Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 20, 2002, 145–149

HARDY–SOBOLEV INEQUALITIES WITH REMAINDER TERMS

Vicenţiu Rădulescu — Didier Smets — Michel Willem

Dedicated to Andrzej Granas

Abstract. We prove two Hardy–Sobolev type inequalities in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, resp. in $H_0^1(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$. The framework involves the singular potential $|x|^{-a}$, with $a \in (0,1)$. Our paper extends previous results established by Bianchi and Egnell ([2]), resp. by Brezis and Lieb ([3]), corresponding to the case $a = 0$.

1. Introduction

Let $\mathcal{D}^{1,2}(\mathbb{R}^N)$ be the completion of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm $\|\nabla u\|_2$. Consider the Hardy–Sobolev inequality on $\mathcal{D}^{1,2}(\mathbb{R}^N)$:

$$\|\nabla u\|_2^2 - S_a \| \, |x|^{-a} u\|_p^2 \geq 0,$$

where $N \geq 3$, $0 < a < 1$ and $p = 2N/(N-2+2a)$.

The minimizers of

$$S_a = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ap}} \, dx \right)^{-2/p} : u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \ u \neq 0 \right\}$$

are given by

$$CU_\lambda(x) = C\lambda^{(N-2)/2} U(\lambda x),$$

2000 *Mathematics Subject Classification.* Primary 35J85, 49J40; Secondary 35R45, 47J20.

Key words and phrases. Hardy–Sobolev inequality, minimization problem, singular potential, Schwartz symmetrization.

where $C \in \mathbb{R}$, $\lambda > 0$ and

$$(1)\qquad U(x) = k_0(1+|x|^\alpha)^{-\beta}, \quad \alpha = \frac{2(N-2)(1-a)}{N-2+2a}, \quad \beta = \frac{N-2+2a}{2(1-a)}.$$

We choose k_0 such that $\|\nabla u\|_2 = S_a$ (see [4]). Hence the minimizers of S_a consist of a 2 dimensional manifold $\mathcal{M} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$. The distance between $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\mathcal{M}$ is defined by

$$d(u,\mathcal{M}) = \inf\{\|\nabla(u - cU_\lambda)\|_2 : c \in \mathbb{R},\ \lambda > 0\}.$$

We prove the following result.

Theorem 1.1. *For $N \geq 3$ and $0 < a < 1$, there exists $A = A(N,a)$ such that, for every $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$,*

$$\|\nabla u\|_2^2 - S_a \|\,|x|^{-a}u\|_p^2 \geq A\, d(u,\mathcal{M})^2.$$

A similar result was proved by Bianchi and Egnell when $a = 0$ (see [2]).

The weak L^p norm is defined by

$$\|u\|_{p,w} = \sup_S |S|^{-1/p'} \int_S |u(x)|\,dx,$$

with S being a set of finite measure $|S|$. Let us recall that the conjugate exponent p' of p is defined by $1/p + 1/p' = 1$.

We deduce from Theorem 1.1 the following result.

Theorem 1.2. *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 3$. For $0 < a < 1$, there exists $B = B(\Omega,a)$ such that, for every $u \in H_0^1(\Omega)$,*

$$\|\nabla u\|_2^2 - S_a \|\,|x|^{-a}u\|_p^2 \geq B\|u\|_{N/(N-2),w}^2.$$

A similar result was proved by Brezis and Lieb when $a = 0$ (see [3]).

In Theorem 1.2 it is not possible to replace $\|u\|_{N/(N-2),w}$ by $\|u\|_{N/(N-2)}$. It suffices to use the function U of (1) and a truncation argument.

It is interesting to compare Theorem 1.2 and the improved Hardy–Poincaré inequality due to Vazquez and Zuazua ([7]).

Theorem. *Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 3$. For $1 \leq q < 2$, there exists $C = C(\Omega,q)$ such that, for every $u \in H_0^1(\Omega)$,*

$$\|\nabla u\|_2^2 - S_1 \|\,|x|^{-1}u\|_2^2 \geq C\|u\|_{W^{1,q}(\Omega)}^2.$$

Let us recall that $S_1 = ((N-2)/2)^2$ is not attained on $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

2. Proof of Theorem 1.1

We follow the argument of [1]. Consider the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda |x|^{-ap} U^{p-2} v, \\ v \in \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases} \tag{2}$$

LEMMA 2.1. *The first two eigenvalues of* (2) *are given by* $\lambda_1 = S_a$ *and* $\lambda_2 = S_a(p-1)$. *The eigenspaces are spanned by* U *and* $\frac{d}{d\lambda}|_{\lambda=1} U_\lambda$, *respectively.*

PROOF. See [6]. □

LEMMA 2.2. *For any sequence* $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \mathcal{M}$ *such that* $\inf_n \|\nabla u_n\|_2 > 0$ *and* $d(u_n, \mathcal{M}) \to 0$ *we have*

$$\liminf_{n\to\infty} \frac{\|\nabla u_n\|_2^2 - S_a \| \, |x|^{-a} u_n\|_p^2}{d(u_n, \mathcal{M})^2} \geq 1 - \frac{\lambda_2}{\lambda_3}. \tag{3}$$

PROOF. We first assume that, for any $n \in \mathbb{N}$, $d(u_n, \mathcal{M}) = \|\nabla(u_n - U)\|_2$. Since $\mathcal{M}$ is a smooth manifold, $v_n = u_n - U$ is orthogonal to the tangent space

$$T_U \mathcal{M} = \operatorname{span} \left\{ U, \frac{d}{d\lambda}\bigg|_{\lambda=1} U_\lambda \right\}.$$

Therefore Lemma 2.1 yields

$$\lambda_3 \int U^{p-2} v_n^2 \frac{dx}{|x|^{ap}} \leq \|\nabla v_n\|^2 = d^2(u_n, \mathcal{M}).$$

Moreover, we have that

$$\int U^{p-1} v_n \frac{dx}{|x|^{ap}} = -S_a^{-1} \int \Delta U v_n \, dx = 0.$$

Setting $d_n = d(u_n, \mathcal{M})$, we obtain

$$\begin{aligned} \int |u_n|^p \frac{dx}{|x|^{ap}} &= \int U^p \frac{dx}{|x|^{ap}} + p \int U^{p-1} v_n \frac{dx}{|x|^{ap}} \\ &\quad + \frac{p(p-1)}{2} \int U^{p-2} v_n^2 \frac{dx}{|x|^{ap}} + o(d_n^2) \\ &\leq 1 + p(p-1) d_n^2 + o(d_n^2) = 1 + \frac{p}{2} \frac{\lambda_2}{\lambda_3} \frac{d_n^2}{S_a} + o(d_n^2) \end{aligned}$$

and

$$\| \, |x|^{-a} u_n\|_p \leq 1 + \frac{\lambda_2}{\lambda_3} \frac{d_n^2}{S_a} + o(d_n^2).$$

Since $\|\nabla u_n\|_2^2 = S_a + d_n^2$, we obtain

$$\|\nabla u_n\|_2^2 - S_a \| \, |x|^{-a} u_n\|_p^2 \geq \left(1 - \frac{\lambda_2}{\lambda_3}\right) d_n^2 + o(d_n^2)$$

and (3) follows immediately.

In the general case, for every n, there exist $c_n \in \mathbb{R}$ and $\lambda_n > 0$ such that $d(u_n, \mathcal{M}) = \|\nabla(u_n - c_n U_{\lambda_n})\|_2$. Setting $w_n(x) = c_n^{-1}\lambda_n^{(2-N)/2} u_n(x/\lambda_n)$, we obtain $\|\nabla(u_n - c_n U_{\lambda_n})\|_2 = |c_n|\, \|\nabla(v_n - U)\|_2 = |c_n|\, d(v_n, \mathcal{M})$. By assumption, $|c_n|$ is bounded away from 0 and

$$\|\nabla(v_n - U)\|_2 = d(v_n, \mathcal{M}) = |c_n|^{-1} d(u_n, \mathcal{M}) \to 0.$$

Using the first part of the proof and the invariance of the quotient in (3), it is easy to conclude. □

Proof of Theorem 1.1. If the theorem is false, there exists a sequence $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \mathcal{M}$ such that

$$\frac{\|\nabla u_n\|_2^2 - S_a \|\, |x|^{-a} u_n\|_p^2}{d(u_n, \mathcal{M})^2} \to 0.$$

We can assume that $\|\nabla u_n\|_2 = 1$ and $d(u_n, \mathcal{M}) \to L \in [0,1]$. It follows that $\|\, |x|^{-a} u_n\|_p^2 \to S_a^{-1}$. By Theorem 2.4 in [5], going if necessary to a subsequence, we can assume the existence of $\lambda_n > 0$ such that $\lambda_n^{(N-2)/2} u_n(\lambda_n x) \to V \in \mathcal{M}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. This implies that $L = 0$. By Lemma 2.2, we have a contradiction. □

3. Proof of Theorem 1.2

We deduce theorem 1.2 from Theorem 1.1 by adapting the argument of [2].

It suffices to prove the theorem when $\Omega = B(0,1)$ and $u = u^*$, where u^* denotes the Schwartz symmetrization of u. Indeed, we have that

$$\|\nabla u\|_2 \geq \|\nabla u^*\|_2, \quad \|\, |x|^{-a} u\|_p = \|\, |x|^{-a} u^*\|_p, \quad \|u\|_{N/(N-2),w} = \|u^*\|_{N/(N-2),w}.$$

If Theorem 1.2 is false, there exists a sequence $(u_n) \subset H_0^1(\Omega)$ such that $u_n = u_n^*$ and

$$\frac{\|\nabla u_n\|_2^2 - S_a \|\, |x|^{-a} u_n\|_p^2}{\|u_n\|_{N/(N-2),w}^2} \to 0. \tag{4}$$

We can assume that $\|\nabla u_n\|_2 = 1$. Since $\|u_n\|_{N/(N-2),w}^2$ is bounded by Sobolev's inequality, we must have $\|\, |x|^{-a} u_n\|_p^2 \to S_a^{-1}$.

By Theorem 1.1, there exists a sequence $(c_n, \lambda_n) \to (1, \infty)$ such that

$$d(u_n, \mathcal{M}) = \|\nabla(u_n - c_n U_{\lambda_n})\|_2 \to 0, \quad \text{as } n \to \infty.$$

It is clear that

$$\begin{aligned} d(u_n, \mathcal{M})^2 &\geq c_n^2 \int_{|x|>1} |\nabla U_{\lambda_n}|^2\, dx \\ &= k_0^2 c_n^2 \lambda_n^{N-2+2\alpha} \alpha^2 \beta^2 \int_1^\infty (1 + \lambda_n^\alpha r^\alpha)^{-2\beta-2} r^{2\alpha+N-3}\, dr \\ &= C_1 c_n^2 \int_{\lambda_n}^\infty (1 + s^\alpha)^{-2\beta-2} s^{2\alpha+N-3}\, ds \geq C_2 c_n^2 \lambda_n^{2-N}. \end{aligned}$$

Hence we obtain

$$\begin{aligned}(5)\quad \|u_n\|_{N/(N-2),w} &\leq \|u_n - c_n U_{\lambda_n}\big|_{\Omega}\|_{N/(N-2),w} + \|c_n U_{\lambda_n}\|_{N/(N-2),w} \\ &\leq C_3 \|u_n - c_n U_{\lambda_n}\|_{2N/(N-2)} + c_n \lambda_n^{(2-N)/2} \|U\|_{N/(N-2),w} \\ &\leq C_4 d(u_n, \mathcal{M}).\end{aligned}$$

But (4) and (5) contradict Theorem 1.2.

References

[1] T. Bartsch, T. Weth and M. Willem, *A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator* (to appear).

[2] G. Bianchi and H. Egnell, *A Note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), 18–24.

[3] H. Brezis and E. Lieb, *Inequalities with remainder terms*, J. Funct. Anal. **62** (1985), 73–86.

[4] E. Lieb, *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*, Ann. of Math. **118** (1983), 349–374.

[5] P. L. Lions, *The concentration–compactness principle in the calculus of variations. The limit case, Part* 2, Rev. Mat. Iberoamericana **1** (1985), 45–121.

[6] D. Smets and M. Willem, *Partial symmetry and asymptotic behaviour for some elliptic variational problems*, work in progress, Calc. Var. Partial Differential Equations (to appear).

[7] J. Vazquez and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse square potential*, J. Funct. Anal. **173** (2000), 103–153.

Manuscript received April 18, 2002

Vicenţiu Rădulescu
Department of Mathematics
University of Craiova
1100 Craiova, ROMANIA

E-mail address: vicrad@yahoo.com

Didier Smets
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
4 place Jussieu
75 252 Paris Cedex 05, FRANCE

E-mail address: smets@ann.jussieu.fr

Michel Willem
Institut des Mathématiques Pures et Appliquées
Université Catholique de Louvain
Chemin du Cyclotron 2
1348 Louvain-La-Neuve, BELGIUM

E-mail address: willem@amm.ucl.ac.be