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THE LEFSCHETZ FIXED POINT THEORY FOR MORPHISMS IN TOPOLOGICAL VECTOR SPACES

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Dedicated to our teacher Professor Andrzej Granas

Abstract. The Lefschetz Fixed Point Theorem for compact absorbing contraction morphisms ($\mathbb{CAC}$-morphisms) of retracts of open subsets in admissible spaces in the sense of Klee is proved. Moreover, the relative version of the Lefschetz Fixed Point Theorem and the Lefschetz Periodic Theorem are considered. Additionally, a full classification of morphisms with compact attractors in the non-metric case is obtained.

1. Vietoris mappings; admissibility in the sense of Klee

We are interested in theory of homology such that Vietoris theorem is satisfied for any topological space. In this paper we use a definition of Čech theory of homology with compact carriers and coefficients in the field of rationals $\mathbb{Q}$ given in [15] (see also [18]).

A space X is acyclic if: (a) X is non-empty, (b) $H_q(X) = 0$ for every $q \geq 1$ and (c) $H_0(X) \approx \mathbb{Q}$.

A continuous mapping $f: X \to Y$ of Hausdorff topological spaces X and Y is called *perfect* if f is closed and for every $y \in Y$ a set $f^{-1}(y)$ is compact.

Definition 1.1. A mapping of pair of spaces $p: (\Gamma, \Gamma_0) \to (X, X_0)$ is called *Vietoris mapping* provided it is a perfect surjection such that a set $p^{-1}(x)$ is acyclic for any $x \in X$ and $\Gamma_0 = p^{-1}(X_0)$.

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A Vietoris mapping p will be denoted as follows: $p\colon (\Gamma,\Gamma_0) \Rightarrow (X, X_0)$.

We remind that (comp. [15]):

Theorem 1.2. *If* $p\colon (\Gamma,\Gamma_0) \Rightarrow (X, X_0)$, *then an induced mapping*

$$p_*\colon H(\Gamma,\Gamma_0) \to H(X, X_0)$$

is a linear isomorphism.

Now, we recall the well-known properties of Vietoris mappings (see [15], [17], [28]).

Theorem 1.3.

(a) *If* $p\colon \Gamma \Rightarrow X$ *and* $p'\colon X \Rightarrow X'$, *then* $p' \circ p\colon \Gamma \Rightarrow X'$ *is a Vietoris mapping.*
(b) *If* $p\colon \Gamma \Rightarrow X$, *then for any* $A \subset X$ *a mapping* $p\colon p^{-1}(A) \to A$ *is Vietoris mapping.*
(c) *If* $p\colon \Gamma \Rightarrow X$ *and* $p_1\colon \Gamma_1 \Rightarrow X_1$, *then a mapping* $p \times p_1\colon \Gamma \times \Gamma_1 \to X \times X_1$, $(p \times p_1)(q, w) = (p(q), p_1(w))$ *is also Vietoris mapping.*
(d) *“Pull-back”*[1] *of Vietoris mapping is also Vietoris mapping.*

The following part is devoted to topological vector spaces admissible in the sense of Klee.

Let X be topological Hausdorff space. If $Y \subset X$, then we use a symbol $\mathrm{Cov}_X(Y)$ to denote a set of all open (in X) coverings of Y ($\mathrm{Cov}(X) = \mathrm{Cov}_X(X)$).

Definition 1.4. If $\alpha \in \mathrm{Cov}(Y)$, then continuous mappings $f, g\colon X \to Y$ will be called *α-close* if for any $x \in X$ there exists $U_x \in \alpha$ such that $f(x), g(x) \in U_x$.

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[1]For a diagram $X \xrightarrow{f} \Gamma \xleftarrow{g} Y$, of topological spaces X, Y and continuous mappings f, g we define

$$X \boxtimes Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

If at least one mapping is a surjection, then $X \boxtimes Y$ is a non-empty space and a diagram

$$\begin{tikzcd}
 & \Gamma & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "g"'] \\
 & X \boxtimes Y \arrow[ul, "\overline{g}"] \arrow[ur, "\overline{f}"'] \arrow[uu, "u"] &
\end{tikzcd}$$

where $\overline{g}(x, y) = x$, $\overline{f}(x, y) = y$, $u(x, y) = f(x)$ commutes. A diagram

$$X \xleftarrow{\overline{g}} X \boxtimes Y \xrightarrow{\overline{f}} Y$$

is called a “pull-back” of a diagram $X \xrightarrow{f} \Gamma \xleftarrow{g} Y$ and a mapping $\overline{f}$ (a mapping $\overline{g}$) is called “pull-back” of f (respectively, g).

Definition 1.5. Let E be a topological vector space. A space E is called *admissible in a sense of Klee* if for any of its compact subset K and any covering $\alpha \in \mathrm{Cov}_E(K)$ there exists a mapping $\pi_\alpha\colon K \to E$ such that:

(a) $\pi_\alpha(K)$ is included in a finite dimensional subspace of E, and
(b) the mapping $\pi_\alpha\colon K \to E$ and an inclusion $i\colon K \to E$ are α-close.

We shall be using the following characterization of a space admissible in a sense of Klee (see [13], comp. also [5], [6], [11], [22], [32], [33], [35]):

Proposition 1.6. *Topological vector space E is admissible in a sense of Klee if and only if for every of its compact subset K and any neighbourhood of zero U in this space there exists a mapping $\pi\colon K \to E$ such that:*

(a) $\pi(K)$ *is included in a finite dimensional subspace of E, and*
(b) $(\pi(x) - x) \in U$ *for every $x \in K$.*

Proof. Suppose E is admissible in a sense of Klee. Let K be a compact set and U a neighbourhood of zero in E. Let $\widetilde{U}$ be open in topological vector space E such that $\widetilde{U} - \widetilde{U} \subset U$. We use a symbol α to denote a covering of K by the sets of form $z + \widetilde{U}$, where $z \in K$. By assumption, there exists a mapping π_α such that for every $x \in K$ there exists $z \in K$, for which both x and $\pi_\alpha(x)$ belong to $z + \widetilde{U}$. Therefore, by the choice of $\widetilde{U}$, an element $\pi_\alpha(x) - x$ belongs to U.

Now, let K be a compact set of E and $\alpha \in \mathrm{Cov}_E(K)$. There exists an open subset U of E such that a family $\{x + U\}_{x \in K}$ (a covering of K) refines α. Indeed, for any $x \in K$ we can choose such a neighbourhood of zero U_x that $x + U_x$ refines α. Since E is topological vector space, we can take V_x, for which $V_x + V_x \subset U_x$. From a covering $\{x + V_x\}_{x \in K}$ of a compact set K we choose a finite covering $\{x_i + V_{x_i}\}_{i=1,\dots,n}$ and we put $U := \bigcap_{i=1}^n V_{x_i}$. Let $x \in K$. There exists $i \in \{1, \dots, n\}$ such that $x \in x_i + V_{x_i}$. Hence, $x + U \subset x_i + V_{x_i} + U \subset x_i + V_{x_i} + V_{x_i} \subset x_i + U_{x_i}$.

By assumption, there exists a mapping $\pi\colon K \to E$ such that: (a) $\pi(K)$ is included in a finite dimensional subspace of E and (b) $\pi(x) \in x + U$ for every $x \in K$. Thus, an inclusion $i\colon K \to E$ and a mapping $\pi\colon K \to E$ are α-close. $\square$

2. Morphisms

We begin this paragraph with the remark that topological spaces together with acyclic mappings do not form a category. This gives a good motivation to consider a notion of morphism. In this paper we follow the definition of morphism introduced in [16] (comp. also [18], [27], [28], [35]).

For pairs of Hausdorff topological spaces (X, A), (Y, B) by $D((X, A), (Y, B))$ we denote the set of all diagrams: $(X, A) \overset{p}{\Longleftarrow} (\Gamma, T) \xrightarrow{q} (Y, B)$, where p is a Vietoris mapping (see Definition 1.1), and q is continuous. Every such a diagram will be denoted by the symbol (p,q) (comp. [16], [27]). In $D((X,A),(Y,B))$ we define a relation: $(p_1,q_1) \sim (p_2,q_2)$ (where (p_1,q_1) denotes a diagram $(X,A) \xLeftarrow{p_1} (\Gamma_1,T_1) \xrightarrow{q_1} (Y,B)$ and (p_2,q_2) a diagram $(X,A) \overset{p_2}{\Leftarrow} (\Gamma_2,T_2) \xrightarrow{q_2} (Y,B)$, respectively) if and only if, there exists a homeomorphism of pairs $f\colon (\Gamma_1,T_1) \to (\Gamma_2,T_2)$ such that the following diagram commutes:

$$\begin{tikzcd} & (\Gamma_1,T_1) \arrow[dl, Rightarrow, "p_1"'] \arrow[dr, "q_1"] \arrow[dd, "f"] & \\ (X,A) & & (Y,B) \\ & (\Gamma_2,T_2) \arrow[ul, Rightarrow, "p_2"] \arrow[ur, "q_2"'] & \end{tikzcd}$$

that is $p_2 \circ f = p_1$ and $q_2 \circ f = q_1$.

It is easy to see that this relation is reflexive, symmetric and transitive. Therefore we have the following definition:

Definition 2.1. Let $M((X,A),(Y,B)) = D((X,A),(Y,B))/\sim$. The elements of this set are called *morphisms of* (X,A) *to* (Y,B).

An equivalence class of an element $(p,q) \in D((X,A),(Y,B))$ in relation $\sim$ will be denoted by greek letters: $\varphi,\psi,\dots$ We write: $\varphi = \{(X,A) \overset{p}{\Leftarrow} (\Gamma,T) \overset{q}{\to} (Y,B)\}\colon (X,A) \to (Y,B)$ or $\varphi = [(p,q)]$ (thus $(p,q) \in \varphi$ and a pair (p,q) is called a representative of morphism φ).

Definition 2.2. For any $\varphi \in M((X,A),(Y,B))$ a set $\varphi(x) = q(p^{-1}(x))$, where $\varphi = [(p,q)]$, is called an image of x under a morphism φ.

Additionally, $q(p^{-1}(x))$ is a compact set (because p is Vietoris mapping). Consequently, *a pre-image* of $U \subset Y$ under a morphism $\varphi \in M((X,A),(Y,B))$ is a set $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$. We say that morphism φ *upper semi-continuous* (shortly, u.s.c.), if for each open subset U in Y a set $\varphi^{-1}(U)$ is open.

Moreover, let $\varphi \in M(X,Y)$, $A \subset X$ and $B \subset Y$ are such that $\varphi(x) \subset B$ for every $x \in A$. Then $\varphi \in M((X,A),(Y,B))$.

Composition of morphisms. Let

$$\begin{aligned} \varphi &= \{(X,A) \xLeftarrow{p} (\Gamma,B) \xrightarrow{q} (Y,T)\}\colon (X,A) \longrightarrow (Y,B), \\ \psi &= \{(Y,B) \xLeftarrow{p'} (\Gamma',T') \xrightarrow{q'} (Z,C)\}\colon (Y,B) \longrightarrow (Z,C). \end{aligned}$$

Definition 2.3. By a composition of morphisms φ and ψ we mean the morphism

$$\psi \circ \varphi = \{(X,A) \overset{p\circ\overline{p'}}{\Longleftarrow} (\Gamma \boxtimes \Gamma', \widetilde{T}) \xrightarrow{q'\circ\overline{q}} (Z,C)\}\colon (X,A) \to (Z,C),$$

where $\overline{p'}$ (a mapping $\overline{q}$) denotes "pull-back" of p' (q, respectively) and $\widetilde{T} = \overline{p'}^{-1}(T)$.

A mapping $f\colon X \to Y$ may be identified with the morphism:

$$f = \{X \overset{\mathrm{Id}_X}{\Longleftarrow} X \xrightarrow{f} Y\}\colon X \longrightarrow Y.$$

A composition $\varphi \circ i_{(X_1,A_1)}$ of an inclusion $i_{(X_1,A_1)}\colon (X_1,A_1) \to (X,A)$ and a morphism $\varphi \in M((X,A),(Y,B))$ will be denoted by $\varphi|(X_1,A_1)$ and called *a restriction* of morphism φ to (X_1,A_1).

Treating Hausdorff spaces as objects and morphisms as category mappings we obtain a category of morphisms Mor (note that category Top is a subcategory of Mor).

We would like to emphasize that a notion of morphism is closely related to a notion of multivalued mapping. Namely, each morphism $\varphi \in M(X,Y)$ determines a multivalued mapping

$$X \ni x \mapsto q(p^{-1}(x)) \subset Y$$

for any pair $(p,q) \in \varphi$.

We call a multivalued mapping $\phi\colon X \multimap Y$ with compact values *determined by morphism*, if there exists a morphism $\varphi \in M(X,Y)$ such that $\phi(x) = \varphi(x)$ for every $x \in X$. Obviously, a multivalued mapping may be determined by more than one morphism (comp. [28]).

Note that, a multivalued mapping determined by morphism is u.s.c. and with compact values.

We remind:

Theorem 2.4. *Homology functor $H\colon \mathrm{Top} \to \mathrm{Vect}_G$ extends over* Mor *to a functor $\widetilde{H}\colon \mathrm{Mor} \to \mathrm{Vect}_G$, where $\widetilde{H}(\varphi) = q_* \circ p_*^{-1}$ for $\varphi = [(p,q)]$. We use standard notation $\varphi_* = \widetilde{H}(\varphi)$.*

Besides,

Definition 2.5. Morphisms $\varphi_0, \varphi_1 \in M((X,A),(Y,B))$ are called *homotopic* (notation: $\varphi_0 \approx \varphi_1$), if there exists a morphism $\varphi \in M((X \times [0,1], A \times [0,1]),(Y,B))$ such that $\varphi \circ i_j = \varphi_j$, where $j = 0,1$, and i_t denotes a function $i_t\colon X \to X \times [0,1]$ defined by $i_t(x) = (x,t)$, for $x \in X$, and $t \in [0,1]$.

A morphism φ will be called *homotopy* joining φ_0 and φ_1. A relation $\approx$ is reflexive, symmetric and transitive (comp. [28, I.5.6]). We also have that if $\psi_0 \approx \psi_1$ then $(\psi_0)_* = (\psi_1)_*$.

With a definition of morphism considered in the paper the operation of piecing morphisms together can be defined ([28, I (4.2)]). Let X, Y be topological spaces, X_1, X_2 closed subsets in X such that $X_1 \cup X_2$. If $\varphi_i \in M(X_i,Y)$, $i = 1,2$, and $\varphi_1|X_1 \cap X_2 = \varphi_2|X_1 \cap X_2$, then there exists a morphism $\varphi \in M(X,Y)$ such that $\varphi|X_i = \varphi_i$, $i = 1,2$. Take $(p_i, q_i) \in \varphi_i$, where $X_i \xleftarrow{p_i} \Gamma_i \xrightarrow{q_i} Y$. The following diagram commutes:

$$\begin{tikzcd}
& p_1^{-1}(X_1 \cap X_2) \arrow[dl, Rightarrow, "p_1"'] \arrow[dd, "f"'] \arrow[dr, "q_1"] & \\
X_1 \cap X_2 & & Y \\
& p_2^{-1}(X_1 \cap X_2) \arrow[ul, Rightarrow, "p_2"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

Put $\Gamma = \Gamma_1 \cup_f \Gamma_2$ i.e. Γ is a result of piecing together spaces Γ_1 and Γ_2 along the mapping f. By $h_i\colon \Gamma_i \to \Gamma$, $i = 1,2$, we denote natural quotient map. Now define $p\colon \Gamma \to X$ by $p(h_i(g_i)) = p_i(g_i)$ for $g_i \in \Gamma_i$, $i = 1,2$. Analogously, we define the mapping $q\colon \Gamma \to Y$. It is easily seen that morphism $[(p,q)]$ has a desired property and is independent of a choice of (p_i, q_i), $i = 1,2$.

Recall that for morphisms of a form

$$\varphi = \{X \overset{p}{\Longleftarrow} \Gamma \xrightarrow{q} E\}\colon X \to E,$$

where E is a topological vector space some operations (addition, multiplication by scalar) are defined in the same manner as for functions.

Definition 2.6. Morphism $\varphi \in M(X,Y)$ is called *compact* if a multivalued mapping determined by this morphism is compact (i.e. a set $\overline{\varphi(X)}$ is compact).

We use a notation $\varphi \in \mathbb{K}(X)$ for a compact morphism $\varphi \in M(X,X)$.

3. Non-compact morphisms

Now, we remind definitions of some classes of mappings which are acceptable in the fixed point theory. For metric spaces these classes were considered in [9]–[12]. A systematic study of non-metric case is presented in unpublished Phd Thesis of the second author (see [36]). Below we present these results.

Let X be a Hausdorff space. By a mapping we mean *multivalued mapping*.

The following definition was introduced in [11] in order to simplify a proof of Lefschetz Theorem.

Definition 3.1. A mapping $\varphi\colon X \multimap X$ is called *compact absorbing contraction* if there exists an open subset U of X such that $\overline{\varphi(U)}$ is a compact subset of U and for all $x \in X$ there exists $n_x \in \mathbb{N}$ such that $\varphi^{n_x}(x) \subset U$.

We write: $\varphi \in \mathbb{CAC}(X)$.

Definition 3.2. A mapping $\varphi\colon X \multimap X$ is called *eventually compact*, if there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}\colon X \multimap X$ is compact.

We write: $\varphi \in \mathbb{EC}(X)$.

DEFINITION 3.3. A mapping $\varphi\colon X \multimap X$ such that

$$\bigcup_{n=1}^{\infty}\{\varphi^n(x)\} \text{ is relative compact for every } x \in X,$$

is called *asymptotically compact*, if a set $C_\varphi = \bigcap_{i=0}^{\infty} \varphi^i(X)$ is non-empty, relative compact subset of X; C_φ is called *the core of* φ.

We use a notation: $\varphi \in \mathbb{ASC}(X)$.

DEFINITION 3.4. A mapping $\varphi\colon X \multimap X$ is called *a compact attraction* if there exists compact subset K of X such that for every open neighbourhood V of K, we have: for all $x \in X$ there exists $n_x \in \mathbb{N}$ such that $\varphi^n(x) \subset V$, for every $n \geq n_x$, we say that K is an *attractor* for φ.

We write: $\varphi \in \mathbb{CA}(X)$.

It is obvious that $\mathbb{K}(X) \subset \mathbb{CAC}(X)$, $\mathbb{EC}(X) \subset \mathbb{CA}(X)$. In fact, it may be proved that:

$$\begin{array}{ccccc}
\mathbb{EC}(X) & \subset & \mathbb{ASC}(X) & \subset & \mathbb{CA}(X) \\
\cup & & & & \cup \\
\mathbb{K}(X) & & \subset & & \mathbb{CAC}(X)
\end{array}$$

Under an assumption of local compactness[2] we additionally[3] have $\mathbb{EC}(X) \subset \mathbb{CAC}(X)$. Furthermore,

THEOREM 3.5. *If $\varphi \in \mathbb{CA}(X)$ is locally compact then $\varphi \in \mathbb{CAC}(X)$.*

We give an example of a mapping of class $\mathbb{CA}$ which is neither of class $\mathbb{CAC}$ nor $\mathbb{EC}$.

EXAMPLE 3.6. A mapping $F\colon c_0 \to c_0$ defined by: $F(\{x_n\}) = \{x_2, x_3, \ldots\}$ on a space c_0 of sequences convergent to zero with a norm $\|\{x_n\}\| = \sup_{n\in\mathbb{N}} |x_n|$ has got a compact attractor $\{0\}$.

In a class of locally compact mappings we have $\mathbb{CA} = \mathbb{CAC}$. Still there exist some mappings of class $\mathbb{CAC}$ not being locally compact.

EXAMPLE 3.7. Let c denote a space of all bounded sequences with a norm $\|\{x_n\}\| = \sup_{n\in\mathbb{N}} |x_n|$. We define a map $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x-1 & \text{if } x > 1, \\ 0 & \text{if } -1 \leq x \leq 1, \\ x+1 & \text{if } x < -1. \end{cases}$$

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[2] A mapping $\varphi\colon X \multimap Y$ is called *locally compact*, if for every $x \in X$ there exists an open subset V of X such that $x \in V$ and a contraction $\varphi|_V$ is compact

[3] This assumption cannot be omitted: a mapping $F\colon c \to c$, where $F(\{x_n\}) = \{0, x_1, 0, x_3, 0, \ldots\}$, defined on a space c of bounded sequences with a norm $\|\{x_n\}\| = \sup_{n\in\mathbb{N}} |x_n|$, is of class $\mathbb{EC}$ ($F^2 = 0$) but not of class $\mathbb{CAC}$, because it is not compact on any open neighbourhood of zero.

Now let $F: c \to c$ be given by $F(\{x_n\}) = \{f(x_1), f(x_2), \ldots\}$. F is of class $\mathbb{CAC}$ (with attractor $\{0\}$) but it is not locally compact.

Eventually, we have a diagram for classes of locally compact mappings:

$$\mathbb{K}(X) \subset \mathbb{EC}(X) \subset \mathbb{ASC}(X) \subset \mathbb{CA}(X) = \mathbb{CAC}(X).$$

We would like to emphasize that the above inclusions cannot be replaced by equalities. We give suitable examples (of single-valued mappings).

Example 3.8. Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$f(x) = \begin{cases} 1/x & \text{if } x < 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f(\mathbb{R}_+) = [1, \infty)$ and f is not compact. On the other hand, $f^2(\mathbb{R}_+) = \{1\}$ which shows that f is of class $\mathbb{EC}$.

Example 3.9. Let

$$A = \{(x, y) \in \mathbb{R}^2 \mid (x \leq 0 \vee y \leq 0) \wedge x^2 + y^2 \leq 1\} \cup (0, 1] \times \mathbb{R}_+$$

A map $f: A \to A$ given by $f(x, y) = (x/2, y/2)$ satisfies the property $f^n(x, y) = (x/2^n, y/2^n)$. But f^n is not of class $\mathbb{EC}(A)$. Still, for every $(x, y) \in A$ a set $\overline{\bigcup_{n=0}^{\infty} f^n(x, y)}$ is compact. Moreover, $\bigcap_{n=0}^{\infty} f^n(A) = \{0\}$. Thus $f \in \mathbb{ASC}(A)$.

Example 3.10. Consider $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x/2$. Obviously, $f \in \mathbb{CA}(\mathbb{R})$ but $f \notin \mathbb{ASC}(\mathbb{R})$, because it is a surjection and $\bigcap_{n=0}^{\infty} f^n(\mathbb{R}) = \mathbb{R}$.

Morphism $\varphi \in M(X, X)$ is called *compact absorbing contraction* if it determines a mapping which is a compact absorbing contraction. We use a similar notation: $\varphi \in \mathbb{CAC}(X)$.

A morphism $\varphi \in M((X, A), (X, A))$ is called $\mathbb{CAC}$-morphism of pairs provided both morphisms $\varphi_X \in M(X, X)$ and $\varphi_A \in M(A, A)$ induced by φ are compact absorbing contractions.

Morphism $\varphi \in M(X, X)$ is called locally compact if a mapping determined by this morphism is locally compact.

We end this paragraph reminding (comp. [16]) those properties of morphisms of class $\mathbb{CAC}$ which will be useful in a proof of Lefschetz Theorem.

Fact 3.11. *Let X be a retract of an open subset V in a Hausdorff topological space. If $\varphi \in \mathbb{CAC}(X)$, then for a retraction $r: V \to X$, and an inclusion $i: X \to V$ a morphism $\widetilde{\varphi} = i \circ \varphi \circ r \in M(V, V)$ is of class $\mathbb{CAC}$.*

Theorem 3.12. *Let $\varphi \in \mathbb{CAC}(X)$ and U be an open subset of X as in Definition 3.1. If K is compact subset of X, then there exists $n \in \mathbb{N}$ such that $\varphi^n(K) \subset U$.*

4. Fixed points and α-fixed points of morphisms

The spaces considered in this paragraph are regular.

Definition 4.1. Let $X \subset Y$. A point $x \in X$ is called a *fixed point of morphism* $\varphi \in M(X, Y)$, if $x \in \varphi(x)$.

A symbol $\mathrm{Fix}(\varphi)$ denotes a set of all fixed points of morphism φ.

Definition 4.2. Let $\alpha \in \mathrm{Cov}(X)$. A point $x \in X$ is called *α-fixed point of morphism* $\varphi \in M(X, X)$ if there exists $U \in \alpha$ such that $x \in U$ and $\varphi(x) \cap U \neq \emptyset$.

Definition 4.3. Let α be an open covering of a Hausdorff space Y.

(a) Morphisms φ, ψ are called α-close, if for every $x \in X$ there exists $U_x \in \alpha$ such that $\varphi(x) \cap U_x \neq \emptyset$ and $\psi(x) \cap U_x \neq \emptyset$.
(b) A homotopy $\psi \in M(X \times [0, 1], Y)$ is called α-homotopy, if for every $x \in X$ there exists $U_x \in \alpha$ such that $\psi(x, t) \cap U_x \neq \emptyset$ for every $t \in [0, 1]$.

Note that if β is a covering which refines α and two morphisms are β-close then they are α-close.

We remind that a family of coverings $D = \{\alpha\} \subset \mathrm{Cov}(X)$ is called *cofinal* provided for every covering $\alpha \in \mathrm{Cov}(X)$ there exists a covering $\beta \in D$ that refines α.

Lemma 4.4 (comp. [15]). *Let X be a regular topological space and let $\varphi \in M(X, X)$ be a morphism. Assume that there exists cofinal family of coverings $D = \{\alpha\} \subset \mathrm{Cov}(X)$ such that φ has got an α-fixed point for every $\alpha \in D$. Then a morphism φ has a fixed point.*

At the end of this paragraph we recall (comp. [12, (5.1)]):

Lemma 4.5. *Let U be an open subset of topological vector space E. Then for every $\alpha \in \mathrm{Cov}(U)$ there exists a covering $\beta \in \mathrm{Cov}(U)$ which refines α such that every two β-close maps f and g of any space X into U are stationary α-homotopic i.e. there exists α-homotopy h joining them such that $h_t(x)$ is constant map $(0 \leq t \leq 1)$ if $f(x) = g(x)$.*

Proof. Let $z \in U$. There exist $W \in \alpha$ and an open neighbourhood of zero W_z such that $z + W_z \subset W$. For W_z we have a set V_z for which $V_z + V_z \subset W_z$. Additionally, we find a set $U_z \subset V_z$ such that $[0, 1]U_z \subset V_z$. Let β be a covering consisting of sets of a form $z + U_z, z \in U$. Since the maps f and g are β-close, for every $x \in X$ there exists $z \in U$ such that $f(x), g(x) \in z + U_z \in \beta$. Now, an α-homotopy h is given by a standard definition: $h(t, x) = (1 - t)f(x) + tg(x)$. Hence $h(t, x) \in z + (1 - t)U_z + tU_z \subset z + V_z + V_z \subset z + W_z$. □

5. Homological invariants

In this section, we shall define homological invariants for admissible mappings which will guarantee the existence of fixed and periodic points.

We shall start from some algebraical operators. We recall that, for an endomorphism $T: E \to E$ of a vector space E (over $\mathbb{Q}$), we let $\tau: \widetilde{E} \to \widetilde{E}$ be induced by T on $\widetilde{E} = E/N(T)$, where $N(T) = \bigcup_{n=0}^{\infty} \operatorname{Ker}(T^n)$, T is called *Leray endomorphism*, provided $\dim \widetilde{E} < \infty$.

For Leray endomorphism $T: E \to E$, the map $\tau: \widetilde{E} \to \widetilde{E}$ is an automorphism and we denote by $w(T)$ the characteristic polynomial of τ (see [22] or [7] for details).

Since $N(T) = N(T^n)$, we have:

(5.1) T is a Leray endomorphism if and only if T^n is a Leray endomorphism for some natural n.

An endomorphism $T = \{T_q\}$ of a graded vector space $E = \{E_q\}$ is called a *Leray endomorphism* of a graded vector spaces if $T_q: E_q \to E_q$ is a Leray endomorphism for every q and $\widetilde{E}_q = 0$ for almost all q.

Recall (see [28] or [7]) that for a Leray endomorphism $T = \{T_q\}$ the *Lefschetz number* $\Lambda(T)$ of T is defined as follows:

$$\Lambda(T) = \sum_q (-1)^q \operatorname{tr}(\tau_q)$$

where $\operatorname{tr}(\tau_q)$ is the trace of τ_q and the Euler characteristic $\chi(T)$ of T is defined by:

$$\chi(T) = \chi(\widetilde{E}) = \sum_q (-1)^q \dim(\widetilde{E}_q).$$

Now from (5.1) immediately follows

(5.2) $T = \{T_q\}$ is a Leray endomorphism if and only if $T^n = \{T_q^n\}$ is a Leray endomorphism for some natural n and, in that case, $\chi(T) = \chi(T^n)$.

Let $Q\{x\}$ denote the integral domain consisting of all formal power series:

$$a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

with coefficients $a_n \in Q$, where Q is a fixed field. Then $Q\{x\}$ contains the polynomial ring $Q[x]$, the field Q and $1 \in Q$.

Definition 5.3. The *Lefschetz power series* $L(T)$ of Leray endomorphism $T = \{T_q\}$ is an element of $Q\{x\}$ defined by

$$L(T) = \chi(T) + \sum_{n=1}^{\infty} \Lambda(T^n) x^n = \sum_{n=0}^{\infty} \lambda(\tau^n) x^n,$$

where $\lambda(\tau^n)$ stands for the ordinary Lefschetz number of τ^n.

We have the following (see [7]).

Proposition 5.4. *The Lefschetz power series $L(T)$ of T admits a representation of the form:*

$$L(T) = u \cdot v^{-1},$$

where u and v are relatively prime polynomials with $\deg u < \deg v$ $(u \neq 0)$.

Proposition 5.4 allows us to define an algebraic invariant $P(T)$ of a Leray endomorphism $T = \{T_q\}$ to be the degree of the polynomial v, i.e.

$$P(T) = \deg(v),$$

where $L(T) = u \cdot v^{-1}$.

We shall summarize our considerations in the following proposition (see [28] or [7]).

Proposition 5.5. *Let $T = \{T_q\}$ be a Leray endomorphism. Then we have:*

(a) $\chi(T) \neq 0$ *implies* $P(T) \neq 0$,

(b) $P(T) \neq 0$ *if and only if* $\Lambda(T^n) \neq 0$, *for some natural* n,

(c) $P(T) = k \neq 0$, *then for any natural* $m \geq 0$, *one of the coefficients*

$$\Lambda(T^{m+1}), \dots, \Lambda(T^{m+k})$$

is different from zero.

Theorem 5.6.

(a) *Let the diagram*

$$\begin{tikzcd} E' \arrow[r, "u"] & E'' \arrow[dl, "v"'] \\ E' \arrow[u, "T'"] \arrow[r, "u"'] & E'' \arrow[u, "T''"'] \end{tikzcd}$$

commute. If any of the mappings T' and T'' is Leray endomorphism then so is the second one and then $\Lambda(T') = \Lambda(T'')$.

(b) *If*

$$\begin{tikzcd} \cdots \arrow[r] & E'_q \arrow[r] \arrow[d, "T'_q"] & E_q \arrow[r] \arrow[d, "T_q"] & E''_q \arrow[r] \arrow[d, "T''_q"] & E'_{q-1} \arrow[r] \arrow[d, "T'_{q-1}"] & \cdots \\ \cdots \arrow[r] & E'_q \arrow[r] & E_q \arrow[r] & E''_q \arrow[r] & E'_{q-1} \arrow[r] & \cdots \end{tikzcd}$$

is a commuting diagram of vector spaces with exact rows and both mappings $T = \{T_q\}$ and $T' = \{T'_q\}$ are Leray endomorphisms, respectively on $E = \{E_q\}$ and $E' = \{E'_q\}$, then so is $T'' = \{T''_q\}$ on $E'' = \{E''_q\}$. Then

$$\Lambda(T) = \Lambda(T') + \Lambda(T'').$$

Definition 5.7. An endomorphism $T: E \to E$ is called *weakly nilpotent* if for every $x \in E$ there exists a natural number $n = n_x$ such that $T^n(x) = 0$.

It is easily seen that $T: E \to E$ is weakly nilpotent if and only if $N(T) = E$. Thus

Theorem 5.8. *Any weakly nilpotent endomorphism $T: E \to E$ is admissible and $\Lambda(T) = 0$.*

Let $E = \{E_q\}$ be graded vector space and let $T = \{T_q\}: E \to E$ be an endomorphism. We say that T is *weakly nilpotent* if T_q is weakly nilpotent for any q.

By Theorem 5.8 we get:

Theorem 5.9. *Any weakly nilpotent endomorphism $T: E \to E$ of a graded vector space E is a Leray endomorphism and $\Lambda(T) = 0$.*

With Čech homology functor extended to a category of morphisms we have the following definition (see Theorem 2.4):

Definition 5.10. A morphism $\varphi \in M((X,A),(X,A))$ is called *Lefschetz morphism* if a linear mapping $\varphi_*: H(X,A) \to H(X,A)$ is Leray endomorphism. For a Lefschetz morphism $\varphi \in M((X,A),(X,A))$ the *Lefschetz number* $\Lambda(\varphi)$ is defined by $\Lambda(\varphi) = \Lambda(\varphi_*)$.

As a straightforward consequence of Theorems 2.4, 5.6 and Definition 5.10, axiom of exactness and a definition of Lefschetz number of morphism we get (see [7], [18]):

Theorem 5.11.

(a) *If morphisms φ and ψ are homotopic then their Lefschetz numbers (if defined) are equal i.e. $\Lambda(\varphi) = \Lambda(\psi)$.*

(b) *Let a commuting diagram in a category of pairs of spaces and continuous mappings be given:*

$$\begin{tikzcd}
(X,A) \arrow[r, "\varphi"] & (Y,B) \\
(X,A) \arrow[u, "\varphi_1"] \arrow[r, "\varphi"'] & (Y,B) \arrow[u, "\varphi_2"'] \arrow[ul, "\psi"']
\end{tikzcd}$$

Then:

(i) *if one of morphisms φ_1 or φ_2 is Lefschetz morphism then so is the second one. In this case $\Lambda(\varphi_1) = \Lambda(\varphi_2)$,*

(ii) *φ_1 has a fixed point if and only if φ_2 has such point.*

(c) *Let $\varphi \in M((X,A),(X,A))$ be a morphism and $\varphi_X \in M(X,X)$, $\varphi_A \in M(A,A)$ be induced by φ. If two of morphisms φ, φ_X and φ_A are*

Lefschetz morphisms then so is the third one and then

$$\Lambda(\varphi) = \Lambda(\varphi_X) - \Lambda(\varphi_A).$$

We remind:

THEOREM 5.12. *Let $\varphi_X \in M(X,X)$ be a morphism of class $\mathbb{CAC}$. Moreover, let $\varphi \in M((X,U),(X,U))$, where U is an open set from a definition of class $\mathbb{CAC}$. Then φ_* is weakly nilpotent endomorphism.*

In spite of the Lefschetz number $\Lambda(\varphi)$ we can define:

$$\chi(\varphi) = \chi(\varphi_*), \quad L(\varphi) = L(\varphi_*), \quad P(\varphi) = P(\varphi_*).$$

Now the Proposition 5.2 can be formulated as follows:

PROPOSITION 5.13. *A morphism $\varphi \in M(X,X)$ is a Lefschetz morphism if and only if any iterate φ^n of φ is a Lefschetz morphism and in such a case we have $\chi(\varphi) = \chi(\varphi^n)$.*

Finally, the preceding discussion can be summarized as follows (cf. Proposition 5.5).

THEOREM 5.14. *Let $\varphi \in M(X,X)$ be a Lefschetz morphism. We have:*

(a) $\chi(\varphi) \neq 0$ *implies* $P(\varphi) \neq 0$,
(b) $P(\varphi) \neq 0$ *if and only if* $\Lambda(\varphi^n) \neq 0$, *for some natural* n,
(c) $P(\varphi) = k \neq 0$, *then for any natural* $m \geq 0$ *at least one of the coefficients* $\Lambda(\varphi^{m+1}), \dots, \Lambda(\varphi^{m+k})$ *of the series* $L(\varphi)$ *must be different from zero.*

6. Lefschetz Theorem

We recall a well-known coincidence theorem (see [18, (12.8)], comp. also [16], [11], [12], [20]–[22]).

THEOREM 6.1. *Let U be an open subset of $\mathbb{R}^n$. For a diagram $U \xLeftarrow{p} Y \xrightarrow{q} U$, where p is a Vietoris mapping and q is compact, $q_*p_*^{-1}$ is Leray endomorphism. Moreover, a condition $\Lambda(q_*p_*^{-1}) \neq 0$ implies an existence of x such that $x \in q(p^{-1}(x))$.*

The main result of this section is the following

THEOREM 6.2. *Let X be a retract of an open set in a space admissible in sense of Klee E and let $\varphi \in M(X,X)$ be a morphism of a class $\mathbb{CAC}(X)$. Then:*

(a) *Lefschetz number* $\Lambda(\varphi)$ *of morphism* φ *is well-defined,*
(b) *if* $\Lambda(\varphi) \neq 0$ *then* φ *has a fixed point.*

First we prove following lemma.

Lemma 6.3. *Let U be an open subset of a space admissible in a sense of Klee E. If $\varphi \in M(U,U)$ is a compact morphism then:*

(a) *Lefschetz number $\Lambda(\varphi)$ of a morphism φ is well-defined,*
(b) *if $\Lambda(\varphi) \neq 0$ then φ has a fixed point.*

Proof. Let $\varphi = \{U \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} U\}: U \to U$ and let K denote a compact set in which $\varphi(U)$ is included. Take $\alpha \in \mathrm{Cov}_V(K)$. By a Lemma 4.5 there exists a covering $\beta \in \mathrm{Cov}_V(K)$ which refines α such that any β-close mappings are stationary α-homotopic. By a definition of a space admissible in a sense of Klee we obtain a mapping $\pi_\beta: K \to E$ such that $i: K \to E$ and π_β are β-close (hence α-close). Furthermore, $\pi_\beta(K)$ is included in finite dimensional subspace E^n of E. Therefore, $q_\beta = \pi_\beta \circ q: \Gamma \to U$ satisfies the following:

(1) q_β is compact map,
(2) q and q_β are β-close,

thus

(3) q and q_β are stationary α-homotopic,
(4) $\overline{q_\beta(\Gamma)} \subset U_\beta$, where $U_\beta = U \cap E^n$ is an open subset of a finite dimensional subspace.

By (3) the following diagram commutes

$$\begin{tikzcd} H(U_\beta) \arrow[r, "i_*"] & H(U) \\ H(U_\beta) \arrow[u, "(\overline{q_\beta})_*(p_\beta)_*^{-1}"] \arrow[r, "i_*"'] & H(U) \arrow[u, "q_* p_*^{-1}"'] \arrow[ul, "(q'_\beta)_* p_*^{-1}"'] \end{tikzcd}$$

where $\overline{q_\beta}$, q'_β, p_β denote respective contractions of q_β and p. By (1) we deduce that a morphism $[(p_\beta, \overline{q_\beta})]$ is compact. From (4) and Theorem 6.1 we obtain that it is Lefschetz morphism. Thus, by Theorem 5.6 we have that φ is Lefschetz morphism and $\Lambda(q_* p_*^{-1}) = \Lambda((\overline{q_\beta})_*(p_\beta)_*^{-1})$.

Now if $\Lambda(q_* p_*^{-1}) \neq 0$ then by Theorem 6.1 we deduce that a morphism $[(p_\beta, \overline{q_\beta})]$ has a fixed point. It is an α-fixed point of morphism φ, by (2). A desired result is then obtained using Lemma 4.4. □

Following [12] we see that the above theorem shows that any space admissible in a sense of Klee, its open acyclic subset have a fixed point property in a class of compact morphisms.

Now we consider another class of morphisms:

Lemma 6.4. *Let X be an open subset in a space admissible in a sense of Klee E. If $\varphi \in M(X,X)$ is a morphisms of a class $\mathbb{CAC}(X)$ then*

(a) *$\Lambda(\varphi)$ is well defined,*
(b) *if $\Lambda(\varphi) \neq 0$, then φ has a fixed point.*

Proof. Let U denote an open subset X from a definition of a morphism of a class $\mathbb{CAC}$. We consider $\widetilde{\varphi} \in M((X,U),(X,U))$ and a contraction of φ to U i.e. $\varphi_U \in M(U,U)$. Note that φ_U is compact and therefore, by Lemma 6.3, it is Lefschetz morphism. By Theorem 5.12 we get $\Lambda(\widetilde{\varphi}) = 0$. Hence, $\Lambda(\varphi) = \Lambda(\varphi_U)$. If $\Lambda(\varphi) \neq 0$ then this together with Lemma 6.3 gives an existence of a fixed point of morphism φ_U and thus of a morphism φ. $\square$

We now give a proof of Theorem 6.2.

Proof of Theorem 6.2. Let X be a retract of an open subset V in a space E admissible in a sense of Klee. From Fact 3.11 we deduce that a morphism $\widetilde{\varphi} = i \circ \varphi \circ r \in M(V,V)$ is of a class $\mathbb{CAC}$. Thus, by Lemma 6.4, $\widetilde{\varphi}$ is a Lefschetz morphism. Commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r, "i"] & V \\
X \arrow[u, "\varphi"] \arrow[r, "i"'] & V \arrow[u, "\widetilde{\varphi}"'] \arrow[ul, "\varphi r"']
\end{tikzcd}$$

implies that φ is Lefschetz morphism and $\Lambda(\varphi) = \Lambda(\widetilde{\varphi})$. Therefore if $\Lambda(\varphi) \neq 0$ then $\Lambda(\widetilde{\varphi}) \neq 0$. Again, using Lemma 6.4 we obtain: $\mathrm{Fix}(\widetilde{\varphi}) \neq \emptyset$. Let $v \in \widetilde{\varphi}(v)$. Since $\varphi \in M(X,X)$, by a definition of retraction and inclusion: $\widetilde{\varphi}(v) \subset X$ thus $v \in X$. Eventually, $v \in \mathrm{Fix}(\varphi)$. $\square$

Note, that by a fact that spaces admissible in a sense of Klee and their retracts are contractible to a point, we have:

Corollary 6.5. *If X is a space admissible in a sense of Klee or X is a retract of such a space then any morphism $\varphi \in M(X,X)$ of class $\mathbb{CAC}(X)$ has a fixed point.*

7. The relative version of the Lefschetz Fixed Point Theorem

Here we give a slight generalization of a notion of fixed point index for admissible maps ([18], also [14]). The the classical version treated a case of compact morphisms and arbitrary ANR. Following construction given in [14] for compact morphisms and open subsets of locally convex space it is possible to obtain results for a case of compact morphisms and open subsets of a space admissible in a sense of Klee.

Now, let M denote the class of all triples (X, W, φ), where X is a retract of an open subset in an admissible space in the sense of Klee, W is open in X, $\varphi \in \mathbb{CAC}(X)$ and $\mathrm{Fix}(\varphi) \cap \partial W = \emptyset$, where ∂W denotes the boundary of W in X. The aim of this section is to generalize the fixed point index over M.

Take a triple (X, W, φ). Since $\varphi \in \mathbb{CAC}(X)$, we consider an open $U \subset X$ given in the definition of compact absorbing contraction. Let us observe that

$\mathrm{Fix}(\varphi)$ is a compact subset of U and $U \in \mathrm{ANR}$ as an open subset of $X \in \mathrm{ANR}$. Therefore, the fixed point index $\mathrm{ind}(U, U \cap W, \widetilde{\varphi})$ of the triple $(U, U \cap W, \widetilde{\varphi})$, where $\widetilde{\varphi} \in M(U,U)$ denotes a restriction of φ[4], is well-defined, according to [14] and [18].

We define the fixed point index $\mathrm{Ind}(X, W, \varphi)$ of the triple (X, W, φ) as follows:

(7.1) $\mathrm{Ind}(X, W, \varphi) = \mathrm{ind}(U, U \cap W, \widetilde{\varphi})$.

The above definition is correct, i.e. it does not depend on the choice of U. In fact, it follows immediately from the additivity property (or, more precisely, from the localization or excision properties of the fixed point index for compact morphisms, see [18] or [14]).

The fixed point index Ind defined above satisfies all the usual properties (see again [18] or [14]). Below, we list the properties which are necessary in what follows:

(7.2) (Excision) If $(X, W, \varphi) \in M$ and $\mathrm{Fix}\,(\varphi) \subset W$, then

$$\mathrm{Ind}(X, W, \varphi) = \mathrm{Ind}(X, X, \varphi).$$

(7.3) (Contraction) If $\varphi(W) \subset A$, $A \in \mathrm{ANR}$ and the restriction $\overline{\varphi}\colon A \multimap A$, $\overline{\varphi}(x) = \varphi(x)$, for every $x \in A$, is compact absorbing contraction, then

$$\mathrm{Ind}(X, W, \varphi) = \mathrm{Ind}\, A, A \cap W, \overline{\varphi}).$$

(7.4) (Normalization) $\mathrm{Ind}(X, X, \varphi) = \Lambda(\varphi)$, where $\Lambda(\varphi)$ denotes the generalized Lefschetz number of φ.

Formulations of further properties of the fixed point index defined in this paragraph are left to the reader. Let us only mention (comp. [14]) that the index depends only on the retraction map chosen according to X.

Now, we are able to prove the following generalization of (7.4)

Theorem 7.5 (The relative version of the Lefschetz Fixed Point Theorem). *Let (X, A) be a pair in which X and A are retracts of some open sets in an admissible space in the sense of Klee. Assume further that $\varphi \in M((X, A), (X, A))$ is $\mathbb{CAC}$-morphism of pairs. Then φ is a Lefschetz morphism and $\Lambda(\varphi) \neq 0$ implies that φ has a fixed point in $\overline{X \setminus A}$.*

Proof. At first, in view of Theorems 5.11 and 6.2, we see that φ is a Lefschetz morphism, and

$$\Lambda(\varphi) = \Lambda(\varphi_X) - \Lambda(\varphi_A).$$

To prove the second part of our theorem, assume that $\Lambda(\varphi) \neq 0$ and φ has no fixed points in $\overline{X \setminus A}$, i.e. $\mathrm{Fix}(\varphi) \subset X \setminus (\overline{X \setminus A})$. Let $W = X \setminus (\overline{X \setminus A}) (= \mathrm{Int}\, A)$.

[4]More precisely, $\widetilde{\varphi}$ is determined by a restriction of a morphism φ to U.

Obviously, W is an open subset of X and, moreover, $W \subset A$. Therefore, from (7.2) and (7.4) we get

$$\mathrm{Ind}(X, W, \varphi_X) = \mathrm{Ind}(X, X, \varphi_X) = \Lambda(\varphi_X).$$

Analogously, since $W = \mathrm{Int}_X A \subset A$, we obtain

$$\mathrm{Ind}(A, W, \varphi_A) = \mathrm{Ind}(A, A, \varphi_A) = \Lambda(\varphi_A).$$

Now, using the contraction property of the fixed point index (7.3), we get

$$\mathrm{Ind}(X, W, \varphi_X) = \mathrm{Ind}(A, W, \varphi_A).$$

From this we finally obtain that $\Lambda(\varphi) = 0$. The desired contradiction ends the proof. $\square$

8. Existence of periodic points

Let $\varphi \in M(X, X)$ be a morphism. A point $x \in X$ is called *periodic* for φ with period n provided $x \in \varphi^n(x)$. Observe that any fixed point of φ is periodic with the period n, for arbitrary $n \geq 1$.

In what follows we shall assume that X is a retract of an open subset in a space admissible in the sense of Klee. We are able to prove:

Theorem 8.1 (Periodic Point Theorem). *Let $\varphi \in \mathbb{CAC}(X)$. If $\chi(\varphi) \neq 0$ or $P(\varphi) \neq 0$, then φ has a periodic point with period n, where $m+1 \leq n \leq m+P(\varphi)$ and m is an arbitrary natural number $(m \geq 0)$.*

Proof. It follows from Theorem 6.2 that φ is a Lefschetz morphism. In view of Theorem 5.14 it is sufficient to assume that $P(\varphi) \neq 0$. Applying Theorem 5.14(c), for any $m \geq 0$, we get n such that $\Lambda(\varphi^n) \neq 0$, where $m + 1 \leq n \leq m + P(\varphi)$. Since the composition of $\mathbb{CAC}$-morphisms is $\mathbb{CAC}$-morphism again, we deduce from Theorem 6.2 that the set $\mathrm{Fix}(\varphi^n)$ of fixed points of φ^n is non-empty. Of course, any $x \in \mathrm{Fix}(\varphi^n)$ is n-periodic point of φ. Hence the proof is completed. $\square$

Using Theorem 7.5 (instead of Theorems 6.2 and 5.14) we get:

Theorem 8.2. *Assume that $A \subset X$ is a retract of an open set in a space admissible in the sense of Klee. Further, assume that $\varphi \in M((X, A), (X, A))$ is a $\mathbb{CAC}$-morphism of pairs. If $\chi(\varphi) \neq 0$ or $P(\varphi) \neq 0$, then φ has n-periodic point in $\overline{X \setminus A}$, where $m + 1 \leq n \leq m + P(\varphi)$ and $m \geq 0$ is an arbitrary natural number.*

The proof of Theorem 8.2 is strictly analogous to the proof of Theorem 8.1.

Remark 8.3. It can be easily checked that $\chi(\varphi)$ and $P(\varphi)$ are homotopic invariants.

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