Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 20, 2002, 179–193

THE EXISTENCE OF MINIMIZERS OF THE ACTION FUNCTIONAL WITHOUT CONVEXITY ASSUMPTION

Aleksandra Orpel

Abstract. We shall prove the existence of minimizers of the following functional $f(u) = \int_0^T L(x, u(x), u'(x))\,dx$ without convexity assumption. As a consequence of this result and the duality described in [10] we derive the existence of solutions for the Dirichlet problem for a certain differential inclusion being a generalization of the Euler–Lagrange equation of the functional f.

1. Introduction

We shall consider the following integral functional

$$f(u) = \int_0^T L(x, u(x), u'(x))\,dx \tag{1}$$

defined on the set $A(\mathbb{R}^n)$ of absolutely continuous functions $u: [0, T] \to \mathbb{R}^n$ satisfying the below boundary condition

$$u(T) = u(0) = 0. \tag{2}$$

2000 *Mathematics Subject Classification.* Primary 49J40, 49J35; Secondary 34B15.
Key words and phrases. Dirichlet problem, duality, variational principle, Euler–Lagrange equation.

We will prove the existence of a minimizer of (1) satisfying (2). Moreover, we will derive a new result concerning the Dirichlet problem for the differential inclusion

$$0 \in \partial_y(-L(x,u(x),u'(x))) + \frac{d}{dx}\partial_w L(x,u(x),u'(x)), \tag{3}$$

where $\partial_y(-L(x,y,w))$ and $\partial_w L(x,y,w)$ denote subdifferentials of the functions $-L(x,\cdot,w)$, $w \in \mathbb{R}^n$, $x \in [0,T]$, and $L(x,y,\cdot)$, $y \in \mathbb{R}^n$, $x \in [0,T]$, respectively. The above inclusion is a natural generalization of the Euler–Lagrange equation of (1).

The work has been intended as an attempt to extend the results presented in [10] to the case when the action functional is not additive separated in u and u'. Section 2 is devoted to the study of the problem concerning the existence of an argument from $A(\mathbb{R}^n)$, at which f attains its infimum. We shall derive an interesting formula for this element and we will show that (2) holds for the minimizer. The proof of these facts is similar in spirit to the one presented in [3] where L has a special form: $L(x,u,w) = -g(x,u)+h(x,w)$. Section 3 establishes the relation between critical points of (1) and solutions of inclusion (3). We shall also present some applications of this theory.

Now we repeat the relevant material from [3] and [6] without proofs. Let $F\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$. We will denote by F^{**} the bipolar of the function $w \to F(x,w)$ and by $\partial F(x,w)$ its subdifferential. Let us recall their properties, which will be used later:

Lemma 1 ([6]). *Assume that $F\colon[0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies the Caratheodory condition and $\int_0^T F(x,v(x))\,dx < \infty$ for a certain $v \in L^\infty(0,T,\mathbb{R})$. Suppose additionally that there exist $a \in L^1([0,T],\mathbb{R})$ and constants $b \ge 0$ and $1 < k < \infty$ such that*

$$a(x) + b|w|^k \le F(x,w)$$

for all $w \in \mathbb{R}^n$, a.e. $x \in [0,T]$. Then

(i) *for all $w \in \mathbb{R}^n$ and $x \in [0,T]$ we have*

$$F^{**}(x,w) = \min\left\{ \sum_{i=1}^{n+1} \lambda_i F(x,\varsigma_i) : \sum_{i=1}^{n+1} \lambda_i\varsigma_i = w,\ \sum_{i=1}^{n+1}\lambda_i = 1,\ \lambda_i \ge 0 \right\}.$$

(ii) *for every measurable function $z\colon[0,T] \to \mathbb{R}^n$ there exist measurable $v_i\colon[0,T] \to \mathbb{R}^n$ and $p_i\colon[0,T]\to[0,1]$, $i = 1,\dots,n+1$, such that for a.e. $x \in [0,T]$*

$$\sum_{i=1}^{n+1} p_i(x) = 1, \tag{4}$$

$$\sum_{i=1}^{n+1} p_i(x)v_i(x) = z(x), \tag{5}$$

$$F^{**}(x, z(x)) = \sum_{n=1}^{n+1} p_i(x) F(x, v_i(x)). \tag{6}$$

Lemma 2 ([3]). *Let $F\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions*

(i) *there exist $b \in L^1(0,T,\mathbb{R})$, $m \in (1,\infty)$ and $k > 0$ such that*

$$F(x, w) \le k|w|^m + b(x) \tag{7}$$

for all $w \in \mathbb{R}^n$ and a.e. $x \in [0,T]$,

(ii) *$[0,T] \ni x \to F(x,w)$ is measurable for every $w \in \mathbb{R}^n$,*

(iii) *F is convex with respect to the second variable for a.e. $x \in [0,T]$.*

Then for any continuous function $u\colon [0,T] \to \mathbb{R}^n$ the set valued map $x \to \partial_u(F(x,u(x))$ admits a selection $\delta \in L^1(0,T,R)$.

2. The existence of the critical points of the action functional f

In this section we will be looking for the critical points of (1) defined on the set $A(\mathbb{R}^n)$. To this effect it is necessary to put some restrictions on L. It is required that

Hypothesis. The map $L\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumptions:

(H1) $L(x, \cdot, w)$ is concave for a.e. $x \in [0,T]$ and all $w \in \mathbb{R}^n$,

(H2) $L(\cdot, u, w)$ is measurable for all $u, w \in \mathbb{R}^n$,

(H3) $L(x, \cdot, \cdot)$ is continuous for a.e. $x \in [0,T]$,

(H4) there exist $m, p, s \in (1,\infty)$, $p \le m$, $s \ge p'$, where $p' = p/(p-1)$, and $d \in L^1(0,T,\mathbb{R})$, $k_1 \in L^s(0,T,\mathbb{R}_+)$, constants $l, k > 0$ such that for $u, w \in \mathbb{R}^n$ and a.e. $x \in [0,T]$

$$-d(x) + \frac{1}{m} l^{1-m} |w|^m - \frac{1}{p} k |u|^p \le L(x,u,w) \le g(x,w) - \langle k_1(x), u\rangle,$$

where $g\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is measurable with respect to the first variable, convex with respect to the second one and for a certain ball $K(\overline{w}, r) \subset \mathbb{R}^n$, centered at $\overline{w}$ of radius r, the function $x \to \sup_w \in K(\overline{w}, r) g(x,w)$ is summable on $[0,T]$,

(H5) in the case of $p = m$ the following condition takes place

$$Q = \frac{1}{p'} k^{1-p'} - \frac{1}{p'} \int_0^T l x^{p'/p}\, dx > 0.$$

Set $A^m = \{z \in A(\mathbb{R}^n) : z' \in L^m(0,T,\mathbb{R}^n)\}$.

The above assumptions imply that the action functional $f\colon A^m \to \mathbb{R}$ is well-defined. The results presented in [10, Proposition 1 and Theorem 7] imply the existence of a minimizer $\overline{u}$ for f^c described by the bipolar $L^{w**}(x,u,w)$ of $\mathbb{R}^n \ni w \to L(x,u,w) \in \mathbb{R}$, $x \in [0,T]$, $u \in \mathbb{R}^n$ in the following way

$$f^c(u) = \int_0^T L^{w**}(x,u(x),u'(x))\,dx.$$

Moreover, $\overline{u}$ satisfies the equalities $\overline{u}(T) = \overline{u}(0) = 0$.

Now we will consider the function $[0,T] \times \mathbb{R}^n \ni (x,w) \to L(x,\overline{u}(x),w)$. By hypothesis (H1)–(H5) we can use the second part of Lemma 1 for $\overline{u}'$ and obtain that there exist measurable $v_i\colon [0,T] \to \mathbb{R}^n$ and $p_i\colon [0,T] \to [0,1]$, $i = 1,\ldots,n+1$, such that $\sum_{i=1}^{n+1} p_i(x) = 1$ and

$$\sum_{i=1}^{n+1} p_i(x) v_i(x) = \overline{u}'(x), \tag{8}$$

$$L^{w**}(x,\overline{u}(x),\overline{u}'(x)) = \sum_{n=1}^{n+1} p_i(x) L(x,\overline{u}(x),v_i(x)). \tag{9}$$

Let us denote by $\aleph_B$ the characteristic function of a set $B \subset \mathbb{R}^n$.

LEMMA 3. *Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of disjoint compact subsets of $[0,T]$ and let $\mathbf{N} \subset [0,T]$ be a set, whose measure is equal to zero, such that $[0,T] = \mathbf{N} \cup \bigcup_{j=1}^{\infty} K_j$. Then if $(E_j^i)_{i=1,\ldots,n+1}$ is a measurable partition of K_j with the property that for every $j \in \mathbb{N}$*

$$\int_{K_j} \sum_{i=1}^{n+1} p_i(x) L(x,\overline{u}(x),v_i(x))\,dx = \int_{K_j} \sum_{i=1}^{n+1} \aleph_{E_j^i}(x) L(x,\overline{u}(x),v_i(x))\,dx \tag{10}$$

then a function

$$x \to \sum_{j=1}^{\infty} \sum_{i=1}^{n+1} \aleph_{E_j^i}(x) L(x,\overline{u}(x),v_i(x))$$

and a function

$$x \to \sum_{j=1}^{\infty} \sum_{i=1}^{n+1} \aleph_{E_j^i}(x) v_i(x)$$

belong to $L^1(0,T,\mathbb{R})$ and $L^m(0,T,\mathbb{R}^n)$, respectively.

PROOF. By (9) and hypothesis (H1)–(H5) we can assert that the map

$$x \to \sum_{i=1}^{n+1} p_i(x) L(x,\overline{u}(x),v_i(x))$$

is integrable on $[0,T]$. Set $\overline{S}_q = \bigcup_{j\le q} K_j$, $q = 1,2,\ldots$

Let us consider the sequence of maps:

$$S_q(x) = \sum_{j\le q} \sum_{i=1}^{n+1} \aleph_{E_j^i}(x)[L(x,\overline{u}(x),v_i(x)) + \gamma(x)], \tag{11}$$

where $\gamma(x) = d(x) + (1/p)k|\overline{u}(x)|^p$. It is easily seen that the following equality holds

$$\int_0^T S_q(x)\,dx = \sum_{j\le q}\int_{K_j}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)[L(x,\overline{u}(x),v_i(x))+\gamma(x)]\,dx.$$

Taking into account (10), (9) we can compute:

$$\begin{aligned}\sum_{j\le q}\int_{K_j}&\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)[L(x,\overline{u}(x),v_i(x))+\gamma(x)]\,dx\\ &=\sum_{j\le q}\int_{K_j}\sum_{i=1}^{n+1}p_i(x)[L(x,\overline{u}(x),v_i(x))+\gamma(x)]\,dx\\ &=\sum_{j\le q}\int_{K_j}[L^{w**}(x,\overline{u}(x),\overline{u}'(x))+\gamma(x)]\,dx\\ &=\int_0^T\aleph_{\overline{S}_q}(x)[L^{w**}(x,\overline{u}(x),\overline{u}'(x))+\gamma(x)]\,dx\\ &\le\int_0^T[L^{w**}(x,\overline{u}(x),\overline{u}'(x))+\gamma(x)]\,dx<\infty.\end{aligned}$$

According to the assumptions concerning L we have that the sequence (11) is nondecreasing. Thus the previous chain of relations gives

$$\begin{aligned}\int_0^T\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)[L(x,\overline{u}(x),v_i(x))+\gamma(x)]\,dx&=\int_0^T\lim_{q\to\infty}S_q(x)\,dx\\ =\lim_{q\to\infty}\int_0^T S_q(x)\,dx&\le\int_0^T[L^{w**}(x,\overline{u}(x),\overline{u}(x))+\gamma(x)]\,dx.\end{aligned}$$

We have proved the integrability of $x\to\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)L(x,\overline{u}(x),v_i(x))$ on $[0,T]$.

It can be noticed that almost every x from $[0,T]$ belongs to exactly one of E_j^i, which implies that for a.e. $x\in[0,T]$ there exists $i_0\in\{1,\dots,n+1\}$ such that $\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_i^j}(x)v_i(x)=v_{i_0}(x)$. Using this fact and hypothesis (H1)–(H5) we obtain:

$$\begin{aligned}\Big|\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_i^j}(x)v_i(x)\Big|^m&\le\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_i^j}(x)|v_i(x)|^m\\ &\le\overline{c}\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_i^j}(x)[L(x,\overline{u}(x),v_i(x))+\gamma(x)],\end{aligned}$$

where $\overline{c}=ml^{m-1}$. By the above chain of relations and the integrability of $[0,T]\ni x\to\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_i^j}(x)[L(x,\overline{u}(x),v_i(x))+\gamma(x)]\in\mathbb{R}^n$, the function $[0,T]\ni x\to\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)v_i(x)\in\mathbb{R}^n$ belongs to $L^m(0,T,\mathbb{R}^n)$. □

Theorem 4. *There exists* $\overline{\overline{u}} \in A^m$ *such that* $f(\overline{\overline{u}}) = \inf_{u\in A^m} f(u)$.

Proof. Using the Lusin's theorem we infer the existence of a sequence $(K_j)_{j\in\mathbb{N}}$ of disjoint compact subsets of $[0,T]$ and a set $\mathbf{N}\subset[0,T]$, whose measure is equal to zero, such that $[0,T]=\mathbf{N}\cup\bigcup_{j=1}^{\infty}K_j$ and the following condition takes place: for every $j\in\mathbb{N}$ the restriction of the map $x\to\sum_{i=1}^{n+1}L(x,\overline{u}(x),v_i(x))$, $x\in[0,T]$, to K_j is continuous. For $i\in\{1,\dots,n+1\}$ we shall define the vector measures ϑ_i as follows

$$\vartheta_i(E)=\int_E v_i(x)dx. \tag{12}$$

From an extension of the Liapunov's theorem on the range of vector measures for each K_j there exists a measurable partition $(E_j^i)_{i=1,\dots,n+1}$ of K_j such that

$$\sum_{i=1}^{n+1}\int_{K_j}\aleph_{E_j^i}(x)v_i(x)\,dx=\sum_{i=1}^{n+1}\int_{K_j}p_i(x)v_i(x)\,dx. \tag{13}$$

Let us define the function $\overline{\overline{u}}'\colon[0,T]\to\mathbb{R}^n$

$$\overline{\overline{u}}'(x)=\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)v_i(x). \tag{14}$$

It can be noticed that almost every x from $[0,T]$ belongs to exactly one of E_j^i, which means: for a.e. $x\in[0,T]$ there exists $i\in\{1,\dots,n+1\}$ such that $\overline{\overline{u}}'(x)=v_i(x)$. It follows that

$$\begin{aligned} L(x,\overline{u}(x),\overline{\overline{u}}'(x)) &= L\Big(x,\overline{u}(x),\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)v_i(x)\Big)\\ &=\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)L(x,\overline{u}(x),v_i(x)). \end{aligned} \tag{15}$$

By (13) and the above consideration we obtain for every $j\in\mathbb{N}$

$$\sum_{i=1}^{n+1}\int_{K_j}\aleph_{E_j^i}(x)L(x,\overline{u}(x),v_i(x))\,dx=\sum_{i=1}^{n+1}\int_{K_j}p_i(x)L(x,\overline{u}(x),v_i(x))\,dx. \tag{16}$$

Now we may state that the assumptions of Lemma 3 are satisfied. It implies that the function $x\to\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)L(x,\overline{u}(x),v_i(x))$ is integrable on $[0,T]$ and the map $x\to\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)v_i(x)$ belongs to $L^m(0,T,\mathbb{R}^n)$, so that, $\overline{\overline{u}}'\in L^m(0,T,\mathbb{R}^n)$.

Set $\overline{\overline{u}}(x) = \int_0^x \overline{\overline{u}}'(s)ds$, $x \in [0,T]$. Of course $\overline{\overline{u}}(0) = 0$ and further, by (14), (13) and (8)

$$\begin{aligned}\overline{\overline{u}}(T) &= \int_0^T \overline{\overline{u}}'(x)\,dx = \int_0^T \sum_{j=1}^{\infty}\sum_{i=1}^{n+1} \aleph_{E_j^i}(x)v_i(x)\,dx \\ &= \sum_{j=1}^{\infty}\int_{K_j}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)v_i(x)\,dx \\ &= \sum_{j=1}^{\infty}\int_{K_j}\sum_{i=1}^{n+1}p_i(x)v_i(x)\,dx = \int_0^T \overline{u}'(x)\,dx = 0.\end{aligned}$$

We shall show now that

$$\int_0^T L(x,\overline{u}(x),\overline{\overline{u}}'(x))\,dx = \int_0^T L^{w**}(x,\overline{u}(x),\overline{u}'(x))\,dx. \tag{17}$$

From (9), (16), (15) and (14) we have

$$\begin{aligned}&\int_0^T L^{w**}(x,\overline{u}(x),\overline{u}'(x))\,dx \\ &\quad= \int_0^T \sum_{i=1}^{n+1}p_i(x)L(x,\overline{u}(x),v_i(x))\,dx = \sum_{j=1}^{\infty}\int_{K_j}\sum_{i=1}^{n+1}p_i(x)L(x,\overline{u}(x),v_i(x))\,dx \\ &\quad= \sum_{j=1}^{\infty}\int_{K_j}\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)L(x,\overline{u}(x),v_i(x))\,dx \\ &\quad= \sum_{j=1}^{\infty}\int_{K_j}L\Big(x,\overline{u}(x),\sum_{i=1}^{n+1}\aleph_{E_j^i}(x)v_i(x)\Big)\,dx = \int_0^T L(x,\overline{u}(x),\overline{\overline{u}}'(x))\,dx.\end{aligned}$$

Set

$$\partial_u L(x,u,\overline{\overline{u}}'(x)) = -\partial_u(-L(x,u,\overline{\overline{u}}'(x)))$$

for $x \in [0,T]$, $u \in \mathbb{R}^n$, where $\partial_u(-L(x,u,\overline{\overline{u}}'(x)))$ is the subdifferential of the function $u \to -L(x,u,\overline{\overline{u}}'(x))$. Applying hypothesis (H1)–(H5) and Lemma 2 we obtain the existence of an integrable selection δ of the set-valued map $[0,T] \ni x \to \partial_u L(x,\overline{u}(x),\overline{\overline{u}}'(x))$. Let

$$B(x) = \int_0^x \delta(s)\,ds,$$

for $x \in [0,T]$. By (13) and the definition of $(E_j^i)_{i=1,\dots,n+1}$ we obtain for every $j \in \mathbb{N}$

$$\begin{aligned}\sum_{i=1}^{n+1}\int_{K_j}\aleph_{E_j^i}(x)\langle v_i(x),B(T)-B(x)\rangle\,dx \\ = \sum_{i=1}^{n+1}\int_{K_j}p_i(x)\langle v_i(x),B(T)-B(x)\rangle\,dx.\end{aligned} \tag{18}$$

The description of $\delta(\,\cdot\,)$ and the properties of subdifferential give that for a.e. $x \in [0,T]$

$$-L(x,y,\overline{\overline{u}}'(x)) \ge -L(x,\overline{u}(x),\overline{\overline{u}}'(x)) + \langle -\delta(x), y-\overline{u}(x)\rangle,$$

where $y \in \mathbb{R}^n$, and further that

$$L(x,\overline{u}(x),\overline{\overline{u}}'(x)) \ge L(x,\overline{\overline{u}}(x),\overline{\overline{u}}'(x)) + \langle \delta(x), \overline{u}(x)-\overline{\overline{u}}(x)\rangle.$$

Now we claim that

$$\int_0^T \langle \delta(x), \overline{u}(x)-\overline{\overline{u}}(x)\rangle\,dx = 0.$$

Indeed, recalling the definition of B and denoting by u_l the l-th component of a vector u, we can compute

$$\begin{aligned}\int_0^T \langle \delta(x),&\overline{u}(x)-\overline{\overline{u}}(x)\rangle\,dx = \int_0^T \sum_{l=1}^n \delta_l(x)(\overline{u}_l(x)-\overline{\overline{u}}_l(x))\,dx\\ &= \sum_{l=1}^n \int_0^T \delta_l(x)\Big(\int_0^x \overline{u}_l'(s)\,ds - \int_0^x \overline{\overline{u}}_l'(s)\,ds\Big)\,dx\\ &= \sum_{l=1}^n \int_0^T (\overline{u}_l'(s)-\overline{\overline{u}}_l'(s))\int_s^T \delta_l(x)\,dx\,ds\\ &= \sum_{l=1}^n \int_0^T (\overline{u}_l'(s)-\overline{\overline{u}}_l'(s))(B_l(T)-B_l(s))\,ds\\ &= \int_0^T \langle \overline{u}'(s)-\overline{\overline{u}}'(s), B(T)-B(s)\rangle\,ds\\ &= \sum_{j=1}^\infty \int_{K_j} \sum_{i=1}^{n+1} (p_i(s)-\aleph_{E_j^i}(s))\langle v_i(s), B(T)-B(s)\rangle\,ds = 0.\end{aligned}$$

The last equality follows from (18). By the above relation we obtain that

$$\int_0^T L(x,\overline{u}(x),\overline{\overline{u}}'(x))\,dx \ge \int_0^T L(x,\overline{\overline{u}}(x),\overline{\overline{u}}'(x))\,dx. \tag{19}$$

Combining (17) with (19) we have

$$\begin{aligned}\int_0^T & L^{w**}(x,\overline{u}(x),\overline{u}'(x))\,dx\\ &= \int_0^T L(x,\overline{u}(x),\overline{\overline{u}}'(x))\,dx \ge \int_0^T L(x,\overline{\overline{u}}(x),\overline{\overline{u}}'(x))\,dx\\ &\ge \int_0^T L^{w**}(x,\overline{\overline{u}}(x),\overline{\overline{u}}'(x))\,dx \ge \int_0^T L^{w**}(x,\overline{u}(x),\overline{u}'(x))\,dx.\end{aligned}$$

The above chain of inequalities and the duality principle from [10, Theorem 6] give

$$f(\overline{\overline{u}}) = f^c(\overline{u}) = \inf_{u\in A^m} f^c(u) = \inf_{u\in A^m} f(u). \qquad \square$$

3. Necessary conditions and regularity

Now we will take up an existence of solutions of the differential inclusion (3) with the boundary condition (2). In the following section we shall apply the results presented in the paper [10]. We will assume that the conditions of hypothesis (H1)–(H5) are satisfied.

Let $B^m = \mathbb{R}^n \oplus L^m(0,T,\mathbb{R}^n)$. We use the fact that the space A^m can be identified with $B^m = \mathbb{R}^n \oplus L^m(0,T,\mathbb{R}^n)$ normed by $\|u\|_{A^m} = |u(0)| + \|u'\|_{L^m(0,T,\mathbb{R}^n)}$. The dual $(A^m)^*$ of A^m will be identified with $B^{m'} = \mathbb{R}^n \oplus L^{m'}(0,T,\mathbb{R}^n)$, $1/m + 1/m' = 1$, under the pairing

$$\langle z, (d,v)\rangle = \langle z(0), d\rangle + \int_0^T \langle z'(x), v(x)\rangle \, dx,$$

where $z \in (A^m)^*, (d,v) \in B^m$.

For $u \in A^m$ we shall define the perturbation $f_u\colon B^p \to \overline{\mathbb{R}}$ of

$$f(u) = \int_0^T L(x, u(x), u'(x))\, dx + l_0(u(0)) + l_0(u(T))$$

by

$$f_u(a,h) = -l_0(u(0)+a) - l_0(u(T)) - \int_0^T L(x, u(x)+h(x), u'(x))\, dx, \tag{20}$$

where $B^p = \mathbb{R}^n \oplus L^p(0,T,\mathbb{R}^n)$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and $l_0\colon \mathbb{R}^n \to \mathbb{R}$

$$l_0(a) = \begin{cases} 0 & \text{for } a = 0, \\ \infty & \text{for } a \neq 0. \end{cases}$$

It can be noticed that for all $u \in A^m$ we have $f_u(0,0) = -f(u)$.

Let $A^{p'} = \{z \in A(\mathbb{R}^n) : z' \in L^{p'}(0,T,\mathbb{R}^n)\}$. We will consider the functional $f_u^{\#}\colon A^{p'} \to \overline{\mathbb{R}}$, where $u \in A^m$, defined in the following way

$$\begin{aligned} f_u^{\#}(v) = \sup_{h \in L^p(0,T,\mathbb{R}^n)} &\left\{ \int_0^T \langle v'(x), h(x)\rangle \, dx - \int_0^T -L(x, u(x)+h(x), u'(x))\, dx \right\} \\ &+ l_0(u(T)) + \inf_{a \in \mathbb{R}^n} \{\langle a, v(0)\rangle + l_0(u(0)+a)\}. \end{aligned}$$

Let $G(x,y,w) = \sup_{u \in \mathbb{R}^n} \{\langle y, u\rangle + L(x,u,w)\}$ for a.e. $x \in [0,T]$ and all $y, w \in \mathbb{R}^n$. Using hypothesis (H1)–(H5) and the description of G we obtain a simpler form of the previous assertion

$$\begin{aligned} f_u^{\#}(v) = &\int_0^T G(x, v'(x), u'(x))\, dx \\ &+ \int_0^T \langle v(x), u'(x)\rangle \, dx + -\langle u(T), v(T)\rangle + l_0(u(T)). \end{aligned} \tag{21}$$

Put

$$H(x,u,v)=\sup_{w\in\mathbb{R}^n}\{\langle v,w\rangle - L(x,u,w)\},\quad L_D(x,v,y)=\sup_{u\in\mathbb{R}^n}\{\langle y,u\rangle - H(x,u,v)\},$$

where $x\in[0,T]$ and $u,y,v\in\mathbb{R}^n$.

Now we shall define the functional $f_D\colon A^{p'}\to\overline{\mathbb{R}}$, dual to f, by

$$f_D(v)=\int_0^T L_D(x,v(x),-v'(x))\,dx. \tag{22}$$

It can be shown that the definition of L_D is equivalent to the following one

$$L_D(x,v,y)=-\sup_{w\in\mathbb{R}^n}\{\langle v,w\rangle - G(x,y,w)\},$$

for all $y,v\in\mathbb{R}^n$ and a.e. $x\in[0,T]$ (see [7]).

The conditions (H1)–(H5) imply that $f_D\colon A^{p'}\to\overline{\mathbb{R}}$ is well-defined. Moreover, for $v\in A^{p'}$ we get ([10])

$$\sup_{u\in A^m}\{-f_u^{\#}(-v)\}=-f_D(v). \tag{23}$$

Theorem 5. *There exists $\overline{v}\in A^{p'}$, $\overline{v}(x)=\overline{v}(0)+\int_0^x\overline{v}'(s)\,ds$ for a.e. $x\in[0,T]$, $-\overline{v}'\in\partial f_{\overline{\overline{u}}}(0,0)$, (where $\partial f_{\overline{\overline{u}}}(0,0)$ is the subdifferential of $L^p(0,T,\mathbb{R}^n)\ni g\to f_{\overline{\overline{u}}}(0,g)$ at zero) such that $f_D(\overline{v})=\inf_{v'\in L^{p'}(0,T,\mathbb{R}^n)}\sup_{v(0)\in\mathbb{R}^n}f(v)$. Moreover,*

$$f_{\overline{\overline{u}}}(0,0)+f_{\overline{\overline{u}}}^{\#}(-\overline{v})=0. \tag{24}$$

Proof. The definitions of $\overline{\overline{u}}$, $f_{\overline{\overline{u}}}$ and hypothesis (H1)–(H5) imply that the function $g\to f_{\overline{\overline{u}}}(0,g)$ is lower semicontinuous, convex and finite on $L^p(0,T,\mathbb{R}^n)$, hence continuous. That is why $\partial f_{\overline{\overline{u}}}(0,0)$ is not empty. It means there exists $-\overline{v}'$, which belongs to $\partial f_{\overline{\overline{u}}}(0,0)$. Thus we have, for all $g\in L^p(0,T,\mathbb{R}^n)$,

$$f_{\overline{\overline{u}}}(0,g)\ge f_{\overline{\overline{u}}}(0,0)+\langle g,-\overline{v}'\rangle,$$

where $\langle g,-\overline{v}'\rangle=\int_0^T\langle g(x),-\overline{v}'(x)\rangle\,dx$. Since $f_u(0,0)=-f(u)$, we get

$$f(\overline{\overline{u}})\ge -f_{\overline{\overline{u}}}(0,g)+\int_0^T\langle g(x),-\overline{v}'(x)\rangle\,dx$$

for all $g\in L^p(0,T,\mathbb{R}^n)$ and further, for $d\in\mathbb{R}^n$,

$$f(\overline{\overline{u}})\ge\sup_{g\in L^p(0,T,\mathbb{R}^n)}\left\{\int_0^T\langle g(x),-\overline{v}'(x)\rangle - f_{\overline{\overline{u}}}(0,g)\right\}=f_{\overline{\overline{u}}}^{\#}(-\overline{v}_d),$$

where $\overline{v}_d(x)=d+\int_0^x\overline{v}'(s)\,ds$. Hence for all $d\in\mathbb{R}^n$ the following chain of relations holds

$$-f(\overline{\overline{u}})\le -f_{\overline{\overline{u}}}^{\#}(-\overline{v}_d)\le\sup_{u\in A^m}\{-f_u^{\#}(-\overline{v}_d)\}=-f_D(\overline{v}_d).$$

This implies that $\sup_{d\in\mathbb{R}^n} f_D(\overline{v}_d) \le f(\overline{\overline{u}})$.

Using the assumptions made in hypothesis (H1)–(H5) we may state that the functional $\mathbb{R}^n \ni d \to \int_0^T -L_D(x,\overline{v}_d(x),\overline{v}'(x))\,dx$ is convex, lower semicontinuous and coercive. Thus, its infimum is attained at some $\overline{d}$. This leads to $\sup_{d\in\mathbb{R}^n} f_D(\overline{v}_d) = f_D(\overline{v})$, where $\overline{v}(x) = \overline{d} + \int_0^x \overline{v}'(s)\,ds$. Theorem 6 from [10] and the last equality yield $f_D(\overline{v}) = \inf_{v'\in L^{p'}(0,T,\mathbb{R}^n)} \sup_{v(0)\in\mathbb{R}^n} f_D(v)$.

From the definitions of $f_{\overline{\overline{u}}}$ and $f^{\#}_{\overline{\overline{u}}}$ we have that the conjugate of $L^p(0,T,\mathbb{R}^n) \ni g \to f_{\overline{\overline{u}}}(0,g)$ at $-\overline{v}'$ is equal to $f^{\#}_{\overline{\overline{u}}}(-\overline{v})$. Thus, by $-\overline{v}' \in \partial f_{\overline{\overline{u}}}(0,0)$ and the properties of subdifferential ([6]), we have assertion (24). □

Let us define the functional $f_c\colon A^m \times A^{p'} \to \mathbb{R}$

$$f_c(u,v) = \int_0^T [\langle v'(x),u(x)\rangle + G(x,-v'(x),u'(x))]\,dx. \tag{25}$$

Theorem 6. *For $\overline{\overline{u}} \in A^m$ from Theorem 4 and $\overline{v} \in A^{p'}$ from Theorem 5, the following equality holds*

$$\begin{aligned} f_D(\overline{v}) &= f(\overline{\overline{u}}) = f_c(\overline{\overline{u}},\overline{v}) \\ &= \inf\{f_c(u,v) : u \in A^m,\ u(0) = u(T) = 0,\ v \in A^{p'}\}. \end{aligned} \tag{26}$$

Moreover, there exists $\overline{d}_1$ such that $\overline{v}_1(x) = \overline{d}_1 + \int_0^x \overline{v}'(s)\,ds$, for a.e. $x \in [0,T]$, satisfies (26) *and the inclusion:*

$$(\overline{\overline{u}}(x),\overline{v}_1(x)) \in \partial G(x,-\overline{v}'(x),\overline{u}'(x)) \tag{27}$$

for a.e. $x \in [0,T]$.

Proof. It can be noticed that the first assertion of (26) is a consequence of descriptions of $\overline{v}$ and $\overline{\overline{u}}$. On the other hand, by (24), we obtain

$$\int_0^T L(x,\overline{\overline{u}}(x),\overline{\overline{u}}'(x))\,dx = \int_0^T G(x,-\overline{v}'(x),\overline{\overline{u}}'(x))\,dx + \int_0^T \langle\overline{\overline{u}}(x),\overline{v}'(x)\rangle\,dx,$$

and further $f(\overline{\overline{u}}) = f_c(\overline{\overline{u}},\overline{v})$. By (25), we get that for $u \in A^m$, $v \in A^{p'}$ the following chain of relations takes place

$$\begin{aligned} f_c(u,v) &= \int_0^T \langle v'(x),u(x)\rangle\,dx + \int_0^T G(x,-v'(x),u'(x))\,dx \\ &\ge \int_0^T L(x,u(x),u'(x))\,dx = f(u) \ge f(\overline{\overline{u}}) \end{aligned}$$

which implies $f_c(u,v) \ge f(\overline{\overline{u}}) = f_c(\overline{\overline{u}},\overline{v})$. This gives (26).

By hypothesis (H1)–(H5) we can state that the functional

$$\mathbb{R}^n \ni d \to \int_0^T H(x,\overline{\overline{u}}(x),d+\overline{v}_0(x))\,dx,$$

where $\overline{v}_0(x) = \int_0^x \overline{v}'(s)\,ds$, is convex, lower semicontinuous and coercive. Thus, there exists $\overline{d}_1$ such that:

$$\int_0^T H(x, \overline{\overline{u}}(x), \overline{d}_1 + \overline{v}_0(x))\,dx = \inf_{d\in\mathbb{R}^n} \int_0^T H(x, \overline{\overline{u}}(x), d + \overline{v}_0(x))\,dx.$$

Using the fact that for $u, v \in \mathbb{R}^n$ and a.e. $x \in [0, T]$

$$H(x, u, v) = \sup_{y\in\mathbb{R}^n,\ w\in\mathbb{R}^n} \{\langle (y, w) \mid (u, v)\rangle - G(x, y, w)\},$$

where $\langle (y, w) \mid (u, v)\rangle = \langle y, u\rangle + \langle w, v\rangle$, $y, w \in \mathbb{R}^n$ and the above equality we have (see [2])

$$\int_0^T H(x, \overline{\overline{u}}(x), \overline{v}_1(x))\,dx = \sup_{u'\in L^m} \sup_{v'\in L_0^{p'}} \bigg\{ \int_0^T \langle \overline{\overline{u}}(x), -v'(x)\rangle\,dx + \int_0^T [\langle \overline{v}_1(x), u'(x)\rangle - G(x, -v'(x), u'(x))]\,dx \bigg\},$$

where $\overline{v}_1(x) = \overline{d}_1 + \int_0^x \overline{v}'(x)\,dx$, $L_0^p = \{z \in L^p(0, T, \mathbb{R}^n) : \int_0^T z(s)\,ds = 0\}$.

Now basing ourselves on the ideas of the proof of Theorem 8 from [10] we can show the following assertion

$$\int_0^T H(x, \overline{\overline{u}}(x), \overline{v}_1(x))\,dx = \int_0^T \langle \overline{\overline{u}}(x), -\overline{v}'(x)\rangle\,dx + \int_0^T \langle \overline{v}_1(x), \overline{\overline{u}}'(x)\rangle\,dx - \int_0^T G(x, -\overline{v}'(x), \overline{\overline{u}}'(x))\,dx, \tag{28}$$

which leads to (27):

$$(\overline{\overline{u}}(x), \overline{v}_1(x)) \in \partial G(x, -\overline{v}'(x), \overline{\overline{u}}'(x))$$

for a.e. $x \in [0, T]$. Combining (25) with the equality (28) gives $f_c(\overline{\overline{u}}, \overline{v}) = f_c(\overline{\overline{u}}, \overline{v}_1)$.

Now we claim, that $f_c(\overline{\overline{u}}, \overline{v}_1) = f_D(\overline{v}_1)$. Using the definitions of f_D and f_c, we get

$$f_D(\overline{v}_1) \le \sup_{d\in\mathbb{R}^n} f_D(d + \overline{v}_0) = f_D(\overline{v}) = f_c(\overline{\overline{u}}, \overline{v}_1).$$

Substituting (25) into (28) we obtain

$$\begin{aligned} f_c(\overline{\overline{u}}, \overline{v}_1) &= \int_0^T [\langle \overline{\overline{u}}(x), -\overline{v}'(x)\rangle - H(x, \overline{\overline{u}}(x), \overline{v}_1(x))]\,dx \\ &\le \int_0^T L_D(x, \overline{v}_1(x), -\overline{v}(x))\,dx = f_D(\overline{v}_1). \end{aligned}$$

Both inequalities give the required condition. We have just shown, that $(\overline{\overline{u}}, \overline{v}_1)$ satisfies (26). □

COROLLARY 7. *For* $(\overline{\overline{u}}, \overline{v})$ *from Theorem* 6 *there exists* $\overline{d}_1$ *such that* $\overline{v}_1(x) = \overline{d}_1 + \int_0^x \overline{v}'(s)\,ds$ *for a.e.* $x \in [0,T]$, *satisfies* (26) *and*

$$-\overline{v}'(x) \in \partial_y[-L(x, \overline{\overline{u}}(x), \overline{\overline{u}}'(x))], \tag{29}$$

$$\overline{v}_1(x) \in \partial_w L(x, \overline{\overline{u}}(x), \overline{\overline{u}}'(x)), \tag{30}$$

for a.e. $x \in [0,T]$.

PROOF. Theorem 6 implies the existence of $\overline{d}_1$ such that $\overline{v}_1(x) = \overline{d}_1 + \int_0^x \overline{v}'(s)\,ds$, $x \in [0,T]$, satisfies (26) and (27). Combining (27) with (24) we obtain, for a.e. $x \in [0,T]$,

$$H(x, \overline{\overline{u}}(x), \overline{v}_1(x)) + L(x, \overline{\overline{u}}(x), \overline{\overline{u}}'(x)) = \langle \overline{v}_1(x), \overline{\overline{u}}'(x)\rangle.$$

This gives the second assertion. Applying the description of $\overline{\overline{u}}$ and (24) yields

$$-f(\overline{\overline{u}}) + f^{\#}_{\overline{\overline{u}}}(-\overline{v}) = 0$$

and further

$$\int_0^T [G(x, -\overline{v}_1'(x), \overline{\overline{u}}'(x)) - L(x, \overline{\overline{u}}(x), \overline{\overline{u}}'(x)) + \langle \overline{v}_1'(x), \overline{\overline{u}}(x)\rangle]\,dx = 0.$$

So we conclude that (29) holds. □

4. Applications

Now we shall apply this theory to derive the existence results for solutions of the Dirichlet problem for a certain class of the second order ordinary differential equations.

PROPOSITION 8. *Let* $T > 0$ *and let* $a\colon [0,T] \to \mathbb{R}$ *be a differentiable function. Assume that the function* $G\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *satisfies the Caratheodory condition, G is concave and Gateaux differentiable with respect to the second variable. Suppose additionally that there exist* $k, k_1 \in \mathbb{R}_+, l_1 \in L^1(0,T,\mathbb{R}_+)$ *such that*

$$-d(x) - \frac{1}{3}k|u|^3 \le G(x,u) \le -\frac{1}{2}k_1|u|^2 - \langle l_1(x), u\rangle$$

for all $u \in \mathbb{R}^n$ *and a.e.* $x \in [0,T]$. *Then there exists a solution* $\overline{\overline{u}}$ *of the Dirichlet problem for the following differential equation*

$$\frac{d}{dx}[|u'(x)|^2 u'(x) + a(x)u(x)] - u''(x) = a(x)u'(x) + G_u(x, u(x)). \tag{31}$$

Moreover, $\overline{\overline{u}}$ *is the minimizer of the functional f given by*

$$f(u) = \int_0^T \left[\frac{1}{4}|u'(x)|^4 - \frac{1}{2}|u'(x)|^2 + a(x)\langle u'(x), u(x)\rangle + G(x, u(x))\right] dx. \tag{32}$$

PROOF. Let us define $L: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows

$$L(x,u,w) = \frac{1}{4}|w|^4 - \frac{1}{2}|w|^2 + a(x)\langle w,u\rangle + G(x,u). \tag{33}$$

From the above assumptions L satisfies the conditions of hypothesis (H1)–(H5). Applying Theorem 4 we obtain the existence of a minimizer $\overline{\overline{u}}$ of (32). By Corollary 7 we infer that $\overline{\overline{u}}$ is the solution of the Dirichlet problem for (31). □

Now we present the example of the referee, which shows that we can not omit assumption (H4).

REMARK. Let us consider the following functional

$$f(u) = \int_0^T [(u'(x)-1)^2(u'(x)+1)^2 - xu'(x)u(x)]\,dx$$

defined on the set $A(\mathbb{R}^n)$ of absolutely continuous functions $u: [0,T] \to \mathbb{R}^n$ satisfying the below boundary condition $u(T) = u(0) = 0$. This minimization problem has no solution. We prove that in this case Theorem 4 can not be applied.

PROOF. Indeed, we shall show that the assumption (H4) (the growth condition) is not satisfied by the given functional. Suppose, contrary to our claim, that there exist $m,p,s \in (1,\infty)$, $p \le m$, $s \ge p'$, where $p' = p/(p-1)$, and $d \in L^1(0,1,R)$, $k_1 \in L^s(0,1,\mathbb{R}_+)$, constants $l,k > 0$ such that for $u,w \in \mathbb{R}$ and a.e. $x \in [0,1]$

$$-d(x) + \frac{1}{m}l^{1-m}|w|^m - \frac{1}{p}k|u|^p \le -xuw + (w-1)^2(w+1)^2 \le g(x,w) - k_1(x)u,$$

where $g: [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to the first variable, convex with respect to the second one and for a certain ball $K(\overline{w},r) \subset \mathbb{R}$, centered at $\overline{w}$ of radius r, the function $x \to \sup_{w\in K(\overline{w},r)} g(x,w)$ is summable on $[0,T]$.

Let $\overline{k}_1$be in the equivalence class of k_1. From the assumption there exists a measurable set $A \subset [0,1]$ such that $|[0,1] \setminus A| = 0$ and for all $x \in A$, $u,w \in \mathbb{R}$ the following condition holds

$$-xuw + (w-1)^2(w+1)^2 \le g(x,w) - \overline{k}_1(x)u. \tag{34}$$

In particular, (34) is satisfied for a certain $x_0 \in A \setminus \{0\}$ and all $u,w \in \mathbb{R}$. If $\overline{k}_1(x_0) > x_0$ then for $w = 1$ we get, by (34),

$$u \le \frac{g(x_0,1)}{(\overline{k}_1(x_0) - x_0)} \quad \text{for all } u \in \mathbb{R}. \tag{35}$$

When $\overline{k}_1(x_0) < x_0$ we obtain for $w = 1$

$$u \ge \frac{g(x_0,1)}{(\overline{k}_1(x_0) - x_0)} \quad \text{for all } u \in \mathbb{R}. \tag{36}$$

In the case $\overline{k}_1(x_0) = x_0$ we can choose $w = 1/2$. Then from (34) we derive

$$u \le \left(g\left(x_0, \frac{1}{2}\right) - \frac{9}{16}\right)\frac{2}{x_0} \quad \text{for all } u \in \mathbb{R}. \tag{37}$$

Each of assertions (35)–(37) contradicts the unboundedness of $\mathbb{R}$.

Summarizing: assumption (H4) is violated by the lagrangean function

$$L(x, u, w) = -xuw + (w-1)^2(w+1)^2$$

and, in consequence, Theorem 4 can not be applied in this case. □

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] J. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley, New York, 1984.

[3] A. Cellina and G. Colombo, *On a classical problem in the calculus of variations without convexity assumptions*, Ann. Inst. H. Poincaré **7** (1990), 97–106.

[4] L. Cesari, *Optimization – Theory and Applications*, Springer, New York, 1983.

[5] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Willey, New York, 1983.

[6] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Springer, North-Holand, Amsterdam, 1976.

[7] A. Granas and F. Liu, *Some minimax theorems without convexity*, Nonlinear and Convex Analysis. Lecture Notes in Pure and Applied Mathematics (Bor-Luh Lin and S. Simons, eds.), vol. 107, 1989, pp. 61–145.

[8] J. Mawhin, *Metody Wariacyjne dla Nieliniowych Problemów Dirichleta*, Wydawnictwo Naukowo Techniczne, Warszawa, 1994. (Polish)

[9] J. Mawhin and M. Willem, *Critical Points Theory and Hamiltonian Systems*, Springer, New York, 1989.

[10] A. Nowakowski, *Metody wariacyjne dla nieliniowych problemów Dirichleta,* (Chapter 6), Wydawnictwo Naukowo Techniczne, Warszawa, 1994.

[11] ______, *Duality and clasical problem of the calculus of variations without convexity assumptions*, Optimization **21** (1990), 575–584.

[12] A. Orpel, *Periodic solutions of differential equations and nonconvex functionals in calculus of variations*, Optimization **40** (1997), 313–333.

[13] ______, *The existence of the critical points for the action functional without convexity assumption nonconvex*, Optimization **44** (1998), 105–122.

[14] R. T. Rockafellar, *Conjugate convex functions in optimal control and the calculus of variations*, J. Math. Anal. Appl. **32** (1970), 174–222.

Manuscript received November 14, 2000

Aleksandra Orpel
Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, POLAND