a_1, a_2, a_3, o)$ *and* $\mathrm{cr}(b_1, b_2, b_3, o)$ *are equal.*

PROOF. We choose a projective chart such that the line passing through o and $A_1 \cap A_2$ is the line at infinity. The the affine lines $A^0 = A \cap \mathbb{C}^2$ and $B^0 = B \cap \mathbb{C}^2$ are parallel, similarly parallel are the lines A_1^0 and A_2^0. The property that the projective lines A_1, A_2 and A_3 intersect at one point is equivalent to the property that the affine lines A_1^0, A_2^0, A_3^0 are parallel . This is true iff the following condition is fulfilled:

$$\frac{a_1 - a_2}{a_2 - a_3} = \frac{b_1 - b_2}{b_2 - b_3},$$

which exactly means that the cross-ratios $\mathrm{cr}(a_1, a_2, a_3, o)$ and $\mathrm{cr}(b_1, b_2, b_3, o)$ are equal. (Here $o = \infty$, see (2.2)). □

The geometrical proof of the Pascal theorem uses also the following result about 4 points in a projective conic.

Let $C \subset \mathbb{C}P^2$ be a smooth conic, i.e. an algebraic curve of degree two, which is not a sum of two lines. Any point $m \in C$ defines a pencil m^* of projective lines through m. The pencil $m^* \simeq \mathbb{C}P^1$ is a projective line in the dual projective space $(\mathbb{C}P^2)^* \simeq \mathbb{C}P^2$.

We are given a map

$$\pi_m : C \to m^*, \tag{2.4}$$

which associates with a point $c \in C$ the line $\pi_m(c)$ passing through m and c. $\pi_m(m)$ is the line tangent to C at m. The map (2.4) defines a biholomorphism between the conic C and $\mathbb{C}P^1 \equiv m^*$. Thus, any smooth conic is a rational curve.

Given a map π_m we are able to define a cross-ratio of a quadruple of points a_1, a_2, a_3, a_4 on C. We define it to be the cross-ratio of the points $\pi_m(a_1), \dots,$ $\pi_m(a_4)$ in m^*. In fact,

$$\mathrm{cr}(a_1, \dots, a_4) = \mathrm{cr}(b_1, \dots, b_4), \tag{2.5}$$

where $b_j = \pi_m(c_j) \cap L$, for some fixed line L in $\mathbb{C}P^2$.

Proposition 2.3. *The number* $\mathrm{cr}(a_1, a_2, a_3, a_4)$ *is well defined. It does not depend neither on $m \in C$ nor on the line L.*

Proof. A change of the point m or the line L results as an automorphism of $\mathbb{C}P^1$. The thesis follows from Lemma 2.1. □

Let us recall Pascal's result. Let $C \subset \mathbb{C}P^2$ be a smooth conic. Let $a_1, \dots, a_6$ be different points on C. We define the lines $A_1 = a_1a_2$, i.e. the line that passes through a_1 and a_2, $A_2 = a_5a_6$, $A_3 = a_3a_4$ $B_1 = a_4a_5$, $B_2 = a_2a_3$, $B_3 = a_6a_1$. The curves $A = A_1 + A_2 + A_3 \overset{\mathrm{def}}{=} A_1 \cup A_2 \cup A_3$, and $B = B_1 + B_2 + B_3$ intersect at all 6 points $a_1, \dots, a_6$, and, besides, at the points $d_1 = A_1 \cap B_1$, $d_2 = A_2 \cap B_2$, $d_3 = A_3 \cap B_3$ laying outside C (see Figure 2).

Tyheorem 2.4 (Pascal theorem). *The points d_1, d_2 and d_3 lie on a straight line.*

First Proof. (We are following [2]). Let $x = B_1 \cap B_2$, $y = A_2 \cap A_3$ (see Figure 2) The points d_1, x, a_4, a_5 lie on the line B_1. The map π_{a_2} associates with each of them a line in a_2^*, namely lines A_1, B_2, a_2a_4 and a_2a_5. These lines intersect the conic C in the points a_1, a_3, a_4 and a_5, respectively.

On the other hand, the points d_3, a_3, a_4, y lie on the line A_3. They define four lines in a_6^*. These intersect C at a_1, a_3, a_4 and a_5. By Proposition 2.3 we have

$$\mathrm{cr}(d_1, x, a_4, a_5) = \mathrm{cr}(a_1, a_3, a_4, a_5) = \mathrm{cr}(d_3, a_3, a_4, y).$$

Now we apply Proposition 2.2 to the lines $A = A_3$, and $B = B_1$, intersecting at $a_4 = o$. Thus the lines d_3d_1, $a_3x = B_2$ and $ya_5 = A_2$ intersect at a single point d_2. □

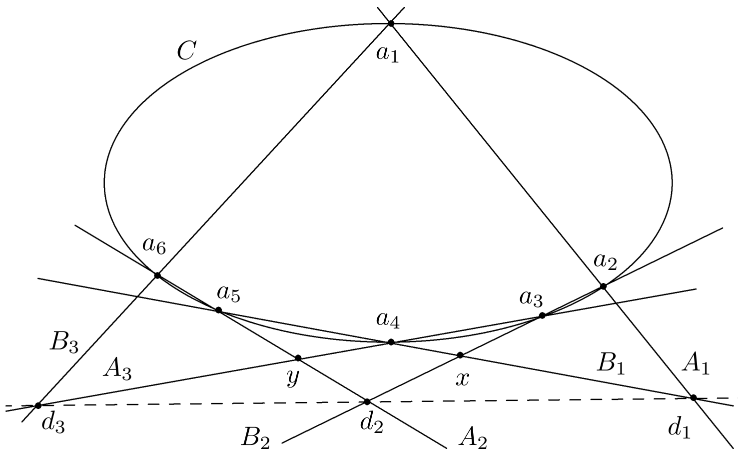

FIGURE 2

REMARK 2.5. The above proof can be repeated in case when the conic $C = C' + C''$ is a union of two lines. It is then called the *Pappus theorem*:

Let C', C'' be two different lines and $a_1, a_2, a_3 \in C'$, whereas $b_1, b_2, b_3 \in C''$. Let us define the lines $A_1 = a_1b_1$, $A_2 = a_2b_3$, $A_3 = a_3b_2$, $B_1 = b_3a_3$, $B_2 = b_2a_1$, $B_3 = b_1a_2$ and the cubics $A = A_1 + A_2 + A_3$, $B = B_1 + B_2 + B_3$. The cubics provide us then with three additional, i.e. laying outside C, intersection points d_1, d_2 and d_3. Then these points are colinear.

There exists also another proof of the Pappus theorem. We can assume the lines A_2 and A_3 to be parallel, as well as lines A_3 and B_3. Let us also assume that $C' \cap C'' = o$ is a finite point. By the Tales theorem there exist homotheties f and g with centre at o such that $B_2 = f(A_2)$ and $B_3 = g(A_3)$. The homotheties commute and their composition is again a homothety, that sends the line A_1 to B_1. Therefore the latter lines are parallel. In case o lies on the line at infinity, we use translations instead of homotheties.

3. Local residuum and the Cayley–Bacharach theorem

Let us begin with recalling the Cauchy integral formula:

$$\frac{1}{2\pi i}\int_{|z-a|=\varepsilon} g(z)\,dz = c_{-1} \stackrel{\mathrm{df}}{=} \mathrm{res}_a g(z), \tag{3.1}$$

where $g(z)$ is a meromorphic function, that expands in a Laurent series $\sum_{j>j_0} c_j (z-a)^j$ at a and ε is sufficiently small. However, in higher dimensions there is no similar formula.

Example 3.1 ([10]). Consider a rational function

$$g(z,w) = \frac{h(z,w)}{zw(z-w)},$$

where h is a polynomial. The three lines $z=0$, $w=0$ and $z=w$ correspond to the point a in one-dimensional case. Let us integrate g over the following two-dimensional cycles:

$$\Gamma_1 = \{|z|=\varepsilon_1,\ |w|=\varepsilon_2>\varepsilon_1\},\quad \Gamma_2 = \{|z|=\varepsilon_1,\ |w|=\varepsilon_2<\varepsilon_1\}.$$

We expand g at the cycles $\Gamma_{1,2}$ in the Laurent series:

$$g|_{\Gamma_1} = \frac{-h}{zw^2}\sum_{k\geq 0}\left(\frac{z}{w}\right)^k, \qquad g|_{\Gamma_2} = \frac{h}{z^2w}\sum_{k\geq 0}\left(\frac{w}{z}\right)^k.$$

Both series are uniformly convergent on the cycles. After integrating them, we obtain:

$$\frac{1}{(2\pi i)^2}\int_{\Gamma_1} g\,dz\wedge dw = -\frac{\partial h}{\partial w}(0,0),$$
$$\frac{1}{(2\pi i)^2}\int_{\Gamma_2} g\,dz\wedge dw = \frac{\partial h}{\partial z}(0,0).$$

The difference, we have just observed, results from a fact that the cycles Γ_1 and Γ_2 are not homologous in $\mathbb{C}^2\setminus\{zw(z-w)=0\}$.

The higher dimensional approach to residues consists on the notion of local residuum at a point a of a meromorphic form of type:

$$\omega = \frac{h(z)}{f_1(z)\dots f_n(z)}\,dz_1\wedge\dots\wedge dz_n, \tag{3.2}$$

where h and f_i's are holomorphic functions.

The definition given below agrees with one given by A. Grothendieck as $\mathrm{Res}\left[\begin{smallmatrix} h\,dz \\ f_1\dots f_n\end{smallmatrix}\right]$ (cf. [6], [8]).

DEFINITION 3.2. Let $a \in \mathbb{C}^n$ be an isolated zero of the holomorphic map $(f_1, \dots, f_n)$. The *local residue of the n-form* (3.2) is the integral:

$$\operatorname{res}_a \omega = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma(\varepsilon)} \omega, \tag{3.3}$$

where $\Gamma(\varepsilon) = \{z\colon |f_j(z)| = \varepsilon_j\}$ and $\varepsilon_j > 0$ are small numbers such that $\Gamma(\varepsilon) \subset \{|z| < \varepsilon_0\}$ is a compact non-singular cycle oriented in such a way that $d \arg f_1 \wedge \dots \wedge d \arg f_n > 0$.

In general, it is not easy to compute the local residue of a given non-trivial form. Below some calculations to be used in the sequel are presented.

EXAMPLE 3.3. If

$$\mathcal{J}(a) = \det \left\{ \frac{\partial f_i}{\partial z_j} \right\} (a) \neq 0, \tag{3.4}$$

i.e. the hypersurfaces $\{f_i = 0\}$ intersect transversely at a, then

$$\operatorname{res}_a \omega = \frac{h(a)}{\mathcal{J}(a)}. \tag{3.5}$$

This follows directly from the Cauchy formula (3.1), after changing coordinates from $(x_1, \dots, x_n)$ to $(f_1, \dots, f_n)$.

EXAMPLE 3.4. Let $f_1 = P(z,w) + \dots$, $f_2 = Q(z,w) + \dots$, $h(z,w) = R(z,w) + \dots$, where P, Q, R are homogeneous polynomials of degrees p, q, r respectively, and the dots denote higher order terms. Assume also that

$$P = \prod_{i=1}^{p} (z - a_i w),\ Q = \prod_{i=1}^{q} (z - b_i w),\ R = \prod_{i=1}^{r} (z - c_i w), \tag{3.6}$$

where

$$a_i \neq a_j \ (i \neq j), \quad b_i \neq b_j \ (i \neq j), \quad a_i \neq b_j, \tag{3.7}$$

$$r + 2 = p + q. \tag{3.8}$$

Then we have

$$\operatorname{res}_0 \frac{h\, dz \wedge dw}{f_1 f_2} = \sum_{i=1}^{p} \operatorname{res}_{a_i} \frac{\widetilde{R}(u)}{\widetilde{P}(u)\widetilde{Q}(u)} = -\sum_{j=1}^{q} \operatorname{res}_{b_j} \frac{\widetilde{R}(u)}{\widetilde{P}(u)\widetilde{Q}(u)}, \tag{3.9}$$

where $\widetilde{P}(u) = \prod (u - a_i)$, $\widetilde{Q}(u) = \prod (u - b_i)$, $\widetilde{R}(u) = \prod (u - c_i)$.

PROOF. By the assumptions (3.7) and (3.8) it suffices to consider the integral

$$\frac{1}{(2\pi i)^2} \iint_{\substack{|P| = \varepsilon_1 \\ |Q| = \varepsilon_2}} \frac{R\, dz\, dw}{PQ}.$$

Putting $z = uw$, which corresponds to the blow-up at 0, we obtain the integral

$$\frac{1}{(2\pi i)^2}\iint \frac{\widetilde{R}(u)\,du\,dw}{\widetilde{P}(u)\widetilde{Q}(u)w}. \tag{3.10}$$

along the 2-cycle

$$\widetilde{\Gamma}(\varepsilon) = \{|w|^p\cdot|\widetilde{P}(u)| = \varepsilon_1,\ |w|^q\cdot|\widetilde{Q}(u)| = \varepsilon_2\}.$$

The projection of the $\widetilde{\Gamma}(\varepsilon)$ onto the u-plane gives the curve (1-cycle)

$$\Delta(\delta) = \{|\widetilde{P}^{\widetilde{q}}(u)\widetilde{Q}^{-\widetilde{p}}(u)| = \delta\}, \tag{3.11}$$

where $\delta = \varepsilon_1^{\widetilde{q}}\varepsilon_2^{-\widetilde{p}} = const$, and $\widetilde{p} = p/\gcd(p,q)$, $\widetilde{q} = q/\gcd(p,q)$. It is then clear that (3.10) equals

$$\pm\frac{1}{2\pi i}\int_{\Delta(\delta)} \frac{\widetilde{R}\,du}{\widetilde{P}\widetilde{Q}}.$$

Here the sign and the orientation of $\Delta(\delta)$ should be properly chosen. We take δ positive and small, such that $\Delta(\delta)$ is an union of small cycles around a_i, i.e. $\Delta_i(\delta) \approx \{|u - a_i| = \text{const}\}$. The 2-cycle $\widetilde{\Gamma}(\varepsilon)$ becomes sum of cycles $\widetilde{\Gamma}_i(\varepsilon)$ that are approximate tori $\Delta_i(\delta)\times\{|w| = \text{const}\}$. The orientation is given by $d\arg P\wedge d\arg Q = d\arg u\wedge d\arg w$, provided that $\Delta_i(\delta)$ and$\{|w| = \text{const}\}$ are oriented in the standard way; (it is because $Q \approx \text{const}\cdot w^q$). Therefore

$$\frac{1}{(2\pi i)^2}\iint_{\widetilde{\Gamma}_i(\varepsilon)}\omega = \operatorname{res}_{a_i}\frac{\widetilde{R}}{\widetilde{P}\widetilde{Q}} = \frac{\widetilde{R}(a_i)}{\widetilde{Q}(a_i)\cdot\prod_{j\neq i}(a_i - a_j)}.$$

Note, that when we choose $\delta\to\infty$ in (3.11), we obtain $\Gamma(\varepsilon)$ as a union of tori around the points $(b_i, 0)$, but with reversed orientation. This agrees with the formula $\sum \operatorname{res}\widetilde{R}/(\widetilde{P}\widetilde{Q}) = 0$.

The formula (3.9) holds also in case when some of the points a_i coincide, as well as when some of b_j's do. However, it becomes (in general) false, when some a_i equals b_j. □

The next result is fundamental in our paper. Its proof is rather long and can be found in [6] and [10].

Theorem 3.5 (Residue theorem). *Let M be an analytic complex manifold without boundary and $X_1,\ldots,X_n$ be a system of effective divisors on M with a discrete intersection $Y = X_1\cap\ldots\cap X_n$. Then, for any meromorphic n-form ω with poles along $X_1,\ldots,X_n$ we have*

$$\sum_{a\in Y}\operatorname{res}_a\omega = 0.$$

In this theorem we have used a notion of *divisor*, i.e. a finite formal sum $\sum n_\alpha V_\alpha$, $n_\alpha\in\mathbb{Z}$, of hypersurfaces V_α, and of *effective divisor*, i.e. a divisor with all $n_\alpha \geq 0$. In fact, in some charts (e.g. near infinity) we can have $f_i = \prod g_\alpha^{n_{i\alpha}}$, where g_α are reduced functions defining V_α.

The following theorem is a very important application of the residue theorem.

Thorem 3.6 (Cayley [3], Bacharach [1]). *Let A and B be two algebraic curves in $\mathbb{C}P^2$ of degrees p and q which intersect at $p \cdot q$ different points. Let $E \subset \mathbb{C}P^2$ be a curve of degree $r \leq p+q-3$ passing through $pq-1$ points of $A \cap B$. Then E passes also through the last point of intersection.*

Proof. Let $A = \{f_1 = 0\}$, $B = \{f_2 = 0\}$, $E = \{h = 0\}$. Let us consider the form $\omega = h\, dx \wedge dy / f_1 f_2$. The condition imposed on the degrees guarantees that ω has no poles on the line at infinity. In fact, near infinity we have $x = 1/z$, $y = u/z$ and $dx \wedge dy \sim z^{-3}$, $h \sim z^{-r}$, $f_1 \sim z^{-p}$, $f_2 \sim z^{-q}$; so $\omega \sim z^{p+q-r-3}$. Therefore all the possible residual points are finite, and the formula (3.5) holds. For any $a_j \in A \cap B$ we have $\mathrm{res}_{a_j} \omega = h(a_j)/\mathcal{J}(a_i)$, and $\mathcal{J}(a_i) \neq 0$. Thus if $a_j \in E$ then $\mathrm{res}_{a_j} \omega = 0$; and conversely, if $\mathrm{res}_{a_j} \omega = 0$ then $a_j \in E$. By assumption, $a_1, \ldots, a_{pq-1} \in E$. Since $0 = \sum_{a_j} \mathrm{res}_{a_j} \omega = \mathrm{res}_{a_{pq}} \omega$, we obtain that $a_{pq} \in E$. □

4. Three applications of the Cayley–Bacharach theorem

4.1. Second proof of the Pascal theorem. Let $A = A_1 + A_2 + A_3$ and $B = B_1 + B_2 + B_3$ be the unions of three lines from the Pascal theorem, $D = d_1 d_2$ be the line through d_1 and d_2 and let E be $C + D$. We know that $\deg E = 3 = \deg A + \deg B - 3$ and $A \cap B = \{a_1, \ldots, a_6, d_1, d_2, d_3\}$, where $a_i \in C$. The curve E passes through all points of $A \cap B$ possibly but one, d_3. By the Cayley–Bacharach theorem E must pass also through d_3. But the point d_3 cannot lie on the conic C, in which case A_3 would intersect C at three points. Therefore $d_3 \in D$. □

Remark 4.2. The reader can observe that in the above proof one could choose A and B as arbitrary cubics passing through 6 points $a_1, \ldots, a_6$. They do not have to be unions of three lines. In this approach the uniqueness of A and B is lost, as well as the simple geometrical meaning.

Theorem 4.3 (Inverse Pascal theorem). *If a hexagon has the property that its opposite sides intersect at three colinear points d_1, d_2 and d_3, then its vertices lie on a conic.*

Proof. Let $H = A_1 A_2 A_3 B_1 B_2 B_3$ be the hexagon, where A_i, B_j's are the lines containing sides of H. Denote $a_1 = A_1 \cap B_3$, $a_2 = A_1 \cap A_2$, $a_3 = A_2 \cap A_3$, $a_4 = A_3 \cap B_1$, $a_5 = B_1 \cap B_2$, $a_6 = B_2 \cap B_3$. Let C be the (unique) conic through 5 points $a_1, \ldots, a_5$, and D – the line through d_1, d_2 and d_3.

As before, we put $A = A_1 + A_2 + A_3$, $B = B_1 + B_2 + B_3$ and $E = C + D$. Theorem 3.5 implies then that E contains a_6. Hence $a_6 \in C$. □

The first new result in our work is the following theorem.

Theorem 4.4. *Let $C \subset \mathbb{C}P^2$ be a general cubic. Take three general lines A_1, A_2, A_3 intersecting C at points a_1, a_2, a_3, b_1, b_2, b_3 and c_1, c_2, c_3, respectively. Let B_1 be a conic through a_1, a_2, b_1, c_1, c_2, B_2 – a line through a_3 and b_2 and B_3 a line through b_3 and c_3. Define $A = A_1 + A_2 + A_3$, $B = B_1 + B_2 + B_3$. Then the additional 3 points of the intersection A with B (i.e. $d_1 = A_2 \cap B_1 \setminus \{b_1\}$, $d_2 = A_3 \cap B_2$, $d_3 = A_1 \cap B_3$) are colinear.*

Proof. Denote by D the line through d_1 and d_2 and define the quartic $E = C + D$. Then the assumption of Theorem 3.5 are satisfied, $\deg E = 4 = 3 + 4 - 3 = \deg A + \deg B - 3$. Hence $d_3 \in E$. Since $d_3 \notin C$, we conclude that $d_3 \in D$, i.e. d_1, d_2, d_3 are colinear. □

Remark 4.5. As we have already mentioned in Introduction, it is possible to formulate other theorems similar to the Theorem 4.4. Since at the moment we do not have a complete classification of the cases, where the Cayley–Bacharach theorem can be directly applied, we postpone presentation of those results to further publication.

Remark 4.6. In the proofs of Theorems 2.4, 4.3 and 4.4 three properties of constructed curves were important:

(i) the curves A and B have regular intersections,
(ii) there is only one line passing through 2 different points,
(iii) there is only one conic passing through 5 points in general position.

In fact, the condition (i) can be weakened. The case when A and B have tangency point is the limit of regular cases. One of the additional points d_i tends to a point $a_j \in C$. In the residuum integral the order of pole increases, but the line D passes through $d_i = a_j$. So the residuum still remains zero.

The property (ii) does not hold when the two points coincide. In that case either the line is fixed by the tangency (e.g. to C at $a_i = a_j$) or, when choosing $D = d_i d_j$ one has a possibility to fix the pair (d_i, d_j).

The condition (iii) does not hold only when 4 points (e.g. $e_1, \ldots, e_4$ of $e_1, \ldots, e_5$) lie on one line L. If the latter point e_5 does not belong to L, we have a pencil of conics with the base set $L \cup \{e_5\}$. If all $e_1, \ldots, e_5 \in L$, then we are given a net of conics with the base L. In the above application we have not encountered such degeneracies. But in the next section this phenomenon will play a crucial role.

5. Eight points on a rational cubic

A rational algebraic curve C is a curve which admits a parametrization $\mathbb{C}P^1 \to C$, which doesn't have to be one-to-one. It means that the normalization of C is diffeomorphic to the projective line. A rational cubic, for instance, must have a singular points (otherwise it would be an elliptic curve of genus 1). It is isomorphic either to the quasi-homogeneous curve

$$y^2 = x^3, \tag{5.1}$$

or to the curve

$$y^2 = x^3 + x^2, \tag{5.2}$$

that has one simple double point. In the latter case the parametrization is given by $x = -1 + t^2, y = t(t^2 - 1)^2$. The curves (5.1) and (5.2) have only one singular point, namely $o = (0, 0)$.

Let C be a rational cubic with a singular point o. Take 8 points $a_1, \ldots, a_8$ on C in general position. Let A_1 be the unique conic through $o, a_1, \ldots, a_4$ and let A_2 be the conic through $o, a_5, \ldots, a_8$. We shall denote by A the sum $A_1 + A_2$.

We define a curve $B = B_1 + B_2$ in two ways. Either

(a) B_1 is the conic through o, a_1, a_2, a_5, a_6 and B_2 the conic through o, a_3, a_4, a_7, a_8,

or

(b) B_1 is the conic that passes through o, a_1, a_2, a_3, a_5 and B_2 the conic through o, a_4, a_6, a_7, a_8.

Other possibilities are provided by permutations of the set $\{a_1, \ldots, a_8\}$. We shall not threat them as different.

The curves A and B intersect in the points $a_1, \ldots, a_8$ with multiplicity 1, in o with multiplicity 4 and in 4 additional points d_1, d_2, d_3, d_4.

The following result is the just generalization of the Pascal theorem to the case of rational cubic curves.

Theorem 5.1. *The points d_1, d_2, d_3, d_4 lie on one line.*

Proof. The cases (a) and (b) are particular cases of the following situation: A is a quartic passing through $a_1, \ldots, a_8$ with a double point at o, while B is another quartic with the same properties. We prove the theorem first in that situation. The proof of Theorem 5.1 follows, since the property that 4 points are colinear is closed.

From now on we shall assume that both A and B are generic in the linear system $\mathcal{L}$ of quartic passing through $a_1, \ldots, a_8$ that have double point at o. The following lemma will be proved later.

Lemma 5.2. *If the points $a_1, \ldots, a_8 \in C$ are in general position and the quartics $A, B \in \mathcal{L}$ are typical, then:*

(i) *the intersections of A and B at $a_1, \ldots, a_8$, d_1, d_2, d_3, d_4 are non-degenerate,*

(ii) *the 4 tangent directions of $A\cup B$ at o are different.*

Consider the 2-form

$$\omega = \frac{gh\,dx\wedge dy}{f_1f_2},$$

where $f_1(x,y)$, and $f_2(x,y)$ define respectively quartics A and B, whereas $g(x,y)$ defines the cubic C and $h(x,y)$ is a quadratic polynomial. Lemma 5.2 implies that the local residua of ω can be calculated using the formulae (3.5) and (3.9) (from Examples 3.3 and 3.4). In particular, if $h(d_i)=0$ then $\mathrm{res}_{a_i}\omega=0$. Analogously, if $h(o)=0$ then $\mathrm{res}_o\omega=0$. Let us denote also by ω_0 the form $g\,dx\wedge dy/f_1f_2$. We have two possibilities:

(α) $\mathrm{res}_o\omega_0\neq 0$,
(β) $\mathrm{res}_o\omega_0=0$.

We claim that there may hold only (β). Suppose conversely, i.e. $\mathrm{res}_o\omega_0\neq 0$. We assume in (5.3) h to vanish at $d_1,\dots d_4$. We have then $\mathrm{res}_{a_i}\omega=0$, $\mathrm{res}_{d_i}\omega=0$ and $\mathrm{res}_o\omega=h(o)\cdot\mathrm{res}_o\omega_0$. By virtue of the residue Theorem 3.4 the equality $h(o)=0$ must hold for any quadratic polynomial vanishing at $d_1,\dots,d_4$. This implies that three of the points $d_1,\dots,d_4$ lie on one line L passing through o. Let us suppose that these are d_1, d_2, d_3. Choose now h to be a linear polynomial vanishing at d_1, d_2, d_3, o. Applying the residue theorem, we obtain that h vanishes also at d_4. It would mean that all five points d_1, d_2, d_3, d_4, o lie on L. But than L would intersect the quartic A at five points, what contradicts the genericity. So the case (α) has been excluded. In particular, $\mathrm{res}_o\omega=0$, whatever h is.

Let us now take h quadratic and vanishing at three points of d_1, d_2, d_3, d_4. By residue theorem h has to be zero also at the fourth one. The only outcome from this seemingly tangible situation is the fact that all four points d_1, d_2, d_3, d_4 lie on one line D; it, of course, does not pass through o. The geometrical picture is presented at Figure 3.

(It is worth to mention that Figure 3 was made using the computer programm PASCAL, which uses the inverse Pascal theorem in construction of a conic through 5 points.)

The proof of Theorem 5.1 has been completed. □

Proof of the Lemma 5.2. It is enough to find two quartics A and B that satisfy (i) and (ii), without specifying a priori the 8–ple $a_1,\dots,a_8$. Take $A=A_1+A_2$, where A_1 and A_2 are two conics through o that intersect transversely $C\setminus\{o\}$ at 8 different points, and with different tangent directions at o and not tangent to any branch of C at o. As B we shall take $C+M$, where M is some line avoiding $o,a_1,\dots,a_8$. Since the conditions (i) and (ii) are (Zariski) open, they hold for generic 8-ple $(a_1,\dots,a_8)$ and generic quartics $A,B\in\mathcal{L}$.

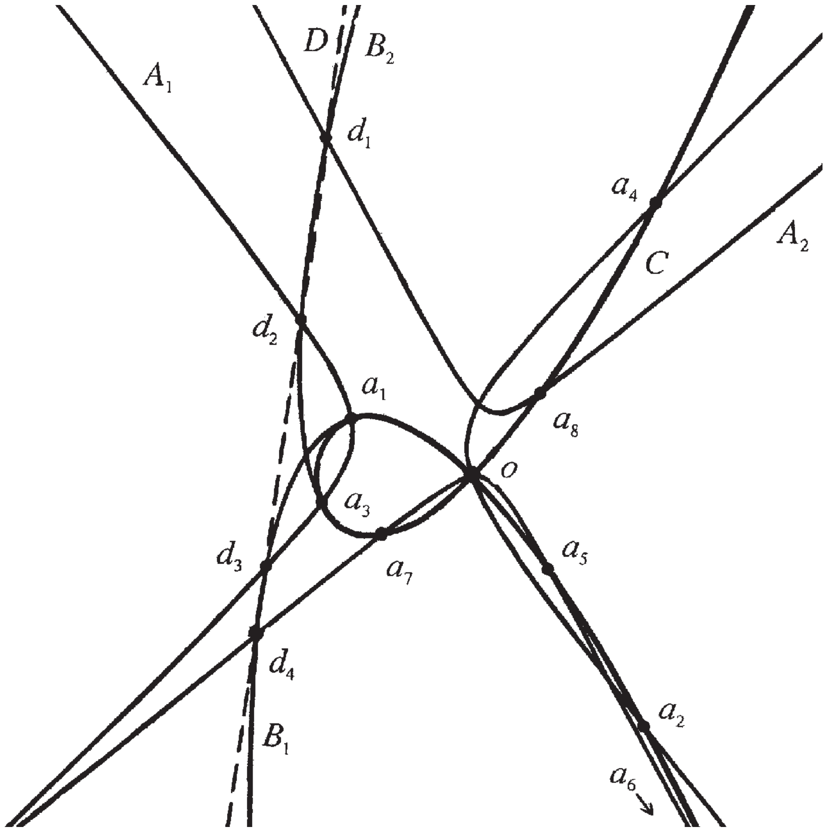

Figure 3

Note that in the latter example the points $d_1, \dots, d_4$ lie on M. □

We finish this section by proving a theorem dual to Theorem 5.1. Since the dual curve to a generic cubic curve is a curve of higher degree, we restrict our considerations to the case of the quasi-homogeneous cubic (5.1). This curve is simply connected, has exactly one singular point and exactly one inflection point ∞ (at infinity). We denote by L_∞ the line tangent to C at ∞.

Theorem 5.3. *Let $C \subset \mathbb{C}P^2$ be a simply connected cubic with the inflection point ∞. Let $a_1, \dots, a_8 \in C \setminus \infty$ and let $L_1, \dots, L_8$ be the lines tangent to C at a_i. Define A_1 as the conic tangent to L_∞, L_1, L_2, L_3, L_4, A_2 – the conic tangent to L_∞, L_5, L_6, L_7, L_8, B_1 – the conic tangent to L_∞, L_1, L_2, L_5, L_6 and B_2 – the conic tangent to L_∞, L_3, L_4, L_7, L_8. Denote $A = A_1 + A_2$ and $B = B_1 + B_2$. There exist* 4 *additional lines M_1, M_2, M_3, M_4 which are tangent to A and to B. Then the lines M_j intersect at one point. The same statement* *holds when we replace B_1 by the conic tangent to L_∞, L_1, L_2, L_3, L_5 and B_2 by the conic tangent to L_∞, L_4, L_6, L_7, L_8.*

Proof. The dual curve to the quasi-homogeneous curve $y^2 = x^3$ is the cubic $27q + 4p^3 = 0$; (where $y = px + q$ is the equation for lines tangent to C). The cusp point $x = y = 0$ corresponds to the inflection point $\infty : p = q = 0$.

The dual to a conic is a conic. The dual to a point is a line (of lines through it). In particular, the point of intersection of two curves corresponds to a line tangent to the two dual curves.

Finally, the dual to a line (e.g. the line D from the proof of Theorem 5.1) is a point (i.e. the common point of the lines M_j). □

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