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MASSERA'S THEOREM FOR QUASI-PERIODIC DIFFERENTIAL EQUATIONS

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Dedicated to Professor Granas

Abstract. For a scalar, first order ordinary differential equation which depends periodically on time, Massera's Theorem says that the existence of a bounded solution implies the existence of a periodic solution. Though the statement is false when periodicity is replaced by quasi-periodicity, solutions with some kind of recurrence are anyway expected when the equation is quasi-periodic in time. Indeed we first prove that the existence of a bounded solution implies the existence of a solution which is quasi-periodic in a weak sense. The partial differential equation, having our original equation as its equation of characteristics, plays a key role in the introduction of this notion of weak quasi-periodicity. Then we compare our approach with others already known in the literature. Finally, we give an explicit example of the weak case, and an extension to higher dimension for a special class of equations.

1. Introduction

Let $\omega_1, \ldots, \omega_N$ be real numbers which are linearly independent over the rationals and let us consider the scalar differential equation

$$\dot{x} = F(\omega_1 t, \ldots, \omega_N t, x). \tag{1.1}$$

Here $F = F(\theta_1, \ldots, \theta_N, x)$ is a continuous function which is 1-periodic with respect to each θ_i. A solution $x(t)$ is said to be quasi-periodic if it can be

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expressed in the form

$$x(t) = u(\omega_1 t, \dots, \omega_N t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $u = u(\theta_1, \dots, \theta_N)$ is continuous and 1-periodic in each θ_i. The function u satisfies the partial differential equation

$$\omega_1 \frac{\partial u}{\partial \theta_1} + \dots + \omega_N \frac{\partial u}{\partial \theta_N} = F(\theta_1, \dots, \theta_N, u) \tag{1.3}$$

in a distributional sense. Since the linear flow

$$\dot{\theta}_1 = \omega_1, \dots, \dot{\theta}_N = \omega_N \tag{1.4}$$

is minimal on the torus, one proves easily that (1.2) defines a one-to-one correspondence between the quasi-periodic solutions of (1.1) and the periodic solutions of (1.3) which are continuous. When F is monotone in x the existence problem can be analyzed using techniques coming from the theory of almost periodic functions [15], [11], [6] and also from the theory of semilinear elliptic partial differential equations [3]. When F is not monotone it is still possible to obtain some existence results using KAM theory (see for instance [14]). However this approach seems to require strong restrictions on the frequencies $\omega_1, \dots, \omega_N$ and on the regularity and size of F.

To understand the intrinsic difficulties of the quasi-periodic problem for (1.1) it can be useful to go back to an example due to Opial [16]. This example consists in an equation of the type (1.1) with two frequencies ($N = 2$) and such that all solutions are bounded but none of them is quasi-periodic. This is in contrast with the periodic case ($N = 1$). In this simpler situation Massera's Theorem [12] says that the existence of a bounded solution implies the existence of a periodic solution. Opial's example suggests that many of the results valid for scalar periodic differential equations should not have an extension to the quasi-periodic case. This is the line of thought in the paper [7], where Fink and Frederickson modified the example in [16] in order to construct a dissipative equation without quasi-periodic solutions. The same example of [7] can be used to show that the method of upper and lower solutions does not work in the quasi-periodic case. Other related and interesting examples can be seen in [13], [23], [25], [10].

From the point of view of the partial differential equation one can say that the main difficulty lies in the lack of regularity of the solutions. In [3] Brezis and Nirenberg obtained a result on the existence of L^∞-solutions of (1.3) on the torus. Their result can be applied to the example in [7] to construct an equation (1.3) having solutions on the torus but such that all of them are discontinuous.

In this paper we are interested in the effect produced by the discontinuous solutions of (1.3) on the ordinary differential equation. This will lead us to a notion of weak quasi-periodicity and to a version of Massera's Theorem valid for quasi-periodic equations. In contrast to [3] we shall consider the equation (1.3) in a classical setting and not in the sense of distributions. To do this we shall interpret the differential operator $\omega_1\partial/\partial\theta_1 + \ldots + \omega_N\partial/\partial\theta_N$ as the directional derivative. Namely,

$$D_{\overrightarrow{\omega}}u(\theta_1,\ldots,\theta_N) = \lim_{h\to 0}\frac{1}{h}\{u(\theta_1+\omega_1 h,\ldots,\theta_N+\omega_N h) - u(\theta_1,\ldots,\theta_N)\}$$

with $\overrightarrow{\omega} = (\omega_1,\ldots,\omega_N)$. A solution on the torus of (1.3) will be a function $u = u(\theta_1,\ldots,\theta_N)$ which is bounded, upper or lower semi-continuous, 1-periodic in each variable, and such that $D_{\overrightarrow{\omega}}u$ exists everywhere and satisfies

$$D_{\overrightarrow{\omega}}u(\theta_1,\ldots,\theta_N) = F(\theta_1,\ldots,\theta_N, u(\theta_1,\ldots,\theta_N)).$$

With this definition, given any $(\theta_1,\ldots,\theta_N)$, the function

$$x_{(\theta_1,\ldots,\theta_N)}(t) = u(\omega_1 t+\theta_1,\ldots,\omega_N t+\theta_N)$$

is a bounded solution of the translated equation

$$\dot{x} = F(\omega_1 t+\theta_1,\ldots,\omega_N t+\theta_N, x). \tag{1.5}$$

The family $\{x_{(\theta_1,\ldots,\theta_N)}\}$ will be called a weak quasi-periodic family of solutions of (1.5). When u is continuous the classical concept of quasi-periodic solution of (1.1) is recovered. In fact in this case it is obvious how to reconstruct the whole family from one of its elements, say from $\theta_i = 0$. On the contrary, when u is only semi-continuous, we shall see that it is not always possible to recover the whole family from one function $x_{(\theta_1,\ldots,\theta_N)}$.

The rest of the paper is organized in four sections and we pass to discuss their contents. In Section 2 we prove that (1.3) has a solution on the torus as soon as (1.1) has a bounded solution. This will allow us to give a new proof of a result in [3] and also to deduce that the equation coming from Opial's example has infinitely many solutions on the torus. Notice that all of them must be discontinuous. To prove the result of Section 2 we shall reformulate our problem on the partial differential equation (1.3) with the terminology of topological dynamics. After this is done the proof follows from ideas which are well known to people in the field of dynamics. In particular, the same type of arguments were employed by Shen and Yi in [21] to study almost automorphic solutions of monotone flows. Section 3 is devoted to discuss the connections between the notions of almost automorphy and weak quasi-periodic families. Almost automorphic functions play a role in the theory of almost periodic equations and, in particular, they appear in the quasi-periodic case. See [6] and the more recent paper [21]. We shall prove that in a weak quasi-periodic family the function $x_{(\theta_1,\ldots,\theta_N)}$ is almost automorphic for a residual set of $(\theta_1,\ldots,\theta_N)$. As a consequence of an example constructed in Section 4 we shall show that two different solutions of (1.3) can produce the same almost automorphic solutions of (1.5). All this indicates that, in the context of quasi-periodic equations, the class of weak quasi-periodic families is wider than the class of almost automorphic solutions. In Section 4 we construct an equation (1.3) having a solution which is discontinuous on a set of full measure. The construction is based on some results by Johnson on the primitive of an almost periodic function (see [8], [9]). It only requires two frequencies ($N = 2$) and we shall show how this discontinuous solution produces some complexity in the dynamics of the three dimensional system

$$\dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \omega_2, \quad \dot{x} = F(\theta_1, \theta_2, x).$$

Finally in Section 5 we extend the theorem of Section 2 to certain classes of higher order equations. For instance, the results of this section will apply to

$$\ddot{x} + c\dot{x} = F(\omega_1 t, \dots, \omega_N t, x), \tag{1.6}$$

where $c \in \mathbb{R}$ and F is smooth and satisfies

$$F_x(\theta_1, \dots, \theta_N, x) \leq \frac{c^2}{4}. \tag{1.7}$$

The key property of first order scalar equations is the monotonicity of the flow; for an equation such as (1.6)–(1.7) this is no longer true. However they satisfy a property that will be sufficient for our purposes: the set of bounded solutions is totally ordered. The results of this section are inspired by [20]. In that paper R. A. Smith obtained an extension of Massera's Theorem valid for certain periodic differential equations of higher order. We finish the paper with an Appendix on an example by R. A. Johnson.

Notations. We shall work on the torus

$$\mathbb{T}^N = (\mathbb{R}/\mathbb{Z}) \times \ldots \times (\mathbb{R}/\mathbb{Z}), \quad N \geq 1.$$

A point in $\mathbb{T}^N$ will be denoted by

$$\Theta = (\overline{\theta_1}, \dots, \overline{\theta_N}), \quad \overline{\theta_i} = \theta_i + \mathbb{Z}.$$

Haar measure on $\mathbb{T}^N$ is indicated by μ and satisfies $\mu(\mathbb{T}^N) = 1$.

The flow associated to (1.4) is

$$\Theta \cdot t = (\overline{\theta_1 + \omega_1 t}, \dots, \overline{\theta_N + \omega_N t}).$$

We recall that this flow is ergodic with respect to μ.

A function $u = u(\theta_1, \dots, \theta_N)$ which is 1-periodic in each θ_i will be interpreted as a function on the torus, $u : \mathbb{T}^N \to \mathbb{R}$. The derivative in the direction $\overrightarrow{\omega}$ can be expressed as

$$D_{\overrightarrow{\omega}} u(\Theta) = \lim_{h \to 0} \frac{1}{h} \{u(\Theta \cdot h) - u(\Theta)\}, \quad \Theta \in \mathbb{T}^N.$$

2. First order equations on the torus

A function $u : \mathbb{T}^N \to \mathbb{R}$ belongs to the class $\mathcal{S}_+(\overrightarrow{\omega})$ it if is upper semi-continuous, the directional derivative $D_{\overrightarrow{\omega}} u$ exists everywhere and both functions u and $D_{\overrightarrow{\omega}} u$ are bounded on the torus. The class $\mathcal{S}_-(\overrightarrow{\omega})$ is defined in an analogous way if one replaces upper semi-continuity by lower semi-continuity. Notice that $\mathcal{S}_-(\overrightarrow{\omega}) = -\mathcal{S}_+(\overrightarrow{\omega})$.

Let $F : \mathbb{T}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function and let us consider the equation (1.3) on the torus. A solution of this equation will be a function $u \in \mathcal{S}_+(\overrightarrow{\omega}) \cup \mathcal{S}_-(\overrightarrow{\omega})$ such that

$$D_{\overrightarrow{\omega}} u(\Theta) = F(\Theta, u(\Theta)) \quad \text{for all } \Theta \in \mathbb{T}^N. \tag{2.1}$$

Our first result shows the connection of this definition with the concept of quasi-periodic solution.

Proposition 2.1. *Assume that $u \in C(\mathbb{T}^N)$ is such that the function $x(t)$ defined by* (1.2) *is a solution of* (1.1). *Then u is in $\mathcal{S}_+(\overrightarrow{\omega}) \cap \mathcal{S}_-(\overrightarrow{\omega})$ and satisfies* (2.1).

Proof. For each $s \in \mathbb{R}$ the function $x(t+s) = u(\omega_1 t + \omega_1 s, \ldots, \omega_N t + \omega_N s)$ solves (1.5) with $\theta_i = \omega_i s$. The continuity of u allows us to pass to the limit and prove that $u(\Theta \cdot t)$ solves (1.5) for arbitrary $\Theta = (\overline{\theta_1}, \ldots, \overline{\theta_N})$. Thus $D_{\overrightarrow{\omega}} u$ exists and (2.1) is satisfied. From this equation we deduce that $D_{\overrightarrow{\omega}} u$ is bounded and so $u \in \mathcal{S}_+(\overrightarrow{\omega}) \cap \mathcal{S}_-(\overrightarrow{\omega})$. $\square$

Next we discuss the connection between the previous definition and the method of characteristic lines. Let $u \in C(\mathbb{T}^N)$ be a solution of (2.1), then for each $\Theta \in \mathbb{T}^N$ the function $t \in \mathbb{R} \mapsto u(\Theta \cdot t) \in \mathbb{R}$ is a quasi-periodic solution of

$$\dot{x} = F(\Theta \cdot t, x). \tag{2.2}$$

When u is not continuous we can only say that it produces a family of bounded solutions of (2.2). Next we show that this process can be reversed.

Theorem 2.2. *Assume that for some $\Theta_0 \in \mathbb{T}^N$ the function $\varphi(t)$ is a bounded solution of* (2.2) *with $\Theta = \Theta_0$. Then there exist $u^\star \in \mathcal{S}_+(\overrightarrow{\omega})$ and $u_\star \in \mathcal{S}_-(\overrightarrow{\omega})$ which are solutions of* (2.1) *and satisfy*

$$\inf_{t \in \mathbb{R}} \varphi(t) \le u_\star(\Theta) \le u^\star(\Theta) \le \sup_{t \in \mathbb{R}} \varphi(t) \quad \textit{for } \Theta \in \mathbb{T}^N, \tag{2.3}$$

$$u_\star(\Theta_0 \cdot t) \le \varphi(t) \le u^\star(\Theta_0 \cdot t) \quad \textit{for } t \in \mathbb{R}. \tag{2.4}$$

Proof. Consider the Fréchet space $BC(\mathbb{R})$. This is the vector space of bounded and continuous functions from $\mathbb{R}$ into $\mathbb{R}$ endowed with the topology induced by uniform convergence on bounded intervals. Given $s \in \mathbb{R}$ we consider the linear operator

$$T_s : BC(\mathbb{R}) \to BC(\mathbb{R}), \quad T_s x(t) = x(t+s).$$

This map is an order-preserving isomorphism and $\{T_s x\}_{s\in\mathbb{R}}$ can be seen as an orbit of the so-called Bebutov flow (see [19] for more details).

In the space $\mathbb{T}^N \times BC(\mathbb{R})$ consider now the product flow

$$(\Theta, x) \cdot s = (\Theta \cdot s, T_s x) \tag{2.5}$$

and denote by Γ the closure of the orbit starting at (Θ_0, φ). Clearly it is an invariant subset of $\mathbb{T}^N \times BC(\mathbb{R})$; since φ is uniformly continuous, it is also a compact set. It is easy to prove that the projection of Γ on $\mathbb{T}^N$ is surjective. The other projection on $BC(\mathbb{R})$ can be described as the closure of $\{T_s\varphi \mid s \in \mathbb{R}\}$. This is the so-called hull of φ and will be denoted by H. For any $\Theta \in \mathbb{T}^N$ define

$$H_\Theta = \{x \in BC(\mathbb{R}) \mid (\Theta, x) \in \Gamma\}.$$

Notice that H_Θ is a compact and non–empty subset of the hull H. Moreover, every $x \in H_\Theta$ satisfies (2.2). When there is uniqueness for the initial value problem associated to this equation then H_Θ is totally ordered, and the idea is to consider its minimal and maximal elements. In general it is not so but it is not too difficult to prove that the functions

$$v_\Theta(t) = \sup\{x(t) \mid x \in H_\Theta\}, \quad w_\Theta(t) = \inf\{x(t) \mid x \in H_\Theta\}$$

are also bounded solutions of (2.2). Since $H_{\Theta\cdot s} = T_s(H_\Theta)$ we deduce that

$$v_{\Theta\cdot s} = T_s(v_\Theta), \quad w_{\Theta\cdot s} = T_s(w_\Theta). \tag{2.6}$$

If we define

$$u^\star(\Theta) = v_\Theta(0), \quad u_\star(\Theta) = w_\Theta(0)$$

then it is clear that $u^\star$ is upper semi-continuous and $u_\star$ is lower semi-continuous. Moreover, from (2.6), we deduce that $u^\star(\Theta \cdot t) = v_\Theta(t)$ and $u_\star(\Theta \cdot t) = w_\Theta(t)$. Thus $D_{\overrightarrow{\omega}} u^\star(\Theta) = dv_\Theta(t)/dt|_{t=0}$, $D_{\overrightarrow{\omega}} u_\star(\Theta) = dw_\Theta(t)/dt|_{t=0}$ and both functions satisfy (2.1). The rest of the proof is immediate. $\square$

Remark 2.3. It is interesting to discuss the previous proof when the solution $\varphi(t)$ is already quasi-periodic, say $\varphi(t) = u(\Theta_0 \cdot t)$ for a suitable $u \in C(\mathbb{T}^N)$. In such a case the space Γ is given by

$$\Gamma = \{(\Theta, \varphi_\Theta) \mid \Theta \in \mathbb{T}^N\},$$

where $\varphi_\Theta(t) = u(\Theta \cdot t)$. Since H_Θ is a singleton one has $v_\Theta = w_\Theta = \varphi_\Theta$. Thus $u_* \equiv u \equiv u^*$.

More discussions on the previous proof will be presented in the next section.

The equation (1.3) on the torus can also be understood in the sense of distributions. A function $u \in L^\infty(\mathbb{T}^N)$ satisfies

$$\overrightarrow{\omega} \cdot \nabla u = F(\Theta, u) \quad \text{in } \mathcal{D}'(\mathbb{T}^N) \tag{2.7}$$

if

$$\int_{\mathbb{T}^N} \{u D_{\overrightarrow{\omega}} \phi + F(\Theta, u)\phi\} = 0 \quad \text{for all } \phi \in C^\infty(\mathbb{T}^N).$$

Next result shows the connection between this concept of solution and the notion previously introduced.

Proposition 2.4.

(i) *Let $u \in \mathcal{S}_+(\overrightarrow{\omega}) \cup \mathcal{S}_-(\overrightarrow{\omega})$ be a solution of* (2.1). *Then u is also a solution of* (2.7).

(ii) *Let $u \in C(\mathbb{T}^N)$ be a solution of* (2.7). *Then u is in $\mathcal{S}_+(\overrightarrow{\omega}) \cap \mathcal{S}_-(\overrightarrow{\omega})$ and satisfies* (2.1).

Proof. Semi-continuous functions can be characterized as limits of monotone sequences of continuous functions. This implies that any function in $\mathcal{S}_+(\overrightarrow{\omega})$ or $\mathcal{S}_-(\overrightarrow{\omega})$ is measurable. In consequence, given any $u \in \mathcal{S}_+(\overrightarrow{\omega}) \cup \mathcal{S}_-(\overrightarrow{\omega})$, we know that u and $D_{\overrightarrow{\omega}} u$ belong to $L^\infty(\mathbb{T}^N)$. To complete the proof of (i) we notice that the identity

$$\int_{\mathbb{T}^N} \phi D_{\overrightarrow{\omega}} u = -\int_{\mathbb{T}^N} u D_{\overrightarrow{\omega}} \phi \quad \text{for all } \phi \in C^\infty(\mathbb{T}^N)$$

follows, after a passage to the limit, from

$$\int_{\mathbb{T}^N} \phi(\Theta) u(\Theta \cdot h)\, d\Theta = \int_{\mathbb{T}^N} \phi(\Theta \cdot (-h)) u(\Theta)\, d\Theta.$$

For this passage to the limit one can apply Lebesgue Dominated Convergence Theorem and it is convenient to notice that the quotient $[u(\Theta \cdot h) - u(\Theta)]/h$ can be dominated by any upper bound of $|D_{\overrightarrow{\omega}} u|$. This follows from the Mean Value Theorem.

To prove (ii) we first notice that the class of test functions in the definition of solution of (2.7) can be changed. Namely, $\mathcal{D}(\mathbb{T}^N) = C^\infty(\mathbb{T}^N)$ can be replaced by $\mathcal{D}(\mathbb{R}^N) = C_0^\infty(\mathbb{R}^N)$. This follows from standard arguments in the theory of distributions. Next we consider the linear mapping in $\mathbb{R}^N$,

$$C(\theta_1, \dots, \theta_N) = (\theta_1 + \omega_1 \theta_N, \dots, \theta_{N-1} + \omega_{N-1}\theta_N, \omega_N \theta_N).$$

Then $v = u \circ C$ is a function in $L^\infty(\mathbb{R}^N)$ satisfying

$$\frac{\partial v}{\partial \theta_N} = F(C(\theta_1, \dots, \theta_N), v) \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

We can now apply Fubini Theorem to deduce that for almost every $(\theta_1, \dots, \theta_N)$ in $\mathbb{R}^N$ the function $t \mapsto v(\theta_1, \dots, \theta_N + t)$ satisfies

$$\dot{x} = F(C(\theta_1, \dots, \theta_N) + (0, \dots, 0, t), x).$$

In principle it is a solution in the sense of distributions but, since the equation is ordinary, it can also be understood in the classical sense. Undoing the change of variables we notice that, for almost every Θ in $\mathbb{T}^N$, the function $x_\Theta(t) = u(\Theta \cdot t)$ is a solution of (2.2). We can now finish the proof by applying Proposition 2.1.□

As a first application of the previous results we go back to Opial's example in [16]. In this example all the solutions of (1.1) satisfy $|x(t) - x(0)| \leq 2$ for all $t \in \mathbb{R}$ but none of them is quasi-periodic. Thus we can apply Theorem 2.2 to deduce the existence of infinitely many solutions of (2.1) or (2.7). On the other hand the discussion after Proposition 2.1 implies that these solutions must be discontinuous.

Next we consider the example in [7]. Now the equation (1.1) is dissipative but has no quasi-periodic solutions. The construction in [7] implies that every constant c with $|c| \geq 2$ is an upper or a lower solution of (1.1) depending on the sign of c. Since there is a bounded solution of (1.1) with $|\varphi| \leq 2$ we can apply Theorem 2.2 to deduce the existence of a solution of (2.1) satisfying $|u| \leq 2$. Again this solution must be discontinuous. This example shows that the method of upper and lower solutions could not work for the quasi-periodic problem associated to (1.1) or for the search of continuous solutions of (2.1). In this last case we interpret the constants $c_+ \geq 2$ and $c_- \leq -2$ as functions defined on the torus.

We shall also mention the example constructed by Zhikov and Levitan in [25]. They found functions $a, b \in C(\mathbb{T}^2)$ for which the linear equation

$$\dot{x} + a(\Theta \cdot t)x = b(\Theta \cdot t)$$

has a bounded solution but no quasi-periodic solutions. This shows that even in the linear case one can find discontinuous solutions of (1.3). It is worth noticing that, in the same paper [25], the authors introduce a notion that can be interpreted in terms of the partial differential equation. They call it an invariant section and it can be seen as a solution of (1.3) in the almost everywhere sense. We thank Prof. R. A. Johnson for informing us about this interesting example.

To finish the section we discuss the connections with Theorem 2' in [3]. In that theorem Brezis and Nirenberg considered the equation

$$\overrightarrow{\omega} \cdot \nabla u + g(u) = p(\Theta) \quad \text{in } \mathcal{D}'(\mathbb{T}^N), \tag{2.8}$$

where g was continuous and locally of bounded variation and $p \in C(\mathbb{T}^N)$. The result in [3] applies for an arbitrary frequency vector $\overrightarrow{\omega}$. When $\omega_1, \dots, \omega_N$ are linearly independent over $\mathbb{Q}$ it says that (2.8) has a solution $u \in L^\infty(\mathbb{T}^N)$ if

$$\limsup_{u\to-\infty} g(u) < \int_{\mathbb{T}^N} p < \liminf_{u\to\infty} g(u). \tag{2.9}$$

It is well known that the characteristic equation $\dot{x}+g(x) = p(\Theta\cdot t)$ has a bounded solution if (2.9) holds. This follows from arguments like in [24] or [1] and one does not need to assume that g is locally of bounded variation. In consequence one can apply Theorem 2.2 to deduce the existence of solutions of (2.8) in $\mathcal{S}_\pm(\overrightarrow{\omega})$.

3. Quasi-periodicity in the weak sense and almost automorphic solutions

In this section we shall assume that the initial value problem associated to (1.5) has a unique solution for every $\Theta \in \mathbb{T}^N$. This will simplify the comparison with other results in the literature.

Given a solution of (2.1), $u \in \mathcal{S}_\pm(\overrightarrow{\omega})$, the function

$$x_\Theta(t) = u(\Theta \cdot t)$$

is a bounded solution of (2.2) for each $\Theta \in \mathbb{T}^N$. We shall say that the family $\{x_\Theta\}_{\Theta\in\mathbb{T}^N}$ is quasi-periodic in the weak sense. With this definition we can reinterpret Theorem 2.2 as a result of Massera type for (2.2). The proof of this Theorem suggests a characterization of the notion of weak quasi-periodic family which does not involve the partial differential equation. In fact we can define such a family as a function

$$\mathcal{F} : (\Theta, t) \in \mathbb{T}^N \times \mathbb{R} \mapsto x_\Theta(t) \in \mathbb{R}$$

which is bounded, upper or lower continuous and satisfies the properties

(3.1) For each $\Theta \in \mathbb{T}^N$ the function $t \in \mathbb{R} \mapsto x_\Theta(t)$ is a solution of (2.2).

(3.2) For each $\Theta \in \mathbb{T}^N$ and $t, s \in \mathbb{R}$ one has $x_{\Theta\cdot s}(t) = x_\Theta(t + s)$.

When $\mathcal{F}$ is continuous the functions x_Θ are quasi-periodic and we recover the classical situation. In this case the knowledge of a single x_Θ allows to reconstruct the whole family by density and continuity. This is no longer true when $\mathcal{F}$ (or u) is discontinuous. In the next section we shall construct an example of the type (2.2) with two families of solutions, $\{x_\Theta\}$ and $\{y_\Theta\}$, having the following properties: $\{x_\Theta\}$ is quasi-periodic in the classical sense and $\{y_\Theta\}$ is quasi-periodic in the weak sense, $x_\Theta \equiv y_\Theta$ on a residual set in $\mathbb{T}^N$, $x_\Theta \not\equiv y_\Theta$ on a set of full measure in $\mathbb{T}^N$. This shows that the notion of weak quasi-periodic family is collective.

Let $\{x_\Theta\}$ be a family of solutions which is quasi-periodic in the weak sense but not in the classical one. It seems natural to ask about the recurrence properties of the functions x_Θ. In principle one could think of a situation where the functions x_Θ are almost periodic and have a module which is not contained in $\langle \omega_1, \dots, \omega_N \rangle$. However this can be excluded because the equation is scalar and there is uniqueness (use [6, p. 231, Theorem 12.10] and then [24, p. 30, Theorem 4.1]).

There are several extensions of the notion of almost periodicity which have been applied to differential equations. We recall the almost periodicity in the sense of Besicovitch (employed in [25]) and also the concept of almost automorphic function (see [2], [22] and [21] for applications to differential equations). The rest of the section will be devoted to discuss the connections between weak quasi-periodicity and almost automorphy.

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost automorphic if from every sequence $(s_n) \subset \mathbb{R}$ it is possible to extract a subsequence (s_{n_k}) such that the translates $T_{s_{n_k}} f$ converge pointwise to a certain function g, while $T_{-s_{n_k}} g \to f$ also in a pointwise sense. The following result on the existence of almost automorphic solutions is an immediate consequence of Chapter 3 in [21]. From a different perspective it could also be thought as a theorem of Massera type.

Theorem 3.1. *Assume that for some $\Theta_0 \in \mathbb{T}^N$ the function $\varphi(t)$ is a bounded solution of* (2.2) *with $\Theta = \Theta_0$. Then there exists a residual set G in $\mathbb{T}^N$ such that for each $\Theta \in G$ the equation* (2.2) *has an almost automorphic solution $x_\Theta(t)$ satisfying*

$$\inf_{t \in \mathbb{R}} \varphi(t) \le x_\Theta(t) \le \sup_{t \in \mathbb{R}} \varphi(t), \quad t \in \mathbb{R}.$$

The ideas in [21] can also be applied to construct certain weak quasi-periodic families. To this end we consider the flow on $\mathbb{T}^N \times \mathbb{R}$ associated to the differential equations

$$\dot{\theta}_1 = \omega_1, \dots, \dot{\theta}_N = \omega_N, \quad \dot{x} = F(\theta_1, \dots, \theta_N, x).$$

Namely,

$$\Pi_t(\Theta, x_0) = (\Theta \cdot t, x(t; x_0, \Theta))$$

where $x(t; x_0, \Theta)$ is the solution of (2.2) with $x(0) = x_0$. Given a bounded solution $\varphi(t)$, the ω-limit set of $(\Theta_0, \varphi(0))$ is compact and invariant. This implies that there is a non-empty minimal set included in it. Now one could define

$$u^{\#}(\Theta) = \sup\{x/(\Theta, x) \in E\}, \quad u_{\#}(\Theta) = \inf\{x/(\Theta, x) \in E\}$$

and these functions would be solutions of (1.3) satisfying (2.3). However with this construction one cannot guarantee that (2.4) holds. In the example of the next section one can select $\varphi > 0$ in such a way that $u_{\#} = u^{\#} = 0$. Now we observe that it is possible to construct many other solutions of (1.3) by keeping the definition of $u^{\#}$ and $u_{\#}$ and allowing a more general class of sets E. In fact

$u^{\#}$ and $u_{\#}$ will be solutions as soon as E is a compact and invariant set. If we go back to the proof of Theorem 2.2 we notice that $u_\star$ and $u^\star$ were defined as

$$u^\star(\Theta) = \sup\{x/(\Theta, x) \in \Gamma\}, \quad u_\star(\Theta) = \inf\{x/(\Theta, x) \in \Gamma\},$$

where Γ was the closure of the orbit passing through $(\Theta_0, \varphi(0))$. From the point of view of topological dynamics this is perhaps a difference of our approach. Instead of looking for minimal sets we construct our solutions of the partial differential equation from compact invariant sets. This is important because there are equations with simple minimal sets but complicated attractors. Again we refer to the example of the next section, where the only minimal set is a torus but the attractor has a complicated topological structure.

Now we shall go in the opposite direction and deduce Theorem 3.1 from Theorem 2.2. Let $\{x_\Theta\}$ be a weak quasi-periodic family of solutions of (2.2). Define $\mathcal{C} \subset \mathbb{T}^N$ as the set of points of continuity of the map

$$\Theta \in \mathbb{T}^N \mapsto x_\Theta(0) \in \mathbb{R}.$$

The theory of semi-continuous functions implies that $\mathcal{C}$ is residual in $\mathbb{T}^N$ (see [4, p. 111, Corollary 7.6]). We will prove later on that $\mathcal{C}$ is invariant for the linear flow on the torus, this implies that its measure is either 0 or 1. In the example of Section 4 the set $\mathcal{C}$ does not coincide with $\mathbb{T}^N$ but has full measure. We shall prove the following characterization of $\mathcal{C}$.

Proposition 3.2. *In the previous setting the function x_Θ is almost automorphic if and only if $\Theta \in \mathcal{C}$.*

We need some preliminary lemmas. First of all, let us discuss the role played by the uniqueness of the initial value problem.

Lemma 3.3. *In the conditions of the previous proposition the following statements hold:*

(i) *if $\Theta_n \to \Theta$ with $\Theta \in \mathcal{C}$ then $x_{\Theta_n} \to x_\Theta$ uniformly on compact sets,*
(ii) *$\Theta \in \mathcal{C}$ implies $\Theta \cdot s \in \mathcal{C}$ for all s.*

Proof. (i) The definition of $\mathcal{C}$ implies that $x_{\Theta_n}(0) \to x_\Theta(0)$. Now, by continuous dependence with respect to parameters and initial conditions, we deduce that $x_{\Theta_n}(t) \to x_\Theta(t)$ uniformly on compact intervals.

The proof of (ii) follows from (i) and the property (3.2). □

The following lemma gives a criterion to decide whether a point $\Theta \in \mathbb{T}^N$ belongs to $\mathcal{C}$.

Lemma 3.4. *Let (X, d) be a metric space and D a dense subset of X. Assume that $f : X \to \mathbb{R}$ is a semi-continuous function. Then f is continuous at a point*

$x \in X$ if for each sequence $x_n \in D$ such that $x_n \to x$, there exists a subsequence x_{n_k} with $f(x_{n_k}) \to f(x)$.

Proof. Let $x \in X$ be a point where f is not continuous. There exist an $\varepsilon > 0$ and a sequence $y_n \in X$ such that $y_n \to x$ and $|f(y_n) - f(x)| \geq \varepsilon$ holds for all n. Assume for instance that f is an upper semi-continuous function. Then we must have $f(y_n) \leq f(x) - \varepsilon$ for large n. By density we can find $x_n \in D$ with $d(x_n, y_n) \to 0$, so that $x_n \to x$. Moreover, using the semi-continuity of f at y_n, we may choose x_n in such a way that $f(x_n) < f(y_n) + \varepsilon/2$ holds for all n. Summing up, $x_n \in D$, $x_n \to x$ and $f(x_n) < f(x) - \varepsilon/2$. The sequence $f(x_n)$ does not admit any subsequence converging to $f(x)$. $\square$

We shall apply this Lemma when $X = \mathbb{T}^N$, $D = \{\Theta_0 \cdot s \mid s \in \mathbb{R}\}$ for a given Θ_0, and $f(\Theta) = x_\Theta(0)$. We are now ready to prove the above stated proposition. The proof will follow along the lines of Chapter 2 in [21].

Proof of Proposition 3.2. We will use the same notations as in the proof of Theorem 2.2. Fix a point $\Theta_0 \in \mathcal{C}$ (we have residually many) and repeat the construction with $\varphi = x_{\Theta_0}$. We claim that, if $\Theta \in \mathcal{C}$, then $H_\Theta = \{x_\Theta\}$.

To prove it, assume that $(\Theta, x) \in \Gamma$. By definition there exists a sequence s_n such that $\Theta_0 \cdot s_n \to \Theta$ and $T_{s_n} x_{\Theta_0} \to x$. On the other hand, $T_{s_n} x_{\Theta_0} = x_{\Theta_0 \cdot s_n}$, and Lemma 3.3 says that $x_{\Theta_0 \cdot s_n} \to x_\Theta$. Thus $x \equiv x_\Theta$, proving the claim.

Coming back to the statement of the proposition, let us assume that $\Theta \in \mathcal{C}$ and prove that x_Θ is almost automorphic. Take a sequence s_n. Again due to standard compactness arguments, there exists a subsequence s_{n_k} such that $\Theta \cdot s_{n_k} \to \Theta_*$, $\Theta_* \cdot (-s_{n_k}) \to \Theta$, $T_{s_{n_k}} x_\Theta \to y$ and $T_{-s_{n_k}} y \to z$, for some suitable $\Theta_* \in \mathbb{T}^N$ and $y, z \in BC(\mathbb{R})$. As a consequence of the previous claim, $(\Theta, x_\Theta) \in \Gamma$; since Γ is closed, the same is true for (Θ_*, y) and (Θ, z). Again using the claim, $z \equiv x_\Theta$, which proves the almost automorphy of x_Θ.

To prove the converse, let us now assume that x_Θ is almost automorphic and prove that $\Theta \in \mathcal{C}$. To this aim we shall use Lemma 3.4. Take any s_n such that $\Theta_0 \cdot s_n \to \Theta$, and notice that $\Theta \cdot (-s_n) \to \Theta_0 \in \mathcal{C}$. By Lemma 3.3, $T_{-s_n} x_\Theta = x_{\Theta \cdot (-s_n)} \to x_{\Theta_0}$. Moreover, since x_Θ is almost automorphic, there exist a subsequence s_{n_k} such that $T_{s_{n_k}} x_{\Theta_0} \to x_\Theta$ pointwise. In particular $x_{\Theta_0 \cdot s_{n_k}}(0) = T_{s_{n_k}} x_{\Theta_0}(0) \to x_\Theta(0)$, and Lemma 3.4 applies to reach the conclusion. $\square$

Summing up, in a weak quasi-periodic family a residual set is made by almost automorphic solutions. Moreover, the solution is quasi-periodic if and only if all the functions in the family are almost automorphic (see [22, p. 738, Theorem 3.3.1] for a similar characterization of almost periodic functions).

4. An example of the weak case

We shall work with the linear flow on $\mathbb{T}^2$

$$\dot{\overline{\theta_1}} = 1, \quad \dot{\overline{\theta_2}} = \overline{\xi} \quad \text{where } \xi \notin \mathbb{Q}, \tag{4.1}$$

and we will be interested in the bounded solutions of the parametric equation

$$\dot{x} + G(x) + A(\Theta \cdot t)x = 0 \tag{4.2}$$

for suitable $G \in C^\infty(\mathbb{R})$ and $A \in C(\mathbb{T}^2)$. Here $\Theta = (\overline{\theta_1}, \overline{\theta_2}) \in \mathbb{T}^2$ and $\Theta \cdot t = (\overline{\theta_1 + t}, \overline{\theta_2 + \xi t})$.

The nonlinearity G will vanish in a neighborhood of the origin and will behave at infinity in such a way that the system is dissipative. Precisely we will assume that it satisfies

$$G \text{ is odd and nonnegative on } [0, \infty), \tag{4.3}$$

$$G(x) = 0 \quad \text{if } x \in [-1, 1], \tag{4.4}$$

$$G(x) \geq \|A\|_\infty \, x + 1 \quad \text{if } x \in [2, \infty). \tag{4.5}$$

As a consequence of (4.3), (4.5) every solution of (4.2) will eventually enter in a compact region inside the strip $|x| < 2$. In particular every bounded solution must satisfy $\|x\|_\infty < 2$. Define K_Θ to be the class of bounded solutions of (4.2). K_Θ can be seen as a compact subset of $BC(\mathbb{R})$ which always contains the trivial solution. It is totally ordered and we will denote its maximal element by v_Θ. By symmetry the minimal one is $-v_\Theta$.

From the definition, it is easy to verify that

$$0 \leq v_\Theta(t) < 2, \quad v_\Theta(t + \tau) = v_{\Theta \cdot \tau}(t)$$

hold for every $\Theta \in \mathbb{T}^2$ and all $t, \tau \in \mathbb{R}$. In addition the map $(\Theta, t) \mapsto v_\Theta(t)$ is upper-semicontinuous. Summing up, the family $\{v_\Theta\}_{\Theta \in \mathbb{T}^2}$ is quasi-periodic in the weak sense. In other words, the function on $\mathbb{T}^2$ defined by

$$u(\Theta) = v_\Theta(0) \tag{4.6}$$

is a solution of the partial differential equation

$$D_{\overrightarrow{\omega}} u + G(u) + A(\Theta)u = 0, \tag{4.7}$$

where $\overrightarrow{\omega} = (1, \xi)$. This solution will be continuous if and only if the set $\mathcal{C}$ coincides with $\mathbb{T}^2$. Next result will show that it is possible to select A so that $\mathcal{C}$ has measure zero. This implies that u is not continuous and not even integrable in the Riemann sense. We recall that $\mathcal{C}$ is not negiglible from the topological point of view because it is always residual.

Proposition 4.1. *Assume that* (4.3)–(4.5) *hold. Then there exists a function* $A \in C(\mathbb{T}^2)$ *such that the solution of* (4.7) *defined by* (4.6) *satisfies*

- $\mathcal{C} = \{\Theta \in \mathbb{T}^2 \mid u(\Theta) = 0\}$,
- $u(\Theta) > 0$ *for almost every* $\Theta \in \mathbb{T}^2$.

At the end of the section we shall show that this proposition yields some complexity in the dynamics for the equation (4.2). For the moment we discuss some consequences of more analytic nature.

Equation (4.7) also admits the trivial solution $w = 0$, which is of course a classical solution. The solution u coincides with w on a residual set but it is positive almost everywhere. This implies that u and w are also different solutions when the equation is understood in a distributional sense. If we consider the proposition from the point of view of the ordinary differential equation (4.2), we see that the weak quasi-periodic family of solutions $\{v_\Theta\}_{\Theta\in\mathbb{T}^2}$ produces the same set of almost automorphic solutions, i.e. the trivial ones, as the trivial classical quasi-periodic solution. Roughly speaking, the theory of weak quasi-periodic solutions is not a subset of the theory of almost automorphic ones.

Proof of Proposition 4.1. Denote by $x(t; x_0, \Theta)$ the solution of (4.2) which takes the value x_0 at time $t = 0$. Since G vanishes on $[-1, 1]$, any solution satisfying $\|x\|_\infty \le 1$ will also solve the linear differential equation

$$\dot{y} = -A(\Theta \cdot t)y. \tag{4.8}$$

Note that the general solution of (4.8) is

$$y(t; x_0, \Theta) := x_0 e^{-\int_0^t A(\Theta\cdot s)\,ds}.$$

Since $G(x) \ge 0$ if $x \ge 0$, any positive solution of (4.2) will be a lower solution of (4.8). From here we deduce the following consequence:

- (Comparison Principle) $x(t; x_0, \Theta) \ge y(t; x_0, \Theta)$ if $t \le 0$ and $x_0 > 0$.

The rest of the proof will be organized in two claims where we impose certain conditions to the function A.

Claim 1. Assume that A has mean value zero and an unbounded primitive. Then

$$\mathcal{C} = \{\Theta \in \mathbb{T}^N \mid u(\Theta) = 0\}.$$

The inclusion $u^{-1}(0) \subset \mathcal{C}$ holds because u is upper semi-continuous and non-negative. To prove the other inclusion we notice that $u^{-1}(0)$ is residual in $\mathbb{T}^N$. Thus, given any $\Theta \in \mathcal{C}$, we can find a sequence Θ_n converging to Θ and such that $u(\Theta_n) = 0$. By the definition of $\mathcal{C}$ we conclude that $u(\Theta) = 0$.

The proof of the claim is not complete because the residual character of $u^{-1}(0)$ is not obvious. To prove this we first consider the set

$$\Omega = \{\Theta \in \mathbb{T}^N \mid \liminf_{t\to-\infty} a_\Theta(t) = -\infty\},$$

where $a_\Theta(t) := \int_0^t A(\Theta \cdot s)\,ds$. We can apply Theorem 3.7 of [8] to deduce that Ω is residual. The definition of Ω says that any positive solution of (4.8) will satisfy

$$\limsup_{t\to-\infty} y(t;x_0,\Theta) = \infty, \quad \Theta \in \Omega.$$

From the Comparison Principle we deduce that any positive solution of (4.2) will also be unbounded. In consequence $u(\Theta) = v_\Theta(0) = 0$ if $\Theta \in \Omega$. Thus Ω is contained in $u^{-1}(0)$ and so $u^{-1}(0)$ is residual.

Claim 2. Assume that there is a set $S \subset \mathbb{T}^N$ of full measure, $\mu(S) = 1$, such that

$$\inf_{t\in\mathbb{R}} a_\Theta(t) > -\infty \quad \text{if } \Theta \in S.$$

Then $u(\Theta) > 0$ for almost every $\Theta \in \mathbb{T}^N$.

If $\Theta \in S$ then $y(t;x_0,\Theta)$ will be bounded. For small x_0 this function will remain in $|x| \le 1$ and so it will coincide with $x(t;x_0,\Theta)$. This implies that $u(0) = v_\Theta(0)$ is strictly positive.

The proof is now finished because there are functions A satisfying simultaneously the requirements of the two claims. This was proved by Johnson in Example 3.12 of [9]. There it was assumed that the number ξ was of constant type. This is not really needed in Johnson's construction and we give the details in an appendix. $\square$

Next we show that the choice of A made in Proposition 4.1 produces some complexity in the skew-product flow on $\mathbb{T}^2 \times \mathbb{R}$ associated to equation (4.2). Consider the system

$$\dot{\overline{\theta_1}} = 1, \quad \dot{\overline{\theta_2}} = \overline{\xi}, \quad \dot{x} = G(x) - A(\overline{\theta_1}, \overline{\theta_2})x.$$

In principle we do not have information about the regularity of the function A and we cannot say that the associated vector field is smooth. However, its special form guarantees the uniqueness of solution for the initial value problem. Assuming that G satisfies the additional condition

$$\limsup_{x\to\infty} \frac{G(x)}{x} < \infty \tag{4.9}$$

the solutions are globally defined in $(-\infty,\infty)$ and our system defines a flow on $\mathbb{T}^2 \times \mathbb{R}$. The set $\Sigma = \{(\Theta, x) \mid \overline{\theta_1} = 0\}$ is a global section and we identify it to the cylinder $\mathbb{T} \times \mathbb{R}$. The associated Poincaré map $P : \Sigma \to \Sigma$ can be expressed as

$$(\overline{\theta}, x_0) \mapsto (\overline{\theta + \xi}, x(1; x_0, \widehat{\theta})), \quad \widehat{\theta} = (\overline{0}, \overline{\theta}) \in \mathbb{T}^2.$$

Here we follow the notation introduced in the proof of Proposition 4.1. Note that P is a skew-product homeomorphism with a dissipative structure. This means that, given an arbitrary $R > 0$, one can find an integer $N \geq 1$ such that

$$|p_2 P^n(\overline{\theta}, x_0)| \leq 2 \quad \text{if } n \geq N \text{ and } |x_0| \leq R$$

(p_2 is the projection of $\mathbb{T} \times \mathbb{R}$ onto $\mathbb{R}$).

We are interested in the structure of the set of bounded orbits, namely

$$\mathcal{B} = \{(\overline{\theta}, x_0) \in \mathbb{T} \times \mathbb{R} \mid \sup_{n \in \mathbb{Z}} |x(n; x_0, \widehat{\theta})| < \infty\}.$$

It is not difficult to prove that the intersection of $\mathcal{B}$ with the fiber $\{\overline{\theta}\} \times \mathbb{R}$ is

$$\mathcal{B}_{\overline{\theta}} = \{\overline{\theta}\} \times [-u(\widehat{\theta}), u(\widehat{\theta})].$$

Proposition 4.2. *Assume that G satisfies* (4.3)–(4.5) *and* (4.9) *and A is chosen as in Proposition* 4.1. *Then the set $\mathcal{B}$ is connected and locally connected at $\mathbb{T} \times \{0\}$, but it is not locally connected on a set of positive measure.*

This is the same kind of topological situation that Johnson found in [10].

Proof. The first two statements are a consequence of the structure of $\mathcal{B}_{\overline{\theta}}$. The last statement needs some work. We start with a preliminary observation. Let $E \subset \mathbb{T}^2$ be a set which is invariant with respect to the linear flow (4.1); that is, $E \cdot t = E$ for all $t \in \mathbb{R}$. Define $E_0 = E \cap (\{\overline{0}\} \times \mathbb{T})$. Then E has measure zero in $\mathbb{T}^2$ (resp. is dense in $\mathbb{T}^2$) if and only if E_0 has measure zero in $\mathbb{T}$ (resp. is dense in $\mathbb{T}$). Consider the sets $D = \{\overline{\theta} \in \mathbb{T} \mid u(\widehat{\theta}) > 0\}$ and, for any $\eta > 0$, $D_\eta = \{\overline{\theta} \in \mathbb{T} \mid u(\widehat{\theta}) \geq \eta\}$. Note that, due to the semi-continuity of u, D_η is a closed set for any $\eta > 0$. Moreover, from Proposition 4.1 and the previous remark with $E = u^{-1}(0)$, $E_0 = \mathbb{T} \setminus D$, we deduce that D has full measure in $\mathbb{T}$ and $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$. In particular $\mu(D_\eta) > 0$ for some $\eta > 0$.

The picture is then the following: $\mathcal{B}_{\overline{\theta}} \supset \{\overline{\theta}\} \times [-\eta, \eta]$ if $\overline{\theta} \in D_\eta$, and $\mathcal{B}_{\overline{\theta}} = \{(\overline{\theta}, 0)\}$ if $\overline{\theta} \notin D$. Since $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, the proof is complete if we show that the set $D'_\eta \subset D_\eta$ of the accumulation points of D_η has a positive measure. To this aim, just note that $D_\eta \setminus D'_\eta$ is at most countable, since $\mathbb{T}$ is a second-countable space; thus $\mu(D'_\eta) = \mu(D_\eta) > 0$. $\square$

5. An extension to higher order equations

We consider a fixed polinomial of degree $p \geq 2$

$$\mathcal{P}(\lambda) = \lambda^p + a_{p-1}\lambda^{p-1} + \ldots + a_1\lambda$$

with $a_1, \ldots, a_{p-1} \in \mathbb{R}$, and we shall study the partial differential equation

$$\mathcal{P}(D_{\vec{\omega}})u := D^p_{\vec{\omega}}u + a_{p-1}D^{p-1}_{\vec{\omega}}u + \ldots + a_1 D_{\vec{\omega}}u = F(\Theta, u), \quad \Theta \in \mathbb{T}^N, \tag{5.1}$$

with $F \in C(\mathbb{T}^N \times \mathbb{R})$. We also assume that F is locally Lipschitz-continuous with respect to u.

To make precise the concept of solution for this equation we introduce the classes $\mathcal{S}_+^p(\overrightarrow{\omega})$ and $\mathcal{S}_-^p(\overrightarrow{\omega})$. They are composed by the functions in $\mathcal{S}_+(\overrightarrow{\omega})$ or $\mathcal{S}_-(\overrightarrow{\omega})$ such that the directional derivatives $D^k_{\overrightarrow{\omega}}u$, $k = 1, \dots , p$ exist everywhere and are bounded on the torus. A solution of (5.1) is a function in $\mathcal{S}_+^p(\overrightarrow{\omega}) \cup \mathcal{S}_-^p(\overrightarrow{\omega})$ satisfying (5.1) in a pointwise sense.

We associate to (5.1) the family of ordinary differential equations

$$\mathcal{P}\left(\frac{d}{dt}\right)x := x^{(p)} + a_{p-1}x^{(p-1)} + \ldots + a_1 x' = F(\Theta \cdot t, x), \quad \Theta \in \mathbb{T}^N. \tag{5.2}$$

We would like to prove a result in the line of Theorem 2.2, connecting the existence of solutions of (5.1) with the bounded solutions of (5.2). To do this we need further assumptions.

First we introduce some terminology about linear homogeneous equations of the type

$$\mathcal{P}\left(\frac{d}{dt}\right)x = \sigma(t)x, \tag{5.3}$$

where $\sigma \in L^\infty(\mathbb{R})$. By a bounded solution $x(t)$ of this equation we understand a solution such that $x, x', \dots , x^{(p-1)}$ are in $L^\infty(\mathbb{R})$. We say that the equation (5.3) is *partially disconjugate* if every non-trivial bounded solution $x(t)$ satisfies

$$x(t) \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

As an example consider $x'' + cx' = -x$, $c \in \mathbb{R}$. This equation is partially disconjugate if $c \neq 0$ but not for $c = 0$.

Let σ_- and σ_+ be numbers satisfying $-\infty \leq \sigma_- < \sigma_+ \leq \infty$. We shall say that the couple (σ_-, σ_+) is *admissible* if every equation of the type (5.3) with

$$\sigma \in L^\infty(\mathbb{R}), \quad \sigma_- \leq \sigma(t) \leq \sigma_+ \quad \text{for almost every } t \in \mathbb{R}$$

is partially disconjugate.

Theorem 5.1. *Assume that for some $\Theta_0 \in \mathbb{T}^N$ there is a solution $\varphi(t)$ of* (5.2) *satisfying*

$$\sup_{t\in\mathbb{R}} |\varphi(t)| < \infty.$$

In addition assume that there is an admissible couple (σ_-, σ_+) such that

$$\sigma_- \leq \frac{F(\Theta, u_1) - F(\Theta, u_2)}{u_1 - u_2} \leq \sigma_+$$

if $\Theta \in \mathbb{T}^N$ and $\inf_{t\in\mathbb{R}} \varphi \leq u_2 < u_1 \leq \sup_{t\in\mathbb{R}} \varphi$. Then there exist solutions of (5.1)*, $u^\star \in \mathcal{S}_+^p(\overrightarrow{\omega})$ and $u_\star \in \mathcal{S}_-^p(\overrightarrow{\omega})$, satisfying the conditions* (2.3) *and* (2.4) *of Theorem* 2.2.

Proof. The function $\varphi(t)$ is a solution of the linear differential equation

$$\mathcal{P}\bigg(\frac{d}{dt}\bigg)\varphi = p(t)$$

with $p(t) := F(\Theta_0 \cdot t, \varphi(t))$. It follows from Esclangon Theorem (see [11, p. 9]) that the successive derivatives of φ, $\varphi', \ldots, \varphi^{(p)}$, are also bounded. From here the proof is very similar to the proof of Theorem 2.2. The main difference is that the hull of φ, denoted again by H, is now inmersed in $BC^{p-1}(\mathbb{R})$. This is the Fréchet space of functions $x \in C^{p-1}(\mathbb{R})$ such that $x, x', \ldots, x^{(p-1)}$ are in $L^\infty(\mathbb{R})$. The topology is induced by the uniform convergence on bounded intervals for all derivatives up to the order $p-1$. The key observation is that H_Θ is totally ordered. In fact, given x_1 and x_2 in H_Θ, they are solutions of (5.2) and the difference $x_1 - x_2$ solves an equation of the type (5.3). Since F is locally Lipschitz-continuous in u we know that $\sigma \in L^\infty(\mathbb{R})$ and, from the assumptions of the Theorem, $\sigma_- \le \sigma \le \sigma_+$. This equation is partially disconjugate and so x_1 and x_2 cannot intersect unless they coincide. Now one uses that H_Θ is a compact subset of $BC^{p-1}(\mathbb{R})$ to prove that there is a maximal and a minimal element, say v_Θ and w_Θ. The rest is almost the same as in the proof of Theorem 2.2. In the process one obtain the estimate

$$|D^k_{\overrightarrow{\omega}} u(\Theta)| \le \sup_{t\in\mathbb{R}} |\varphi^{(k)}(t)|, \quad \Theta \in \mathbb{T}^n,\ k = 1\ldots, p-1. \qquad \square$$

Of course we can also understand the partial differential equation in the sense of distributions. A solution of

$$\mathcal{P}(D_{\overrightarrow{\omega}})u = F(\Theta, u) \quad \text{in } \mathcal{D}'(\mathbb{T}^N)$$

is a function $u \in L^\infty(\mathbb{T}^N)$ such that

$$\int_{\mathbb{T}^N} u\mathcal{P}^\star(D_{\overrightarrow{\omega}})\phi = \int_{\mathbb{T}^N} F(\Theta, u)\phi \quad \text{for all } \phi \in C^\infty(\mathbb{T}^N),$$

where $\mathcal{P}^\star(\lambda) := \mathcal{P}(-\lambda)$. As in Proposition 3 one can prove that a solution $u \in \mathcal{S}^p_+(\overrightarrow{\omega}) \cup \mathcal{S}^p_-(\overrightarrow{\omega})$ of (5.1) is also a solution in the sense of distributions. To do this it is convenient to notice that the successive derivatives $D^k_{\overrightarrow{\omega}} u$ belong to $L^\infty(\mathbb{T}^N)$ and can also be understood in the sense of distributions.

After Theorem 5.1 it is natural to ask when do admissible couples exist and how to compute them. An easy way to obtain an admissible interval is the following. Let $\sigma_0 \in \mathbb{R}$ be such that the polynomial $\mathcal{P}(\lambda) - \sigma_0$ has no roots on the imaginary axis. Then the equation

$$x^{(p)} + a_{p-1}x^{(p-1)} + \ldots + a_1 x' = \sigma_0 x$$

has an exponential dichotomy (see [5]). The Roughness Theorem implies that also (5.3) has an exponential dichotomy if $\|\sigma - \sigma_0\|_{L^\infty(\mathbb{R})}$ is small. In such a case the only bounded solution of (5.3) is $x \equiv 0$ and so the equation is partially disconjugate. Next we present a result on how to compute an optimal admissible interval in a more delicate situation. It concerns quadratic polynomials and, in particular, it shows that the Theorem 5.1 can be applied to the equations (1.6)–(1.7) of the introduction.

LEMMA 5.2. *Consider the polynomial*

$$\mathcal{P}(\lambda) = \lambda^2 + c\lambda.$$

Then $\sigma_- = -c^2/4$, $\sigma_+ = \infty$ *is an admissible couple, while any couple of the type* $\sigma_- < -c^2/4$, $\sigma_+ = \infty$ *is not admissible.*

PROOF. We first prove that $(-c^2/4, \infty)$ is admissible. Assume that $x \not\equiv 0$ is a bounded solution of

$$x'' + cx' = \sigma(t)x \tag{5.4}$$

with $\sigma \in L^\infty(\mathbb{R})$, $\operatorname{essinf}_t \sigma(t) \geq -c^2/4$. The function $y(t) = e^{c/2t}x(t)$ solves $y'' = (c^2/4 + \sigma(t))y$ and we can apply Sturm comparison theory to deduce that y vanishes at most once. Assume for the moment that y has a zero, say $\tau \in \mathbb{R}$ with $y(\tau) = 0$, $y'(\tau) < 0$. Then y is positive in $(-\infty, \tau)$ and, from the equation, we can deduce that the same happens to y''. From here we deduce that y' is negative in $(-\infty, \tau)$. This is not compatible with $y(\tau) = y(-\infty) = 0$. Hence y cannot vanish and the same applies to x.

To prove that $\sigma_- < -c^2/4$, $\sigma_+ = \infty$ is not admissible we adapt the Example 1.3 in [17]. Letting the period T to go to infinity one can construct an equation (5.4) having an anti-periodic solution and satisfying $\operatorname{essinf} \sigma \geq \sigma_-$. $\square$

We finish this section with a result which extends to second order equations the results that we discussed at the end of Section 2. It refers to the equation

$$D^2_{\overrightarrow{\omega}} u + cD_{\overrightarrow{\omega}} u + g(u) = p(\Theta), \tag{5.5}$$

where g is a bounded function in $C^1(\mathbb{R})$ and $p \in C(\mathbb{T}^N)$.

COROLLARY 5.3. *Assume that* $g'(u) \leq c^2/4$ for all $u \in \mathbb{R}$ *and*

$$\limsup_{u \to -\infty} g(u) < \int_{\mathbb{T}^N} p < \liminf_{u \to \infty} g(u).$$

Then (5.5) *has a solution in* $\mathcal{S}^2_+(\overrightarrow{\omega})$ *(resp. in* $\mathcal{S}^2_-(\overrightarrow{\omega})$*).*

In view of Theorem 5.1 and Lemma 5.2 we must prove that the equation

$$\frac{d^2x}{dt} + c\frac{dx}{dt} + g(x) = p(\Theta \cdot t)$$

has a bounded solution. This is a consequence of the results in [1] or in [18].

6. Appendix: a remark on an example by R. A. Johnson

On the footsteps of the example 3.5 in [9], we will prove that, for any given $\xi \in \mathbb{R}\backslash\mathbb{Q}$ there exists a function $A : \mathbb{T}^2 \to \mathbb{R}$ which fulfills the following conditions

$$\begin{aligned}
&A \in C(\mathbb{T}^2) \quad \text{with mean value zero,}\\
&\sup_{t\in\mathbb{R}} \left| \int_0^t A(\Theta\cdot s)\,ds \right| = \infty \quad \text{for all } \Theta \in \mathbb{T}^2,\\
&\inf_{t\in\mathbb{R}} \int_0^t A(\Theta\cdot s)\,ds > -\infty \quad \text{for almost every } \Theta \in \mathbb{T}^2,
\end{aligned}$$

where $\Theta = (\overline{\theta_1}, \overline{\theta_2}) \in \mathbb{T}^2$ and $\Theta\cdot t = (\overline{\theta_1 + t}, \overline{\theta_2 + \xi t})$ is the linear flow on $\mathbb{T}^2$ associated to ξ.

Lemma 6.1. *Assume that it is possible to find a sequence of functions* $\{L_n\}_{n\geq 1}$ *satisfying the conditions below*

(1) $L_n \in C(\mathbb{T}^2)$, $L_n \geq 0$ *on* $\mathbb{T}^2$,
(2) $\|L_n\|_\infty \to \infty$ *as* $n \to \infty$,
(3) $\sum_{n=1}^\infty \mu(J_n) < \infty$ *where* $J_n = \operatorname{supp}(L_n)$,
(4) L_n *has directional derivative* $D_{\overrightarrow{\omega}} L_n$ *along* $\overrightarrow{\omega} = (1,\xi)$, *and it belongs to* $C(\mathbb{T}^2)$. *We shall employ the notation* $A_n := D_{\overrightarrow{\omega}} L_n$,
(5) $\sum_{n=1}^\infty \|A_n\|_\infty < \infty$.

Then the function

$$A := \sum_{n=1}^\infty A_n$$

has the required properties.

Proof. Since $\int_{\mathbb{T}^2} A_n = \int_{\mathbb{T}^2} D_{\overrightarrow{\omega}} L_n = 0$ then $\int_{\mathbb{T}^2} A = \sum_{n=1}^\infty \int_{\mathbb{T}^2} A_n = 0$. Define

$$L(\Theta) = \sum_{n=1}^\infty L_n(\Theta).$$

The sum $L(\Theta)$ belongs to $[0,\infty]$ and we define the set

$$\Omega = \{\Theta \in \mathbb{T}^2 \mid L(\Theta) < \infty\}.$$

The set Ω is invariant since, if $\Theta \in \Omega$, then for every t and any integer N we have

$$\sum_{n=1}^N L_n(\Theta\cdot t) = \sum_{n=1}^N L_n(\Theta) + \int_0^t \sum_{n=1}^N A_n(\Theta\cdot s)\,ds \leq L(\Theta) + \bigg(\sum_{n=1}^\infty \|A_n\|_\infty\bigg)|t| < \infty.$$

Next we show that $\mu(\Omega) = 1$. It is clear that Ω contains the set

$$V = \{\Theta \in \mathbb{T}^2 : \exists n_\Theta \mid \Theta \notin J_n \text{ if } n \geq n_\Theta\}.$$

Since the complement of V is the set $\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} J_m$ we have

$$\mu(\mathbb{T}^2 \setminus V) \le \mu\left(\bigcup_{m=n}^{\infty} J_m\right) \le \sum_{m=n}^{\infty} \mu(J_m) \to 0 \quad \text{as } n \to \infty.$$

Now it is easy to show that for each $\Theta \in \Omega$,

$$L(\Theta \cdot t) - L(\Theta) = \int_0^t A(\Theta \cdot s)\, ds \quad \text{for all } t \in \mathbb{R}.$$

Thus it is clear that the primitive of $A(\Theta \cdot t)$ has a lower bound if $\Theta \in \Omega$. Namely,

$$\int_0^t A(\Theta \cdot s)\, ds \ge -L(\Theta) \quad \text{for all } t \in \mathbb{R}.$$

It remains to prove that the primitive is unbounded for any $\Theta \in \mathbb{T}^2$. It is sufficient to prove it for some Θ_0 (see for instance [8]). By a contradiction argument, assume that for a given $\Theta_0 \in \Omega$ one has

$$\int_0^t A(\Theta_0 \cdot s)\, ds \le M_0 \quad \text{for all } t \in \mathbb{R}.$$

Then

$$L_n(\Theta_0 \cdot t) \le L(\Theta_0 \cdot t) = L(\Theta_0) + \int_0^t A(\Theta \cdot s)\, ds \le L(\Theta_0) + M_0.$$

Since the orbit of Θ_0 is dense in $\mathbb{T}^2$ and L_n is continuous,

$$\|L_n\|_\infty \le L(\Theta_0) + M_0$$

and this contradicts our assumptions. $\square$

It remains to show that the assumptions of the above lemma may be realized. Let us begin by localizing the support J_n of the function L_n. Consider the orbit on $\mathbb{T}^2$ starting form $(\overline{0}, \overline{0})$, namely $\{\overline{(t, \xi t)} \mid t \in \mathbb{R}\}$. Since $\xi \notin \mathbb{Q}$, the orbit does not have self-intersections and, for any given $T_n > 0$, we can find an $\varepsilon_n > 0$ such that the same happens to the strip $\{\overline{(t, \theta_2 + \xi t)} \mid |\theta_2| \le \varepsilon_n \quad |t| \le T_n\}$. We shall employ (θ_2, t), $\theta_2 \in Q_n = [-\varepsilon_n, \varepsilon_n]$, $t \in I_n = [-T_n, T_n]$ as local coordinates for the strip. The flow transforms diffeomorphically the region $Q_n \times I_n$ (lying on the plane) onto the strip in the torus. Moreover, this transformation transports the Lebesgue measure on the plane to the Haar measure on the torus. Finally, the directional derivative $D_{\overrightarrow{\omega}}$ on the torus becomes $\partial/\partial t$ in the plane.

We shall construct L_n with support on the strip, so that

$$\mu(J_n) \le 4\varepsilon_n T_n \quad \text{for all } n.$$

By using the local coordinates we define

$$L_n(\Theta) = \Gamma_n\, r\left(\frac{\theta_2}{\varepsilon_n}\right) g\left(\frac{t}{T_n}\right)$$

where Γ_n is a positive real number, and $r \geq 0$, $g \geq 0$ are C^∞ functions supported in $(-1,1)$ and satisfying $\|r\|_\infty = \|g\|_\infty = 1$. Thus $\|L_n\|_\infty = \Gamma_n$.

To conclude the construction, choose Γ_n such that $\Gamma_n \to \infty$ as $n \to \infty$, T_n such that $\sum_{n\geq 1} \Gamma_n/T_n < \infty$ and ε_n such that $\sum_{n\geq 1} \varepsilon_n T_n < \infty$.

Before ending this appendix we make some remarks concerning the regularity of A. The previous construction does not allow to obtain a function in $C^1(\mathbb{T}^2)$ and we do not know which is the optimal regularity for a function A satisfying the conditions stated at the beginning of this appendix. Certainly A cannot be very smooth if ξ satisfies an arithmetic condition of diophantine type. On the other hand it is easy to check that for each $\Theta \in \mathbb{T}^2$ the function $t \mapsto A(\Theta \cdot t)$ belongs to $C^\infty(\mathbb{R})$. This follows from the previous construction because

$$\|D^k_{\vec{\omega}} L_n\|_\infty \leq \frac{\Gamma_n}{T_n^k}\|g^k\|_\infty.$$

Thus, in the example of Section 4, the nonlinearity for the partial differential equation (4.7) is not smooth. The ordinary differential equation (4.2) is smooth but does not depend smoothly on the parameter Θ.

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