Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 20, 2002, 1–14

A STRONGLY NONLINEAR NEUMANN PROBLEM AT RESONANCE WITH RESTRICTIONS ON THE NONLINEARITY JUST IN ONE DIRECTION

Jean Mawhin — David Ruiz

Dedicated to Andrzej Granas

Abstract. Using topological degree techniques, we state and prove new sufficient conditions for the existence of a solution of the Neumann boundary value problem

$$(|x'|^{p-2}x')' + f(t,x) + h(t,x) = 0, \quad x'(0) = x'(1) = 0,$$

when h is bounded, f satisfies a one-sided growth condition, $f + h$ some sign condition, and the solutions of some associated homogeneous problem are not oscillatory. A generalization of Lyapunov inequality is proved for a p-Laplacian equation. Similar results are given for the periodic problem.

1. Introduction

It is well known (see [4], [6]) that, if the L^1-Caratheodory function f is such that, for some $\mu^+, \mu^- \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{f(t,x)}{x} = \mu^+, \quad \lim_{x\to-\infty} \frac{f(t,x)}{x} = \mu^- \tag{1.1}$$

2000 *Mathematics Subject Classification.* 34B15, 34C25, 34C10.

Key words and phrases. Neuman problem, periodic solutions, p-Laplacian equation, topological degree, Lyapunov inequality.

The second author has been supported in part by D.G.E.S., Ministry of Education and Culture (Spain), under grant number PB98-1343 and by J. Andalucía (FQM 116).

then, using topological degree, one can reduce the existence of a solution of the Neumann problem

$$x'' + f(t, x) = 0, \tag{1.2}$$

$$x'(0) = x'(1) = 0, \tag{1.3}$$

to the nonexistence of nonzero solutions of the "limit" problem

$$x'' + \mu^+ x^+(t) - \mu^- x^-(t) = 0, \tag{1.4}$$

$$x'(0) = x'(1) = 0, \tag{1.5}$$

which is related with the Fučik spectrum (see [6], [4]). Even if the limits (1.1) are not constants, but L^1-functions, it is possible to state a similar result in the same way. Note that condition (1.1) fixes the behaviour of the function f at ∞ and $-\infty$. A number of papers, for example [11], [15], [3], have considered the case in which one has restrictions only in one direction. The aim of this paper is to give some new results and generalizations in this type of problems. In addition we work in the more general setting of perturbations the p-Laplacian.

The hypothesis we assume may be separated into two groups: conditions of change of sign for f (i.e. the nonlinearity crosses the zero level, which is resonant with Neumann boundary condition), and growth restrictions in one direction. The first type of conditions will be explained in Section 2, when we state our main result. Notice that these conditions are weaker than those imposed in [3], [15] or [11] (in this last paper, periodic boundary conditions are considered). The growth restrictions generalize those given in [3]:

(1) *There exists* $q \in L^1(0,1)$, $q \geq 0$, $r > 0$, *such that*

$$\frac{f(t,x)}{|x|^{p-2}x} \leq q(t) \quad \textit{for all } x \leq -r. \tag{1.6}$$

(2) *The solutions of the initial value problems*

$$\begin{gathered} (|x'|^{p-2}x')' + q(t)|x|^{p-2}x = 0, \\ x(0) = -1,\ x'(0) = 0 \quad (\textit{resp. } x(1) = -1,\ x'(1) = 0) \end{gathered} \tag{1.7}$$

have no zeros in $[0,1]$.

The main tool used in the proof is a classical argument of continuation of solutions based upon topological degree. Some ideas of integral conditions related with hypothesis (1.7) are also used (see for example [10]).

It is not difficult to show (see [3, Section 2]) that if the function q is such that the solution x of the initial value problem (1.7) vanishes in one point in $(0,1)$, then our problem may have no solutions. But until now it was not known, even when $p = 2$, what happens in the critical case (that is, for instance, $x(1) = 0$, $x(t) > 0$ if $t < 1$). Although the paper [3] seems to deal also with this situation, its arguments do not work in this case. In Section 3 we prove the existence of solution in this last case.

In Section 4 we discuss condition (1.7), trying to make it more explicit in terms of conditions upon the function q.

2. The main result

In this section we state and prove our main result. Let us consider the Neumann boundary value problem

$$(|x'|^{p-2}x')' + f(t,x) + h(t,x) = 0, \tag{2.1}$$

$$x'(0) = x'(1) = 0, \tag{2.2}$$

where $p \geq 2$, $f, h\colon (0,1) \times \mathbb{R} \to \mathbb{R}$ are L^1-Caratheodory functions, and there exists $\overline{h} \in L^1$ such that

$$|h(t,x)| \leq \overline{h}(t) \quad \text{for a.e. } t \in (0,1).$$

We write

$$g(t,x) = f(t,x) + h(t,x).$$

Theorem 2.1. *Suppose that conditions* (1.6), (1.7) *are verified, and assume also that the following hypotheses hold.*

(1) *There exists* $r > 0$ *such that*

$$\int_0^1 g(t, x_1(t))\,dt \geq 0 \geq \int_0^1 g(t, x_2(t))\,dt \tag{2.3}$$

whenever $x_1, x_2 \in W^{1,p}(0,1)$, *satisfy boundary condition* (2.2), *and* $x_1(t) > r > -r > x_2(t)$ *for all* $t \in [0,1]$.

(2) *There exists a nonnegative function* $j \in L^1(0,1)$, *such that*

$$\int_0^1 j(t)\,dt \leq 1 \quad \text{and} \quad \frac{f(t,x)}{|x|^{p-2}x} \geq -j(t) \tag{2.4}$$

for all $x > r$.

Then, problem (2.1) *has at least one solution.*

Proof. Obviously, we can suppose throughout the proof that the nonnegative functions q, j are not identically zero. Using the Leray–Schauder principle ([9]), we only need to prove that the solutions of the following problem

$$(|x'|^{(p-2)}x')' + \lambda q(t)|x|^{p-2}x + (1-\lambda)g(t,x) = 0, \quad \lambda \in [0,1], \tag{2.5}$$

$$x'(0) = x'(1) = 0, \tag{2.6}$$

are bounded in the norm of $W^{1,p}$. By contradiction, suppose that x_n is an unbounded sequence in $W^{1,p}$ of solutions of (2.5), (2.6) for $\lambda = \lambda_n$, and let $y_n = x_n/||x_n||$. Taking a convenient subsequence, we can suppose that $y_n \rightharpoonup y_0$

in $W^{1,p}$, $y_n \to y_0$ uniformly and $\lambda_n \to \lambda_0$. We denote the norm if f in $W^{1,p}$ by $||f||$ and the uniform norm by $|f|$.

We define

$$f_n(t,x) = \lambda_n q(t)|x|^{p-2}x + (1-\lambda_n)f(t,x),$$
$$g_n(t,x) = f_n(t,x) + (1-\lambda_n)h(t,x) = \lambda_n q(t)|x|^{p-2}x + (1-\lambda_n)g(t,x).$$

Dividing (2.5) by $\|x_n\|$ we see that the functions y_n are solutions of the problem

$$\begin{gathered}(|y_n'|^{p-2}y_n')' + ||x_n||^{1-p}g_n(t,\ x_n(t)) = 0,\\ y_n'(0) = y_n'(1) = 0.\end{gathered} \tag{2.7}$$

Let us prove now some inequalities which will be useful later. Integrating the differential equation in (2.7), we obtain

$$\int_0^1 g_n(t,x_n(t))\,dt = 0. \tag{2.8}$$

Multiplying the differential equation in (2.7) by y_n and integrating over $(0,1)$ gives

$$||x_n||^{1-p}\int_0^1 g_n(t,x_n(t))y_n(t)\,dt = \int_0^1 |y_n'(t)|^p\,dt. \tag{2.9}$$

From (2.8), we can write

$$0 = \int_{x_n>r} g_n(t,x_n(t))\,dt + \int_{x_n<-r} g_n(t,x_n(t))\,dt + \int_{|x_n|\le r} g_n(t,x_n(t))\,dt.$$

Hence

$$\begin{aligned}\int_{x_n>r} g_n(t,x_n(t))\,dt &\le C + \int_{x_n<-r} g_n(t,x_n(t))\,dt\\ &\le \int_{x_n<-r} |f_n(t,x_n(t))|\,dt + C' \le \int_{x_n<-r} q(t)|x_n(t)|^{p_1}\,dt + C'.\end{aligned}$$

Thus, we finally have that

$$\int_{x_n>r} f_n(t,x_n(t))\,dt \le C'' + \int_0^1 q(t)|x_n^-(t)|^{p-1}\,dt. \tag{2.10}$$

For clarity in the exposition, the rest of the proof is divided in three steps.

Step 1. There exists $t_n \in (0,1)$ such that $|x_n(t_n)| \le r$.

Suppose that $x_n(t) > r$ for all $t \in (0,1)$. Integrating (2.5) over $(0,1)$, we have

$$\lambda_n \int_0^1 q(t)|x_n(t)|^{p-2}x_n(t)\,dt + (1-\lambda_n)\int_0^1 g(t,x_n(t))\,dt = 0,$$

where $\lambda_n \in (0,1)$, $q \ge 0$ and $q \not\equiv 0$. This is a contradiction with our hypothesis (2.3). If $x_n(t) < -r$, we can argue in an analogous manner.

Observe that, in particular, we can assure the existence of a point $t_0 \in [0,1]$ such that $y_0(t_0) = 0$. This will be useful in the following.

Step 2. $y_0^- \neq 0$.

Suppose that $y_0^- = 0$. Recall that $y_n \rightharpoonup y_0$ in $W^{1,p}(0,1)$ and $y_n \to y_0$ uniformly. Then, from the weak lower semicontinuity of the norm, we know that

$$\liminf_{n\to\infty} \int_0^1 |y_n'(t)|^p \, dt \geq \int_0^1 |y_0'(t)|^p \, dt.$$

Taking into account formula (2.9), we have

$$\begin{aligned}
&\int_0^1 |y_n'(t)|^p \, dt = ||x_n||^{1-p} \int_0^1 g_n(t, x_n(t)) y_n(t) \, dt \\
&\leq ||x_n||^{1-p} \left\{ \int_0^1 f_n(t, x_n(t)) y_n(t) \, dt + C \right\} \\
&\leq ||x_n||^{1-p} \left\{ \int_{x_n > r} f_n(t, x_n(t)) y_n(t) \, dt + \int_{x_n < -r} f_n(t, x_n(t)) y_n(t) \, dt \right\} + o(1) \\
&\leq ||x_n||^{1-p} \int_{x_n > r} f_n(t, x_n(t)) y_n(t) \, dt + \int_{x_n < -r} \frac{f_n(t, x_n(t))}{|x_n(t)|^{p-1}} |y_n^-(t)|^p \, dt + o(1).
\end{aligned}$$

Since $f_n(t, x_n(t))/|x_n(t)|^{p-1} \leq q(t)$ when $x_n < -r$, and $y_n^- \to 0$, we get that the second term of the above expression tends to zero. Thus,

$$\begin{aligned}
\int_0^1 |y_n'(t)|^p \, dt &\leq ||x_n||^{1-p} \int_{x_n > r} f_n(t, x_n(t)) y_n(t) \, dt + o(1) \\
&\leq ||x_n||^{1-p} \int_{x_n > r} [f_n(t, x_n(t))]^+ y_n(t) \, dt + o(1) \\
&\leq ||x_n||^{1-p} |y_n| \int_{x_n > r} [f_n(t, x_n(t))]^+ \, dt + o(1) = (*).
\end{aligned}$$

Notice that we have been able to consider only the positive part of $f_n(t, x_n(t))$ provided that $y_n(t) = x_n(t)/||x_n|| > r/||x_n|| > 0$. Now we apply the general equality $f^+ = f + f^-$ in our equation, and use formula (2.10)

$$\begin{aligned}
&(*) = ||x_n||^{1-p} |y_n| \int_{x_n > r} [f_n(t, x_n(t))] \, dt + \int_{x_n > r} [f_n(t, x_n(t))]^- \, dt + o(1) \\
&\leq |y_n| \left\{ \int_0^1 q(t) \frac{(x_n^-(t))^{p-1}}{||x_n||^{p-1}} \, dt + ||x_n||^{1-p} \int_{x_n > r} [f_n(t, x_n(t))]^- \, dt \right\} + o(1) \\
&\leq ||x_n||^{1-p} |y_n| \int_{x_n > r} [f_n(t, x_n(t))]^- \, dt + o(1) = (**).
\end{aligned}$$

From (2.4), we know that if $x_n(t) > r$, then $f_n(t, x_n(t))/|x_n(t)|^{p-1} \geq -j(t)$, which implies $[f_n(t, x_n(t))]^-/|x_n(t)|^{p-1} \leq j(t)$. Then

$$\begin{aligned}
(**) &= |y_n| \int_{x_n > r} \frac{[f_n(t,\ x_n(t))]^-}{(x_n(t))^{p-1}} (y_n(t))^{p-1} \, dt + o(1) \\
&\leq |y_n| \int_{x_n > r} j(t) (y_n(t))^{p-1} \, dt + o(1).
\end{aligned}$$

If now $n \to \infty$, we get

$$\int_0^1 |y_0'(t)|^p\,dt \le |y_0| \int_0^1 j(t)(y_0(t))^{p-1}\,dt. \tag{2.11}$$

First, y_0 cannot be identically zero because, if y_0 is zero, the previous arguments show that $\lim_{n\to\infty}\int_0^1 |y_n'(t)|^p\,dt = 0$. Moreover, $y_n \to 0$ uniformly, but, by definition, $||y_n|| = 1$.

From (2.11) it follows that

$$\int_0^1 |y_0'(t)|^p\,dt \le |y_0|^p \int_0^1 j(t)\,dt, \tag{2.12}$$

and if the equality is verified then $y_0(t) = ||y_0||$ when $t \in A = \{t \in [0,1] : j(t) \neq 0\}$, which is a set with positive measure.

But, applying Hölder inequality and the mean value theorem, we obtain the other inequality

$$\int_0^1 |y_0'(t)|^p\,dt \ge \bigg(\int_0^1 |y_0'(t)|\,dt\bigg)^p \ge |y_0|^p$$

and the equality is verified if y_0' is constant. So, we arrive again to a contradiction, which proves that $y_0^- \neq 0$.

Step 3. End of the proof of Theorem 2.1.

If we multiply equation (2.7) by y_n^-, integrate, and recall the definition of g_n, we have

$$\int_0^1 |(y_n^-)'(t)|^p\,dt - \int_0^1 \lambda_n q(t)|y_n^-(t)|^p + (1-\lambda_n)\frac{g(t,x_n(t))}{||x_n||^{p-1}} y_n^-(t)\,dt = 0. \tag{2.13}$$

The idea is to pass to the limit when $n \to \infty$. Since $y_n \rightharpoonup y_0$ in $W_0^{1,p}(0,1)$, then $y_n^- \rightharpoonup y_0^-$, and because of the weak lower semicontinuity of the norm, we have:

$$\liminf_{n\to\infty} \int_0^1 |(y_n^-)'(t)|^p\,dt \ge \int_0^1 |(y_0^-)'(t)|^p\,dt. \tag{2.14}$$

Now, we consider the second term of (2.13). For simplicity, we write

$$I_n(t) = \lambda_n q(t)|y_n^-(t)|^p + (1-\lambda_n)\frac{g(t,x_n(t))}{||x_n||^{p-1}} y_n^-(t).$$

We define also

$$A_n = \{t \in (0,1) : x_n(t) \in [-r,0]\}, \quad B_n = \{t \in (0,1) : x_n(t) < -r\}.$$

Then the second term in (2.13) can be written as follows

$$\int_0^1 I_n(t)\,dt = \int_{A_n} I_n(t)\,dt + \int_{B_n} I_n(t)\,dt = (***).$$

The first term in the above equation tends to zero as n tends to ∞. Then, we can write

$$(***) = \int_{B_n} \left[\lambda_n q(t) + (1-\lambda_n)\frac{f(t,x_n(t))}{|x_n(t)|^{p-2}x_n(t)}\right]|y_n^-(t)|^p\,dt + o(1)$$
$$\leq \int_0^1 q(t)|y_n^-(t)|^p\,dt + o(1).$$

In conclusion, in this last step we have proved that

$$\int_0^1 |(y_0^-)\prime(t)|^p\,dt \leq \int_0^1 q(t)|y_0^-(t)|^p\,dt. \tag{2.15}$$

Hypothesis (1.7) is equivalent to the following one (see [10])

$$\int_0^1 |y'(t)|^p\,dt \geq \int_0^1 q(t)|y(t)|^p\,dt \tag{2.16}$$

for all $y \in W^{1,p}$ such that y vanishes at some point, and equality holds if and only if $y = 0$. Thus, we arrive to a contradiction, which ends the proof. □

Remarks. (1) The same arguments work if we have periodic instead of Neumann boundary conditions (see [3], [11]). In this case, the hypothesis (1.7) has to be replaced by the following one:

For each $t_0 \in [0,1]$, the solutions of the initial value problem

$$\begin{gathered}(|x'|^{p-2}x')' + q(t)|x|^{p-2}x = 0,\\ x(t_0) = 0, \quad x'(t_0) = 1,\end{gathered} \tag{2.17}$$

have no zero in $(t_0, t_0+1]$.

(We extend the function q to $[0,2]$ by defining $q(t+1) = q(t)$).

(2) If q is a constant, then it is easily checked that hypothesis (1.7) is equivalent to $q < (\pi_p/2)^p$ (see for example [8] for the description of the spectrum of the p-Laplacian and the definition of π_p). In Section 3 we will show that, with slightly stronger hypothesis, we can also take $q = (\pi_p/2)^p$.

(3) All these reasonings can be applied in the PDE case, in a bounded domain, if $p > n$ (that condition is necessary to assure continuity of solutions, which is essential in the proof). In that case we should replace hypothesis (1.7) on q by the integral condition (2.16).

3. The critical case

We now consider the following problem

$$(|x'(t)|^{p-2}x'(t))' + f(t,x(t)) = 0, \tag{3.1}$$

$$x'(0) = x'(1) = 0, \tag{3.2}$$

where f is a L^1-Caratheodory function verifying the following properties:

(1) *There exists* $r > 0$ *such that*

$$f(t,x)x \geq 0 \quad \text{if } |x| \geq r. \tag{3.3}$$

(2) *There exists* $q \in L^1(0,1)$, $q \geq 0$, $r > 0$, *such that*

$$\frac{f(t,\ x)}{|x|^{p-2}x} \leq q(t) \quad \textit{for all } x \leq -r. \tag{3.4}$$

(3) *The solutions of the initial value problems*

$$\begin{gathered}(|x'|^{p-2}x')' + q(t)|x|^{p-2}x = 0,\\ x(0) = -1,\ x'(0) = 0 \quad (\text{resp. } x(1) = -1,\ x'(1) = 0)\end{gathered} \tag{3.5}$$

have no zero in $[0,1)$ (*resp. in* $(0,1]$).

The condition (3.3) is slightly stronger than hypothesis (2.3), (2.4), but they keep the same idea: the function f changes of sign in some sense. The last condition (3.5) is equivalent to the following integral condition:

$$\int_0^1 |y'(t)|^p\,dt \geq \int_0^1 q(t)|y(t)|^p\,dt \tag{3.6}$$

for all $y \in W^{1,p}$ such that y is zero in some point, and equality holds if and only if y is proportional to a solution of (3.5) verifying $y(0)y(1) = 0$.

Theorem 3.1. *If the above hypotheses hold, problem* (3.1), (3.2) *has at least one solution.*

Proof. The idea is to come back to the functions x_n and to prove that they must be uniformly bounded from above; from this fact it is easy to obtain a contradiction.

If we follow the arguments showed in Section 1, we get that y_0 is a positive multiple of a solution of (3.5), and verifies $y_0(0)y_0(1) = 0$. Suppose, for instance, that $y_0(1) = 0$. No contradiction follows now, at least, not immediately.

According to Step 1 in Section 2, we can take $t_n \in (0,\ 1)$ to be the first point such that $x_n(t_n) = -r$. Clearly, $t_n \to 1$. Choose $m_n > 0$ conveniently such that $m_n y_0(t_n) = -r$. It is easy to show, from (3.4), that $m_n y_0(t) \leq x_n(t)$ for all $t \in [0, t_n]$, and observe that both functions are convex in this interval. It follows also that

$$x_n'(t_n) \leq m_n y_0'(t_n). \tag{3.7}$$

Since f is a L^1-Caratheodory function, there exists $C > 0$ not depending on n such that

$$\left| \int_{|x_n| \leq r} f(t, x_n(t))\,dt \right| < C.$$

Denote by $s_n > t_n$ the first point such that $x_n'(s_n) = 0$ (observe that $x_n \in C^1(0,1)$ and that, by assumption, $x_n'(1) = 0$) .

In order to obtain bounds on the function x_n, let us study its derivative. If we take $t \in (t_n, s_n)$ arbitrary, we have

$$\begin{aligned}
&|x_n'(t)|^{p-2}x_n'(t) - |x_n'(t_n)|^{p-2}x_n'(t_n) \\
&\quad = \int_{t_n}^{t} (|x_n'(s)|^{p-2}x_n'(s))'\,dt = -\int_{t_n}^{t} f(s, x_n(s))\,ds \\
&\quad = -\int_{|x_n(t)|\le r,\ t\in(t_n,t)} f(s, x_n(s))\,dt - \int_{x_n(t)>r,\ t\in(t_n,t)} f(s, x_n(s))\,dt \le C.
\end{aligned}$$

From (3.7), the following inequality is then verified

$$x_n'(t) \le [(m_n y_0'(t_n))^{p-1} + C]^{1/(p-1)}, \quad t \in (t_n, s_n).$$

Since $x_n(t_n) = -r < 0$, we have that $x_n(t) \le d_n(t)$ for all $t \in [t_n, s_n]$, where d_n is defined as follows

$$d_n(t) = [(m_n y_0'(t_n))^{p-1} + C]^{1/(p-1)}(t - t_n).$$

Recall that our aim is to get an upper bound (independent of n) of the functions x_n. To do that, it suffices to get a bound of the sequence:

$$[(m_n y_0'(t_n))^{p-1} + C]^{1/(p-1)}(1 - t_n).$$

Recall that $m_n = -r/y_0(t_n)$; we can write the above expression as

$$\left[\left(\frac{-r y_0'(t_n)(1-t_n)}{y_0(t_n)}\right)^{p-1} + C(1-t_n)^{p-1}\right]^{1/(p-1)}.$$

Since the function y_0 is convex, we have that

$$\frac{-y_0'(t_n)(1-t_n)}{y_0(t_n)} \le 1.$$

Then, the functions x_n are uniformly bounded from above and let $H > 0$ be such a bound. We can write

$$\begin{aligned}
0 &= \int_0^{s_n} f(t, x_n(t))\,dt = \int_0^{t_n} f(t, x_n(t))\,dt + \int_{t_n}^{s_n} f(t, x_n(t))\,dt, \\
0 &= -|x_n'(t_n)|^{p-1} + \int_{t_n}^{s_n} f(t, x_n(t))\,dt.
\end{aligned}$$

When $t \in (t_n, s_n)$, $x_n(t)$ is always in the interval $(-r, H)$; then, the second term in the above expression is bounded when $n \to \infty$. But x_n is also convex in $(0, t_n)$, and then we arrive to a contradiction

$$x_n'(t_n) \ge \frac{x_n(t_n) - x_n(0)}{t_n} = \frac{-r - x_n(0)}{t_n} \to \infty. \qquad \square$$

4. Non-oscillation conditions

In the previous section we have proved the existence of solution for (2.1) and (2.2). As we have already said, we made assumptions on the change of sign of the nonlinearity and on the boundedness of the function f when $x < -r$. The aim of this section is to discuss this last condition. Actually, we are now interested in the functions $q \in L^1(0,1)$ verifying the following property:

The solution of the initial value problem

$$\begin{gathered}(|x'|^{p-2}x')' + q(t)|x|^{p-2}x = 0,\\ x(0) = 1, \quad x'(0) = 0,\end{gathered} \tag{4.1}$$

has no zero in [0, 1].

When $p = 2$, this problem has been studied since a long time ago. For more information about this classical problem and its relations with stability, see [5], [12]–[14], [16].

We denote $A = \{q \in L^1(0,1),\ q \geq 0 : (4.1) \text{ is verified}\}$. We are interested in studying necessary or sufficient conditions on q to belong to A. The first proposition is an adaptation of Theorem 5.1, of [7, Chapter XI], and is, in some sense, a generalization of Lyapunov inequality ([2]).

Proposition 4.1. *Suppose that $q \in L^1(0,1)$ is nonnegative and verifies the inequality*

$$\int_0^1 q(t)(1-t)\,dt \leq 1. \tag{4.2}$$

Then, $q \in A$.

Proof. Given q verifying the above assumptions, take one solution x of equation (4.1). Suppose that there exists $t \in (0,1]$ such that $x(t) = 0$, $x(s) > 0$, for all $s < t$. By integrating equation (4.1), we obtain:

$$|x'(t)|^{p-2}x'(t) = -\int_0^t q(s)|x(s)|^{p-2}x(s)\,ds,$$

and hence

$$0 = x(t) = 1 - \int_0^t \left[\int_0^s q(r)|x(r)|^{p-2}x(r)\,dr\right]^{1/(p-1)} ds.$$

We apply Hölder inequality and Fubini's theorem, to get

$$\begin{aligned} 0 &\geq 1 - \left[\int_0^t \int_0^s q(r)|x(r)|^{p-2}x(r)\,dr\,ds\right]^{1/(p-1)} \\ &= 1 - \left[\int_0^t \int_r^t q(r)|x(r)|^{p-2}x(r)\,ds\,dr\right]^{1/(p-1)} \end{aligned} \tag{4.3}$$

$$= 1 - \left[\int_0^t q(r)|x(r)|^{p-2}x(r)(t-r)\,dr \right]^{1/(p-1)}$$
$$\geq 1 - \left[\int_0^1 q(r)(1-r)\,dr \right]^{1/(p-1)}.$$

In the last inequality, we used the fact that x is a decreasing function and, therefore, $x(t) \leq 1$. Moreover, the equality $x(t) = 1$ can only hold in an interval containing zero if $q(t) = 0$ in this interval. If q is not identically zero, then the strict inequality holds in (4.3). If $q = 0$ the thesis (4.1) is clearly verified. □

In Proposition 4.1, we have given a sufficient condition on q to belong to A. Now, we want to give necessary conditions, as well as to study the boundedness of the set A. In order to do that, we define the functional

$$\Psi_{r,t} \colon A \to \mathbb{R}, \quad q \mapsto \int_0^t |q(s)|^r\,ds,$$

where $t, r \in (0,\ 1]$. Our approach is to find out the values of r, t for which the range of the functional $\Psi_{r,t}$ is bounded or not.

It seems clear that considering different values of r could be interesting; it will give us an idea of "how much unbounded" is the set A. It is less clear why it is interesting to consider different values of t and not only $t = 1$. As we shall see, there is a qualitative difference between the values $t_0 \in (0,1)$ and $t_0 = 1$. Namely, functions in A are allowed to have stronger singularities in $t_0 = 1$ than in any other point. This is suggested by intuition and by Proposition 4.1.

Proposition 4.2. *Suppose that $t < 1$. Then, if $r \leq 1$, $\Psi_{r,t}$ is bounded. Suppose that $t = 1$. Then*

(1) *if $r < 1/3(p-1)$, $\Psi_{r,t}$ is bounded,*
(2) *if $r \geq 1/p$, $\Psi_{r,t}$ is not bounded.*

Proof. First, we prove (1). Let $t \in (0,1)$, $q \in A$, x one solution of (4.1). Then, $q(s)$ can be written:

$$q(s) = -\frac{(|x'(s)|^{p-2}x'(s))'}{|x(s)|^{p-2}x(s)},$$

for all $s \leq t$, and hence

$$\int_0^t |q(s)|\,ds = -\int_0^t \frac{(|x'(s)|^{p-2}x'(s))'}{|x(s)|^{p-2}x(s)}\,ds.$$

Since the function x is concave, it is easy to prove that $x(s) \geq 1 - s$ for all $s \in [0,1]$. We substitute in the above expression and get

$$\int_0^t |q(s)|\,ds \leq -\frac{1}{|x(t)|^{p-2}x(t)} \int_0^t (|x'(s)|^{p-2}x'(s))'\,ds = \frac{|x'(t)|^{p-1}}{(1-t)^{p-1}}.$$

Using again that the function x is concave, we have

$$0 > x'(t) \geq \frac{x(1)-x(t)}{1-t} \geq \frac{-x(t)}{1-t} \geq \frac{-1}{1-t}.$$

Thus, we finally get a bound for the norm L^1 of q, namely

$$\int_0^t |q(s)|\,ds \leq \frac{1}{(1-t)^{2p-2}}.$$

To prove (2), we recall from the proof of (4.1) that if $q \in A$ and x is a solution of (4.1) then

$$0 < x(1) = 1 - \int_0^t \bigg[\int_0^s q(r)|x(r)|^{p-2}x(r)\,dr\bigg]^{1/(p-1)} ds.$$

Applying again Hölder inequality and Fubini's theorem, we get

$$0 < 1 - \int_0^t \int_0^1 q(r)^{1/(p-1)} x(r)\,dr\,ds = \int_0^1 q(r)^{1/(p-1)} x(r)(1-r)\,dr.$$

As $x(r) \geq 1-r$, we have

$$\int_0^1 (1-r)^2 q(r)^{1/(p-1)}\,dr < 1.$$

Now we apply the reverse Hölder inequality in the above expression (see [1]). We take $u \in (0,1/3)$, $v = u/(u-1) > -1/2$, and obtain

$$\begin{aligned} 1 > \int_0^1 (1-r)^2 q(r)^{1/(p-1)}\,dr &\geq \bigg(\int_0^1 (q(r))^{u/(p-1)}\,dr\bigg)^{1/u} \bigg(\int_0^1 (1-r)^{2v}\,dr\bigg)^{1/v} \\ &= \bigg(\int_0^1 (q(r))^{u/(p-1)}\,dr\bigg)^{1/u} \bigg(\frac{1}{2v+1}\bigg)^{1/v}. \end{aligned}$$

Then, for each $w \in (0, 1/3(p-1))$, we have

$$\bigg(\int_0^1 (q(r))^w\,dr\bigg)^{1/w} < (2\alpha+1)^{(p-1)/\alpha},$$

where $\alpha = w(p-1)/(w(p-1)-1) > -1/2$.

Finally, we will see that the functional $\Psi_{r,1}$ is not bounded when $r = 1/p$. Let $x\colon [0,1] \to \mathbb{R}$ be a continuous function such that $C^\infty[0,1)$ and

(1) x is concave,
(2) $x(0) = 1$, $x'(0) = 0$,
(3) $x(t) = (1-t)^a$ for all $t > 1/2$, where $a \in (0,1)$.

We get

$$[|x'(t)|^{p-2}x'(t)]' = a^{p-1}(a-1)(p-1)(1-t)^{(a-1)(p-1)-1},$$

for all $t \in (1/2, 1)$. For $\varepsilon > 0$, we define

$$q_\varepsilon(t) = \begin{cases} -\dfrac{(|x'(t)|^{p-2}x'(t))'}{|x(t)|^{p-2}x(t)} & \text{for } t \in [0, 1-\varepsilon], \\ 0 & \text{for } t \in (1-\varepsilon, 1]. \end{cases} \tag{4.4}$$

It is clear, by the definition of x, that $q_\varepsilon \in L^1(0,1)$, $q_\varepsilon \geq 0$. Note also that if we call x_ε the solution of the problem

$$\begin{gathered} (|x'|^{p-2}x')' + q_\varepsilon(t)|x|^{p-2}x = 0, \\ x(0) = 1, \quad x'(0) = 0, \end{gathered} \tag{4.5}$$

then $x_\varepsilon(t) = x(t)$ for all $t \in [0, 1-\varepsilon]$, and, since x is concave, $x_\varepsilon(t) \geq x(t)$ for all $t > 1-\varepsilon$. It follows that $q_\varepsilon \in A$ and

$$\int_0^1 |q_\varepsilon(t)|^{1/p}\,dt \geq \int_{1/2}^{1-\varepsilon} |q_\varepsilon(t)|^{1/p}\,dt = K\int_{1/2}^{1-\varepsilon} (1-t)^{-1}\,dt \to \infty$$

when $\varepsilon \to 0$. Here K stands for a positive constant, and, in this last step of the proof, we have used the concrete expression of $x(t)$ when $t > 1/2$. □

Remark. In the remaining cases, we think that the functional should remain bounded, but we have not been able to prove it.

Acknowledgements. The second author visited the Department of Mathematics of the Université Catolique de Louvain for six months during 2001, and would like to thank this Department for the warm hospitality.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] R. C. Brown and D. Hinton, *Lyapunov inequalities and their applications*, Survey on Classical Inequalities (T. M. Rassias, ed.), Kluwer, Dordrecht, 2000, pp. 1–25.

[3] Y. Dong, *A Neumann problem at resonance with the nonlinearity restricted in one direction*, Nonlinear Anal. (to appear).

[4] P. Drabek, *Solvability and Bifurcations of Nonlinear Equations*, U.K. Longman Scientific and Technical, J. Wiley & Sons Inc, New York, 1992.

[5] A. M. Fink and D. F. St Mary, *On an inequality of Nehari*, Proc. Amer. Math. Soc. **21** (1969), 640-642.

[6] S. Fučik, *Solvability of Nonlinear Equations and Boundary Value Problems*, D. Reidel, Boston, 1980.

[7] P. Hartman, *Ordinary Differential Equations*, J. Wiley & Sons Inc, New York, 1964.

[8] R. Manásevich and J. Mawhin, *The spectrum of p-Laplacian systems with various boundary conditions and applications*, Advances in Differential Equations **5** (2000), 1289–1318.

[9] R. Manásevich and J. Mawhin, *Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators*, J. Korean Math. Soc. **37** (2000), 665–685.

[10] R. Marik, *A half linear differential equation and variational problem*, Nonlinear Anal. **45** (2001), 203–211.

[11] J. Mawhin and J. R. Ward, *Periodic solutions of some forced Lienard differential equations at resonance*, Arch. Math. **41** (1983), 337–351.

[12] Z. Nehari, *On an Inequality of Liapunov*, Stanford Univ. Press, Stanford, California, 1962, pp. 256-261.

[13] C. R. Putnam, *On the first stability interval of the Hill equation*, Quart. Appl. Math. **16** (1959), 421–422.

[14] W. T. Reid, *A Generalized Liapunov inequality*, J. Differential Equations **13** (1973), 182–196.

[15] S. Villegas, *A Neumann problem with asymmetric nonlinearity and related minimizing problem*, J. Differential Equations **145** (1998), 145–155.

[16] A. Wintner, *On an inequality of Liapunov*, Quart. Appl. Math. **16** (1958), 175–178.

Manuscript received February 25, 2002

Jean Mawhin
Départamento de Mathématique
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, BELGIUM

E-mail address: mawhin@math.ucl.ac.be

David Ruiz
Department de Análisis Matemático
Universidad de Granada
18071 Granada, SPAIN

E-mail address: daruiz@ugr.es