Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 20, 2002, 261–273

FIRST NOETHER-TYPE THEOREM FOR THE GENERALIZED VARIATIONAL PRINCIPLE OF HERGLOTZ

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We wish to dedicate this paper to our friend and mentor, Professor Andrzej Granas.

Abstract. In this paper we formulate and prove a theorem, which provides the conserved quantities of a system described by the generalized variational principle of Herglotz. This new theorem contains as a special case the classical first Noether theorem. It reduces to it when the generalized variational principle of Herglotz reduces to the classical variational principle. Several examples for applications to physics are given.

1. Introduction

In the middle of the nineteenth century Sophus Lie discovered that the various techniques for solving differential equations, were in fact all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of transformations. These groups, now known as Lie groups, have had a profound impact on all areas of mathematics, as well as physics, engineering and other mathematically based sciences.

Lie's discovery was that the complicated *nonlinear* conditions of invariance of the system under the group transformations could, in the case of a continuous group, be replaced by equivalent, but far simpler, *linear* conditions reflecting a form of "infinitesimal" invariance of the system under the generators of the group. In many physically important systems of differential equations, these

2000 *Mathematics Subject Classification.* 58E30, 70H20.
Key words and phrases. Noether, variational, conserved, conservation.

infinitesimal symmetry conditions, the so-called defining equations of the symmetry group of the system, can be explicitly solved in closed form and thus the most general continuous symmetry group of the system can be explicitly determined.

The knowledge of a one-parameter symmetry group of an ordinary differential equation allows one to reduce the order of the equation by one. If the problem is variational in nature and the symmetry group leaves the variational integral invariant, then the order of the equation can be reduced by two.

Symmetries and their properties were subsequently investigated by Herglotz ([8]), Klein ([11]), Kneser ([12]), and Noether ([16], [17]). In 1918, she proved two remarkable theorems relating symmetry groups of a variational integral to properties of its associated Euler–Lagrange equations, see Noether in [16], [17]. In these papers both the concept of a variational symmetry group and the connection with conservation laws were given in complete generality. For modern derivations and discussions of these theorems see Logan ([14]), Olver ([18]), Bluman and Kumei ([1]). In the first of these theorems, Noether shows how each one-parameter variational symmetry group gives rise to a conservation law of the Euler–Lagrange equations. Conservation of energy, for example, arises from the invariance of the problem under time translations, while conservation of linear and angular momenta reflect invariance of the system under spatial translations and rotations. Each one-parameter group of symmetries of the classical variational problem gives rise to a conservation law and, conversely, every conservation law arises in this manner.

Noether's theorems are applicable only to the classical variational principle, in which the functional is defined by an integral. They do not apply to functionals defined by differential equations. The Generalized Variational Principle, proposed by Gustav Herglotz in 1930 (see [9]), generalizes the classical variational principle by defining the functional, whose extrema are sought, by a differential equation. It reduces to the classical variational integral under classical conditions. Herglotz's original idea was published in 1979 in his collected works (see [8] and [10]). Herglotz reached the idea of the Generalized Variational Principle through his work on contact transformations and their connections with Hamiltonian systems and Poisson brackets. His work was motivated by ideas from S. Lie and others. For historical remarks through 1935, see C. Caratheodory ([3]). Herglotz's formulation is found in [8] together with references to other applications.

The generalized variational principle is important for a number of reasons:

(1) The solutions of the equations, which give the extrema of the functional defined by the generalized variational principle, when written in terms of the dependent variables x_i and the associated momenta $p_i = \partial L/\partial \dot{x}_i$, determine

a family of *contact transformations*. This family is a one-parameter group in certain cases. See Guenther et al ([7]), as well as Caratheodory ([2]) and Eisenhart ([4]).

(2) The generalized variational principle gives a *variational* description of nonconservative processes. Unlike the classical variational principle, the generalized one provides such a description *even when the Lagrangian is independent of time*.

(3) For a process, conservative or nonconservative, which can be described with the generalized variational principle, one can *systematically derive conserved quantities*, as shown in this paper, by applying the first Noether-type theorem.

(4) The generalized variational principle provides a link between the mathematical structure of control and optimal control theories and contact transformations (see Furta, et al. in [5]).

(5) The contact transformations, which can always be derived from the generalized variational principle, have found applications in thermodynamics. Mrugała in [15] shows that the processes in equilibrium thermodynamics can be described by successions of contact transformations acting in a suitably defined *thermodynamic phase space*. The latter is endowed with a *contact structure*, which is closely related to the *symplectic structure* (occurring in mechanics).

The significance of the generalized variational principle of Herglotz and the fact that the classical first Noether theorem does not apply to it motivated the work in this paper. For a system of differential equations derivable from the generalized variational principle of Herglotz we prove a first Noether-type theorem, which provides explicit conserved quantities corresponding to the symmetries of the functional defined by the generalized variational principle of Herglotz. This theorem reduces to the classical first Noether theorem in the case when the generalized variational principle of Herglotz reduces to the classical variational principle. Thus, it contains the classical first Noether theorem as a special case.

Applications of this new result are shown and specific examples are provided.

2. First Noether-type theorem for the generalized variational principle of Herglotz

The generalized variational principle of Herglotz defines the functional z, whose extrema are sought, by the differential equation

$$\frac{dz}{dt} = L\left(t, x(t), \frac{dx(t)}{dt}, z\right), \tag{2.1}$$

where t is the independent variable, $x \equiv (x^1, \dots, x^n)$ stands for the dependent variables and $dx(t)/dt$ denotes the derivatives of the dependent variables. In order for the equation (2.1) to define a functional, z, of $x(t)$ we must solve equation

(2.1) with the same fixed initial condition $z(0)$ for all argument functions $x(t)$, and evaluate the solution $z(t)$ at the same fixed final time $t = T$ for all argument functions $x(t)$.

The generalized variational principle of Herglotz requires that the first variations of the functional z, defined by (2.1), vanish at $t = 0$ and $t = T$. The equations which produce the extrema of this functional are

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, \dots, n.$$

Herglotz called them the *generalized Euler–Lagrange equations*. See Guenther et al. ([7]) for a derivation of this system.

Consider the one-parameter group of invertible transformations

$$\begin{aligned} \bar{t} &= \phi(t, x, \varepsilon), \\ \overline{x}_k &= \psi_k(t, x, \varepsilon), \quad k = 1, \dots, n, \end{aligned} \tag{2.2}$$

where ε is the parameter, $\phi(t, x, 0) = t$, and $\psi_k(t, x, 0) = x_k$. Let the generators of the corresponding infinitesimal transformation be

$$\tau(t, x) = \frac{d\phi}{d\varepsilon}(t, x, 0) \quad \text{and} \quad \xi_k(t, x) = \frac{d\psi_k}{d\varepsilon}(t, x, 0). \tag{2.3}$$

Denote by $\zeta = \zeta(t)$ the total variation of the functional $z = z[x; t]$ produced by the family of transformations (2.2), i.e.

$$\zeta(t) = \frac{d}{d\varepsilon} z[x; t, \varepsilon]\bigg|_{\varepsilon=0}.$$

Throughout this paper we assume that the summation convention on repeated indices holds and that "$\cdot$" denotes differentiation with respect to t.

Theorem 2.1. *If the functional $z = z[x(t); t]$ defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ is invariant under the one-parameter group of transformations* (2.2) *then the quantities*

$$\exp\bigg(-\int_0^t \frac{\partial L}{\partial z}\, d\theta\bigg)\bigg(\bigg(L - \dot{x}_k \frac{\partial L}{\partial \dot{x}_k}\bigg)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k\bigg) \tag{2.4}$$

are conserved along the solutions of the generalized Euler–Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, \dots, n. \tag{2.5}$$

Proof. First note that

$$\begin{aligned} \bar{t} &= t + \tau(t, x)\, \varepsilon + o(\varepsilon), \\ \overline{x}_k &= x_k + \xi_k(t, x)\, \varepsilon + o(\varepsilon). \end{aligned} \tag{2.6}$$

Let us apply the transformation (2.6) to the differential equation (2.1). Since $d\bar{z}/d\bar{t} = (d\bar{z}/dt)\,(dt/d\bar{t}\,)$, we get

$$\begin{aligned} \frac{d\bar{z}}{d\bar{t}} &= L\Big(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\Big), \\ \frac{d\bar{z}}{dt} &= L\Big(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\Big)\frac{d\bar{t}}{dt}. \end{aligned} \tag{2.7}$$

Differentiate (2.7) with respect to ε and set $\varepsilon = 0$ to obtain

$$\frac{d}{d\varepsilon}\left(\frac{d\bar{z}}{dt}\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\left(\frac{d\bar{z}}{d\varepsilon}\right)\bigg|_{\varepsilon=0} = \frac{dL}{d\varepsilon}\bigg|_{\varepsilon=0}\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} + L\frac{d}{d\varepsilon}\left(\frac{d\bar{t}}{dt}\right)\bigg|_{\varepsilon=0}. \tag{2.8}$$

Next, we use the fact that $\varphi(t, x, 0) = t$ to conclude that

$$\frac{d\bar{t}}{dt}\bigg|_{\varepsilon=0} = 1.$$

Similarly, we have

$$\frac{d}{d\varepsilon}\left(\frac{d\bar{t}}{dt}\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\left(\frac{d}{d\varepsilon}\varphi(t, x, \varepsilon)\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\left(\frac{d\varphi}{d\varepsilon}(t, x, 0) + o(\varepsilon)\right)\bigg|_{\varepsilon=0} = \frac{d}{dt}\tau(t, x).$$

Thus, equation (2.8) becomes

$$\frac{d\zeta}{dt} = \frac{dL}{d\varepsilon}\bigg|_{\varepsilon=0} + L\frac{d\tau}{dt}.$$

Expanding the derivative $dL/d\varepsilon$ and setting $\varepsilon = 0$, we obtain

$$\begin{aligned} \frac{d\zeta}{dt} &= \frac{\partial L}{\partial t}\frac{d\bar{t}}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{\partial L}{\partial x_k}\frac{d\bar{x}_k}{d\varepsilon}\bigg|_{\varepsilon=0} \\ &\qquad + \frac{\partial L}{\partial \dot{x}_k}\frac{d}{d\varepsilon}\left(\frac{d\bar{x}_k}{d\bar{t}}\right)_{\varepsilon=0} + \frac{\partial L}{\partial z}\frac{d\bar{z}}{d\varepsilon}\bigg|_{\varepsilon=0} + L\frac{d\tau}{dt}, \\ \frac{d\zeta}{dt} &= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\frac{d}{d\varepsilon}\left(\frac{d\bar{x}_k}{d\bar{t}}\right)_{\varepsilon=0} + \frac{\partial L}{\partial z}\zeta + L\frac{d\tau}{dt}. \end{aligned} \tag{2.9}$$

We need to calculate and insert in equation (2.9) the expression

$$\frac{d}{d\varepsilon}\left(\frac{d\bar{x}_k}{d\bar{t}}\right)\bigg|_{\varepsilon=0}.$$

For this we proceed as follows:

$$\frac{d\bar{x}_k}{dt} \equiv \frac{\partial \bar{x}_k}{\partial t} + \frac{\partial \bar{x}_k}{\partial x_h}\dot{x}_h = \frac{d\bar{x}_k}{d\bar{t}}\frac{d\bar{t}}{dt} \equiv \frac{d\bar{x}_k}{d\bar{t}}\left(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x_h}\dot{x}_h\right). \tag{2.10}$$

Set $\varepsilon = 0$ to obtain

$$\frac{d\bar{x}_k}{d\bar{t}}\bigg|_{\varepsilon=0} = \delta^k_h \dot{x}_h = \dot{x}_k. \tag{2.11}$$

Differentiate (2.10) with respect to ε and expand both sides to get

$$\begin{aligned}(2.12)\quad &\frac{d}{d\varepsilon}\left(\frac{\partial \overline{x}_k}{\partial t}\right)+\frac{d}{d\varepsilon}\left(\frac{\partial \overline{x}_k}{\partial x_h}\right)\dot{x}_h\\ &\quad=\frac{d\overline{x}_k}{d\overline{t}}\left(\frac{d}{d\varepsilon}\left(\frac{\partial \overline{t}}{\partial t}\right)+\frac{d}{d\varepsilon}\left(\frac{\partial \overline{t}}{\partial x_h}\right)\dot{x}_h\right)+\frac{d}{d\varepsilon}\left(\frac{d\overline{x}_k}{d\overline{t}}\right)\left(\frac{\partial \overline{t}}{\partial t}+\frac{\partial \overline{t}}{\partial x_h}\dot{x}_h\right).\end{aligned}$$

We set $\varepsilon = 0$ in this equation, substitute in it (2.11) and use the following relations:

$$\frac{d}{d\varepsilon}\left(\frac{\partial \overline{x}_k}{\partial t}\right)\bigg|_{\varepsilon=0}=\frac{\partial \xi_k}{\partial t},\quad \frac{d}{d\varepsilon}\left(\frac{\partial \overline{t}}{\partial t}\right)\bigg|_{\varepsilon=0}=\frac{\partial \tau}{\partial t},\quad \frac{d}{d\varepsilon}\left(\frac{\partial \overline{t}}{\partial x_h}\right)\bigg|_{\varepsilon=0}=\frac{\partial \tau}{\partial x_h},$$
$$\frac{\partial \overline{t}}{\partial t}\bigg|_{\varepsilon=0}=1,\quad \frac{\partial \overline{t}}{\partial x_h}\bigg|_{\varepsilon=0}=0.$$

Then equation (2.12) becomes

$$\frac{\partial \xi_k}{\partial t}+\frac{\partial \xi_k}{\partial x_h}\dot{x}_h=\dot{x}_k\left(\frac{\partial \tau}{\partial t}+\frac{\partial \tau}{\partial x_h}\dot{x}_h\right)+\frac{d}{d\varepsilon}\left(\frac{d\overline{x}_k}{d\overline{t}}\right)\bigg|_{\varepsilon=0}.$$

Observe that the total derivatives of ξ_k and τ appear in the last equation. Solving for the last term in it, we obtain

$$\frac{d}{d\varepsilon}\left(\frac{d\overline{x}_k}{d\overline{t}}\right)\bigg|_{\varepsilon=0}=\frac{d\xi_k}{dt}-\dot{x}_k\frac{d\tau}{dt}.$$

We insert the last result in equation (2.9) to obtain the differential equation

$$(2.13)\qquad \frac{d\zeta}{dt}=\frac{\partial L}{\partial t}\tau+\frac{\partial L}{\partial x_k}\xi_k+\frac{\partial L}{\partial \dot{x}_k}\left(\frac{d\xi_k}{dt}-\dot{x}_k\frac{d\tau}{dt}\right)+\frac{\partial L}{\partial z}\zeta+L\frac{d\tau}{dt}$$

for the variation ζ of the functional z. The solution ζ of (2.13) is given by

$$\begin{aligned}&\exp\left(-\int_0^t\frac{\partial L}{\partial z}\,d\theta\right)\zeta-\zeta(0)\\ &=\int_0^t\exp\left(-\int_0^s\frac{\partial L}{\partial z}\,d\theta\right)\left(\frac{\partial L}{\partial s}\tau+\frac{\partial L}{\partial x_k}\xi_k+\frac{\partial L}{\partial \dot{x}_k}\left(\frac{d\xi_k}{ds}-\dot{x}_k\frac{d\tau}{ds}\right)+L\frac{d\tau}{ds}\right)ds.\end{aligned}$$

Notice that $\zeta(0)=0$. Indeed, as explained earlier, in order to have a well-defined functional z as a functional of $x(t)$, we must evaluate the solution $z(t)$ of the equation (2.1) with the same fixed initial condition $z(0)$, independently of the function $x(t)$. Then $z(0)$ is independent of ε. Hence, the variation of z evaluated at $t=0$ is

$$\zeta(0)=\frac{d}{d\varepsilon}z[x;0,\varepsilon]\bigg|_{\varepsilon=0}=\frac{d}{d\varepsilon}z(0)\bigg|_{\varepsilon=0}=0.$$

Since by hypothesis the one-parameter family of transformations (2.2) leaves the functional $z = z[x(t);t]$ stationary, we have $\zeta(t) = 0$. Thus, one obtains

$$\int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}\,d\theta\Big)\Big(\frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{d\xi_k}{ds} - \dot{x}_k\frac{d\tau}{ds}\Big) + L\frac{d\tau}{ds}\Big)\,ds = 0. \tag{2.14}$$

An integration by parts of the last equation produces

$$\begin{aligned}&\exp\Big(-\int_0^s \frac{\partial L}{\partial z}d\theta\Big)\Big(L\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau\Big)\Big|_{s=0}^{s=t}\\ &\quad+\int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}d\theta\Big)\Big(\frac{\partial L}{\partial s}\tau - \dot{L}\tau + L\frac{\partial L}{\partial z}\tau + \frac{\partial L}{\partial x_k}\xi_k\\ &\quad-\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{x}_k}\Big)\xi_k + \frac{\partial L}{\partial \dot{x}_k}\frac{\partial L}{\partial z}\xi_k - \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau\\ &\quad+\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{x}_k}\Big)\dot{x}_k\tau + \frac{\partial L}{\partial \dot{x}_k}\ddot{x}_k\tau\Big)ds = 0,\end{aligned}$$

which on solutions of the generalized Euler-Lagrange equations becomes

$$\begin{aligned}&\exp\Big(-\int_0^s \frac{\partial L}{\partial z}d\theta\Big)\Big(L\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau\Big)\Big|_{s=0}^{s=t}\\ &\quad+\int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}d\theta\Big)\Big(-\frac{\partial L}{\partial z}\dot{z} + L\frac{\partial L}{\partial z}\\ &\quad-\dot{x}_k\Big(\frac{\partial L}{\partial x_k} - \frac{d}{ds}\frac{\partial L}{\partial \dot{x}_k} + \frac{dL}{dz}\frac{\partial L}{\partial \dot{x}_k}\Big)\Big)\tau\,ds = 0.\end{aligned}$$

Taking into consideration the fact that $\dot{z} = L$, we obtain that along the solutions of the generalized Euler–Lagrange equations (2.5) equation (2.14) reduces to

$$\exp\Big(-\int_0^s \frac{\partial L}{\partial z}\,d\theta\Big)\Big(L\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k - \frac{\partial L}{\partial \dot{x}_k}\dot{x}_k\tau\Big)\Big|_{s=0}^{s=t} = 0.$$

Since the last equation holds for all t, it follows that

$$\exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\Big)\Big(\Big(L - \dot{x}_k\frac{\partial L}{\partial \dot{x}_k}\Big)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k\Big) = \text{constant}$$

along solutions of the generalized Euler–Lagrange equations, as claimed. □

It should be observed that the exponential factor

$$\exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\Big) = \frac{1}{\rho} \tag{2.15}$$

which is present in the conserved quantities (2.4) is the reciprocal of the *multiplier function* ρ which appears in the definition

$$P_i\,dX_i - dZ = \rho\,(p_i\,dx_i - dz)$$

of contact transformations (see Guenther, [7]).

The conserved quantities (2.4) have a remarkable form – they are products of ρ^{-1} with the expressions

$$\Big(L - \dot{x}_k \frac{\partial L}{\partial \dot{x}_k}\Big)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k \tag{2.16}$$

whose form is exactly the same as that of the conserved quantities obtained from the classical first Noether theorem. In the special case $\partial L/\partial z = 0$ the functional z is defined by the integral

$$z = \int_0^t L(t, x, \dot{x})\, d\theta$$

and $\rho = 1$. Hence, in this case Theorem 2.1 reduces to the classical first Noether theorem. Recent references on the classical Noether theorems are Logan ([14]), Olver ([18]) and Bluman, Kumei ([1]). For applications to physics see Goldstein ([6]) and Roman ([19]).

3. Conserved quantities in generative and dissipative systems

Physical systems described by the generalized Euler–Lagrange equations (2.5) or by the corresponding canonical equations (see Guenther in [7]) are not conservative in general. Since the Lagrangian functional of such systems cannot be expressed as an integral, the first Noether theorem cannot be used for finding conserved quantities. Below we show how the first Noether-type theorem can be used to find conserved quantities in non-conservative systems. For this, we must describe the physical system with the generalized Euler–Lagrange equations or the canonical equations and then find symmetries of the functional $z = z[x(t); t]$ defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$, that is, transformations of both dependent and independent variables which leave $z[x(t); t]$ invariant.

To test whether a transformation is a symmetry of the functional $z[x(t); t]$ we use the following

Proposition 3.1. *The transformation*

$$\begin{aligned} \bar{t} &= \varphi(t, x, \varepsilon), \\ \overline{x}_k &= \psi_k(t, x, \varepsilon), \end{aligned} \tag{3.1}$$

leaves the functional z, defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ invariant if and only if

$$L\bigg(\bar{t}, \overline{x}, \frac{d\overline{x}}{dt}\bigg(\frac{d\bar{t}}{dt}\bigg)^{-1}, z\bigg)\frac{d\bar{t}}{dt} = L\bigg(t, x(t), \frac{dx(t)}{dt}, z\bigg) \tag{3.2}$$

holds for all t, x, z in the domain of consideration.

We are now ready to apply the first Noether-type theorem to find specific conserved quantities corresponding to several basic symmetries. Because of their generality and physical significance we state the results as corollaries to the first Noether-type theorem.

COROLLARY 3.2. *Let the functional z defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ be invariant with respect to translation in time, $\overline{t} = t + \varepsilon$, $\overline{x} = x$. Then the quantity*

$$E = \exp\left(- \int_0^t \frac{\partial L}{\partial z}\, d\theta \right) \left(L(x, \dot{x}, z) - \frac{\partial L}{\partial \dot{x}_k} \dot{x}_k \right) \tag{3.4}$$

is conserved on solutions of the generalized Euler–Lagrange equations.

PROOF. By Proposition 3.1 we see that $\partial L/\partial t = 0$. The infinitesimal generator of the group translation in time is $\partial/\partial t$. To obtain the conclusion of the corollary, apply the first Noether-type theorem with

$$\tau = \frac{d\overline{t}}{d\varepsilon}\bigg|_{\varepsilon=0} = 1, \quad \xi_k = \frac{d\overline{x}_k}{d\varepsilon}\bigg|_{\varepsilon=0} = 0. \qquad \square$$

Noticing that the Hamiltonian is $H = p_k \dot{x}_k - L$, we see that on solutions of the generalized Euler–Lagrange equations

$$E = \frac{1}{\rho} H = \text{constant}, \tag{3.5}$$

where ρ is the multiplier function (2.15). We observe a correspondence with the classical law of conservation of energy: If a physical system is described by a time-independent Lagrangian, which does not depend on z, then the Hamiltonian H is conserved and is identified with the energy of the system. If we continue to interpret H as the energy of the system when L does depend on z, then we see from formula (3.5) that H varies proportionally to ρ since E is constant.

A direct computation verifies the statement of the corollary.

COROLLARY 3.3. *Let the functional z defined by the differential equation $\dot{z} = L(t, x, \dot{x}, z)$ be invariant with respect to translation in space direction x_k, i.e. $\overline{t} = t$, $\overline{x}_k = x_k + \varepsilon$, $\overline{x}_i = x_i$ for $i = 1, \ldots, k-1, k+1, \ldots, n$. Then the quantity*

$$M_k = \exp\left(- \int_0^t \frac{\partial L}{\partial z}\, d\theta \right) \frac{\partial L}{\partial \dot{x}_k} \tag{3.6}$$

is conserved on solutions of the generalized Euler-Lagrange equations.

PROOF. By Proposition 3.1 we know that $\partial L/\partial x_k = 0$. The infinitesimal generator of the group of translations in direction x_k is $\partial/\partial x_k$. To get the conserved quantity M_k apply the first Noether-type theorem with

$$\tau = \frac{d\overline{t}}{d\varepsilon}\bigg|_{\varepsilon=0} = 0, \quad \xi_k = \frac{d\overline{x}_k}{d\varepsilon}\bigg|_{\varepsilon=0} = 1, \quad \xi_i = \frac{d\overline{x}_i}{d\varepsilon}\bigg|_{\varepsilon=0} = 0, \quad \text{for } i \neq k. \qquad \square$$

In terms of the multiplier function ρ the expression (3.6) takes the form

$$M_k = \frac{1}{\rho}\frac{\partial L}{\partial \dot{x}_k} = \text{constant}. \tag{3.7}$$

Again, we observe a correspondence with the classical law of conservation of linear momentum. If we retain the definition of linear momentum $\partial L/\partial \dot{x}_k$ then the result (3.7) says that the linear momentum is not conserved, but changes proportionally to ρ.

Although the linear momentum changes in time, its direction remains invariant, since the ratios of its components are constant.

Corollary 3.4. *Let the functional z defined by the equation $\dot{z} = L(t, x, \dot{x}, z)$ be invariant with respect to rotations in the $x_i x_j$-plane. Then the quantity*

$$A_{ij} = \exp\bigg(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\bigg)\bigg(\frac{\partial L}{\partial \dot{x}_i}x_j - \frac{\partial L}{\partial \dot{x}_j}x_i\bigg) \tag{3.8}$$

is conserved along solutions of the generalized Euler–Lagrange equations.

Proof. By Proposition 3.1 we know that the Lagrangian has the form

$$L = L(t, {x_i}^2 + {x_j}^2, x_r, \dot{x}_i^2 + \dot{x}_j^2, \dot{x}_r, z)$$

where x_r stands for all coordinates distinct from x_i or x_j. Indeed, $d\overline{t}/dt = 1$ and the invariance of z under rotations in the $x_i x_j$-plane implies the specific form of L.

The infinitesimal generator of the group of rotations is

$$x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}.$$

Thus, to obtain the conserved quantity A_{ij}, we apply the first Noether-type theorem with $\tau = 0$, $\xi_i = x_j$, $\xi_j = -x_i$, $\xi_r = 0$ for $r \neq i, j$. □

Once again, in the case of z-independent L, we have a correspondence with the classical law of conservation of angular momentum

$$x_j\frac{\partial L}{\partial x_i} - x_i\frac{\partial L}{\partial x_j}.$$

Again, we can verify the result (3.8) by a direct calculation, noticing the use of *both*

$$\frac{\partial L}{\partial \dot{x}_i}\dot{x}_j = \frac{\partial L}{\partial \dot{x}_j}\dot{x}_i \quad \text{and} \quad \frac{\partial L}{\partial x_i}x_j = \frac{\partial L}{\partial x_j}x_i$$

which hold since $L = L(t, {x_i}^2 + {x_j}^2, \dot{x}_i^2 + \dot{x}_j^2, z)$.

It should be observed that, although the components of the angular momentum change according to

$$x_j\frac{\partial L}{\partial x_i} - x_i\frac{\partial L}{\partial x_j} = \rho A_{ij},$$

their ratios remain constant. Thus, the direction of the angular momentum is invariant.

4. Additional applications of the first Noether-type theorem

It is known that dissipation effects in physical processes can often be accounted for in the equations describing these processes by terms which are proportional to the first time derivatives $\dot{x}_i(t) = dx_i/dt$ of the dependent variables (see Goldstein in [6]). For example, the viscous frictional forces acting on an object which is moving in a resistive medium, such as a gas or a liquid, are proportional to the object's velocity. Similarly, the dissipative effects (due to the ohmic resistance) in electrical circuits can often be modeled by including terms which are proportional to the first time-derivative of the corresponding dependent variables, such as the electric charge.

All such dissipative processes can be given a unified description by the generalized variational principle.

For example, let us consider the motion of a small object with mass m (point mass) under the action of some potential $U = U(t, x)$ with $x = (x_1, x_2, x_3)$ in a resistive medium. We assume that the resistive forces are proportional to the velocity. The equations describing the motion of such an object are

$$m\ddot{x}_i = -\frac{\partial U}{\partial x_i} - k\dot{x}_i, \quad i = 1, 2, 3, \tag{4.1}$$

where $k > 0$ is a constant. All equations of this form can be obtained from the generalized variational principle by choosing for the Lagrangian function L the expression

$$L = \frac{m}{2}(\dot{x}_1^2 + \ldots + \dot{x}_n^2) - U(t, x_1, \ldots, x_n) - \alpha z \tag{4.2}$$

where $U = U(t, x_1, \ldots, x_n)$ is the potential energy of the system and $\alpha > 0$ is a constant. From (4.2) we obtain the generalized Euler–Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_i} = -\frac{\partial U}{\partial x_i} - \frac{d}{dt}(m\dot{x}_i) - m\alpha\dot{x}_i = 0$$

which are the same as (4.1) for $n = 3$ and $k = m\alpha$.

Depending on the choice of the function U, equations (4.1) can describe a variety of systems. For instance:

(1) When $U = kr^2 = c(x_1^2 + \ldots + x_n^2)$, with $c > 0$ constant, (4.1) describe one-dimensional or multi-dimensional *isotropic damped harmonic oscillators*.

(2) When $U = -c/r = -k/\sqrt{x_1^2 + x_2^2 + x_3^2}$, equations (4.1) describe the motion of a point mass m under Coulomb (electrostatic) or gravitational forces in a resistive medium characterized by the constant α.

(3) The equations describing the currents, voltages and charges in single or coupled electrical circuits have the same form as equations (4.1) with $i = 1, \dots, n$, where n is the number of *state variables* (currents, voltages and charges) and U is an appropriately chosen function (see Goldstein, [6, p. 52]). Hence, the processes in electrical circuits can also be derived from a Lagrangian function of the form (4.2) via the generalized variational principle. This is not possible with the classical variational principle.

As an illustration of the preceding discussion consider a system whose Lagrangian is of the form (4.2) and assume that the potential U is time-independent. Then, $\partial L/\partial t = 0$ and it follows from the Noether-type theorem that the quantity

$$\exp\left(-\int_0^t \frac{\partial L}{\partial z}\,d\vartheta\right)\left(\dot{x}_i\frac{\partial L}{\partial \dot{x}_i} - L\right) = e^{\alpha t}\left(\frac{m}{2}\left(\dot{x}_1^2 + \ldots + \dot{x}_n^2\right) + U(x) + \alpha z\right)$$

is conserved. Recognizing that $H = \dot{x}_i(\partial L/\partial \dot{x}_i) - L$ is the Hamiltonian of the system, we conclude that the value of the Hamiltonian decreases exponentially in time, i.e.

$$H = e^{-\alpha t}\left(\frac{m}{2}\left(\dot{x}_1^2 + \ldots + \dot{x}_n^2\right) + U(x)\right)\bigg|_{t=0} = e^{-\alpha t}H_0 \tag{4.3}$$

where H_0 is the initial value of the Hamiltonian, that is, the initial total energy of the system.

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Manuscript received September 12, 2001

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