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APPLICATION OF TOPOLOGICAL TECHNOLOGY TO CONSTRUCTION OF A PERTURBATION SYSTEM FOR A STRONGLY NONLINEAR EQUATION

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Abstract. The homotopy perturbation method proposed by the present author is further improved in this paper, which is proved to be effective and convenient to solving nonlinear equations.

Introduction

There exist many difficulties encountered in the application of the traditional perturbation techniques to solving strongly nonlinear equations. Of these, one of the most frustrating is the fact that even a simple practical system may not posses the so-called "small parameter", which is the theoretical base of the perturbation methods. The determination of a small parameter in an equation seems to be a special art requiring special techniques. An appropriate choice of a small parameter leads to an ideal result, however, an unsuitable choice of a small parameter results in badly effects, sometimes seriously. Furthermore, even if there exists a suitable small parameter, the approximate solutions solved by the perturbation methods are valid, in most cases, only for the small values of the parameter.

To overcome the shortcomings of the perturbation techniques, we will apply homotopy in topological technique to construct a perturbation system. The basic

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idea of homotopy method is to continuously deform a simple problem easy to solve into the difficult problem under study.

1. Application homotopy technique to construct a perturbation system

The homotopy method, or the continuous mapping technique, has been generally used to widen the domain of convergence of a given method or as a procedure to obtain sufficiently close starting points (see [3]–[5]). The continuous mapping technique embeds a parameter that typically ranges from zero to one. When the embedding parameter is zero, the equation is one of the linear system. When it is one, the equation is the same as the original.

To illustrate its basic idea, we consider following nonlinear algebraic equation

$$f(x) = 0, \quad x \in \mathbb{R}. \tag{1.1}$$

We construct a homotopy map $\mathbb{R} \times [0,1] \to \mathbb{R}$ which satisfies

$$\mathbb{H}(\xi, p) = pf(\xi) + (1-p)[f(\xi) - f(x_0)] = 0, \quad x \in \mathbb{R},\ p \in [0,1] \tag{1.2a}$$

or, equivalently,

$$\mathbb{H}(\xi, p) = f(\xi) - f(x_0) + pf(x_0) = 0, \quad x \in \mathbb{R},\ p \in [0,1], \tag{1.2b}$$

where p is an imbedding parameter, x_0 is an initial approximation of (1.1).

It is obvious that

$$\mathbb{H}(\xi, 0) = f(\xi) - f(x_0) = 0 \quad \text{and} \quad \mathbb{H}(\xi, 1) = f(\xi) = 0.$$

The embedding parameter p monotonically increases from zero to unit as the trivial problem $f(\xi) - f(x_0) = 0$ is continuously deformed to the problem $f(\xi) = 0$. So, if we can construct an iteration formula for the equation (1.2), the series of approximations comes along the solution path, by incrementing the imbedding parameter from zero to one; this continuously maps the initial solution into the solution of the original equation (1.1). The changing process of p from zero to one is just that of $\mathbb{H}(\xi, p)$ from $f(\xi) - f(x_0)$ to $f(\xi)$. In topology, this is called deformation, and $f(\xi) - f(x_0)$, $f(\xi)$ are homotopic.

Due to the fact that $0 \le p \le 1$, so the embedding parameter can be considered as a "small parameter", and the equation (1.2) is called perturbation equation with an embedding parameter.

Applying the perturbation technique, we can assume that the solution of the equation (1.2) can be expressed as

$$\xi = \xi_0 + p\xi_1 + p^2\xi_2 + p^3\xi_3 + \ldots \tag{1.5}$$

If p increments from zero to one, then the solution (1.5) converges to the solution of the original equation (1.1). This means that the solution of the equation (1.1) can be written in the form

$$x = \lim_{p \to 1} \xi = \xi_0 + \xi_1 + \xi_2 + \xi_3 + \dots \tag{1.6}$$

To obtain the approximate solution of the equation (1.2), we, at first, expand $f(\xi)$ into Taylor series

$$f(\xi) = f(\xi_0) + f'(\xi_0)(p\xi_1 + p^2\xi_2 + \dots) + \frac{1}{2!} f''(\xi_0)(p\xi_1 + p^2\xi_2 + \dots)^2 + \dots \tag{1.7}$$

Substituting (1.7) into (1.2), collecting coefficients of equal powers of p and equating coefficients of like powers of p to zero, we obtain

$$\begin{aligned}
p^0 &: \quad f(\xi_0) - f(x_0) = 0,\\
p^1 &: \quad f'(\xi_0)\xi_1 + f(x_0) = 0,\\
p^2 &: \quad f'(\xi_0)\xi_2 + \frac{1}{2!} f''(\xi_0)\xi_1^2 = 0,\\
p^3 &: \quad f'(\xi_0)\xi_3 + f''(\xi_0)\xi_1\xi_2 + \frac{1}{3!} f'''(\xi_0)\xi_1^3 = 0.
\end{aligned}$$

From the above equations, $\xi_0 \sim \xi_3$ can be solved easily

$$\begin{aligned}
\xi_1 &= -\frac{f(x_0)}{f'(\xi_0)},\\
\xi_2 &= -\frac{1}{2!}\frac{f''(\xi_0)}{f'(\xi_0)}\xi_1^2 = -\frac{1}{2!}\frac{f''(\xi_0)}{f'(\xi_0)}\left\{\frac{f(x_0)}{f'(\xi_0)}\right\}^2,\\
\xi_3 &= -\frac{1}{2!}\frac{f''(\xi_0)}{f'(\xi_0)}2\xi_1\xi_2 - \frac{1}{3!}\frac{f'''(\xi_0)}{f'(\xi_0)}\xi_1^3\\
&= \left\{\frac{f'''(\xi_0)}{3!f'(\xi_0)} - \frac{1}{2}\left[\frac{f''(\xi_0)}{f'(\xi_0)}\right]^2\right\}\left\{\frac{f(x_0)}{f'(\xi_0)}\right\}^3.
\end{aligned}$$

We, therefore, obtain its first-order approximation of the equation (1.1), $x = \xi_0 + \xi_1$, its second-order approximation $x = \xi_0 + \xi_1 + \xi_2$ and its third approximation $x = \xi_0 + \xi_1 + \xi_2 + \xi_3$. We can write down their iteration formulae respectively as follows

$$\begin{aligned}
x_{n+1} &= \xi_n - \frac{f(\xi_n)}{f'(\xi_n)},\\
x_{n+1} &= \xi_n - \frac{f(\xi_n)}{f'(\xi_n)} - \frac{f''(\xi_n)}{2f'(\xi_n)}\left\{\frac{f(\xi_n)}{f'(\xi_n)}\right\}^2,\\
x_{n+1} &= \xi_n - \frac{f(\xi_n)}{f'(\xi_n)} - \frac{f''(\xi_n)}{2f'(\xi_n)}\left\{\frac{f(\xi_n)}{f'(\xi_n)}\right\}^2\\
&\quad + \left\{\frac{f'''(\xi_n)}{3!f'(\xi_n)} - \frac{1}{2}\left[\frac{f''(\xi_n)}{f'(\xi_n)}\right]^2\right\}\left\{\frac{f(\xi_n)}{f'(\xi_n)}\right\}^3.
\end{aligned}$$

The above results are same with those obtained in [7], and [8] when $h = -1$. From $f(\xi_0) - f(x_0) = 0$, we can obtain one of its solutions, $\xi_0 = x_0$, under such condition, the first iteration formula can be re-written down as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which is the well-known Newton iteration formula.

The iteration formulae obtained by the homotopy technique can find all the solutions of an algebraic equation, while the Newton iteration method can only find the solution near the initial solution. For example, we consider a polynomial of third degree

$$f(x) = x^3 - 12x^2 + 21x - 10 = 0.$$

We begin with $x_0 = 0$, from $f(\xi_0) - f(x_0) = 0$ we have

$$\xi_0^{(1)} = 0, \quad \xi_0^{(2)} = 2.13 \quad \text{and} \quad \xi_0^{(3)} = 9.87.$$

By few iterations, we obtain $x_1^{(1)} = 1$, $x_1^{(2)} = 1$, and $x_1^{(3)} = 10$.

2. Solving strongly nonlinear equation by homotopy technology

We consider a general form of a nonlinear system

$$L(u) + N(u) = 0, \tag{2.1}$$

where $L(u) = 0$ is an equation that can be readily solved, in most cases, it is an linear equation.

By the homotopy technique ([9]), we construct a homotopy map, which satisfies

$$\mathbb{H}(\nu, p) = (1 - p)[L(\nu) - L(u_0)] + p[L(\nu) + N(\nu)] = 0, \tag{2.2a}$$

or equivalently

$$\mathbb{H}(\nu, p) = L(\nu) - L(u_0) + pL(u_0) + pN(\nu) = 0, \tag{2.2b}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of the original equation (2.1).

Applying the perturbation technique ([10]), we assume that the solution of the equation (2.2) can be written in the form

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \ldots \tag{2.3}$$

Therefore the approximate solution of the nonlinear system (2.1), can be readily obtained when $p \to 1$

$$u = \lim_{p \to 1} \nu = \nu_0 + \nu_1 + \nu_2 + \ldots$$

Observe that the new perturbation technique needs not possess a "small parameter" in an equation, so it has eliminated limitations of the traditional perturbation methods. On the other hand the proposed technique can take full advantage of the traditional perturbation techniques.

EXAMPLE 1 ([1]). Consider the motion of a ball-bearing oscillating in a glass tube that is bent into a curve such that the restoring force depends upon the cube of the displacement u (Figure 1).

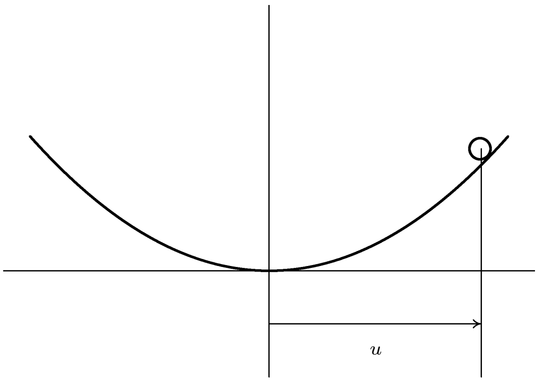

FIGURE 1. A ball-bearing oscillating in a smooth tube bent to produce a restoring force proportional to cube of the displacement

The governing equation, ignoring frictional losses, is

$$u'' + \varepsilon u^3 = 0, \tag{2.4}$$

and the auxiliary conditions are that the ball-bearing is released from rest at a displacement u_0 when $t = 0$. Expressed mathematically, this is $u(0) = A$, $u'(0) = 0$.

In our study, the parameter ε needs not to be small, i.e. it follows $0 < \varepsilon < \infty$. For this special example, the traditional perturbation methods can not be applied even in case $0 < \varepsilon \ll 1$, for the unperturbed equation $u'' = 0$ can not lead to a period solution.

Now we constrict a homotopy map

$$\nu'' + \omega^2\nu + p(\varepsilon\nu^3 - \omega^2\nu) = 0, \qquad \nu(0) = A, \quad \nu'(0) = 0. \tag{2.5}$$

Supposing that the solution of (2.5) can be expressed in the form of (2.3), by simple manipulation, we obtain the following linear equations

$$\nu_0'' + \omega^2\nu_0 = 0, \quad \nu_0(0)= A, \quad \nu_0'(0) = 0, \tag{2.6}$$

$$\nu_1'' + \omega^2\nu_1 + \varepsilon\nu_0^3 - \omega^2\nu_0 = 0, \quad \nu_1(0) = 0, \quad \nu_1'(0) = 0. \tag{2.7}$$

Solving (2.6), we have $\nu_0 = A\cos\omega t$. Substituting u_0 into (2.7) results in

$$\nu_1'' + \omega^2\nu_1 + \frac{1}{4}\varepsilon A^3\cos 3\omega t + \left(\frac{3}{4}\varepsilon A^3 - A\omega^2\right)\cos\omega t = 0. \tag{2.9}$$

Eliminating secular term needs

$$\omega = \frac{\sqrt{3}}{2}\varepsilon^{1/2}A.$$

Solving (2.9) subject to initial conditions $\nu_1(0) = 0$, $\nu_1'(0) = 0$, we have

$$\nu_1 = \frac{1}{32\omega^2}\varepsilon A^3(\cos 3\omega t - \cos\omega t).$$

So we obtain first-order approximate solution of original equation by setting $p = 1$

$$u = \lim_{p\to 1}(\nu_0 + p\nu_1) = A\cos\omega t + \frac{1}{32\omega^2}\varepsilon A^3(\cos 3\omega t - \cos\omega t).$$

We can obtained the same result if we apply various perturbation techniques proposed in [6]. Its period can be written as

$$T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{3}}\varepsilon^{-1/2}A^{-1} = 7.25\varepsilon^{-1/2}A^{-1}.$$

Its exact period can be readily obtained, which reads [1], [2]

$$T = 7.4164\varepsilon^{-1/2}A^{-1}.$$

The maximal relative error is less than 2.2% for all $\varepsilon > 0$!

3. Conclusion

In this paper, we proposed a new approach to finding the periodic solution of a kind of nonlinear oscillations, the obtained results reveal that the proposed method is valid for all $\varepsilon > 0$. Our approach is much more effective and convenient than Liao's homotopy analysis method in [9], where the general Taylor series is used.

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