Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 20, 2002, 195–215

CHAOS ARISING NEAR A TOPOLOGICALLY TRANSVERSAL HOMOCLINIC SET

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Abstract. A diffeomorphism on a C^1-smooth manifold is studied possessing a hyperbolic fixed point. If the stable and unstable manifolds of the hyperbolic fixed point have a nontrivial local topological crossing then a chaotic behaviour of the diffeomorphism is shown. A perturbed problem is also studied by showing the relationship between a corresponding Melnikov function and the nontriviality of a local topological crossing of invariant manifolds for the perturbed diffeomorphism.

1. Introduction

Let $\mathcal{M}$ be a C^1-smooth manifold without boundary. Consider a C^1-smooth diffeomorphism $f:\mathcal{M}\to\mathcal{M}$ possessing a hyperbolic fixed point p and let W^s_p, W^u_p be the global stable and unstable manifolds of p, respectively. Let $\widetilde{W}^s_p$, $\widetilde{W}^u_p$ be open subsets of W^s_p, W^u_p, respectively, which are submanifolds of $\mathcal{M}$, that is the immersed and induced topologies on $\widetilde{W}^s_p$ and $\widetilde{W}^u_p$, respectively, coincide. We assume that $\widetilde{W}^s_p\cap\widetilde{W}^u_p\setminus\{p\}\neq\emptyset$, i.e. there is a point q homoclinic to p. We also suppose the existence of a compact component $K\ni q$ of the set $\widetilde{W}^s_p\cap\widetilde{W}^u_p$, that is a compact subset $K\subset\widetilde{W}^s_p\cap\widetilde{W}^u_p\setminus\{p\}$ such that $q\in K$ and there exists an

2000 *Mathematics Subject Classification.* 37C29, 37D45.
Key words and phrases. Diffeomorphisms, hyperbolic fixed points, invariant manifolds, topological transversality, chaos.
First named author artially supported by G.N.A.F.A. – Indam, Italy.
Second named author partially supported by G.N.A.F.A. – Indam, Italy and Grant GA-SAV 2/1140/22.

open bounded subset $U \subset \overline{U} \subset \mathcal{M} \setminus \{p\}$ satisfying $U \cap \widetilde{W}_p^s \cap \widetilde{W}_p^u = K$. Since K is compact there is an m_0 such that $f^{m_0}(K)$ is in a local chart U_p of p. By shrinking U, we can assume in addition that $\widetilde{W}_p^{s(u)} \cap \overline{U} = \overline{\widetilde{W}_p^{s(u)} \cap U}$ and as well as $f^{m_0}(\overline{U}) \subset U_p$, and consequently, U is orientable. Moreover, since U is bounded and open, $\widetilde{W}_p^{s(u)} \cap U$ are also submanifolds of $\mathcal{M}$ and there is an $N_0 > 0$ such that $\overline{\widetilde{W}_p^{s(u)} \cap U} \subset f^{\mp N_0}(W_{p,\mathrm{loc}}^{s(u)}) \setminus \{p\}$. Hence $\widetilde{W}_p^{s(u)} \cap U$ are also orientable. Then we can define the local intersection number $\#\big(\widetilde{W}_p^s \cap U, \widetilde{W}_p^u \cap U\big)$ of the manifolds $\widetilde{W}_p^s \cap U$ and $\widetilde{W}_p^u \cap U$ in $\mathcal{M}$ (see [5]).

Let $\mathcal{E} = \{0,1\}^{\mathbb{Z}}$ be the set of doubly infinite sequences of 0 and 1 endowed with the metric

$$d(\{e_n\},\{e_n'\}) = \sum_{n\in\mathbb{Z}} \frac{|e_n - e_n'|}{2^{|n|+1}}.$$

On $\mathcal{E}$ it is defined the so called *shift map* $\sigma: \mathcal{E} \to \mathcal{E}$ by $\sigma(\{e_j\}_{j\in\mathbb{Z}}) = \{e_{j+1}\}_{j\in\mathbb{Z}}$. The main purpose of this note is to prove the following result.

Theorem 1.1. *If* $\#(\widetilde{W}_p^s \cap U, \widetilde{W}_p^u \cap U) \neq 0$ *then there exists* $\omega_0 \in \mathbb{N}$ *such that for any* $\mathbb{N} \ni \omega \geq \omega_0$ *there is a set* $\Lambda_\omega \subset \mathcal{M}$ *and a mapping* $\pi_\omega: \Lambda_\omega \to \mathcal{E}$ *such that*

(i) $f^{2\omega}(\Lambda_\omega) = \Lambda_\omega$,
(ii) π_ω *is continuous, one to one and onto,*
(iii) $\pi_\omega \circ f^{2\omega} = \sigma \circ \pi_\omega$, *where* $\sigma: \mathcal{E} \to \mathcal{E}$ *is the shift map.*

Note that we do not know whether π_ω^{-1} is continuous. Thus we cannot say, in general, that Λ_ω is homeomorphic to $\mathcal{E}$. However, if q is a transversal homoclinic point, π_ω is a homeomorphism, since in the considerations that follow we can use the implicit function theorem instead of the Brouwer degree theory, getting the standard Smale horseshoe (see [7]).

Results similar to Theorem 1.1 have been proved by other authors. For example, a semiconjugacy to the shift σ on $\mathcal{E}$ of some power of a given map is proved in [6] provided an isolating neighbourhood of the map satisfies some conditions on the Conley indices of its subsets. On the other hand, Lefschetz Fixed Point Theorem and Topological Principle of Ważewski is applied in [8] to prove the existence of a compact invariant set for the Poincaré map of a time-periodic vector field on which the same map is semiconjugated to the shift σ on $\mathcal{E}$ and the counterimage (by the semiconjugacy) of any periodic point of σ contains a periodic point of the Poincaré map. The notion of periodic isolating segments is an essential tool for the proofs in [8]. Finally, the same situation as in this paper is studied in [3] (that motivated the present work). By using geometric and homological methods, it is proved in [3] that, under the conditions of Theorem 1.1, there is an invariant set of some power of f on which the same power of f is semiconjugated to the shift σ on $\mathcal{E}$. In all these papers by semiconjugacy it is meant that the associated map between the invariant set and the symbolic set (in our paper it is the map π_ω) is shown to be continuous and onto. Hence the semiconjugacy does not directly imply the existence of infinitely many periodic orbits of a given map (apart from the result in [8]), but it implies positive topological entropy of the map. Our approach instead, which is based on an idea in [1], namely on the notion of exponential dichotomies of difference equations, allow us to prove that π_ω is one to one, a result that was not stated in [3], [6], [8]. Consequently, f has infinitely many periodic orbits as well as quasiperiodic ones (this fact has not been proved earlier). Moreover, we are able to identify the periodic points of the map as solutions of a particular equation.

Next, checking the topological transversality of stable and unstable manifold, is not an easy task. This is the reason why in Section 4 we study the case where W_p^s and W_p^u intersect on a homoclinic manifold and consider a C^2-smooth perturbation of f (again, this point was not considered in [3], [6], [8]). Associated to such a perturbation there is a Melnikov function. Then we obtain the following result:

Theorem 4.4. *Let $f(x,\varepsilon)$ be a C^2-map in its arguments, and assume there exist open, connected, bounded subsets $\Omega \subset \overline{\Omega} \subset U_\Omega \subset \mathbb{R}^\mu$ and C^2-smooth mappings $x_n(\alpha)$, $\alpha \in U_\Omega$, such that the following hold:*

- (i) *$x_{n+1}(\alpha) = f(x_n(\alpha),0)$, $n \in \mathbb{Z}$, $\lim_{n\to\pm\infty} x_n(\alpha) = p$ uniformly with respect to $\alpha \in U_\Omega$ for a hyperbolic fixed point p of the mapping $f(x,0)$,*
- (ii) *$\{\partial x_n/\partial\alpha_i(\alpha),\ i = 1,\dots,\mu\}$ are linearly independent and they form a basis for the space of bounded solutions of the equation*

$$v_{n+1} = f_x(x_n(\alpha),0)v_n$$

on $\mathbb{Z}$ for any $\alpha \in U_\Omega$. Moreover, the mapping $x_0 \colon U_\Omega \to \mathbb{R}^N$ is one to one.

Assume, moreover, that the Melnikov function associated to the perturbation satisfies the following conditions:

- (H1) *$M(\alpha) \neq 0$ on $\partial\Omega$,*
- (H2) *$\deg(M,\Omega,0) \neq 0$.*

Then there exists $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| \le \varepsilon_0$, it is nonzero the local intersection number of the stable and unstable manifolds of the hyperbolic fixed point of the map $x_{n+1} = f(x_n,\varepsilon)$ which is located near the fixed point p of the map $x_{n+1} = f(x_n,0)$.

Thus when a map satisfies the conditions of Theorem 4.4, we obtain, thanks to Theorem 1.1, a kind of chaotic behaviour of the perturbed diffeomorphism $f(x,\varepsilon)$, when $\varepsilon \neq 0$.

Finally, at the end of the paper, we make some remarks about certain extensions and consequences of our results.

2. Preliminary results

To avoid the use of either the tangent vector bundle of $\mathcal{M}$ or local charts of $\mathcal{M}$, we assume for simplicity in this section that $\mathcal{M} = \mathbb{R}^N$. This restriction is only technical. Next, for any $\xi \in \widetilde{W}_p^s \cap \overline{U}$ and $\eta \in \widetilde{W}_p^u \cap \overline{U}$ we set $\xi_n = f^n(\xi)$, $n \in \mathbb{Z}_+$, $\eta_n = f^n(\eta)$, $n \in \mathbb{Z}_-$, where $\mathbb{Z}_+ = \{0, 1, \dots\}$ and $\mathbb{Z}_- = \{\dots, -1, 0\}$. Then the linear systems

$$v_{n+1} = Df(\xi_n)v_n, \quad n \in \mathbb{Z}_+, \tag{2.1}$$

$$w_{n+1} = Df(\eta_n)w_n, \quad n \in \mathbb{Z}_-,\ n \neq 0, \tag{2.2}$$

have exponential dichotomies on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, respectively, i.e. there are positive constants $L \geq 1$, $\delta \in (0,1)$ and orthogonal projections $P_\xi : \mathbb{R}^N \to T_\xi \widetilde{W}_p^s$, $Q_\eta : \mathbb{R}^N \to T_\eta \widetilde{W}_p^u$ such that the fundamental solutions $V_\xi(n)$ and $W_\eta(n)$ of (2.1) and (2.2) respectively, satisfy the following conditions:

$$\begin{aligned} \|V_\xi(n)P_\xi V_\xi(m)^{-1}\| &\leq L\delta^{n-m}, \quad m \leq n,\ m, n \in \mathbb{Z}_+, \\ \|V_\xi(n)(I - P_\xi)V_\xi(m)^{-1}\| &\leq L\delta^{m-n}, \quad n \leq m,\ m, n \in \mathbb{Z}_+, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \|W_\eta(n)(I - Q_\eta)W_\eta(m)^{-1}\| &\leq L\delta^{n-m}, \quad m \leq n,\ m, n \in \mathbb{Z}_-, \\ \|W_\eta(n)Q_\eta W_\eta(m)^{-1}\| &\leq L\delta^{m-n}, \quad n \leq m,\ m, n \in \mathbb{Z}_-, \end{aligned} \tag{2.4}$$

respectively (see [7]). We note that L and δ can be chosen to be independent of $\xi \in \widetilde{W}_p^s \cap \overline{U}$ and $\eta \in \widetilde{W}_p^u \cap \overline{U}$. In fact, let $\xi \in \widetilde{W}_p^s \cap \overline{U}$, then $f^n(\xi) \to p$ as $n \to \infty$. From the roughness of exponential dichotomies ([7]), it follows that L_ξ, δ_ξ exist such that $v_{n+1} = f'(\xi_n)v_n$ has an exponential dichotomy on $\mathbb{Z}_+$ with constants L_ξ and δ_ξ. Again from the roughness of exponential dichotomies, given any $\xi \in \widetilde{W}_p^s \cap \overline{U}$ there exists $r_\xi > 0$ such that when $|\widetilde{\xi} - \xi| < r_\xi$, $\widetilde{\xi} \in \widetilde{W}_p^s \cap \overline{U}$, $v_{n+1} = f'(\widetilde{\xi}_n)v_n$ has an exponential dichotomy on $\mathbb{Z}_+$ with constants $2L_\xi$ and $(1 + \delta_\xi)/2$. Covering $\widetilde{W}_p^u \cap \overline{U}$ with a finite number of balls centered at ξ and of radius r_ξ the result follows as far as the dichotomy on $\mathbb{Z}_+$ is concerned. A similar argument applies for the dichotomy on $\mathbb{Z}_-$.

Note that any projection having the same range as P_ξ, (resp. Q_η) satisfies condition (2.3) (resp. (2.4)). Thus, it is the additional requirement that P_ξ and Q_η are orthogonal that makes them unique. This uniqueness also implies that P_ξ and Q_η are continuous in ξ, η, respectively.

In fact let us prove this for P_ξ. Since $\widetilde{W}_p^s$ is C^1, we get that $T_\xi \widetilde{W}_p^s$ depends continuously on ξ and the same holds for its orthogonal complement $(T_\xi \widetilde{W}_p^s)^\perp$ in $\mathbb{R}^n$. So, if $\{v_1(\xi), \dots, v_d(\xi)\}$ is a (local) orthonormal basis of $T_\xi \widetilde{W}_p^s$ that depends continuously on ξ in a neighbourhood of some $\xi_0 \in \widetilde{W}^s_p$, we have $P_\xi v = \sum_{j=1}^d \langle v, v_j(\xi)\rangle v_j(\xi)$ and then P_ξ is continuous in ξ. Note that the uniqueness of P_ξ implies that $P_\xi v$ does not depend on the choice of the basis $\{v_1(\xi), \dots, v_d(\xi)\}$.

A similar argument holds for Q_η. Moreover, note that, when $\mathcal{M}$ is a C^r-manifold and f is a C^r-diffeomorphism, P_ξ and Q_η are of class C^{r-1}.

Now we fix $\omega \in \mathbb{N}$ large and put

$$\begin{aligned} J_\omega &= \{-\omega, \dots, \omega\}, \\ J_\omega^- &= \{-\omega, \dots, 0\}, \quad I_\omega^- = \{-\omega, \dots, -1\}, \\ J_\omega^+ &= \{0, \dots, \omega\}, \qquad I_\omega^+ = \{0, \dots, \omega - 1\}. \end{aligned}$$

Arguing as in Lemma 2 of [1], we can prove the following results. In this paper, $\mathcal{R}L$ and $\mathcal{N}L$ denote, respectively, the range and the kernel of a linear operator L.

Lemma 2.1. *There exist $\omega_0 \in \mathbb{N}$ and a constant $c > 0$ such that given any $\omega \in \mathbb{N}$, $\omega \geq \omega_0$, $(\xi, \eta) \in (\widetilde{W}^s_p \cap \overline{U}) \times (\widetilde{W}^u_p \cap \overline{U})$, and $b, h_n \in \mathbb{R}^N$, $n \in J_\omega$, $\phi \in \mathcal{R}P_\xi$, $\psi \in \mathcal{R}Q_\eta$, there exist unique solutions $\{v_n\}_{n\in J_\omega^+}$ and $\{w_n\}_{n\in J_\omega^-}$ of the linear systems*

$$\begin{aligned} v_{n+1} &= Df(\xi_n)v_n + h_n, \quad n \in I_\omega^+, \\ w_{n+1} &= Df(\eta_n)w_n + h_n, \quad n \in I_\omega^-, \end{aligned}$$

respectively, together with the boundary value conditions

$$P_\xi v_0 = \phi, \quad Q_\eta w_0 = \psi, \quad v_\omega - w_{-\omega} = b.$$

Moreover, such solutions are linear in (b, h, ϕ, ψ), $h = \{h_n\}_{n\in J_\omega}$ and satisfy

$$\max_{n\in J_\omega^+} |v_n|, \max_{n\in J_\omega^-} |w_n| \leq c(\max_{n\in J_\omega^\pm} |h_n| + |b| + |\phi| + |\psi|).$$

Lemma 2.2. *For any $(\xi, \eta) \in (\widetilde{W}^s_p \cap \overline{U}) \times (\widetilde{W}^u_p \cap \overline{U})$, $\phi \in \mathcal{R}P_\xi$, $\psi \in \mathcal{R}Q_\eta$, and for any bounded sequence $\{h_n\}_{n\in\mathbb{Z}}$, there exist unique solutions $\{v_n\}_{n\geq 0}$ and $\{w_n\}_{n\leq 0}$ of the linear systems*

$$\begin{aligned} v_{n+1} &= Df(\xi_n)v_n + h_n, \quad n \geq 0, \\ w_{n+1} &= Df(\eta_n)w_n + h_n, \quad n \leq -1, \end{aligned}$$

respectively, together with the boundary value conditions: $P_\xi v_0 = \phi$, $Q_\eta w_0 = \psi$. Moreover, such solutions are linear in $(h^\pm, \phi, \psi)$, $h^+ = \{h_n\}_{n\geq 0}$, $h^- = \{h_n\}_{n\leq 0}$, and there exists a constant $c > 0$, independent of $(h^\pm, \phi, \psi)$, such that

$$\sup_{n\geq 0} |v_n| \leq c(\sup_{n\geq 0} |h_n| + |\phi|), \qquad \sup_{n\leq 0} |w_n| \leq c(\sup_{n\leq 0} |h_n| + |\psi|).$$

Now we study the nonlinear system

$$x_{n+1} = f(x_n) \tag{2.5}$$

near $\{\xi_n\}_{n\in J_\omega^+}$ and $\{\eta_n\}_{n\in J_\omega^-}$. By arguing as in [1, Theorem 1] we obtain the following result.

Theorem 2.3. *There exist $\omega_0 \in \mathbb{N}$ and a constant $c > 0$ such that, for any $\omega \in \mathbb{N}$, $\omega \geq \omega_0$, and $(\xi,\eta) \in (\widetilde{W}_p^s \cap \overline{U}) \times (\widetilde{W}_p^u \cap \overline{U})$, there exist unique $\{x_n^+(\omega,\xi,\eta)\}_{n\in J_\omega^+}$ and $\{x_n^-(\omega,\xi,\eta)\}_{n\in J_\omega^-}$ which satisfy (2.5) separately on I_ω^+ and I_ω^- such that*

$$P_\xi x_0^+(\omega,\xi,\eta) = P_\xi \xi, \quad Q_\eta x_0^-(\omega,\xi,\eta) = Q_\eta \eta,$$
$$x_\omega^+(\omega,\xi,\eta) = x_{-\omega}^-(\omega,\xi,\eta),$$

together with

$$\max_{n\in J_\omega^+} |x_n^+(\omega,\xi,\eta) - \xi_n| \leq c\delta^\omega, \qquad \max_{n\in J_\omega^-} |x_n^-(\omega,\xi,\eta) - \eta_n| \leq c\delta^\omega.$$

Moreover, $x_n^\pm(\omega,\xi,\eta)$ are continuous with respect to ξ and η, more precisely they are C^{r-1} when $\mathcal{M}$ is a C^r-manifold and f is a C^r-diffeomorphism.

Proof. Putting $x_n^+ = \xi_n + v_n$, $n \in J_\omega^+$ and $x_n^- = \eta_n + w_n$, $n \in J_\omega^-$. We get the systems

$$v_{n+1} = Df(\xi_n)v_n + f(\xi_n + v_n) - f(\xi_n) - Df(\xi_n)v_n = Df(\xi_n)v_n + o(|v_n|), \tag{2.6}$$

for $n \in I_\omega^+$, and

$$\begin{aligned} w_{n+1} &= Df(\eta_n)w_n + f(\eta_n + w_n) - f(\eta_n) - Df(\eta_n)w_n \\ &= Df(\eta_n)w_n + o(|w_n|), \end{aligned} \tag{2.7}$$

for $n \in I_\omega^-$. Since we are looking for solutions of equation (2.5) such that $x_\omega^+ = x_{-\omega}^-$, we add the boundary value conditions:

$$v_\omega - w_{-\omega} = \eta_{-\omega} - \xi_\omega = O(\delta^\omega), \quad P_\xi v_0 = 0, \quad Q_\eta w_0 = 0. \tag{2.8}$$

Let $v = (v_0,\ldots,v_\omega) \in \mathbb{R}^{N(\omega+1)}$, $w = (w_{-\omega},\ldots,w_0) \in \mathbb{R}^{N(\omega+1)}$. To solve equations (2.6)–(2.8), we take the mapping

$$\Gamma_\omega \colon (\widetilde{W}_p^s \cap \overline{U}) \times \widetilde{W}_p^u \cap \overline{U}) \times \mathbb{R}^{2N(\omega+1)} \to \mathbb{R}^{2N(\omega+1)}$$

defined by

$$\Gamma_\omega(\xi,\eta,v,w) = \begin{pmatrix} (v_{n+1} - f(\xi_n + v_n) + f(\xi_n))_{n\in I_\omega^+} \\ (w_{n+1} - f(\eta_n + w_n) + f(\eta_n))_{n\in I_\omega^-} \\ v_\omega - w_{-\omega} - (\eta_{-\omega} - \xi_\omega) \\ P_\xi v_0 \\ Q_\eta w_0 \end{pmatrix}.$$

where $\binom{P_\xi v_0}{Q_\eta w_0}$ has to be meant as a vector in $\mathbb{R}^N = \mathcal{R}P_\xi \times \mathcal{R}Q_\eta$. We have already observed that P_ξ and Q_η are continuous. Thus, for any fixed $\omega \geq \omega_0$, Γ_ω is continuous in (ξ, η, u, v) as well as its derivatives with respect to (v, w) when we take on $\mathbb{R}^{2N(\omega+1)}$ the maximum norm $\max_i\{|v_i|, |w_i|\}$. We have $\Gamma_\omega(\xi, \eta, 0, 0) = O(\delta^\omega)$ uniformly with respect to (ξ, η) and the linearized map $D_{(v,w)}\Gamma_\omega(\xi, \eta, 0, 0)$ has the form

$$D_{(v,w)}\Gamma_\omega(\xi,\eta,0,0)\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} (v_{n+1} - Df(\xi_n)v_n)_{n\in I_\omega^+} \\ (w_{n+1} - Df(\eta_n)w_n)_{n\in I_\omega^-} \\ v_\omega - w_{-\omega} \\ P_\xi v_0 \\ Q_\eta w_0 \end{pmatrix}.$$

Lemma 2.1 implies that the map $D_{(v,w)}\Gamma_\omega(\xi, \eta, 0, 0)$ is invertible and that its inverse is bounded uniformly with respect to (ξ, η). Hence from the implicit function theorem we get that $c > 0$ and $\omega_0 \gg 1$ exist such that for $\omega \geq \omega_0$, the equation $\Gamma_\omega(\xi, \eta, v, w) = 0$ can be solved uniquely for (v, w) in a neighbourhood of $(0, 0)$ in terms of (ξ, η, ω). Moreover, $\max_i\{|v_i|, |w_i|\} < c\delta^\omega$, and the solution is continuous in (ξ, η), for any fixed $\omega \geq \omega_0$. □

We note that if f and $\mathcal{M}$ are C^r-smooth, $r \geq 1$, then $x_0^\pm(\omega, \xi, \eta)$ is C^{r-1} with respect to (ξ, η) for any fixed $\omega \geq \omega_0$.

3. Chaotic iterations

In this section we prove Theorem 1.1. Let $V \subset \mathcal{M}$ be an open subset such that $K \subset V \subset \overline{V} \subset U$ and ω_0 be as in Theorem 2.3. We also assume that ω_0 is large enough that $c\delta^{\omega_0}$ is less than the distance of V from ∂U and for any $\xi \in \widetilde{W}_p^s \cap \overline{V}$, $\eta \in \widetilde{W}_p^u \cap \overline{V}$ and $n \geq \omega_0$ we have $|\xi_n - p|, |\eta_n - p| \leq C\delta^n$ where C can be chosen independent of ξ, η because of the compactness of $\widetilde{W}_p^s \cap \overline{V}$ and $\widetilde{W}_p^u \cap \overline{V}$. Of course, here we assume that ξ_n, η_n, p are in the local chart U_p of p, for any $n \geq \omega_0$ so that we can consider their differences. Note that the solutions $\{x_n^\pm(\omega, \xi, \eta)\}_{n\in J_\omega^\pm}$ are defined for $(\xi, \eta) \in \widetilde{W}_p^s \cap \overline{V} \times \widetilde{W}_p^u \cap \overline{V}$, and $\#\big(\widetilde{W}_p^s \cap V, \widetilde{W}_p^u \cap V\big) = \#\big(\widetilde{W}_p^s \cap U, \widetilde{W}_p^u \cap U\big)$ because $K \subset V$ implies $K \cap \partial V = \emptyset$. The first step is to show that, for $\omega \geq \omega_0$, the map $x_{n+1} = f(x_n)$ has enough periodic orbits. We recall that $f^{m_0}(\overline{U}) \subset U_p$ for some m_0, and then we can assume that U is embedded in $\mathbb{R}^N$, i.e. $U \hookrightarrow \mathbb{R}^N$. Let h, k be non negative integers. For any finite sequence $E = \{e_j\}_{j=-h}^k$, $e_j \in \{0, 1\}$ such that $e_0 = 1$, we set

$$\{j_1, \ldots, j_{i_E}\} = \{j \mid e_j = 1\},$$

where $-h \leq j_1 < j_2 < \ldots < j_{i_E} \leq k$. Note that, being $e_0 = 1$, we have $j_1 \leq 0 \leq j_{i_E}$. Then we set

$$j_0 = j_{i_E} - h - k - 1, \quad j_{i_E+1} = h + k + 1 + j_1. \tag{3.1}$$

Note that $j_0 \leq -h-1 < j_1$ and $j_{i_E+1} \geq k+1 > j_{i_E}$. Moreover,

$$j_{i_E+1} - j_{i_E} = j_1 - j_0. \tag{3.2}$$

Next, for $\omega \in \mathbb{N}$ fixed and large (that is greater than ω_0), we define

$$F^E\colon ((\widetilde{W}_p^s \cap \overline{V}) \times (\widetilde{W}_p^u \cap \overline{V}))^{i_E} \to \mathbb{R}^{Ni_E}, \quad F^E = (F_1^E, \dots, F_{i_E}^E)$$

where

$$F_r^E(\xi^1, \eta^1, \dots, \xi^{i_E}, \eta^{i_E}) = x_0^-((j_r - j_{r-1})\omega, \xi^r, \eta^r) - x_0^+((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1}),$$

and

$$\xi^{i_E+1} = \xi^1, \quad \eta^{i_E+1} = \eta^1, \tag{3.3}$$

and $x_0^\pm((j_r - j_{r-1})\omega, \xi^r, \eta^r)$ are derived as in Theorem 2.3.

We note that $x_0^+((j_r - j_{r-1})\omega, \xi^r, \eta^r)$ is at a distance from $\xi^r \in \widetilde{W}_p^s \cap \overline{V} \subset U \hookrightarrow \mathbb{R}^N$ less than $c\delta^\omega$ and that the same holds for $x_0^-((j_r - j_{r-1})\omega, \xi^r, \eta^r)$ and $\eta^r \in \widetilde{W}_p^u \cap \overline{V} \subset U \hookrightarrow \mathbb{R}^N$. Consequently, $x_0^\pm((j_r - j_{r-1})\omega, \xi^r, \eta^r) \in U$ and we can consider the above differences in the definition of F^E.

Let us now give a brief motivation for such a definition. Assume that the equation $F_r^E(\xi^1, \eta^1, \dots, \xi^{i_E}, \eta^{i_E}) = 0$ has a solution $(\xi^1, \eta^1, \dots, \xi^{i_E}, \eta^{i_E})$. Then, starting from

$$x_0^-((j_r - j_{r-1})\omega, \xi^r, \eta^r) = x_0^+((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1})$$

and using

$$x^+_{(j_{r+1}-j_r)\omega}((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1}) = x^-_{-(j_{r+1}-j_r)\omega}((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1})$$

we obtain

$$\begin{aligned}
f^{2(j_{r+1},-j_r)\omega} & x_0^-((j_r - j_{r-1})\omega, \xi^r, \eta^r) \\
&= f^{2(j_{r+1}-j_r)\omega} x_0^+((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1}) \\
&= f^{(j_{r+1}-j_r)\omega} x^+_{(j_{r+1}-j_r)\omega}((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1}) \\
&= f^{(j_{r+1}-j_r)\omega} x^-_{-(j_{r+1}-j_r)\omega}((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1}) \\
&= x_0^-((j_{r+1} - j_r)\omega, \xi^{r+1}, \eta^{r+1}),
\end{aligned}$$

and then, using the induction,

$$f^{2(j_s-j_r)\omega} x_0^-((j_r - j_{r-1})\omega, \xi^r, \eta^r) = x_0^-((j_s - j_{s-1})\omega, \xi^s, \eta^s), \tag{3.4}$$

for any $0 \leq r \leq s \leq j_{i_E+1}$. Now, from $e_0 = 1$ we see that $\bar\iota \in \{1, \dots, i_E\}$ exists such that $j_{\bar\iota} = 0$. Then we define

$$\begin{aligned}
x_0(\omega, E) &= x_0^-(-j_{\bar\iota-1}\omega, \xi^{\bar\iota}, \eta^{\bar\iota}) = x_0^-((j_{\bar\iota} - j_{\bar\iota-1})\omega, \xi^{\bar\iota}, \eta^{\bar\iota}) \\
&= x_0^+((j_{\bar\iota+1} - j_{\bar\iota})\omega, \xi^{\bar\iota+1}, \eta^{\bar\iota+1}) = x_0^+(j_{\bar\iota+1}\omega, \xi^{\bar\iota+1}, \eta^{\bar\iota+1}),
\end{aligned} \tag{3.5}$$

and note that from (3.1), (3.2) and (3.4) we obtain

$$\begin{aligned}
f^{2(h+k+1)\omega}x_0(\omega,E) &= f^{2(j_{i_E+1}-j_1)\omega}x_0(\omega,E)\\
&= f^{2(j_{\bar\iota}-j_1)\omega}[f^{2(j_{i_E+1}-j_{\bar\iota})\omega}x_0^-((j_{\bar\iota}-j_{\bar\iota-1})\omega,\xi^{\bar\iota},\eta^{\bar\iota})]\\
&= f^{2(j_{\bar\iota}-j_1)\omega}x_0^-((j_1-j_0)\omega,\xi^1,\eta^1)\\
&= x_0^-((j_{\bar\iota}-j_{\bar\iota-1})\omega,\xi^{\bar\iota},\eta^{\bar\iota}) = x_0(\omega,E),
\end{aligned}$$

that is $x_0(\omega,E)$ is a $2(h+k+1)\omega$-periodic point of the map $x_{n+1}=f(x_n)$. Next, using (3.4), for any $r\in\{1,\dots,j_{i_E}\}$ we have

$$\begin{aligned}
f^{2j_r\omega}x_0(\omega,E) &= f^{2(j_r-j_{\bar\iota})\omega}x_0^-((j_{\bar\iota}-j_{\bar\iota-1})\omega,\xi^{\bar\iota},\eta^{\bar\iota})\\
&= x_0^-((j_r-j_{r-1})\omega,\xi^r,\eta^r) = x_0^+((j_{r+1}-j_r)\omega,\xi^{r+1},\eta^{r+1}),
\end{aligned}$$

and then Theorem 2.3 implies that

$$\begin{aligned}
\|f^{2j_r\omega}x_0(\omega,E)-\eta^r\| &\le c\delta^{(j_r-j_{r-1})\omega}\le c\delta^\omega,\\
\|f^{2j_r\omega}x_0(\omega,E)-\xi^{r+1}\| &\le c\delta^{(j_r-j_{r-1})\omega}\le c\delta^\omega,
\end{aligned}$$

that is $f^{2j_r\omega}x_0(\omega,E)$ belongs to a (small when $\omega>\omega_0$ is sufficiently large) neighbourhood of K for any $r=1,\dots,i_E$. Moreover, for any $j\in\mathbb{N}$ such that $0<j<j_{r+1}-j_r$, we have

$$\begin{aligned}
f^{2(j_r+j)\omega}x_0(\omega,E) &= f^{2j\omega}x_0^+((j_{r+1}-j_r)\omega,\xi^{r+1},\eta^{r+1})\\
&= x_{j\omega}^+((j_{r+1}-j_r)\omega,\xi^{r+1},\eta^{r+1}),
\end{aligned}$$

and then, again from Theorem 2.3,

$$\|f^{2(j_r+j)\omega}x_0(\omega,E)-p\|\le\|f^{2(j_r+j)\omega}x_0(\omega,E)-\xi^{r+1}_{j\omega}\|+\|\xi^{r+1}_{j\omega}-p\|\le 2c\delta^\omega.$$

Thus the map F^E is constructed so that if $F^E(\xi^1,\eta^1,\dots,\xi^{i_E},\eta^{i_E})=0$, then the diffeomorphism f has a periodic orbit attracting and repelling several times by the hyperbolic fixed point p. More precisely, if the initial point of this periodic orbit is given by (3.5), the point $f^{2j\omega}x_0(\omega,E)$ is near the set K if $e_j=1$ and it is near the fixed point p if $e_j=0$. Using this it is easy to see that starting from different E we get different periodic orbits. To solve $F_E=0$, we take the simple homotopy

$$\begin{gathered}
H^E\colon((\widetilde{W}_p^s\cap\overline{V})\times(\widetilde{W}_p^u\cap\overline{V}))^{i_E}\times[0,1]\to\mathbb{R}^{Ni_E},\\
H^E=(H_1^E,\dots,H_{i_E}^E)
\end{gathered}$$

given by

$$H_r^E(\xi^1,\eta^1,\dots,\xi^{i_E},\eta^{i_E},\lambda)=\lambda F_r^E(\xi^1,\eta^1,\dots,\xi^{i_E},\eta^{i_E})+(1-\lambda)(\eta^r-\xi^{r+1}),$$

for $0\le\lambda\le1$. Theorem 2.3 gives

$$|F_r^E(\xi^1,\eta^1,\dots,\xi^{i_E},\eta^{i_E})-\eta^r+\xi^{r+1}|\le 2c\delta^\omega,$$

where the constant c is the same as in Theorem 2.3. Hence we get

$$|H_r^E(\xi^1, \eta^1, \dots, \xi^{i_E}, \eta^{i_E}, \lambda) - \eta^r + \xi^{r+1}| \leq 2c\delta^\omega.$$

Consequently $H^E(\,\cdot\,, \lambda) \neq 0$ on the boundary $\partial((\widetilde{W}_p^s \cap V) \times (\widetilde{W}_p^u \cap V))^{i_E}$ for any $0 \leq \lambda \leq 1$. This gives for the Brouwer degree

$$\deg(F^E, ((\widetilde{W}_p^s \cap V) \times (\widetilde{W}_p^u \cap V))^{i_E}, 0) = \pm\#(\widetilde{W}_p^s \cap V, \widetilde{W}_p^u \cap V)^{i_E} \neq 0.$$

Summarizing, we see that, under the assumptions of Theorem 1.1, the equation $F_E = 0$ is always solvable in the set $((\widetilde{W}_p^s \cap V) \times (\widetilde{W}_p^u \cap V))^{i_E}$ for any sequence $E = \{e_j\}_{j=1}^k \in \{0,1\}^k$, $e_1 = 1$ and any $k \in \mathbb{N}$ for a fixed large (i.e. greater than ω_0) $\omega \in \mathbb{N}$. Thus we have seen that the map f has enough periodic orbits. Now let $\sim$ be the equivalence relation on the set $\mathcal{E} = \{0,1\}^{\mathbb{Z}}$ defined as follows:

> let $E, E' \in \mathcal{E}$. We say that $E \sim E'$ if $n_0 \in \mathbb{Z}$ exists such that $E = \sigma^{n_0}(E')$.

Then we choose a unique element for any equivalence class in $\mathcal{E}/\!\sim$ and form a metric subspace $\mathcal{E}_\sim$. Without loss of generality we can also assume that $\mathcal{E}_\sim \subset \mathcal{E}_1 := \{E = \{e_j\}_{j\in\mathbb{Z}} \in \mathcal{E} : e_j = 0$ for any $j \in \mathbb{Z}$ or $e_0 = 1\}$. We obtain in this way a subspace $\mathcal{E}_\sim \subset \mathcal{E}_1$ such that if $E_1, E_2 \in \mathcal{E}_\sim$, then either $E_1 = E_2$ or $E_1 \neq \sigma^n(E_2)$ for any $n \in \mathbb{Z}$.

Now we define a map $E \mapsto \mathcal{O}_E$ from $\mathcal{E}_1$ in the space of orbits of f as follows. If $e_j = 0$, for all $j \in \mathbb{Z}$ then we put $\mathcal{O}_E = \{p\}$ the fixed point orbit of f. On the other hand, if $e_0 = 1$, we have the following two possibilities: either E is periodic with the minimal period m, i.e. $\sigma^m(E) = E$ and $\sigma^k(E) \neq E$ for $1 \leq k < m$, or E is nonperiodic, that is there is no $m \in \mathbb{N}$ such that $\sigma^m(E) = E$.

In the first case we apply the above procedure to the finite sequence $\{e_j\}_{j=0}^{m-1}$ (m being the minimal period of E). We obtain then a $2m\omega$-periodic orbit $\mathcal{O}_E$ such that $f^{2j\omega}(x_0)$ is either near the set K or the point p according to $e_j = 1$ or $e_j = 0$, respectively.

In the second case we consider, for any $m \in \mathbb{N}$, the finite sequence $E_m = \{e_j^m\}_{j=-m}^m := \{e_j\}_{j=-m}^m$, $m \in \mathbb{N}$, to obtain a periodic orbit $\mathcal{O}_{E_m}$ of $x_{n+1} = f(x_n)$ with the same oscillation property between K and p as above.

We set $\mathcal{O}_{E_m} = \{x_n^m\}_{n\in\mathbb{Z}}$. Then take a convergent subsequence $x_0^{m_i}$ of x_0^m and let x_0 be its limit as $i \to \infty$. Note that, $\mathcal{O}_{E_m}$ being an orbit of $x_{n+1} = f(x_n)$, we have $x_j^m = f^j(x_0^m)$ for any $j \in \mathbb{Z}$. Thus $x_j^{m_i}$ converges to $f^j(x_0)$. Hence we set $\mathcal{O}_E = \{f^j(x_0)\}_{j\in\mathbb{Z}}$. Note that $\mathcal{O}_E$ is an orbit of the map f such that $f^{2j\omega}(x_0)$ is either near the set K or the point p according to $e_j = 1$ or $e_j = 0$, respectively. In fact, for any given $j \in \mathbb{Z}$, there exists $m_0 \in \mathbb{N}$ such that $e_j^m = e_j$ for any $m \geq m_0$. Thus the conclusion follows because it is satisfied by $f^{2j\omega}(x_0^{m_i})$ for any i sufficiently large. Observe also that if E is not periodic (that is $\sigma^n(E) \neq E$ for any $n \in \mathbb{Z}$) then $\mathcal{O}_E$ is also a non periodic orbit of f because of the stated oscillation properties. Moreover, if $\mathcal{O}_E = \mathcal{O}_{E'}$ then $E = E'$, that is the map $E \mapsto \mathcal{O}_E$ is one to one. Finally, for $\mathcal{O}_E = \{f^i(x_0)\}_{i\in\mathbb{Z}}$ we set

$$f^{2j\omega}(\mathcal{O}_E) = \{f^{2j\omega+i}(x_0)\}_{i\in\mathbb{Z}}. \tag{3.6}$$

At this point we would like that the following holds: $\mathcal{O}_{\sigma^n(E)} = f^{2\omega n}(\mathcal{O}_E)$ when E and $\sigma^n(E)$ belong to $\mathcal{E}_1$. However this is not generally true even if it is true that $\mathcal{O}_{\sigma^n(E)}$ and $f^{2\omega n}(\mathcal{O}_E)$ have the same oscillating properties between K and p. The point is that in order to define the orbit $\mathcal{O}_E$ we actually use the axiom of choice to choose a convergent subsequence $x_0^{m_i}$ of x_0^m. Thus, in general $\mathcal{O}_{\sigma(E)} \neq f^{2\omega}(\mathcal{O}_E)$, because we can perhaps choose convergent subsequences of x_0^m and x_1^m such that their limits do not satisfy the equality $x_1 = f(x_0)$ (of course, when the sequence x_0^m is itself convergent this does not happen). For this reason, in order to extend the map $E \mapsto \mathcal{O}_E$ to $\mathcal{E}$ we have to pass through $\mathcal{E}_\sim$.

Let $E = \{e_n\}_{n\in\mathbb{Z}} \in \mathcal{E}$ be a doubly infinite sequence of 0 and 1. If $e_j = 0$ for any $j \in \mathbb{Z}$ we set $J_\omega(E) = \{p\}$, the fixed point orbit of f. If $j \in \mathbb{Z}$ exists such that $e_j = 1$, a unique $E' \in \mathcal{E}_\sim$ exists such that $E = \sigma^{n_0}(E')$ for some $n_0 \in \mathbb{Z}$. Such a n_0 is unique when E is nonperiodic and is defined up to a multiple of the least period, when E is periodic. Then we set

$$J_\omega(E) = f^{2\omega n_0}(\mathcal{O}_{E'}). \tag{3.7}$$

This definition does not depend on n_0. We only have to prove this in the case where E is periodic with least period, say, m. We have:

$$f^{2\omega(km+n_0)}(\mathcal{O}_{E'}) = f^{2\omega n_0}(f^{2\omega km}(\mathcal{O}_{E'})) = f^{2\omega n_0}(\mathcal{O}_{E'}),$$

for any $k \in \mathbb{Z}$, since $\mathcal{O}_{E'}$ is $2\omega m$-periodic. Thus the definition (3.7) is independent of n_0. Moreover, if $E = \sigma^{n_0}(E')$, $E' \in \mathcal{E}_\sim$, then $\sigma(E) = \sigma^{n_0+1}(E')$ and

$$J_\omega(\sigma(E)) = f^{2\omega(n_0+1)}(\mathcal{O}_{E'}) = f^{2\omega}(J_\omega(E))$$

that is

$$J_\omega \circ \sigma = f^{2\omega} \circ J_\omega. \tag{3.8}$$

Now we prove that J_ω is one to one. Because of the oscillating property, it follows immediately that $J_\omega(E) \neq \{p\}$ when E is not the identically zero sequence. Now, let $E_1, E_2 \in \mathcal{E}$ be two, non identically zero, sequences such that $J_\omega(E_1) = J_\omega(E_2)$. Write $E_1 = \sigma^{n_1}(E_1')$ and $E_2 = \sigma^{n_2}(E_2')$, with $E_1' = \{{e'}_n^{(1)}\}, E_2' = \{{e'}_n^{(2)}\} \in \mathcal{E}_\sim$. Then $J_\omega(E_1) = J_\omega(E_2)$ implies $\mathcal{O}_{E_1'} = f^{2\omega(n_2-n_1)}(\mathcal{O}_{E_2'})$. From this equation and the oscillating property we see that ${e'}_{(n_2-n_1)}^{(2)} = 1$, that is $\sigma^{n_2-n_1}(E_2') \in \mathcal{E}_1$. Moreover, as we have already observed, $f^{2\omega(n_2-n_1)}(\mathcal{O}_{E_2'})$ has the same oscillating properties between K and p as $\mathcal{O}_{\sigma^{(n_2-n_1)}(E_2')}$. Thus E_1' and $\sigma^{(n_2-n_1)}(E_2')$ are two elements of $\mathcal{E}_1$ such that $\mathcal{O}_{E_1'}$ and $\mathcal{O}_{\sigma^{(n_2-n_1)}(E_2')}$ have the same oscillating properties between K an p. But this means that $E_1' = \sigma^{(n_2-n_1)}(E_2')$ from which we get immediately $E_1 = E_2$. So J_ω is one to one and satisfies (3.8).

Now we consider the map $\mathcal{P}\colon J_\omega(\mathcal{E}) \to \mathbb{R}^N$ given by $\mathcal{P}(J_\omega(E)) = x_0$, where $J_\omega(E) = \{x_j\}_{j\in\mathbb{Z}}$. We set $\Lambda_\omega = \mathcal{P}(J_\omega(\mathcal{E}))$, we define $\mathcal{Q}\colon \Lambda_\omega \to J_\omega(\mathcal{E})$ as $\mathcal{Q}(x_0) = \{f^j(x_0)\}_{j\in\mathbb{Z}}$.

Finally, we define $\pi_\omega\colon \Lambda_\omega \to \mathcal{E}$ as $\pi_\omega(x_0) = J_\omega^{-1}\big(\mathcal{Q}(x_0)\big)$.

Now we state some of the properties of π_ω:

- (i) π_ω is one to one. This easily follows from the fact that different initial points give different orbits (that is $\mathcal{Q}$ is one to one).
- (ii) π_ω is continuous. To show this, let x_0^0, $\{x_0^i\}_{i\in\mathbb{N}} \subset \Lambda_\omega$ and $x_0^i \to x_0^0$ as $i \to \infty$. Then $f^j(x_0^i) \to f^j(x_0^0)$ as $i \to \infty$ for any $j \in \mathbb{Z}$. Hence for any $N_0 \in \mathbb{N}$, and $|j| \le N_0$, the points $f^{2j\omega}(x_0^i)$ of the orbit $\mathcal{Q}(x_0^i) = J_\omega(E^i) \in J_\omega(\mathcal{E})$ and $f^{2j\omega}(x_0^0)$ of the orbit $\mathcal{Q}(x_0^0) = J_\omega(E^0) \in J_\omega(\mathcal{E})$ have, for i large, the same kind of oscillation between K and p. Consequently, the sequences E^i and E^0, for i large, have the same elements in the first j, $|j| \le N_0$ places. This implies that $E^i \to E^0$ as $i \to \infty$.
- (iii) $\sigma(\pi_\omega(x_0)) = \pi_\omega(f^{2\omega}(x_0))$. In fact, we know that $J_\omega(\sigma(E)) = f^{2\omega}(J_\omega(E))$ for any $E \in \mathcal{E}$. Thus if $E = \pi_\omega(x_0)$ we have $J_\omega(E) = \mathcal{Q}(x_0) = \{f^j(x_0)\}_{j\in\mathbb{Z}}$, then

$$J_\omega(\sigma(\pi_\omega(x_0))) = J_\omega(\sigma(E)) = f^{2\omega}(J_\omega(E)) = \{f^j(f^{2\omega}(x_0))\}_{j\in\mathbb{Z}} = \mathcal{Q}(f^{2\omega}(x_0)).$$

Thus $\sigma(\pi_\omega(x_0)) = \pi_\omega(f^{2\omega}(x_0))$ for any $x_0 \in \Lambda_\omega$.

Summarizing, π_ω is continuous, one to one and $\pi_\omega \circ f^{2\omega} = \sigma \circ \pi_\omega$. By the construction it is also clear that π_ω is onto. This result proves the statement of Theorem 1.1.

4. Topological transversality and Melnikov function

In this section, we consider a C^2-smooth perturbation of f on $\mathcal{M} = \mathbb{R}^N$ given by $f(x,\varepsilon)$, $f(x,0) = f(x)$ for $\varepsilon \in \mathbb{R}$. Moreover, we suppose that f has an μ-parametric nondegenerate family of orbits homoclinic to a hyperbolic fixed point p, that is there exist open, connected and bounded subsets $\Omega \subset \overline{\Omega} \subset U_\Omega \subset \mathbb{R}^\mu$ and C^2-smooth mappings $x_n(\alpha)$, $\alpha \in U_\Omega$, $n \in \mathbb{Z}$, such that

- (i) $x_{n+1}(\alpha) = f(x_n(\alpha))$, $n \in \mathbb{Z}$, $\lim_{n\to\pm\infty} x_n(\alpha) = p$ uniformly with respect to $\alpha \in U_\Omega$,
- (ii) $\{(\partial x_n/\partial\alpha_i)(\alpha),\ i = 1,\dots,\mu\}$ are linearly independent and they form a basis for the space of bounded solutions of the equation $v_{n+1} = Df(x_n(\alpha))v_n$ on $\mathbb{Z}$ for any $\alpha \in U_\Omega$. Moreover, the mapping $x_0\colon U_\Omega \to \mathbb{R}^N$ is one to one.

We note that $x_0(\alpha) \in W_p^s \cap W_p^u$, for any $\alpha \in U_\Omega$. In the next constructions of this section, the set Ω is fixed but the neighbourhood U_Ω of $\overline{\Omega}$ could be shrunk by keeping its connectedness. Let $U \subset \mathbb{R}^N$ be an open and bounded subset such that

$$\mathcal{H}_\Omega = \{x_0(\alpha) \mid \alpha \in U_\Omega\} = U \cap \widetilde{W}_p^s \cap \widetilde{W}_p^u, \tag{4.1}$$

where again $\widetilde{W}_p^s$ and $\widetilde{W}_p^u$ are open subsets of W_p^s and W_p^u, respectively, which are submanifolds of $\mathbb{R}^N$.

Let P_ξ and Q_η be the projections of Section 2 for the open subset U, which are now C^1-smooth in ξ and η, respectively. Arguing as in Section 2 (see also [1, Theorem 1]), we get the following result.

Theorem 4.1. *There exist $\varepsilon_0 > 0$ and $\rho > 0$ such that for any $|\varepsilon| < \varepsilon_0$ and $\xi \in \widetilde{W}_p^s \cap U$, $\eta \in \widetilde{W}_p^u \cap U$ the equations*

$$x_{n+1}^+(\varepsilon,\xi) = f(x_n^+(\varepsilon,\xi),\varepsilon), \quad P_\xi x_0^+(\varepsilon,\xi) = P_\xi \xi$$

for $n \geq 0$, and

$$x_{n+1}^-(\varepsilon,\eta) = f(x_n^-(\varepsilon,\eta),\varepsilon), \quad Q_\eta x_0^-(\varepsilon,\eta) = Q_\eta \eta$$

for $n \leq -1$, have unique solutions $\{x_n^+(\varepsilon,\xi)\}_{n\geq 0}$ and $\{x_n^-(\varepsilon,\eta)\}_{n\leq 0}$ respectively, such that

$$\sup_{n\geq 0} |x_n^+(\varepsilon,\xi) - \xi_n| \leq \rho, \quad \sup_{n\leq 0} |x_n^-(\varepsilon,\eta) - \eta_n| \leq \rho. \tag{4.2}$$

Moreover, $\{x_n^+(\varepsilon,\xi)\}_{n\geq 0}$ and $\{x_n^-(\varepsilon,\eta)\}_{n\leq 0}$ are C^1-smooth in their arguments and

$$\lim_{\varepsilon\to 0} \sup_{n\geq 0} |x_n^+(\varepsilon,\xi) - \xi_n| = 0, \quad \lim_{\varepsilon\to 0} \sup_{n\leq 0} |x_n^-(\varepsilon,\eta) - \eta_n| = 0. \tag{4.3}$$

Proof. We give the proof for $n \geq 0$ the case $n \leq 0$ being handled similarly. Let $\xi \in \widetilde{W}_p^s \cap U$, and $x_n = \xi_n + v_n$. Then $\{v_n\}_{n\geq 0}$ satisfies the system

$$\begin{cases} v_{n+1} - f'(\xi_n)v_n = \{f(\xi_n + v_n,\varepsilon) - f(\xi_n) - f'(\xi_n)v_n\}, \\ P_\xi v_0 = 0. \end{cases} \tag{4.4}$$

We are looking for solutions of (4.4) such that $\sup_{n\geq 0} |v_n| \to 0$ as $\varepsilon \to 0$. Let $\rho > 0$ be fixed. From Lemma 2.2 it follows that the map

$$\Gamma_\infty(v) = \begin{pmatrix} \{v_{n+1} - f'(\xi_n)v_n\}_{n\geq 0} \\ P_\xi v_0 \end{pmatrix}$$

has a bounded inverse. So, for any $\{v_n\}_{n\geq 0}$ such that $\sup_{n\geq 0} |v_n| < \rho$ we define $\{\widehat{v}_n\}_{n\geq 0}$ as the unique solution of

$$\Gamma_\infty(\{\widehat{v}_n\}_{n\geq 0}) = \begin{pmatrix} \{f(\xi_n + v_n,\varepsilon) - f(\xi_n) - f'(\xi_n)v_n\}_{n\geq 0} \\ 0 \end{pmatrix}.$$

From Lemma 2.2 it follows that

$$\sup_{n\geq 0}|\widehat{v}_n| \leq c\sup_{n\geq 0}|f(\xi_n+v_n,\varepsilon)-f(\xi_n)-f'(\xi_n)v_n| \leq c\{\Delta(\rho)\sup_{n\geq 0}|v_n|+O(\varepsilon)\}$$

where $\Delta(\rho)\to 0$ as $\rho\to 0$. Thus it is easy to see that the map $\{v_n\}_{n\geq 0}\mapsto\{\widehat{v}_n\}_{n\geq 0}$ is a contraction on the ball $\{\{v_n\}_{n\geq 0} : \sup_{n\geq 0}|v_n|<\rho\}$ provided ρ and ε_0 are sufficiently small. As a consequence there exists a unique fixed point $\{v_n(\varepsilon,\xi)\}_{n\geq 0}$ that gives rise to the solution $x_n(\varepsilon,\xi)=\xi_n+v_n(\varepsilon,\xi)$. From the smoothness of the map $\{v_n\}_{n\geq 0}\mapsto\{f(\xi_n+v_n,\varepsilon)-f(\xi_n)-f'(\xi_n)v_n\}_{n\geq 0}$, we obtain that $x_n(\varepsilon,\xi)$ is smooth and that (4.2), (4.3) hold. □

Now we consider the function $H:\widetilde{W}_p^s\cap U\times\widetilde{W}_p^u\cap U\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^N$ given by

$$H(\xi,\eta,\varepsilon)=x_0^+(\varepsilon,\xi)-x_0^-(\varepsilon,\eta).$$

Note that, because of the hyperbolicity of p, the map $x_{n+1}=f(x_n,\varepsilon)$ has, for small $|\varepsilon|$, a unique hyperbolic fixed point $p(\varepsilon)$ such that $p(\varepsilon)\to p$ as $\varepsilon\to 0$. Such a fixed point is C^2-smooth in ε and the solutions of $H(\xi,\eta,\varepsilon)=0$ give rise to orbits $\{x_n(\varepsilon)\}_{n\in\mathbb{Z}}$ of the map $x_{n+1}=f(x_n,\varepsilon)$ that are homoclinic to $p(\varepsilon)$. Moreover, if $U_1\subset\overline{U}_1\subset U$ is an open, connected subset of U, the functions $x_0^+(\varepsilon,\xi)$, $\xi\in\widetilde{W}_p^s\cap U_1$ and $x_0^-(\varepsilon,\eta)$, $\eta\in\widetilde{W}_p^u\cap U_1$, describe open subsets of the stable and unstable manifolds $W^s_{p(\varepsilon)}$ and $W^u_{p(\varepsilon)}$ of $p(\varepsilon)$ that are also immersed submanifolds in $\mathbb{R}^n$. So, denoting with $\widetilde{W}^s_{p(\varepsilon)}$ and $\widetilde{W}^u_{p(\varepsilon)}$ these submanifolds of $\mathbb{R}^N$, we see that the intersection number $\#(\widetilde{W}^s_{p(\varepsilon)}\cap U_1,\widetilde{W}^u_{p(\varepsilon)}\cap U_1)$ can be studied by looking at the Brouwer degree $\deg(H(\xi,\eta,\varepsilon),(\widetilde{W}_p^s\cap U_1)\times(\widetilde{W}_p^u\cap U_1),0)$.

Thus, let $d_s=\dim\widetilde{W}_p^s$, and $d_u=\dim\widetilde{W}_p^u$. From the hyperbolicity of p we get $d_s+d_u=N$, hence we can write $\mathbb{R}^N=W_\mu\oplus W_s\oplus W_u\oplus V_\mu$ where $\dim W_\mu=\dim V_\mu=\mu$, $V_\mu^\perp=W_\mu\oplus W_s\oplus W_u$, $\dim W_s=d_s-\mu$, $\dim W_u=d_u-\mu$, and U_Ω is an open subset of W_μ. Then, replacing U and U_Ω with smaller, open, connected and bounded subsets of $\mathbb{R}^n$ and $\mathbb{R}^\mu$ respectively, so that (4.1) and $\overline{\Omega}\subset U_\Omega$ are still satisfied, we can find open and convex subsets $O^s\subset W_s$, $O^u\subset W_u$, $O^*\subset V_\mu$, containing 0, and a C^1-diffeomorphism $\Phi: U_\Omega\oplus O^s\oplus O^u\oplus O^*\to U\subset\mathbb{R}^N$ such that the following holds:

$$\begin{aligned}\Phi(\alpha)&=x_0(\alpha),\quad\text{for any }\alpha\in U_\Omega,\\ \Phi(U_\Omega\oplus O^s)&=\widetilde{W}_p^s\cap U,\\ \Phi(U_\Omega\oplus O^u)&=\widetilde{W}_p^u\cap U.\end{aligned}$$

Let $\widetilde{\xi}$, $\widetilde{\eta}$ be the coordinates on $W_\mu\oplus W_s$ and $W_\mu\oplus W_u$ respectively. Then possibly shrinking U_Ω, O^s and O^u we consider, the function

$$\widetilde{H}:\overline{(U_\Omega\oplus O^s)\times(U_\Omega\oplus O^u)}\to\mathbb{R}^N$$

given by

$$\widetilde{H}(\widetilde{\xi},\widetilde{\eta},\varepsilon) := \Phi^{-1}(x_0^+(\varepsilon,\Phi(\widetilde{\xi}))) - \Phi^{-1}(x_0^-(\varepsilon,\Phi(\widetilde{\eta}))).$$

Obviously, $\widetilde{H}(\widetilde{\xi},\widetilde{\eta},\varepsilon) = 0$ if and only if $H(\xi,\eta,\varepsilon) = 0$, and then

$$\begin{aligned}\deg\,&(H(\xi,\eta,\varepsilon),(\widetilde{W}_p^s\cap U)\times(\widetilde{W}_p^u\cap U),0)\\ &= \pm\deg(\widetilde{H}(\widetilde{\xi},\widetilde{\eta},\varepsilon),(U_\Omega\oplus O^s)\times(U_\Omega\oplus O^u),0).\end{aligned}$$

Theorem 4.1 implies that $\widetilde{H}(\widetilde{\xi},\widetilde{\eta},0) = \widetilde{\xi} - \widetilde{\eta}$ from which we get $\widetilde{H}(\alpha,\alpha,0) = 0$ and

$$\begin{aligned}&\widetilde{H}(\widetilde{\xi},\widetilde{\eta},\varepsilon)\\ &\quad = \widetilde{\xi} - \widetilde{\eta} + \varepsilon\bigg\{[\Phi'(\widetilde{\xi})]^{-1}\frac{\partial x_0^+}{\partial\varepsilon}(0,\Phi(\widetilde{\xi})) - [\Phi'(\widetilde{\eta})]^{-1}\frac{\partial x_0^-}{\partial\varepsilon}(0,\Phi(\widetilde{\eta}))\bigg\} + r(\widetilde{\xi},\widetilde{\eta},\varepsilon),\end{aligned}$$

where $\|r(\widetilde{\xi},\widetilde{\eta},\varepsilon)\| = o(\varepsilon)$ uniformly in $(\widetilde{\xi},\widetilde{\eta}) \in \overline{(U_\Omega\oplus O^s)\times(U_\Omega\oplus O^u)}$.

Let $L\colon(W_\mu\oplus W_s)\times(W_\mu\oplus W_u)\to\mathbb{R}^N$ be the linear map defined as $L(\widetilde{\xi},\widetilde{\eta}) = \widetilde{\xi} - \widetilde{\eta}$. We have $L(\widetilde{\xi},\widetilde{\eta}) = 0$ if and only if $\widetilde{\xi} = \widetilde{\eta} \in W_\mu$ and $\mathcal{R}L = W_\mu\oplus W_s\oplus W_u$, so we can write $\mathbb{R}^N = \mathcal{R}L\oplus V_\mu$. Next, let $W_\mu^\perp$ be a fixed subspace of $(W_\mu\oplus W_s)\times(W_\mu\oplus W_u)$ transversal to $\mathcal{N}L = \{(\widetilde{\xi},\widetilde{\xi}) : \widetilde{\xi}\in W_\mu\}$. Then, there exists an open convex set $O_1\subset W_\mu^\perp$ such that $0\in O_1$ and for any $(\widehat{\xi},\widehat{\eta})\in\overline{O}_1$ and $\alpha\in\overline{\Omega}$ the point $(\widetilde{\xi},\widetilde{\eta}) = (\alpha+\widehat{\xi},\alpha+\widehat{\eta})$ belongs to $(U_\Omega\oplus O^s)\times(U_\Omega\oplus O^u)$. We define a map $\widehat{H}\colon\overline{O_1}\times\overline{\Omega}\to\mathbb{R}^N$ as

$$\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon) := \widetilde{H}(\alpha+\widehat{\xi},\alpha+\widehat{\eta},\varepsilon)\,.$$

Let $\mathcal{Q}\colon\mathbb{R}^N\to\mathbb{R}^N$ be the projection that corresponds to the splitting $\mathbb{R}^N = \mathcal{R}L\oplus V_\mu$ that is such that $\mathcal{N}Q = V_\mu$ and $\mathcal{R}Q = \mathcal{R}L$, and set $\widehat{r} = \widehat{r}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon) = \varepsilon^{-1}r(\alpha+\widehat{\xi},\alpha+\widehat{\eta},\varepsilon) = o(1)$. We write $\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon)$ as

$$\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon) = \widehat{\xi} - \widehat{\eta} + \varepsilon\widehat{H}_1(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon) + \varepsilon\widehat{H}_2(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon)$$

where

$$\begin{aligned}\widehat{H}_1(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon) &= \mathcal{Q}\bigg\{[\Phi'(\alpha+\widehat{\xi})]^{-1}\frac{\partial x_0^+}{\partial\varepsilon}(0,\Phi(\alpha+\widehat{\xi}))\\ &\qquad - [\Phi'(\alpha+\widehat{\eta})]^{-1}\frac{\partial x_0^-}{\partial\varepsilon}(0,\Phi(\alpha+\widehat{\eta})) + \widehat{r}\bigg\},\\ \widehat{H}_2(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon) &= (\mathbb{I}-\mathcal{Q})\bigg\{[\Phi'(\alpha+\widehat{\xi})]^{-1}\frac{\partial x_0^+}{\partial\varepsilon}(0,\Phi(\alpha+\widehat{\xi}))\\ &\qquad - [\Phi'(\alpha+\widehat{\eta})]^{-1}\frac{\partial x_0^-}{\partial\varepsilon}(0,\Phi(\alpha+\widehat{\eta})) + \widehat{r}\bigg\}.\end{aligned}$$

Note that $\widehat{H}_1(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon)\in\mathcal{R}\mathcal{Q}$ and $\widehat{H}_2(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon)\in\mathcal{N}\mathcal{Q}$. Thus $\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon) = 0$ if and only if $\widehat{\xi} - \widehat{\eta} + \varepsilon\widehat{H}_1 = 0$ and $\widehat{H}_2 = 0$. Next we introduce the Melnikov function $M:\overline{\Omega}\to\mathcal{N}Q$:

$$M(\alpha)=(\mathbb{I}-\mathcal{Q})\Phi'(\alpha)^{-1}\bigg[\frac{\partial x_0^+}{\partial\varepsilon}(0,x_0(\alpha))-\frac{\partial x_0^-}{\partial\varepsilon}(0,x_0(\alpha))\bigg],$$

whose components with respect to a fixed orthonormal basis $\{e_1,\dots,e_\mu\}$ of V_μ are:

$$\begin{aligned}M_j(\alpha)&=e_j^*\Phi'(\alpha)^{-1}\bigg[\frac{\partial x_0^+}{\partial\varepsilon}(0,x_0(\alpha))-\frac{\partial x_0^-}{\partial\varepsilon}(0,x_0(\alpha))\bigg]\\&=\Big[(\Phi'(\alpha)^{-1})^*e_j\Big]^*\bigg[\frac{\partial x_0^+}{\partial\varepsilon}(0,x_0(\alpha))-\frac{\partial x_0^-}{\partial\varepsilon}(0,x_0(\alpha))\bigg]\\&=\psi_j(\alpha)^*\bigg[\frac{\partial x_0^+}{\partial\varepsilon}(0,x_0(\alpha))-\frac{\partial x_0^-}{\partial\varepsilon}(0,x_0(\alpha))\bigg]\end{aligned}$$

where $\psi_j(\alpha)$ are defined by the equality. Note that for any $v\in T_{x_0(\alpha)}\widetilde{W}_p^s$ we have $\psi_j(\alpha)^*v=e_j^*\Phi'(\alpha)^{-1}v=0$ because $V_\mu^\perp=W_\mu\oplus W_s\oplus W_u$ and $\Phi'(\alpha)(W_\mu\oplus W_s)=T_{x_0(\alpha)}\widetilde{W}_p^s$. Similarly $\psi_j(\alpha)^*w=0$ for any $w\in T_{x_0(\alpha)}\widetilde{W}_p^u$. Thus the vectors $\psi_j(\alpha)$ are exactly the initial conditions to assign to the adjoint of the variational system $v_{n+1}=f'(x_n(\alpha))v_n$ to obtain solutions that are bounded on $\mathbb{Z}$. Thus $M(\alpha)$ is the usual Melnikov function associated to the system $x_{n+1}=f(x_n,\varepsilon)$ (see [1], [4]). We assume that

(H1) $M(\alpha)\neq 0$ on $\partial\Omega$,
(H2) $\deg(M,\Omega,0)\neq 0$.

From the smoothness of the functions $x_0^+(\varepsilon,\xi)$, $x_0^-(\varepsilon,\eta)$, $r(\xi,\eta,\varepsilon)$, $x_0(\alpha)$ and possibly changing O_1, we see that

$$\frac{\partial x_0^+}{\partial\varepsilon}(0,\widehat{\xi}+x_0(\alpha))-\frac{\partial x_0^-}{\partial\varepsilon}(0,\widehat{\eta}+x_0(\alpha))+\widehat{r}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon)$$

is bounded on $\overline{O}_1\times\overline{\Omega}\times[-\varepsilon_0,\varepsilon_0]$. Then we plug $\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon)$ in the homotopy $\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon,t)$, $0\le t\le 1$ given by

$$\widehat{H}(\widehat{\xi},\widehat{\eta},\varepsilon,\alpha,t)=\widehat{\xi}-\widehat{\eta}+\varepsilon t\widehat{H}_1(\widehat{\xi},\widehat{\eta},\varepsilon,\alpha)+k_\varepsilon(t)\widehat{H}_2(t\widehat{\xi},t\widehat{\eta},\varepsilon,\alpha)$$

where $k_\varepsilon(t)=\varepsilon t+1-t$ for $\varepsilon\ge 0$ and $k_\varepsilon(t)=\varepsilon t-1+t$ for $\varepsilon<0$. Note that $|k_\varepsilon(t)|\ge|\varepsilon|$ and then $k_\varepsilon(t)\neq 0$ for $\varepsilon\neq 0$.

We have the following

LEMMA 4.2. *Assume* (H1) *holds. Then, if the neighbourhood O_1 is chosen sufficiently small there is an $\varepsilon_0>0$ such that $\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon,t)\neq 0$ for any $0\le t\le 1$, $0<|\varepsilon|<\varepsilon_0$ and $(\widehat{\xi},\widehat{\eta},\alpha)\in\partial(O_1\times\Omega)$.*

PROOF. We have already seen that $\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon,t)=0$ if and only if $\widehat{\xi}-\widehat{\eta}+\varepsilon t\widehat{H}_1(\widehat{\xi},\widehat{\eta},\varepsilon,\alpha)=0$ and $k_\varepsilon(t)\widehat{H}_2(t\widehat{\xi},t\widehat{\eta},\varepsilon,\alpha)=0$. Now, if $(\widehat{\xi},\widehat{\eta},\alpha)\in\partial(O_1\times\Omega)$ then either $(\widehat{\xi},\widehat{\eta})\in\partial O_1$ or $\alpha\in\partial\Omega$.

If $(\widehat{\xi},\widehat{\eta}) \in \partial O_1$, we have $\widehat{\xi} \neq \widehat{\eta}$ and then $\widehat{\xi} - \widehat{\eta} + \varepsilon t \widehat{H}_1(\widehat{\xi},\widehat{\eta},\varepsilon,\alpha) \neq 0$ for ε_0 sufficiently small, because of the boundedness of

$$\frac{\partial x_0^-}{\partial \varepsilon}(0,\widehat{\xi}+x_0(\alpha)) - \frac{\partial x_0^+}{\partial \varepsilon}(0,\widehat{\eta}+x_0(\alpha)) + \widehat{r}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon)$$

on $\overline{O}_1 \times \overline{\Omega} \times [-\varepsilon_0,\varepsilon_0]$. If $\alpha \in \partial\Omega$ then $M(\alpha) \neq 0$. Since $|k_\varepsilon(t)| \geq |\varepsilon|$, we get

$$k_\varepsilon(t)\widehat{H}_2(t\widehat{\xi},t\widehat{\eta},\varepsilon,\alpha) \neq 0$$

provided O_1 and $|\varepsilon| \neq 0$ are sufficiently small. So again $H(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon,t) \neq 0$ and the proof is finished. □

Lemma 4.2 gives the next result.

THEOREM 4.3. *Let O_1 be as in Lemma* 4.2. *Assume* (H1), (H2). *Then it follows that* $\deg(\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon),O_1\times\Omega,0) \neq 0$, *for any $\varepsilon \neq 0$ sufficiently small.*

PROOF. Lemma 4.2 implies

$$\deg(\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon,1),O_1\times\Omega,0) = \deg(\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon,0),O_1\times\Omega,0).$$

Now

$$\widehat{H}(\widehat{\xi},\widehat{\eta},\varepsilon,\alpha,0) = \begin{pmatrix} L(\widehat{\xi},\widehat{\eta}) \\ \operatorname{sgn}\varepsilon M(\alpha) \end{pmatrix}$$

and $L\colon W_\mu^\perp \to \mathcal{R}L$ is invertible. Thus

$$\deg(\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon,0),O_1\times\Omega,0) = \pm\deg(M,\Omega,0) \neq 0. \qquad \square$$

Possibly shrinking U_Ω, O^s and O^u and using similar arguments like in the proof of Lemma 4.2 along with assumption (H1), we get $\widetilde{H}(\widetilde{\xi},\widetilde{\eta},\varepsilon) \neq 0$ for any $\varepsilon \neq 0$ sufficiently small and $(\widetilde{\xi},\widetilde{\eta}) \in \overline{(U_\Omega \oplus O^s)\times(U_\Omega\oplus O^u)} \setminus \{(\alpha+\widehat{\xi},\alpha+\widehat{\eta}) \mid \alpha\in\Omega,\ (\widehat{\xi},\widehat{\eta})\in O_1\}$. Then, because of definition and the connectedness of U_Ω, we have

$$\begin{aligned}(4.5)\quad \#\Big(\widetilde{W}^s_{p(\varepsilon)}\cap U,\widetilde{W}^u_{p(\varepsilon)}\cap U\Big) &= \deg(H(\xi,\eta,\varepsilon),(\widetilde{W}^s_p\cap U)\times(\widetilde{W}^u_p\cap U),0)\\ &= \pm\deg(\widetilde{H}(\widetilde{\xi},\widetilde{\eta},\varepsilon),(U_\Omega\oplus O^s)\times(U_\Omega\oplus O^u),0)\\ &= \pm\deg(\widehat{H}(\widehat{\xi},\widehat{\eta},\alpha,\varepsilon),O_1\times\Omega,0)\neq 0.\end{aligned}$$

From (4.5) and Theorem 4.3 we finally obtain:

THEOREM 4.4. *Assume* (i), (ii), (H1) *and* (H2). *Then there exists $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| \leq \varepsilon_0$, it is nonzero the local intersection number of the stable and unstable manifolds of the hyperbolic fixed point of the map $x_{n+1} = f(x_n,\varepsilon)$ which is located near the fixed point p of the map $x_{n+1} = f(x_n,0)$.*

Consequently, Theorem 1.1 together with the assumptions (i), (ii), (H1) and (H2) imply chaos for $f(x,\varepsilon)$ with $\varepsilon \neq 0$ small.

Finally, when f and $x_n(\alpha), n \in \mathbb{Z}$ are all C^3-smooth, we note that from the implicit function theorem we usually get that the existence of a simple root α_0 of $M(\alpha)$ (i.e. $M(\alpha_0) = 0$ and $M_\alpha(\alpha_0)$ is invertible), implies the solvability of $H(\xi, \eta, \varepsilon) = 0$ for $\varepsilon \neq 0$ small. So we get a transversal homoclinic orbit of $f(x, \varepsilon)$ for $\varepsilon \neq 0$ small. Indeed, equating to zero the projection onto $\mathcal{R}Q$ of $\widehat{H}(\widehat{\xi}, \widehat{\eta}, \alpha, \varepsilon)$ we obtain the equation

$$\widehat{\xi} - \widehat{\eta} + \varepsilon \widehat{H}_1(\widehat{\xi}, \widehat{\eta}, \alpha, \varepsilon) = 0$$

which can be solved by means of the implicit function theorem and gives unique C^1-smooth functions $\widehat{\xi}(\alpha, \varepsilon)$, $\widehat{\eta}(\alpha, \varepsilon)$. Note that, because of uniqueness, we have $\widehat{\xi}(\alpha, 0) = \widehat{\eta}(\alpha, 0) = 0$. Plugging these solutions in the projection onto $\mathcal{N}Q$ of $\widehat{H}(\widehat{\xi}, \widehat{\eta}, \alpha, \varepsilon)$ we obtain the so called bifurcation function $B\colon \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^\mu$, $(\alpha, \varepsilon) \mapsto B(\alpha, \varepsilon)$, whose components $B_j(\alpha, \varepsilon)$ are:

$$\begin{aligned} B_j(\alpha, \varepsilon) = \psi_j(\alpha)^* \bigg[& \frac{\partial x_0^-}{\partial \varepsilon}(0, \Phi(\alpha + \widehat{\xi}(\alpha, \varepsilon)) \\ & - \frac{\partial x_0^+}{\partial \varepsilon}(0, \Phi(\alpha + \widehat{\eta}(\alpha, \varepsilon)) + r(\widehat{\xi}(\alpha, \varepsilon), \widehat{\eta}(\alpha, \varepsilon), \alpha, \varepsilon) \bigg]. \end{aligned}$$

Now, it is not difficult to see that, for $\varepsilon \to 0$, $B(\alpha, \varepsilon) \to M(\alpha)$, uniformly on compact sets. We conclude this section noting that the condition that $M(\alpha)$ has a simple zero at some α_0 is equivalent to the fact that the function

$$\widetilde{M}(\alpha) := \left(\psi_j(\alpha_0)^* \left[\frac{\partial x_0^+}{\partial \varepsilon}(0, x_0(\alpha)) - \frac{\partial x_0^-}{\partial \varepsilon}(0, x_0(\alpha)) \right] \right)_{j=1,\dots,\mu}$$

has α_0 as a simple zero. In fact both $M(\alpha_0) = 0$ and $\widetilde{M}(\alpha_0) = 0$ mean that $(\partial x_0^- / \partial \varepsilon)(0, \Phi(\alpha_0)) = (\partial x_0^+ / \partial \varepsilon)(0, \Phi(\alpha_0))$ and then the equality $M_\alpha(\alpha_0) = \widetilde{M}_\alpha(\alpha_0)$ easily follows from $\Phi(\alpha) = x_0(\alpha)$.

5. Concluding remarks

Remark 5.1. The diffeomorphism f of Theorem 1.1 has positive topological entropy. This follows from [3, Lemma 1.3].

Remark 5.2. Consider a C^1-smooth diffeomorphism $f\colon \mathcal{M} \to \mathcal{M}$ possessing two hyperbolic fixed points p_1 and p_2, $p_1 \neq p_2$. If $W^s_{p_1}$ and $W^u_{p_2}$, and $W^s_{p_2}$ and $W^u_{p_1}$, are topologically transversal, respectively, then we can prove a similar result for f like in Theorem 1.1.

Remark 5.3. Let $\mathcal{M}$ be a smooth symplectic surface, i.e., $\dim \mathcal{M} = 2$, with the symplectic area form ω. Let $f\colon \mathcal{M} \to \mathcal{M}$ be a smooth area-preserving diffeomorphism homotopic to identity and exactly symplectic, i.e. $f^*(\alpha) = \alpha + dS$ for some smooth function $S\colon \mathcal{M} \to \mathbb{R}$ and α is a differential one-form such that

$d\alpha = \omega$. Time-one-maps of 1-periodic Hamiltonian systems are such diffeomorphisms (see [9]). We note that any exactly symplectic map is also symplectic. If $\mathcal{M}$ is exactly symplectic, i.e. $\omega = d\alpha$, and simply connected then any symplectic map is also exactly symplectic. Assume that f has two hyperbolic fixed points p_1, p_2, $p_1 \neq p_2$. Let us suppose that $W^s_{p_1} \cap W^u_{p_2} \neq \emptyset$ and $W^s_{p_2} \cap W^u_{p_1} \neq \emptyset$. If $W^s_{p_1} \neq W^u_{p_2}$ and $W^s_{p_2} \neq W^u_{p_1}$, and $S(p_1) = S(p_2)$, then we can prove as in [9, Theorem 2.1] that $W^s_{p_1}$ and $W^u_{p_2}$, and $W^s_{p_2}$ and $W^u_{p_1}$, are topologically transversal, respectively. Hence Remark 5.2 gives a chaotic behaviour of f.

This remark can be applied to the results of [2, Section 5, p. 703]. More precisely, let us consider the equation

$$\ddot{u} + W(u,t) = 0, \tag{5.1}$$

where $W: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1-smooth and 1-periodic in t. Suppose that (5.1) has two different hyperbolic periodic solutions u_1 and u_2. Let $\phi: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the flow of $\dot{u} = v$, $\dot{v} = -W(u,t)$. Then $f(x,y) = \phi(x,y,1)$ is exactly symplectic by taking

$$\omega = dx \wedge dy, \quad \alpha = -y\,dx, \quad S(x,y) = -\int_0^1 L(\phi(x,y,s))\,ds,$$

where L is the Lagrangian of (5.1) given by $L(\phi) = \phi_2^2/2 - G(\phi_1,t)$, $\phi = (\phi_1,\phi_2)$, $\partial G/\partial u = W$. Clearly the periodic solutions u_1, u_2 induce hyperbolic fixed points $(u_1(0),\dot{u}_1(0)) = w_1$ and $(u_2(0),\dot{u}_2(0)) = w_2$ of f. Then $S(w_1) = S(w_2)$ is a part of the condition (1) of [2, Definition 2.1]. Hence w_1 and w_2 are on the same action level for f in the terminology of [9, Theorem 8.1]. S is naturally related to the action functional over $H^1_1 = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}) : u(t+1) = u(t) \text{ a.e. in } \mathbb{R}\}$ defined as $u \mapsto \int_0^1 (\dot{u}(s)^2/2 - G(u(s),s))\,ds$ on [2, p. 679].

Hence the assumptions of Section 5 of [2] imply the validity of Theorem 1.1, which is stronger than the results of Section 5 of [2]. On the other hand, the main results of [2] deal with equations like (5.1) under assumptions on u_1 and u_2 weaker than in this paper, namely u_1 and u_2 are not hyperbolic but they are the so-called consecutive minimizers, see [2, Definition 2.1]. Using variational methods, chaotic bumping solutions are shown to exist in [2]. Finally we note that for a C^1-smooth 1-periodic Hamiltonian system

$$\dot{x} = -\frac{\partial H}{\partial y}(x,y,t), \quad \dot{y} = \frac{\partial H}{\partial x}(x,y,t), \tag{5.2}$$

the time-one map is exactly symplectic with

$$\omega = dx \wedge dy, \quad \alpha = x\,dy,$$
$$S(x,y) = \int_0^1 (\psi_1(x,y,t)\dot{\psi}_2(x,y,t) - H(\psi(x,y,t)))\,dt,$$

where $\psi = (\psi_1, \psi_2)$ is the flow of (5.2). The action functional for (5.2) over H_1^1 is given by

$$(x,y) \mapsto \int_0^1 x(t)\dot{y}(t) - H(x(t), y(t), t)\, dt = \int_0^1 \frac{1}{2}\langle J\dot{u}(t), u(t)\rangle - H(u(t), t)\, dt,$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $u = (x, y)$ and $\langle \cdot\,, \cdot \rangle$ is the usual inner product on $\mathbb{R}^2$.

Remark 5.4. Consider the second order equation $\ddot{x} = g(x) + \varepsilon q(t)$, where $x \in \mathbb{R}$, g, q are C^2-smooth and q is 1-periodic. Suppose that the equation $\dot{x} = y$, $\dot{y} = g(x)$ has a homoclinic solution $(p(t), \dot{p}(t))$ to a hyperbolic fixed point. Then the Melnikov function has the form $M(\alpha) = \int_{-\infty}^{\infty} q(t+\alpha)\dot{p}(t)\, dt$, see [1]. $M(\alpha)$ is 1-periodic and $\int_0^1 M(\alpha)\, d\alpha = 0$. Hence if $M \neq 0$ then it changes the sign on $[0, 1]$ and Theorem 4.4 can be applied.

Remark 5.5. We claim that periodic points of f are dense in the set Λ_ω. Let $x_0 \in \Lambda_\omega$. Then there is an $E \in \mathcal{E}$ such that $\mathcal{P}(J_\omega(E)) = x_0$ where $J_\omega(E) = \{x_j\}_{j\in\mathbb{Z}}$. Let $E = \{e_j\}_{j\in\mathbb{Z}}$. If E is periodic then x_0 is a periodic point of f. Let E be non-periodic. There are unique $E' \in \mathcal{E}_\sim$ and $n_0 \in \mathbb{Z}$ such that $E = \sigma^{n_0}(E')$. Now E' is also non-periodic. We have $J_\omega(E') = f^{-2n_0\omega}(J_\omega(E))$. The point $x_0' = \mathcal{P}(J_\omega(E'))$ can be approximated by the proof of Theorem 1.1 with periodic points of f from Λ_ω. Of course the same hold for the point $x_0 = f^{2n_0\omega}(x_0')$. This gives the claim. We also get that the only isolated points of the set Λ_ω could be periodic points of f and f depends sensitively on the set Λ_ω' of all non-isolated points of Λ_ω, that is there is a constant $d > 0$ such that in any neighbourhood of $x_0 \in \Lambda_\omega'$ there are $x_0' \in \Lambda_\omega$ and $n_0' \in \mathbb{N}$ such that the distance between $f^{n_0'}(x_0)$ and $f^{n_0'}(x_0')$ is greater than d. We do not know whether the periodic points of f in Λ_ω are non-isolated or not. On the other-hand, let either $\Upsilon_\omega = \Lambda_\omega$ or $\Upsilon_\omega = \Lambda_\omega'$. We extend the map π_ω on the closure $\overline{\Upsilon}_\omega$ of Υ_ω. So $\overline{\Upsilon}_\omega$ is compact but we do not know whether the unique continuous extension of π_ω is one-to-one or not. The extension is made as follows: For any $x_0 \in \overline{\Upsilon}_\omega \setminus \Upsilon_\omega$, we take a sequence $\{x_j\}_{j\in\mathbb{N}} \subset \Upsilon_\omega$ such that $x_j \to x_0$. Hence $f^{2\omega k}(x_j) \to f^{2\omega k}(x_0)$. Consequently, for any $N \in \mathbb{N}$, the orbits $\{f^{2\omega k}(x_j)\}_{k=-N}^{k=N}$ and $\{f^{2\omega k}(x_0)\}_{k=-N}^{k=N}$ have the same oscillating properties between set K and point p for j large. This implies the existence of the limit $\lim_{j\to\infty} \pi_\omega(x_j) := \pi_\omega(x_0)$, which is independent of $\{x_j\}_{j\in\mathbb{N}}$. The continuity of π_ω follows as in the proof of Theorem 1.1. Clearly the extension π_ω is onto $\mathcal{E}$ for the case $\Upsilon_\omega = \Lambda_\omega$. If $\Upsilon_\omega = \Lambda_\omega'$ then $\pi_\omega(\Upsilon_\omega)$ is dense in $\mathcal{E}$ and $\pi_\omega(\overline{\Upsilon}_\omega)$ is compact in $\mathcal{E}$. This implies $\pi_\omega(\overline{\Upsilon}_\omega) = \mathcal{E}$ also for this case. The property $\pi_\omega \circ f^{2\omega} = \sigma \circ \pi_\omega$ follows from the limit procedure $x_j \to x_0$. Of course, $\overline{\Upsilon}_\omega$ is invariant for $f^{2\omega}$. For the case $\Upsilon_\omega = \Lambda_\omega$, we again have infinitely many periodic points of f which are dense in $\overline{\Upsilon}_\omega$. For the case $\Upsilon_\omega = \Lambda_\omega'$, we have that any point $x_0 \in \overline{\Lambda_\omega'}$ is an accumulating point of periodic points of f with periods tending to infinity. Iterations of those periodic points oscillate differently between the set K and the point p. Consequently, the map f is sensitive on $\overline{\Lambda'_\omega}$ in the following sense: there is a constant $d > 0$ such that in any neighbourhood of $x_0 \in \overline{\Lambda'_\omega}$ there are x'_0 and $n'_0 \in \mathbb{N}$ such that the distance between $f^{n'_0}(x_0)$ and $f^{n'_0}(x'_0)$ is greater than d.

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Manuscript received November 5, 2001

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