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CONTINUITY OF ATTRACTORS FOR NET-SHAPED THIN DOMAINS

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Abstract. Consider a reaction-diffusion equation $u_t = \triangle u + f(u)$ on a family of net-shaped thin domains Ω_ε converging to a one dimensional set as $\varepsilon \downarrow 0$. With suitable growth and dissipativeness conditions on f these equations define global semiflows which have attractors $\mathcal{A}_\varepsilon$. In [4] it has been shown that there is a limit problem which also defines a semiflow having an attractor $\mathcal{A}_0$, and the family of attractors is upper-semi-continuous at $\varepsilon = 0$. Here we show that under a stronger dissipativeness condition the family of attractors $\mathcal{A}_\varepsilon$, $\varepsilon \geq 0$, is actually continuous at $\varepsilon = 0$.

1. Introduction

Consider domains Ω_ε depending on a parameter $\varepsilon > 0$. On Ω_ε we have a reaction-diffusion equation with Neumann boundary condition

$$u_t = \triangle u + f(u) \quad \text{in } \Omega_\varepsilon, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{1.1}$$

This equation generates a dynamical system if we impose suitable growth and dissipativeness conditions on the non linearity f. Then equation (1.1) induces a semiflow π_ε on some functional space, and this semiflow has an attractor $\mathcal{A}_\varepsilon$. Many authors have asked and answered questions regarding the existence of a limiting dynamical system, as $\varepsilon \to 0$. E.g. if there is a equation which induces a semiflow π_0 with attractor $\mathcal{A}_0$, such that the semiflows π_ε and attractors $\mathcal{A}_\varepsilon$

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converge in some sense. We are interested principally in the case that the domains Ω_ε are thin domains, that is they are squeezed in some sense as $\varepsilon \to 0$, collapsing to a lower dimensional set. Among others, Hale and Raugel in [8] and [9], Prizzi and Rybakowski in [14], Prizzi, Rinaldi and Rybakowski in [13] and Elsken in [4] have shown for this type of singular perturbations that a limiting semiflow π_0 exists, it has an attractor $\mathcal{A}_0$, π_ε converges to π_0 in some sense, and the family of attractors $\mathcal{A}_\varepsilon$, $\varepsilon \geq 0$, is upper-semi-continuous at $\varepsilon = 0$. Under the assumption that eigenvalues and eigenvectors converge, and some mild geometrical condition, Arrieta and Carvalho show in [2] that the attractors are even continuous.

We want to extend the results of [2] to the case of squeezed domains. That is we will prove the continuity of the attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$ for the case of thin net-shaped domains. The results in [2] do not include the case where $|\Omega_\varepsilon| \to 0$, in particular they do not apply to the case of squeezed domains.

The fundamental idea we use is the same as in [2], but due to the singular perturbation of collapsing domains there are additional difficulties which have to be overcome. Roughly the argument is as follows. One knows that the attractors are upper-semi-continuous, and shows that the same is true for all points of equilibrium of the semiflows π_ε. We assume that there are only finitely many of these points and that 0 is not in the spectrum of the linearization around each point of equilibrium for the limit flow. Then the same holds for π_ε for $\varepsilon > 0$ small, and the points of equilibrium are continuous at $\varepsilon = 0$. Any point in $\mathcal{A}_0$ which is not a point of equilibrium has to lie on a trajectory which is in the unstable manifold of some point of equilibrium of π_0. Unlike in [2] we use fixed points on spaces of functions with exponential growth to construct the unstable manifolds (see e.g. Schneider [16], Fischer [5] and Rybakowski [15]). We show that given a trajectory $\pi_0(\,\cdot\,, u_0)$ converging exponentially as $t \to -\infty$ to a point of equilibrium of π_0, for $\varepsilon > 0$ small there are trajectories $\pi_\varepsilon(\,\cdot\,, u_\varepsilon)$ converging exponentially (as $t \to -\infty$) to some point of equilibrium of π_ε, and the $\pi_\varepsilon(\,\cdot\,, u_\varepsilon)$ themselves converge (as $\varepsilon \to 0$) in some sense to $\pi_0(\,\cdot\,, u_0)$. This then gives the continuity of the attractors.

Our technique works also in the other cases of thin domains mentioned above. We consider here only the case of net-shaped ones because this is the most general case. Also it presents some features which give rise to technical difficulties which are not present in the remaining cases.

The most important one is related to the weaker convergence we have for this case: in [14] and other papers the semiflows converge with respect to the family of norms $\|A_\varepsilon^{1/2} \cdot\|_{L^2}$, that is the natural norms of fractional power spaces induced by the abstract linear operator of equation (1.1). For net-shaped thin domains this is not true in general, and one has to introduce a second family of norms (defined in (1.3)) for the convergence of the semiflows and attractors.

We will now state our main result. Unfortunately the exact definition of net-shaped domains is rather lengthy, so we shall postpone it to the next section and give only the essential features here.

We assume $\Omega_\varepsilon \subset \mathbb{R}^{M+1}$, $\varepsilon > 0$, $M \in \mathbb{N}$ fixed, to be C^2, bounded, and to consist of K_E edges and K_N nodes:

$$\Omega_\varepsilon = \bigcup_{j=1}^{K_E} \Omega_{\varepsilon,j} \cup \bigcup_{j=K_E+1}^{K_E+K_N} \Omega_{\varepsilon,j},$$

$K_E, K_N \in \mathbb{N}$. All edges and nodes may have holes or multiple branches. Roughly speaking the edges converge to curves and the nodes to points, as $\varepsilon \to 0$. Each edge $\Omega_{\varepsilon,j}$, $j = 1, \ldots, K_E$, is the transformation of a fixed bounded, Lipschitz domain G_j via a map $\Psi_{\varepsilon,j}$. These maps $\Psi_{\varepsilon,j}$ have a special structure and satisfy $|\det D\Psi_{\varepsilon,j}| \leq C\varepsilon^M$. For each node $\Omega_{\varepsilon,j}$, $j = K_E + 1, \ldots, K_E + K_N$, there are bounded $G_{\varepsilon,j}$ and a bijection $\Psi_{\varepsilon,j}\colon G_{\varepsilon,j} \to \Omega_{\varepsilon,j}$ such that $D\Psi_{\varepsilon,j} = \varepsilon E_{M+1}$, where the latter is the unit matrix. Also, each $G_{\varepsilon,j}$ is transformed by bounded diffeomorphisms onto a finite number of fixed domains.

We identify $H^1(\Omega_\varepsilon)$ and $L^2(\Omega_\varepsilon)$ with spaces

$$H_\varepsilon \subset \prod_{j=1}^{K_E} H^1(G_j) \times \prod_{j=K_E+1}^{K_E+K_N} H^1(G_{\varepsilon,j}),$$
$$L_\varepsilon \subset \prod_{j=1}^{K_E} L^2(G_j) \times \prod_{j=K_E+1}^{K_E+K_N} L^2(G_{\varepsilon,j}),$$

respectively (see (2.2), (2.3) below).

The nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ is C^2. We impose two conditions on it:

- (H1) $|f'(s)| \leq C(|s|^{\beta_1}+1)$ for all $s \in \mathbb{R}$, where $C, \beta_1 \geq 0$ are constants; if $M > 1$, then additionally $\beta_1 \leq p^*/2 - 1$, where $p^* = 2(M+1)/(M-1) > 2$.
- (H2) $\limsup_{|s|\to\infty} f(s)/s|s|^{\beta_2} \leq -\xi$, for some $\xi > 0$ and $\beta_2 > 0$.

In this paper we will always impose condition (H1) on f. Condition (H2) will be needed for our central result and in part of section three (see Proposition 3.3). Throughout this paper we shall assume at least the following weaker version of (H2) on f:

- (H2') $\limsup_{|s|\to\infty} f(s)/s \leq -\xi$, for some $\xi > 0$.

It is well known that under these assumptions equation (1.1) can be written as an abstract equation

$$u_t = -A_\varepsilon u + f_\varepsilon(u) \quad t > 0, \tag{1.2}$$

where $A_\varepsilon\colon D(A_\varepsilon) \subset L_\varepsilon \to L_\varepsilon$ is a sectorial operator and $f_\varepsilon\colon H_\varepsilon \to L_\varepsilon$ is the Nemitsky operator of f. (1.2) induces a semiflow π_ε on H_ε, and this semiflow has a global attractor $\mathcal{A}_\varepsilon \subset H_\varepsilon$ (see e.g. [4]).

We need a few notations regarding the limit semiflow. Write (x,y), $x \in \mathbb{R}$, $y \in \mathbb{R}^M$, for a generic point of $\mathbb{R}^{M+1}$, and set $H^1_s(U) := \{u \in H^1(U) : D_y u = 0\}$, $L^2_s(U)$ the closure of $H^1_s(U)$ in $L^2(U)$, for a domain $U \subset \mathbb{R}^{M+1}$. Denote by f_0 and Df_0 the Nemitsky operators of f and f' on $\prod_{j=1}^{K_E} H^1_s(G_j)$, respectively. Define norms $|\cdot|_{\varepsilon,d}$, $\varepsilon > 0$, $0 \le d \le 1$, on H_ε by

$$
(1.3)\quad |(u_1,\dots,u_{K_E+K_N})|^2_{\varepsilon,d} := \sum_{j=1}^{K_E} \|u_j\|^2_{L^2(G_j)} + \|D_x u_j\|^2_{L^2(G_j)} + \frac{1}{\varepsilon^{2d}}\|D_y u_j\|^2_{L^2(G_j)} + \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \|u_j\|^2_{L^2(G_{\varepsilon,j})} + \frac{1}{\varepsilon^2}\|Du_j\|^2_{L^2(G_{\varepsilon,j})}.
$$

For $\varepsilon = 0$ set

$$
|(u_1,\dots,u_{K_E})|^2_{0,d} := \sum_{j=1}^{K_E} \|u_j\|^2_{H^1(G_j)}.
$$

In [4] it is shown that there are linear spaces

$$
H_0 \subset \prod_{j=1}^{K_E} H^1_s(G_j), \quad L_0 = \prod_{j=1}^{K_E} L^2_s(G_j),
$$

a linear embedding $\Phi^H_\varepsilon\colon H_0 \to H_\varepsilon$ and a sectorial operator $A_0\colon D(A_0) \subset L_0 \to L_0$ such that

$$
(1.4)\qquad u_t = -A_0 u + f_0(u) \quad \text{for } t > 0
$$

induces a semiflow π_0 on H_0 which has an attractor $\mathcal{A}_0 \subset H_0$. As $\varepsilon \to 0$, π_ε converges to π_0 with respect to the family of norms $|\cdot|_{\varepsilon,d}$ for $d < 1$, and the family of attractors $\mathcal{A}_\varepsilon$, $\varepsilon \ge 0$, is upper-semicontinuous at $\varepsilon = 0$ (see Theorem 2.2).

In this article we prove the continuity of these attractors. That is we show

Theorem 1.1. *Assume Ω_ε satisfy the conditions of Section 2 and f (H1), (H2) above. Assume also that the limit semiflow π_0 has only finitely many points of equilibrium, say $\{u^0_1,\dots,u^0_{M_0}\} \subset H_0$, and 0 is not in the spectrum of the linear operators*

$$
(1.4)\qquad A_0 - Df_0(u^0_j)\mathrm{id}\colon D(A_0) \to L_0 \quad \textit{for all } j = 1,\dots,M_0.
$$

Then the family of attractors $\mathcal{A}_\varepsilon$, $\varepsilon \ge 0$, is continuous at $\varepsilon = 0$, i.e. for $0 \le d < 1$

$$
\lim_{\varepsilon\to 0} \operatorname{dist}_{\varepsilon,d}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0,
$$

where for $U \subset H_\varepsilon$, $V \subset H_0$ we define

$$\operatorname{dist}_{\varepsilon,d}(U,V) := \sup_{u\in U}\inf_{v\in V} |u - \Phi_\varepsilon^H v|_{\varepsilon,d} + \sup_{v\in V}\inf_{u\in U} |u - \Phi_\varepsilon^H v|_{\varepsilon,d}.$$

In section three we prove the attractors $\mathcal{A}_\varepsilon$, $\varepsilon \geq 0$, to be bounded uniformly in L^∞ (Proposition 3.3). Thus for $u \in \mathcal{A}_0$ we have $Df_0(u)\colon L_0 \to L_0$ and (1.5) in the theorem above makes sense.

This paper is organized as follows. In section two we present our notations, define net-shaped domains and state some results of [4]. In section three we prove the boundedness of the attractors in L^∞ and some auxiliary results we shall need in the next section. There we prove Theorem 1.1.

2. Notations and assumptions on Ω_ε

In this section we will present our notations and state the exact requirements on the domains Ω_ε. We will also bring some results of [4] we shall need.

In the rest of this paper ε will always — unless stated otherwise — denote a number in $]0,1]$.

$M \in \mathbb{N}$ is a fixed positive natural number. We will write (x,y) for a generic point in $\mathbb{R}\times\mathbb{R}^M = \mathbb{R}^{M+1}$. Let $U \subset \mathbb{R}^{M+1}$ then $\operatorname{proj}_x(U)$ is the projection onto the first coordinate.

As in [14], [4] and other papers here also the set of functions on an open set $\Omega \subset \mathbb{R}^{M+1}$ which have derivative 0 in y-direction plays an important role. We define

$$H_s^1(\Omega) := \{u \in H^1(\Omega) : D_y u = 0\}, \quad L_s^2(\Omega) := \overline{H_s^1}^{L^2(\Omega)}(\Omega).$$

$L_s^2(\Omega)$ is a closed subset of $L^2(\Omega)$, hence the orthogonal complement exists. Denote it by $L_\perp^2(\Omega)$.

For $n \in \mathbb{N}$ we denote by $E_n \in \mathbb{R}^{n\times n}$ the unit-matrix and for a vector $x \in \mathbb{R}^n$ $\|x\|$ denotes the Euclidian norm.

Let V be a normed space, $z \in V$ and $\delta > 0$. Then $B_\delta(z) \subset V$ denotes the open ball around z with radius δ.

If $U \subset \mathbb{R}^n$ then $|U|$ is the Lebesgue-measure of U. The closure will be denoted by $\overline{U}$.

We will use the notation $u_0\pi t$ for semiflows $\pi(t,u_0) = u(t)$, u solution of some (abstract) differential equation with initial value u_0.

In proofs we shall often substitute an index ε_n by the simpler n. For example A_{ε_n}, ${H^1}_{\varepsilon_n}$ and $\|\cdot\|_{\varepsilon_n,d}$ will be A_n, ${H^1}_n$ and $\|\cdot\|_{n,d}$. Also we shall assume constants $C_1, C_2, \ldots$ to be independent of ε. If they depend on ε we shall always indicate this, writing $C(\varepsilon)$, or $C(n)$ if $\varepsilon = \varepsilon_n$.

We will start defining the domain Ω_ε which, as already mentioned, will be net like and consists of $K_E \in \mathbb{N}$ edges and $K_N \in \mathbb{N}$ nodes. More in detail we assume

$\Omega_\varepsilon \subset \mathbb{R}^{M+1}$ to be bounded, connected and C^2. $\Omega_\varepsilon = \bigcup_{j=1}^{K_E} \Omega_{\varepsilon,j} \cup \bigcup_{j=K_E+1}^{K_E+K_N} \Omega_{\varepsilon,j}$, where the $\Omega_{\varepsilon,j}$ are mutually disjoint and satisfy the following.

The edges $\Omega_{\varepsilon,j}$, $j = 1, \dots, K_E$, have a description

$$\Omega_{\varepsilon,j} = \Psi_{\varepsilon,j}(G_j),$$

where $G_j \subset \mathbb{R} \times \mathbb{R}^M$ is open, bounded, connected and Lipschitz. To facilitate notation we assume $\mathrm{proj}_x(G_j) =]0,1[$.

The transformation $\Psi_{\varepsilon,j}: \overline{G_j} \to \Psi_{\varepsilon,j}(\overline{G_j}) \supset \Omega_{\varepsilon,j}$ is a C^1-diffeomorphism $T_{\varepsilon,j}$ which is near to the identity, followed by a contraction S_ε in y-direction and a C^1-diffeomorphism T_j which is independent of ε:

$$\Psi_{\varepsilon,j} = T_j \circ S_\varepsilon \circ T_{\varepsilon,j}.$$

Here $T_{\varepsilon,j}: Q_{1,j} \supset \overline{G_j} \to T_{\varepsilon,j}(\overline{G_j}) \subset Q_{2,j}$ is a C^1-diffeomorphism, $Q_{1,j}, Q_{2,j} \subset \mathbb{R}^{M+1}$ fixed, open, bounded sets. $S_\varepsilon(x,y) := (x, \varepsilon y)$ and $T_j: \widetilde{Q}_j \to T_j(\widetilde{Q}_j) \subset \mathbb{R}^{M+1}$ is again a C^1-diffeomorphism, $\widetilde{Q}_j \supset \overline{\bigcup_{0\le\varepsilon\le1} S_\varepsilon(T_{\varepsilon,j}(G_j))}$ open. Roughly speaking $T_{\varepsilon,j}$ is there to give some liberty choosing the nodes, S_ε is the normal squeezing, and T_j moves an edge into the right position (i.e. to $[0,1] \times \mathbb{R}^M$), eventually scaling and deforming it in a way independent of ε.

We want an edge to touch a node only at the sides corresponding to $(\{0\} \times \mathbb{R}^M) \cap \overline{G_j}$ or $(\{1\} \times \mathbb{R}^M) \cap \overline{G_j}$, so we assume

$$\emptyset \neq {\Psi_{\varepsilon,j}}^{-1}(\Omega_{\varepsilon,i} \cap \overline{\Omega_{\varepsilon,j}}) \subset \{0\} \times \mathbb{R}^M,$$

if the edge $\Omega_{\varepsilon,j}$ begins at the node $\Omega_{\varepsilon,i}$, or

$$\emptyset \neq {\Psi_{\varepsilon,j}}^{-1}(\Omega_{\varepsilon,i} \cap \overline{\Omega_{\varepsilon,j}}) \subset \{1\} \times \mathbb{R}^M,$$

if the edge $\Omega_{\varepsilon,j}$ ends at the node $\Omega_{\varepsilon,i}$, for all possible i, j.

We assume also that any edge may only begin or end at a given node, but not both.

Each of the nodes $\Omega_{\varepsilon,j}$, $j = K_E+1, \dots, K_E+K_N$, converges to a one point set, say $\Omega_{0,j} = \{z_{0,j}\} \in T_i(\widetilde{Q}_i) \subset \mathbb{R}^{M+1}$, for all edges $\Omega_{\varepsilon,i}$ which either start or end at the node $\Omega_{\varepsilon,j}$.

We assume the node $\Omega_{\varepsilon,j}$, $j = K_E+1, \dots, K_E+K_N$, has a description $\Omega_{\varepsilon,j} = \Psi_{\varepsilon,j}(G_{\varepsilon,j})$, where $\Psi_{\varepsilon,j}(z) = \varepsilon z + z_{\varepsilon,j}$, $z_{\varepsilon,j} \to z_{0,j}$ as $\varepsilon \to 0$.

Note that since all edges are open, each node is closed in Ω_ε. It may even have empty interior.

Throughout this article we put the following additional conditions (C1)–(C8) on G_j, $T_{\varepsilon,j}$, T_j and $G_{\varepsilon,i}$, where always $j = 1, \dots, K_E$, $i = K_E+1, \dots, K_E+K_N$.

(C1) $\overline{G_j} \cap (\{0\} \times \mathbb{R}^M)$ or $\overline{G_j} \cap (\{1\} \times \mathbb{R}^M)$ has finitely many connected components with positive M-dimensional measure, if the edge G_j begins or ends at some node, respectively.

(C2) There are at most countably many open, connected, pairwise disjoint $U^{j,l} \subset G_j$, $l \in I_j$, such that each $U^{j,l}$ has connected x-crossections and $E := \{x \in \mathbb{R} :$ there exists $y \in \mathbb{R}^M,\ (x,y) \in G_j \setminus \bigcup_{l \in I_j} U^{j,l}\}$ has at most finitely many points of accumulation.

(C3) $T_{\varepsilon,j}(x,y) \to (x,y)$, $\varepsilon \to 0$, pointwise for all $(x,y) \in \overline{G_j}$, and if $(T_{\varepsilon,j})_x$ denotes the x-component of $T_{\varepsilon,j}$, then $(T_{\varepsilon,j})_x \to \operatorname{proj}_x|_{\overline{G_j}}$ uniformly on $\overline{G_j}$.

(C4) There is a $C > 0$ such that, for all $\varepsilon \le 1$, $v \in \mathbb{R}^{M+1}$, $\|v\| = 1$,

$$\sup_{(x,y)\in\overline{G_j}} \|DT_{\varepsilon,j}(x,y)v\|, \qquad \sup_{(x,y)\in T_{\varepsilon,j}(\overline{G_j})} \|DT_{\varepsilon,j}{}^{-1}(x,y)v\| < C.$$

(C5) Define $\mathcal{T}_{\varepsilon,j}$, $\mathcal{T}^*_{\varepsilon,j}$ by

$$\begin{aligned} DT_{\varepsilon,j}(x,y) &= E_{M+1} - \mathcal{T}_{\varepsilon,j}(x,y), \\ (DT_{\varepsilon,j}(x,y))^{-1} &= E_{M+1} + \mathcal{T}^*_{\varepsilon,j}(x,y). \end{aligned}$$

Denote the elements of these matrix-functions by $\mathcal{T}_{\varepsilon,j,l,k}$ and $\mathcal{T}^*_{\varepsilon,j,l,k}$, $l,k = 0,\dots,M$. We assume

$$\sup_{0<\varepsilon\le 1,(x,y)\in G_j} \left(\frac{1}{\varepsilon}|\mathcal{T}_{\varepsilon,j,0,l}(x,y)|, \frac{1}{\varepsilon}|\mathcal{T}^*_{\varepsilon,j,0,l}(x,y)|\right) < \infty,$$
$$\lim_{\varepsilon\to 0} \mathcal{T}_{\varepsilon,j}(x,y) = \lim_{\varepsilon\to 0} \mathcal{T}^*_{\varepsilon,j}(x,y) = 0,$$

and there are maps $\mathcal{T}_j = (\mathcal{T}_{j,1},\dots,\mathcal{T}_{j,M}) \colon \overline{G_j} \to \mathbb{R}^M$ such that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathcal{T}_{\varepsilon,j,0,l}(x,y) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathcal{T}^*_{\varepsilon,j,0,l}(x,y) = \mathcal{T}_{j,l}(x,y)$$

for all $(x,y) \in G_j$, $l = 1,\dots,M$.

(C6) $G_{\varepsilon,i}$ is bounded independent of ε, i.e. there is a positive R_Ω such that $G_{\varepsilon,i} \subset B_{R_\Omega}(0)$ for all $0 < \varepsilon \le 1$.

(C7) Ω_ε is nicely connected, that is Ω_ε connects nicely at all edges.
We say Ω_ε connects nicely at the node $G_{\varepsilon,i}$ if the following is satisfied. There are $\delta, C > 0$, and for all edges G_k which begin or end at the node $G_{\varepsilon,i}$ there are open, connected, Lipschitz, pairwise disjoint $G_{i,k,l} \subset G_k$, connected $\omega_{i,k,l,x} \subset \mathbb{R}^M$, $|\omega_{i,k,l,x}| \ge \delta$ for all $x \in I_{i,k}$, where $I_{i,k} =]0,\delta[$ if G_k begins, $I_{i,k} =]1-\delta,1[$ if it ends at $G_{\varepsilon,i}$, such that

$$\begin{aligned} G_{i,k,l} &= \bigcup_{x\in I_{i,k}} \{x\} \times \omega_{i,k,l,x}, \\ G_k \cap (I_{i,k} \times \mathbb{R}^M) &= \bigcup_{l=1}^{L_{i,k}} G_{i,k,l} \end{aligned}$$

for all possible l, k. Set $S_\Omega := \{(i,k,l) : i = K_E+1,\dots,K_E+K_N,\ l = 1,\dots,L_{i,k},\ k = 1,\dots,K_E,\ G_k$ begins or ends at $G_{\varepsilon,i}\}$.

If there are an $\varepsilon_1 > 0$, $(i, k_m, l_m) \in S_\Omega$, $(x_m, y_m) \in \partial G_{i,k_m,l_m}$, $m = 1, 2$, such that $\Psi_{\varepsilon_1,k_1}(x_1, y_1)$ and $\Psi_{\varepsilon_1,k_2}(x_2, y_2)$ belong to the same connected component of Ω_{i,ε_1}, then there are an open, connected, bounded, Lipschitz $U = U_{i,k_1,l_1,k_2,l_2} \subset {\Psi_{\varepsilon,i}}^{-1}(\Omega_\varepsilon)$, $r > 0$, both independent of ε, and open

$$U_\varepsilon = U_{\varepsilon,i,k_m,l_m} = B_r(z_{\varepsilon,i,k_m,l_m}) \subset U \cap {\Psi_{\varepsilon,i}}^{-1} \circ \Psi_{\varepsilon,k_m}(G_{i,k_m,l_m}),$$
$${\Psi_{\varepsilon,k_m}}^{-1} \circ \Psi_{\varepsilon,i}(U_\varepsilon) \subset (]0, \varepsilon C[\times \mathbb{R}^M)$$

if G_{k_m} begins, and

$${\Psi_{\varepsilon,k_m}}^{-1} \circ \Psi_{\varepsilon,i}(U_\varepsilon) \subset (]1 - \varepsilon C, 1[\times \mathbb{R}^M)$$

if G_{k_m} ends at $G_{\varepsilon,i}$, for all ε and $m = 1, 2$.

(C8) One of the following holds:

(i) $G_{i,\varepsilon}$ has empty interior for all $\varepsilon > 0$.

(ii) There are $G_{i,1}, \dots, G_{i,N_i} \subset \mathbb{R}^{M+1}$ open, bounded, connected, Lipschitz, $C > 0$, $\overline{G_{i,k}} \subset Q_k \subset \mathbb{R}^{M+1}$ open, $\Psi_{\varepsilon,i,k}: Q_k \to \Psi_{\varepsilon,i,k}(Q_k) \subset \mathbb{R}^{M+1}$ C^1-diffeomorphisms, $\Psi_{\varepsilon,i,k}(G_{i,k}) \subset G_{\varepsilon,i}$,

$$\frac{1}{C} \leq |\det D\Psi_{\varepsilon,i,k}(z)|, \quad \|D\Psi_{\varepsilon,i,k}(z)v\|, \leq C,$$
$$\left| G_{\varepsilon,i} \setminus \bigcup_{k=1}^{N_i} \Psi_{\varepsilon,i,k}(G_{i,k}) \right| = 0,$$

for all possible z, k, ε and $v \in \mathbb{R}^{M+1}$, $\|v\| = 1$. For all $k \in \{1, \dots, N_i\}$ exist $l \in \{1, \dots, K_E\}$, open, bounded, connected, Lipschitz $U_{i,k} \subset {\Psi_{\varepsilon,i}}^{-1}(\Omega_{\varepsilon,i} \cup \Omega_{\varepsilon,l})$, $r > 0$, all independent of ε, and open $U_\varepsilon = U_{\varepsilon,i,k,l} = B_r(z_{\varepsilon,i,k,l}) \subset U_{i,k} \cap {\Psi_{\varepsilon,i}}^{-1}(\Omega_{\varepsilon,l})$, such that $|{\Psi_{\varepsilon,i,k}}^{-1}(U_{i,k} \cap \Psi_{\varepsilon,i,k}(G_{i,k}))| \geq 1/C$, ${\Psi_{\varepsilon,l}}^{-1} \circ \Psi_{\varepsilon,i}(U_\varepsilon) \subset]0, \varepsilon C[\times \mathbb{R}^M$ if G_l begins, ${\Psi_{\varepsilon,l}}^{-1} \circ \Psi_{\varepsilon,i}(U_\varepsilon) \subset]1 - \varepsilon C, 1[\times \mathbb{R}^M$ if it ends at $G_{\varepsilon,i}$, for all ε.

Proposition 3.1 in [4] states, that if (C1)–(C8) hold, then the following two conditions hold too:

(C9) Define H_0 as the set of all $[u] = [u_1, \dots, u_{K_E}] \in H_s^1(G_1) \times \dots \times H_s^1(G_{K_E})$ such that there are a constant $\beta > 0$, a sequence $\varepsilon_n \downarrow 0$ (both dependent on $[u]$), and $\widehat{u}_n \in H^1(\Omega_{\varepsilon_n})$ such that $\widehat{u}_n \circ \Psi_{\varepsilon_n,k} \rightharpoonup u_k$ weakly in $H^1(G_k)$, $k = 1, \dots, K_E$,

$$\sum_{k=1}^{K_E} \frac{1}{\varepsilon_n} \|D_y(\widehat{u}_n \circ \Psi_{\varepsilon_n,k})\|_{L^2(G_k)} + \sum_{k=K_E+1}^{K_E+K_N} \varepsilon_n \|\widehat{u}_n \circ \Psi_{\varepsilon_n,k}\|^2_{L^2(G_{\varepsilon_n,k})}$$
$$+ \frac{1}{\varepsilon_n} \|D(\widehat{u}_n \circ \Psi_{\varepsilon_n,k})\|^2_{L^2(G_{\varepsilon_n,k})} < \beta.$$

We assume H_0 is a closed subspace of $H^1_s(G_1) \times \ldots \times H^1_s(G_{K_E})$, and for every $\varepsilon > 0$ there is a linear map

$$\Phi^H_\varepsilon\colon H_0 \to H^1(G_1) \times \ldots \times H^1(G_{K_E}) \times H^1(G_{\varepsilon,K_E+1}) \times \ldots \times H^1(G_{\varepsilon,K_E+K_N})$$

and a constant $C > 0$, independent of ε, such that for all $[u] = [u_1, \ldots, u_{K_E}]$ we have $(\Phi^H_\varepsilon[u])_k = u_k$, $k = 1, \ldots, K_E$. Also

$$(2.1)\quad C \sum_{k=1}^{K_E} \|u_k\|^2_{H^1(G_k)} \geq \sum_{k=K_E+1}^{K_E+K_N} \varepsilon \|(\Phi^H_\varepsilon[u])_k\|^2_{L^2(G_{\varepsilon,k})} + \frac{1}{\varepsilon}\|D(\Phi^H_\varepsilon[u])_k\|^2_{L^2(G_{\varepsilon,k})} \to 0,$$

as $\varepsilon \to 0$, and $\widehat{u}_\varepsilon$ defined by

$$\widehat{u}_\varepsilon := (\Phi^H_\varepsilon[u])_k \circ {\Psi_{\varepsilon,k}}^{-1} \quad \text{on } \Omega_{\varepsilon,k},\ k = 1, \ldots, K_E + K_N,$$

is a function in $H^1(\Omega_\varepsilon)$ (i.e. $\Phi^H_\varepsilon[u]$ comes from the H^1-function $\widehat{u}_\varepsilon$ via the transformations $\Psi_{\varepsilon,k}$).

(C10) If $C > 0$, $\varepsilon_n \to 0$, $[u_n] \in {H^1}_{\varepsilon_n}$, $|[u_n]|_{\varepsilon_n,1} \leq C$ and $\|[u_n]\|_{{L^2}_{\varepsilon_n}} = 1$ for all n, then

$$\varepsilon_n \sum_{k=K_E+1}^{K_E+K_N} \|u_{n,k}\|^2_{L^2(G_{\varepsilon_n,k})} \to 0, \quad \text{as } n \to \infty.$$

To simplify notations we set $\Phi^H_0 := \mathrm{id}$ on H_0.

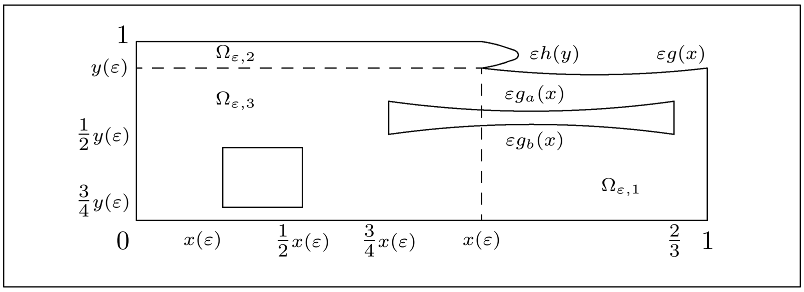

Figure 1. An example for a net-shaped domain. It is an L-shaped one as defined in [9] only that it has holes.

As has already been mentioned in the introduction, we want to identify $H^1(\Omega_\varepsilon)$ with a certain space

$$H_\varepsilon \subset \prod_{j=1}^{K_E} H^1(G_j) \times \prod_{j=K_E+1}^{K_E+K_N} H^1(G_{\varepsilon,j}).$$

To do this define matrix functions $\mathcal{A}_{\varepsilon,j}\colon \overline{G_j} \to \mathbb{R}^{(M+1)\times(M+1)}$, $j = 1, \dots, K_E$, by

$$\mathcal{A}_{\varepsilon,j}(x,y) := \begin{pmatrix} 1 & & & \\ & \varepsilon & & \\ & & \ddots & \\ & & & \varepsilon \end{pmatrix} (DT_{\varepsilon,j}(x,y))^{-1} \cdot \begin{pmatrix} 1 & & & \\ & 1/\varepsilon & & \\ & & \ddots & \\ & & & 1/\varepsilon \end{pmatrix} (DT_j(S_\varepsilon \circ T_{\varepsilon,j}(x,y)))^{-1}.$$

Note that the norms and determinants of all $\mathcal{A}_{\varepsilon,j}$ are bounded from 0 and infinity uniformly in ε and (x,y), and

$$\mathcal{A}_{\varepsilon,j}(x,y) \to \begin{pmatrix} 1 & \mathcal{T}_j(x,0) \\ 0 & E_M \end{pmatrix} DT_j{}^{-1}(x,0)$$

pointwise as $\varepsilon \to 0$ (see [4, Lemma 2.3]).

We divide Ω_ε into the above mentioned K_E edges and K_N nodes, which in turn get transformed by $\Psi_{\varepsilon,j}$ into G_j, $j = 1, \dots, K_E$, and $G_{\varepsilon,j}$, $j = K_E + 1, \dots, K_E + K_N$. Thus we can identify $L^2(\Omega_\varepsilon)$, $H^1(\Omega_\varepsilon)$ with

$$\begin{aligned} \text{(2.2)}\quad L_\varepsilon := \{[u] = [u_1, \dots, u_{K_E+K_N}] : u_j \in L^2(G_j),\ j = 1, \dots, K_E, \\ u_i \in L^2(G_{\varepsilon,i}),\ i = K_E + 1, \dots, K_E + K_N\}, \end{aligned}$$

$$([u],[v])_{L_\varepsilon} := \sum_{j=1}^{K_E} \int_{G_j} u_j v_j \, d\lambda_{\varepsilon,j} + \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \int_{G_{\varepsilon,j}} u_j v_j \, dz,$$

$$\begin{aligned} \text{(2.3)}\quad H_\varepsilon := \{[u] \in L_\varepsilon : u_j \in H^1(G_j),\ j = 1, \dots, K_E, \\ u_i \in H^1(G_{\varepsilon,j}),\ i = K_E + 1, \dots, K_E + K_N, \\ \exists \widehat{u} \in H^1(\Omega_\varepsilon)\ \widehat{u} \circ \Psi_{\varepsilon,j} = u_j,\ j = 1, \dots, K_E + K_N\}, \end{aligned}$$

$$\begin{aligned} ([u],[v])_{H_\varepsilon} &:= ([u],[v])_{L_\varepsilon} + \frac{1}{\varepsilon} \sum_{j=K_E+1}^{K_E+K_N} \int_{G_{\varepsilon,j}} Du_j Dv_j \, dz \\ &+ \sum_{j=1}^{K_E} \int_{G_j} \Big(D_x u_j, \frac{1}{\varepsilon} D_y u_j \Big) \mathcal{A}_{\varepsilon,j}(x,y) \mathcal{A}_{\varepsilon,j}^T(x,y) \Big(D_x v_j, \frac{1}{\varepsilon} D_y v_j \Big)^T d\lambda_{\varepsilon,j}, \end{aligned}$$

and norms $\|\cdot\|_{L_\varepsilon}$, $\|\cdot\|_{H_\varepsilon}$, respectively. Here we used measures on $\mathbb{R}^{M+1}$ defined by

$$\begin{aligned} \lambda_{\varepsilon,j}(A) &:= \int_A |\det DT_{\varepsilon,j}(x,y)| |\det DT_j(S_\varepsilon \circ T_{\varepsilon,j}(x,y))| \, dx \, dy, \\ \lambda_{0,j}(A) &:= \int_A |\det DT_j(x,0)| \, dx \, dy, \end{aligned}$$

for all Lebesgue measurable sets $A \subset \overline{G_j}$, $j = 1, \dots, K_E$.

Note that $|\det DT_{\varepsilon,j}(x,y)||\det DT_j(S_\varepsilon \circ T_{\varepsilon,j}(x,y))|$ is bounded from 0 and infinity uniformly in (x,y) and ε (see [4, Lemma 2.1]). Also above expression tends pointwise to $|\det DT_j(x,0)|$ as $\varepsilon \to 0$.

Given u_j, $j = 1, \dots, K_E$ or $j = 1, \dots K_E + K_N$, we write $[u]$ for $[u_1, \dots u_{K_E}]$ and $[u_1, \dots, u_{K_E+K_N}]$, respectively. It will be clear from the context which case is meant.

The definition of L_ε and H_ε with the respective scalar products in (2.2) and (2.3) is just a change of variables on each subset $\Omega_{\varepsilon,j}$, $j = 1, \dots, K_E + K_N$, the measures $\lambda_{\varepsilon,j}$ being the Jacobian of the respective transformations dropping the common factor ε^M. Thus $\widehat{u} \in L^2(\Omega_\varepsilon)$ if and only if $[u] \in L_\varepsilon$ and $\|\widehat{u}\|^2_{L^2(\Omega_\varepsilon)} = \varepsilon^M \|[u]\|^2_{L_\varepsilon}$; $\widehat{u} \in H^1(\Omega_\varepsilon)$ if and only if $[u] \in H_\varepsilon$ and $\|\widehat{u}\|^2_{H^1(\Omega_\varepsilon)} = \varepsilon^M \|[u]\|^2_{H_\varepsilon}$. Also, if $[u_\varepsilon] \in H_\varepsilon$ is such that $(\|[u_\varepsilon]\|_{\varepsilon,1})_{\varepsilon>0}$ is bounded, then $(\varepsilon^{-M}\|\widehat{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)})_{\varepsilon>0}$ is bounded too.

Note that by Lemma 2.7 in [4] there is a constant $C > 0$, independent of ε, such that for all $[u] \in H_\varepsilon$ we have

$$\frac{1}{C}\|[u]\|_{H_\varepsilon} \le |[u]|_{\varepsilon,1} \le C\|[u]\|_{H_\varepsilon}. \tag{2.4}$$

We have already introduced the space H_0 in (C9), let L_0 be the closure of H_0 in $L^2(G_1) \times L^2(G_2) \times L^2(G_3)$. Then $L_0 = \prod_{j=1}^{K_E} L^2_s(G_j)$ (see [4, Lemma 2.5]). We introduce inner products on them by

$$([u],[v])_{L_0} := \sum_{j=1}^{K_E} \int_{G_j} u_j v_j \, d\lambda_{0,j}, \tag{2.5}$$

$$([u],[v])_{H_0} := ([u],[v])_{L_0} + \sum_{j=1}^{K_E} \int_{G_j} D_x u_j D_x v_j \, d\lambda_{0,j}. \tag{2.6}$$

Denote the respective norms by $\|\cdot\|_{L_0}$ and $\|\cdot\|_{H_0}$.

We need to embed H_0 in H_ε and L_0 in L_ε in order to be able to compare semiflows and attractors. We do this by the linear operator Φ^H_ε given in condition (C9) in the case of the H^1-spaces.

To embed the L^2-spaces define $\Phi^L_\varepsilon : L_0 \to L_\varepsilon$ by

$$\begin{aligned} (\Phi^L_0[u])_j &:= u_j, \quad \text{for } j = 1, \dots, K_E, \\ (\Phi^L_\varepsilon[u])_j &:= 0, \quad \text{for } j = K_E + 1, \dots, K_E + K_N. \end{aligned}$$

Then the $\Phi^L_\varepsilon : L_0 \to L_\varepsilon$, $\Phi^H_\varepsilon : H_0 \to H_\varepsilon$ are both linear and bounded, the bound being independent of $\varepsilon \ge 0$.

We want to write equation (1.1) as an abstract equation. To do so define bilinear forms $a_\varepsilon\colon H_\varepsilon \times H_\varepsilon \to \mathbb{R}$, $\varepsilon \geq 0$, by

$$a_\varepsilon([u],[v]) := \sum_{j=1}^{K_E} \int_{G_j} (D_x u_j, \frac{1}{\varepsilon} D_y u_j) \mathcal{A}_{\varepsilon,j} \mathcal{A}_{\varepsilon,j}^T (D_x v_j, \frac{1}{\varepsilon} D_y v_j)^T \, d\lambda_{\varepsilon,j} + \frac{1}{\varepsilon} \sum_{j=K_E+1}^{K_E+K_N} \int_{G_{\varepsilon,j}} Du_j Dv_j \, dz, \tag{2.7}$$

$$a_0([u],[v]) := \sum_{j=1}^{K_E} \int_{G_j} D_x u_j D_x v_j |(1,0) DT_j^T(x,0)|^{-2} \, d\lambda_{0,j}. \tag{2.8}$$

It is well known (see e.g. [4]) that for $\varepsilon \geq 0$ these bilinear forms a_ε define linear operators $A_\varepsilon\colon D(A_\varepsilon) \subset L_\varepsilon \to L_\varepsilon$. $D(A_\varepsilon) \subset L_\varepsilon$, $D(A_\varepsilon) \subset H_\varepsilon$ densely and the operators A_ε have compact resolvent, are selfadjoint and sectorial. There are complete orthonormal systems (ONS) of L_ε consisting of eigenvectors of A_ε. Note that the fractional power space $X_\varepsilon^{1/2}$ belonging to A_ε is H_ε.

If $f_\varepsilon\colon H_\varepsilon \to L_\varepsilon$ denote the Nemitsky operators of f for $\varepsilon \geq 0$, i.e. f_ε is defined by $f_\varepsilon([u])(z) = f(u_j(z))$ for $z \in G_j$ or $z \in G_{\varepsilon,j}$ for all possible j, then equation (1.1) and — in a certain sense — its limit can be written in an abstract form as

$$[u_t] = -A_\varepsilon[u] + f_\varepsilon([u]), \quad t > 0. \tag{2.9}$$

It is clear that it suffices to investigate the behavior of the semiflow generated by equation (2.9) because a simple transformation changes it into the semiflow generated by (1.1) (for $\varepsilon > 0$).

Henceforth we shall only treat equation (2.9).

We cite now some results from [4] regarding the convergence of eigenvectors and eigenvalues of A_ε, of the existence of semiflows π_ε generated by equation (2.9) and their convergence, and finally of the existence of global attractors $\mathcal{A}_\varepsilon$ and their upper-semicontinuity.

Theorem 2.1 (cf. [4]). *Denote by $\lambda_{\varepsilon,l}$ the eigenvalues of A_ε, $\varepsilon \geq 0$, and assume them to be ordered $0 \leq \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \dots$ Denote by $[u_{\varepsilon,l}] \in H_\varepsilon$ the corresponding eigenvectors which form a complete* ONS *of L_ε. Let $\varepsilon_n \to 0$. Then $\lambda_{\varepsilon_n,l} \to \lambda_{0,l}$, for all $l \in \mathbb{N}$. There is a subsequence, called ε_n too, and a complete* ONS *$([u_l])_l$ of L_0 consisting of eigenvectors belonging to $\lambda_{0,l}$ such that $|[u_{\varepsilon_n,l}] - \Phi_{\varepsilon_n}^H [u_l]|_{\varepsilon_n,d} \to 0$ as $n \to \infty$, for all $0 \leq d < 1$.*

Theorem 2.2 (cf. [4]). *Let $\varepsilon_n \downarrow 0$, $[u_n] \in H_{\varepsilon_n}$, $[u_0] \in H_0$ and $\|[u_n] - \Phi_\varepsilon^L[u_0]\|_{L_\varepsilon} \to 0$, $n \to \infty$. Assume f satisfies* (H1) *and* (H2'). *Then equation* (2.9) *generates a global semiflow, called π_ε, on H_ε, for $\varepsilon \geq 0$. If $t_0, t_n > 0$, $t_n \to t_0$ as $n \to \infty$, and $\sup_{n\in\mathbb{N},\, 0<t\leq 2t_0} |[u_n]\pi_{\varepsilon_n} t|_{\varepsilon_n,1} < \infty$, then for $0 \leq d < 1$*

$$|[u_n]\pi_{\varepsilon_n} t_n - \Phi_{\varepsilon_n}^H([u_0]\pi_0 t_0)|_{\varepsilon_n,d} \to 0, \quad \textit{as } n \to \infty.$$

For $\varepsilon \geq 0$ the semiflows π_ε have attractors $\mathcal{A}_\varepsilon \subset H_\varepsilon$ consisting of all full bounded solutions on H_ε which attract every bounded set $B \subset H_\varepsilon$. The family of attractors is upper-semi-continuous at $\varepsilon = 0$, i.e. for all $0 \leq d < 1$,

$$\lim_{\varepsilon \to 0} \sup_{[u] \in \mathcal{A}_\varepsilon} \inf_{[v] \in \mathcal{A}_0} |[u] - \Phi_\varepsilon^H[v]|_{\varepsilon,d} = 0.$$

3. Boundedness in L^∞ and auxiliary results

We want the Nemitsky operators f_ε to be differentiable on the attractors. On way to get this is to show the attractors to be bounded uniformly in L^∞. Then one can cut f without changing it on the attractor.

In this section we show the attractors to be bounded uniformly in L^∞. For this purpose we need the stronger dissipativeness condition (H2) on f, i.e. we suppose $\beta_2 > 0$, where in many other papers (e.g. [14], [4]) $\beta_2 = 0$ is allowed.

We also provide some results we shall need later, among them a convergence result for eigenvalues and eigenfunctions for the linear problem $[u_t] = -A_\varepsilon[u] + V_\varepsilon[u]$, where V_ε are some potentials. Additionally we define a (the usual) Liapunov-function for these semiflows.

For $\varepsilon > 0$ we can apply Theorem 2.1 from [3]: with this theorem $[u]\pi_\varepsilon t$ is in L^∞ for $t > 0$, and all $[u] \in H_\varepsilon$. In particular all $\mathcal{A}_\varepsilon$, $\varepsilon > 0$, are bounded in L^∞. But we want a uniform bound on $[u]\pi_\varepsilon t$ independent of ε and $[u]$. We cannot apply the results of above paper to this case because Ω_ε collapses to a lower dimensional set, and on the fixed sets G_j the coefficients in the linear operator tend to infinity. Also we do not have similar results for the limiting case since the abstract theorems do not apply to it.

We shall use functions of the form $t \mapsto \|[u]\pi_\varepsilon t\|_{L^p}$ to show that after a certain time (independent of ε) $[u]\pi_\varepsilon t$ is bounded in L^∞ by a bound independent of the initial value $[u]$ and ε. Then the convergence of the semiflows π_ε to π_0 shows a similar result for π_0. Thus all attractors $\mathcal{A}_\varepsilon$, $\varepsilon \geq 0$, are bounded uniformly in L^∞.

We need the spaces $L^p(\Omega_\varepsilon)$, $1 \leq p \leq \infty$. Dividing Ω_ε as before into edges and nodes, and making the transformations via $\Psi_{\varepsilon,j}$, each of these spaces corresponds to an L^p_ε with norm $\|\cdot\|_{L^p_\varepsilon}$ defined by

$$\begin{aligned} L^p_\varepsilon := \{[u] = [u_1, \dots, u_{K_E+K_N}] : u_j \in L^p(G_j),\ j = 1, \dots, K_E,& \\ u_l \in L^p(G_{\varepsilon,l}),\ l = K_E+1, \dots, K_E+K_N\},& \quad \text{for } \varepsilon > 0, \end{aligned}$$

$$\|[u]\|_{L^p_\varepsilon} := \left(\sum_{j=1}^{K_E} \int_{G_j} |u_j|^p d\lambda_{\varepsilon,j} + \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \int_{G_{\varepsilon,j}} |u_j|^p \, dz \right)^{1/p}, \quad p < \infty,\ \varepsilon > 0,$$

$$\begin{aligned} \|[u]\|_{L^\infty_\varepsilon} := \max(\|u_j\|_{L^\infty(G_j)}, \|u_l\|_{L^\infty(G_{\varepsilon,l})},\ j = 1, \dots, K_E,& \\ l = K_E+1, \dots, K_E+K_N),& \quad \varepsilon > 0, \end{aligned}$$

$$L_0^p := \{[u] = [u_1, \dots, u_{K_E}] : u_j \in L^p(G_j),\ j = 1, \dots, K_E\},$$
$$\|[u]\|_{L_0^p} := \bigg(\sum_{j=1}^{K_E} \int_{G_j} |u_j|^p d\lambda_{0,j}\bigg)^{1/p}, \quad p < \infty,$$
$$\|[u]\|_{L_0^\infty} := \max(\|u_j\|_{L^\infty(G_j)},\ j = 1, \dots, K_E).$$

We need a few technical lemmas.

Lemma 3.1. *Let $y\colon [a,b[\to \mathbb{R}_\geq$ be continuous and differentiable on $]a,b[$ with $y' \leq C_1 - C_2 y^{C_3}$, where $C_1 \geq 0$, $C_2 > 0$, $C_3 > 1$ are some given constants. Then for $t \in [a,b[$ we have either*

$$y(t) \leq 2\bigg(\frac{C_1}{C_2}\bigg)^{1/C_3}$$

or

$$y(t) \leq ((t-a)\frac{C_2}{2}(C_3 - 1) + y^{1-C_3}(a))^{-1/(C_3-1)}.$$

Proof. If $y(t) > (C_1/C_2)^{1/C_3}$, then $y' < 0$. Hence if $y(a) \leq 2(C_1/C_2)^{1/C_3}$ $=: C_4$, then $y(t) \leq C_4$ for all t.

Now assume $y(a) > C_4$. Then as long as $y(t) \geq C_4$ we have $1/2 C_2 y^{C_3}(t) > C_1$ and thus

$$\begin{aligned} t - a &\leq \int_a^t \frac{-y'(s)\,ds}{C_2 y^{C_3}(s) - C_1} = \int_{y(t)}^{y(a)} \frac{dy}{C_2 y^{C_3} - C_1} \leq \int_{y(t)}^{y(a)} \frac{2\,dy}{C_2 y^{C_3}} \\ &= \frac{2}{C_2(1 - C_3)}(y^{1-C_3}(a) - y^{1-C_3}(t)). \end{aligned}$$

We get

$$y(t) \leq ((t-a)\frac{C_2}{2}(C_3 - 1) + y^{1-C_3}(a))^{-1/C_3-1}. \qquad \square$$

Lemma 3.2. *Assume f satisfies* (H2). *Let $\varepsilon > 0$, $2 \leq p < \infty$, $[u_0] \in H_\varepsilon$, and set $u(t) := [u_0]\pi_\varepsilon t$, $t \geq 0$. Then $u(t) \in L_\varepsilon^\infty$, for $t > 0$, and we can define $\mathcal{G}_{\varepsilon,p}\colon \mathbb{R}_> \to \mathbb{R}$ by $\mathcal{G}_{\varepsilon,p}(t) := \|u(t)\|_{L_\varepsilon^p}{}^p$.*

(a) *$\mathcal{G}_{\varepsilon,p}$ is is differentiable on $]0,\infty[$. The derivative is*

$$\mathcal{G}'_{\varepsilon,p}(t) = p(|u(t)|^{p-2} u(t), \partial_t u(t))_{L_\varepsilon}.$$

(b) *There are constants $C, T > 0$, independent of p, ε, $[u_0]$, such that $\mathcal{G}_{\varepsilon,p}(t) \leq C^p$ for all $t \geq T$.*

(c) *If f satisfies only* (H2'), *$\widetilde{C} > 0$ is such that $f(s)/s \leq -1/2\xi$, $|s| \geq \widetilde{C}$, and $u(t) = [u_0]$ is constant, then $\|[u_0]\|_{L_\varepsilon^\infty} \leq \widetilde{C}$, for $\varepsilon > 0$.*

Proof. First note that condition (H2) implies there is a constant $C_1 > 0$, such that $f(s)/s|s|^{\beta_2} < -1/2\xi$, for $|s| \geq C_1$. The same holds if f only satisfies (H2') setting $\beta_2 = 0$.

In this proof we will work with Ω_ε rather than the partition into G_j, $G_{\varepsilon,j}$.

If $[u] \in L_\varepsilon$ and $w \in L^2(\Omega_\varepsilon)$ are such that $w \circ \Phi_{\varepsilon,j} = u_j$ for all j, then by construction $w \in L^p(\Omega_\varepsilon)$ if and only if $[u] \in L^p_\varepsilon$, and in this case $\|w\|^p_{L^p(\Omega_\varepsilon)} = \varepsilon^M \|[u]\|_{L^p_\varepsilon}{}^p$.

For $t \geq 0$ set $w(t) := u_j(t) \circ \Phi_{\varepsilon,j}{}^{-1} \colon \Omega_{\varepsilon,j} \to \mathbb{R}$, $j = 1, \ldots, K_E + K_N$, then $w(t)$ is solution of equation (1.1) with initial value $w(0) \in H^1(\Omega_\varepsilon)$. Define $\widetilde{\mathcal{G}}_{\varepsilon,p}(t) := \|w(t)\|^p_{L^p(\Omega_\varepsilon)}$. If we show $\widetilde{\mathcal{G}}_{\varepsilon,p}$ is differentiable with derivative $\widetilde{\mathcal{G}}'_{\varepsilon,p}(t) = p(|w(t)|^{p-2}w(t), w_t(t))_{L^2(\Omega_\varepsilon)}$, then the first part of the lemma follows immediately.

Let $0 < t_0$. Then $w(t_0) \in L^\infty(\Omega_\varepsilon)$, as has already been mentioned. For $t_0 \leq t$ we can view $w(t)$ as the solution of the abstract equation

$$w_t = -\widetilde{A}_\varepsilon w + f_\varepsilon(w),$$

where the linear operator $\widetilde{A}_\varepsilon \colon D(\widetilde{A}_\varepsilon) \subset L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)$ is sectorial. It is well known that the restriction $\widetilde{A}_{\varepsilon,p} \colon D(\widetilde{A}_{\varepsilon,p}) \subset L^p(\Omega_\varepsilon) \to L^p(\Omega_\varepsilon)$ is sectorial (see e.g. [12, Theorem 3.1.3]). Hence there is $T_1 = T_1(w(t_0), \varepsilon, p) > 0$ such that $t \mapsto w(t) \in L^p(\Omega_\varepsilon)$ is continuous on $[t_0, T_1[$ and differentiable on $]t_0, T_1[$. If T_1 is maximal, then either $T_1 = \infty$ or $\|w(t)\|_{L^p(\Omega_\varepsilon)} \to \infty$, as $t \uparrow T_1$.

The differentiability of $t \mapsto w(t) \in L^p(\Omega_\varepsilon)$ implies

$$\widetilde{\mathcal{G}}'_{\varepsilon,p}(t) = p(|w(t)|^{p-2}w(t), w_t(t))_{L^2(\Omega_\varepsilon)}.$$

Hence if we knew $T_1 = \infty$, then the first part of the lemma would have been shown.

To prove $T_1 = \infty$ and the second part, note that $w(t) \in L^\infty(\Omega_\varepsilon)$ for $0 < t < T_1$. Thus $|w(t)|^{p-2}w(t) \in H^1(\Omega_\varepsilon)$ and $\partial(|w(t)|^{p-2}w(t)) = (p-1)|w(t)|^{p-2}\partial w(t)$. If $\widetilde{a}_\varepsilon$ denotes the bilinear form which generates $\widetilde{A}_\varepsilon$, we get for $t_0 < t < T_1$

$$\begin{aligned}
\mathcal{G}'_{\varepsilon,p}(t) &= p\varepsilon^{-M}(-\widetilde{a}_\varepsilon(w(t), |w(t)|^{p-2}w(t)) + (f_\varepsilon(w(t)), |w(t)|^{p-2}w(t))_{L^2(\Omega_\varepsilon)}) \\
&= p\varepsilon^{-M}\int_{\Omega_\varepsilon} (-(p-1)|w(t)|^{p-2}\nabla w(t)\nabla w(t) + f(w(t))w(t)|w(t)|^{p-2})\,dz \\
&\leq p\varepsilon^{-M}\bigg(|\Omega_\varepsilon| C_1^{p-1} \max_{|s|\leq C_1}(|f(s)|) - \frac{1}{2}\xi\|w(t)\|^{p+\beta_2}_{L^{p+\beta_2}(\Omega_\varepsilon)}\bigg) \\
&\leq p\varepsilon^{-M}\bigg(|\Omega_\varepsilon| C_2 C_1^{p-1} - \frac{1}{2}\xi|\Omega_\varepsilon|^{-\beta_2/p}\|w(t)\|^{p+\beta_2}_{L^p(\Omega_\varepsilon)}\bigg),
\end{aligned}$$

where the constant C_2 is independent of ε, p, $[u_0]$, t. The conditions on the transformations $\Phi_{\varepsilon,j}$ imply the existence of a constant C_3 such that $|\Omega_\varepsilon| \leq C_3\varepsilon^M$. Thus there are constants $C_4, C_5 > 0$, also independent of ε, $p \geq \beta_2$, $[u_0]$, t, such that

$$\mathcal{G}'_{\varepsilon,p}(t) \leq p\,C_4 C_1^p - p\,C_5 \mathcal{G}^{(p+\beta_2)/p}_{\varepsilon,p}(t), \quad t_0 < t < T_1. \tag{3.1}$$

By Lemma 3.1 we have either

$$\mathcal{G}_{\varepsilon,t}(t) \le 2\frac{C_4}{C_5} C_1^{p^2/(p+\beta_2)} \le C_6^p,$$

where C_6 is independent of ε, p, $[u_0]$, t, or

$$\mathcal{G}_{\varepsilon,p}(t) \le \left(\frac{t-t_0}{2} p\, C_5 \frac{\beta_2}{p} + \mathcal{G}_{\varepsilon,p}^{-\beta_2/p}(t_0) \right)^{-p/\beta_2}.$$

Thus for $t \uparrow T_1$ $\mathcal{G}_{\varepsilon,p}(t)$ is bounded, and $T_1 = \infty$ follows.

Now if $t \ge T := ((1/2)C_5 C_6^{\beta_2} \beta_2)^{-1} + t_0$, then in any case $\mathcal{G}_{\varepsilon,p}(t) \le C_6^p$ and the the first two parts of the lemma have been proven.

To prove part (c) use again inequality (3.1), only that now $\beta_2 = 0$. Since $\mathcal{G}'_{\varepsilon,p}(t) \equiv 0$ in this case, we have

$$\|[u_0]\|_{L^p_\varepsilon}^p \le \frac{C_4}{C_5} C_1^p.$$

It is well known that if $u: U \to \mathbb{R}$, $U \subset \mathbb{R}^{M+1}$ open, then $\|u\|_\infty \le C$ if and only if $\|u\|_{L^p(U)} \le |U|^{1/p} C$ for all p big enough (see e.g. [6, Problem 7.1]).

Thus $\|[u_0]\|_{L^\infty_\varepsilon} \le C_1$ and the third part is true too. □

Now we can prove that the attractors $\mathcal{A}_\varepsilon$ are bounded uniformly in L^∞.

Proposition 3.3. *Assume f satisfies condition* (H2). *There is a constant $C > 0$ such that $\mathcal{A}_\varepsilon \subset B_C(0) \subset L^\infty_\varepsilon$ for all $\varepsilon \ge 0$.*

Proof. First let $\varepsilon > 0$. Let T and C_1 be as in Lemma 3.2(b). If $[u] \in \mathcal{A}_\varepsilon$, then there is a $[u_0] \in H_\varepsilon$ such that $[u_0]\pi_\varepsilon 2T = [u]$. Thus by Lemma 3.2 $\|[u]\|_{L^p_\varepsilon} \le C_1$ for all $2 \le p < \infty$.

Using the same characterization of L^∞ we used in the proof of Lemma 3.2, there is a $C_2 > 0$ and $\|[u]\|_{L^\infty_\varepsilon} \le C_2$. This proves the uniform bound on $\mathcal{A}_\varepsilon$ for $\varepsilon > 0$.

Now we bound $\mathcal{A}_0$. Let $[u] \in \mathcal{A}_0$. Again there is a $[u_0] \in H_0$ such that $[u_0]\pi_0 2T = [u]$. Let $\varepsilon_n \to 0$. By Theorem 2.2 we have

$$(\Phi_\varepsilon^H[u_0])\pi_\varepsilon 2T)_j(x,y) \to u_j(x,y)$$

for a.a. $(x,y) \in G_j$, and arguing as before we get $\|u_j\|_{L^\infty(G_j)} \le C_2$, for all $j = 1, \dots, K_E$. $\|[u]\|_{L^\infty_0} \le C_2$ follows immediately. □

Now we bring a few lemmas we shall need in later sections. We start by proving a sufficient condition for convergence in $|\cdot|_{\varepsilon,d}$.

LEMMA 3.4. *Let* $\varepsilon_n \to 0$, $0 \le d < 1$, $[u_n] \in D(A_{\varepsilon_n})$, $[u_0] \in H_0$ *and* $\|[u_n]\|_{H_{\varepsilon_n}}, \|A_{\varepsilon_n}[u_n]\|_{L_{\varepsilon_n}} \le C$, $C > 0$ *independent of* n. *Suppose*

$$\|[u_n] - \Phi^L_{\varepsilon_n}[u_0]\|_{L_{\varepsilon_n}} \to 0, \quad \text{as } n \to \infty,$$

then

$$\lim_{n\to\infty} a_{\varepsilon_n}([u_n], \Phi^H_{\varepsilon_n}[v]) = a_0([u_0],[v]) \quad \text{for all } [v] \in H_0. \tag{3.2}$$

If additionally $a_{\varepsilon_n}([u_n],[u_n]) \to a_0([u_0],[u_0])$ *as* $n \to \infty$, *then*

$$|[u_n] - \Phi^H_{\varepsilon_n}[u_0]|_{\varepsilon_n,d} \to 0, \quad \text{as } n \to \infty.$$

PROOF. Assume the situation of the lemma. By Lemma 2.12 in [4] there is a subsequence, called ε_n again, $[v_0] \in H_0$, $\widetilde{u}_j \in (L^2(G_j))^M$, $j = 1, \dots, K_E$, such that $u_{n,j} \rightharpoonup v_{0,j}$ weakly in $H^1(G_j)$ and strongly in $L^2(G_j)$, $(1/\varepsilon_n)D_y u_{n,j} \rightharpoonup \widetilde{u}_{0,j}$ weakly in $L^2(G_j)$, $j = 1, \dots, K_E$, and $a_n([u_n], \Phi^H_n[v]) \to a_0([v_0],[v])$, for all $[v] \in H_0$, as $n \to \infty$. $[v_0] = [u_0]$ and thus (3.2) follows.

Now assume additionally $a_n([u_n],[u_n]) \to a_0([u_0],[u_0])$, $n \to \infty$, then

$$\begin{aligned}
0 &\leftarrow a_n([u_n],[u_n]) - a_0([u_0],[u_0]) \\
&= \sum_{j=1}^{K_E} \underbrace{\int_{G_j} \left| (D_x u_{0,j}, \widetilde{u}_{0,j}) \left(\begin{smallmatrix} 1 & \mathcal{T}_j(x,y) \\ 0 & E_M \end{smallmatrix}\right) DT_j{}^{-1}(x,0) \right|^2 d\lambda_{0,j}}_{=:E_{1,n,j}} \\
&\quad - \int_{G_j} (D_x u_{0,j})^2 |(1,0) DT_j^T(x,0)|^{-2}\, d\lambda_{0,j} \\
&\quad + \int_{G_j} \Bigg(\Bigg| \underbrace{\left(D_x u_{n,j}, \frac{1}{\varepsilon_n} D_y u_{n,j}\right) \mathcal{A}_{n,j}(x,y) \sqrt{|\det DT_{\varepsilon,j}||\det DT_j(S_\varepsilon \circ T_{\varepsilon,j})|}}_{=:E_{2,n,j}} \Bigg|^2 \\
&\quad - \underbrace{\left| (D_x u_{0,j}, \widetilde{u}_{0,j}) \left(\begin{smallmatrix} 1 & \mathcal{T}_j(x,y) \\ 0 & E_M \end{smallmatrix}\right) DT_j{}^{-1}(x,0) \right|^2 |\det DT_j(x,0)|}_{=:E_{3,j}} \Bigg)\, dx\, dy \\
&\quad + \underbrace{\frac{1}{\varepsilon_n} \sum_{j=K_E+1}^{K_E+K_N} \int_{G_{n,j}} |Du_{n,j}|^2\, dz}_{=:E_{4,n} \ge 0}.
\end{aligned}$$

By Lemmas 2.3 and 2.6 in [4]

$$E_{2,n,j} \rightharpoonup (D_x u_{0,j}, \widetilde{u}_{0,j}) \begin{pmatrix} 1 & \mathcal{T}_j \\ 0 & E_M \end{pmatrix} DT_j{}^{-1}(x,0) \sqrt{|\det DT_j(x,0)|}$$

weakly in L^2, thus for all j

$$\liminf_{n\to\infty} \int_{G_j} (|E_{2,n,j}|^2 - E_{3,j})\, dx\, dy \ge 0. \tag{3.3}$$

Recall that the orthogonal complement of $L^2_s(G_j)$ is $L^2_\perp(G_j)$. Decompose $\mathcal{T}_{j,l}$, $\widetilde{u}_{0,j,l}$, by setting $\mathcal{T} = \mathcal{T}_{s,j} + \mathcal{T}_{\perp,j}$, $\widetilde{u}_{0,j} = \widetilde{u}_{s,0,j} + \widetilde{u}_{\perp,0,j}$, where $\mathcal{T}_{s,j,l}, \widetilde{u}_{s,0,j,l} \in L^2_s(G_j)$, $\mathcal{T}_{\perp,j,l}, \widetilde{u}_{\perp,0,j,l} \in L^2_\perp(G_j)$, $j = 1, \dots, K_E$, $l = 1, \dots, M$.

Note that by Proposition 2.1 in [4] for $u \in H^1_s(G_j)$ we have $D_x u \in L^2_s(G_j)$ for all j, and by Lemma 2.12 of the same article

$$\widetilde{u}_{s,0,j} = D_x u_{0,j}(|(1,0)DT_j^T(x,0)|^{-2}(1,0)DT_j^T(x,0)DT_j(x,0)(0,E_M)^T - \mathcal{T}_{s,j}).$$

Proceeding as in the proof of Lemma 2.13 in [4], we get

$$E_{1,n,j} = \int_{G_j} (D_x u_{0,j})^2 |(1,0)DT_j^T(x,0)|^{-2} d\lambda_{0,j} + \underbrace{\int_{G_j} |(0, D_x u_{0,j}\mathcal{T}_{\perp,j} + \widetilde{u}_{\perp,0,j})DT_j{}^{-1}(x,0)|^2 d\lambda_{0,j}}_{=:E_{5,j}\geq 0}.$$

We find

$$0 \leftarrow \sum_{j=1}^{K_E} E_{5,j} + \int_{G_j} (|E_{2,n,j}|^2 - E_{3,j})\,dx\,dy + E_{4,n}$$

and (3.3) implies

$$E_{5,j} = \lim_{n\to\infty} E_{4,n} = \lim_{n\to\infty} \int_{G_j} (|E_{2,n,j}|^2 - E_{3,j})\,dx\,dy = 0$$

for all j. Thus $D_x u_{n,j} \to D_x u_{0,j}$, $(1/\varepsilon_n)D_y u_{n,j} \to \widetilde{u}_{0,j}$ strongly in L^2, which in turn implies $|[u_n] - \Phi_n^H[u_0]|_{n,d} \to 0$, $n \to \infty$, for all $d < 1$. $\square$

We need the uniform boundedness of the attractors $\mathcal{A}_\varepsilon$ in $\|\cdot\|_{H_\varepsilon}$. Since this is not included in [4], we prove it here. For this we use a Liapunov-function often used for such equations.

Let $\varepsilon \geq 0$. Define $F(x) := \int_0^x f(s)\,ds$. Denote by $F_\varepsilon: H_\varepsilon \to L^1_\varepsilon$ the Nemitsky operator of F. It is well known that F_ε is well defined, maps bounded sets of H_ε into bounded sets of L^1_ε, and is Frechét-differentiable with derivative $DF_\varepsilon([u])[v] = f_\varepsilon([u])[v]$.

Define $\mathcal{G}_{\varepsilon,H}: H_\varepsilon \to \mathbb{R}$ by

$$\mathcal{G}_{\varepsilon,H}([u]) := \frac{1}{2} a_\varepsilon([u],[u]) - (F_\varepsilon([u]), 1)_{L_\varepsilon}$$

(here $\varepsilon \geq 0$). It is well known that $\mathcal{G}_{\varepsilon,H}$ is Frechét-differentiable, and if $\sigma_\varepsilon(t)$ is a solution of equation (2.9), then $(t \mapsto \mathcal{G}_{\varepsilon,H}(\sigma_\varepsilon(t)))' = -\|\partial_t\sigma_\varepsilon(t)\|^2_{L_\varepsilon}$. $\mathcal{G}_{\varepsilon,H}$ maps bounded sets of H_ε into bounded sets of $\mathbb{R}$. Since f satisfies condition (H2'), there is a $C_1 > 0$ such that $F(s) \leq -(1/4)\xi s^2 + C_1$ for all $s \in \mathbb{R}$. Thus there are $C_2, C_3 > 0$ such that

$$\|[u]\|^2_{H_\varepsilon} \leq C_2(\mathcal{G}_{\varepsilon,H}([u]) + C_3) \quad \text{for all } [u] \in H_\varepsilon,\ \varepsilon \geq 0. \tag{3.4}$$

By Lemma 3.5 to come the sets of equilibrium points of π_ε is bounded in H_ε, and π_ε is gradient like with respect to $\mathcal{G}_{\varepsilon,H}$, $\varepsilon \geq 0$.

Lemma 3.5. *For every $\delta > 0$ there is a $C = C(\delta) > 0$, independent of $\varepsilon \geq 0$, such that if $[u] \in D(A_\varepsilon)$, $\| -A_\varepsilon[u] + f_\varepsilon([u])\|_{L_\varepsilon} \leq \delta$ implies $\|[u]\|_{H_\varepsilon} < C$.*

Proof. Let $[u_0] \in D(A_\varepsilon)$, $\varepsilon \geq 0$, $\delta > 0$ and assume

$$\| -A_\varepsilon[u_0] + f_\varepsilon([u_0])\|_{L_\varepsilon} \leq \delta.$$

f satisfies (H2') means there are constants $C_1, C_2 > 0$ such that

$$([u], f_\varepsilon([u]))_{L_\varepsilon} \leq (C_1 - C_2\|[u]\|_{L_\varepsilon})\|[u]\|_{L_\varepsilon} \quad \text{for all } [u] \in H_\varepsilon,\ \varepsilon \geq 0.$$

Now if $\|[u_0]\|_{L_\varepsilon} \geq \max((2 + 4/C_2)\delta, C_1/C_2 + 1/2)$, then for $\varepsilon > 0$

$$\begin{aligned}
-\delta\|[u_0]\|_{L_\varepsilon} &\leq (-A_\varepsilon[u_0] + f_\varepsilon([u_0]), [u_0])_{L_\varepsilon} \\
&\leq -\min\left(1, \frac{1}{2}C_2\right)(a_\varepsilon([u_0],[u_0]) + \|[u_0]\|^2_{L_\varepsilon}) \\
&\quad + \left(C_1 - \frac{1}{2}C_2\|[u_0]\|_{L_\varepsilon}\right)\|[u_0]\|_{L_\varepsilon} \\
&\leq -\min\left(1, \frac{1}{2}C_2\right)\|[u_0]\|^2_{H_\varepsilon} \leq -2\delta\|[u_0]\|_{L_\varepsilon} \#.
\end{aligned}$$

If $\|[u_0]\|_{L_\varepsilon} < \max((2 + 4/C_2)\delta, C_1/C_2 + 1/2)$, then

$$\begin{aligned}
-\delta\|[u_0]\|_{L_\varepsilon} &\leq (-A_\varepsilon[u_0] + f_\varepsilon([u_0]), [u_0])_{L_\varepsilon} \\
&\leq -(a_\varepsilon([u_0],[u_0]) + \|[u_0]\|^2_{L_\varepsilon}) + C_3 \leq -\|[u_0]\|^2_{H_\varepsilon} + C_3.
\end{aligned}$$

Hence $\| -A_\varepsilon[u_0] + f_\varepsilon([u_0])\|_{L_\varepsilon} \leq \delta$ implies $\|[u_0]\|^2_{H_\varepsilon} \leq C_4 = C_4(\delta)$.

For $\varepsilon = 0$ we do no longer have $a_\varepsilon([u],[u]) + \|[u]\|^2_{L_\varepsilon} = \|[u]\|^2_{H_\varepsilon}$ but only $a_0([u],[u]) + \|[u]\|^2_{L^\varepsilon} \leq C_5\|[u]\|^2_{H_\varepsilon}$. In this case we can adapt the argument above using $\widetilde{\delta} = \delta/C_5$. □

Lemma 3.6. *Let $C_1 > 0$, $\Omega \subset]0,\infty[\times\mathbb{R}^M$ be open, bounded, Lipschitz. Then there is a constant $C_2 = C_2(C_1) > 0$ such that for all $u \in H^1(\Omega)$ and $\varepsilon > 0$*

$$\frac{1}{\varepsilon}\|u\|^2_{L^2(\{(x,y)\in\Omega: 0<x\leq\varepsilon C_1\})} \leq C_2\|u\|^2_{H^1(\Omega)}.$$

An analogous statement holds if $\Omega \subset]\infty, 1[\times\mathbb{R}^M$.

Proof. Extend $u \in H^1(\Omega)$ to $\widetilde{u} \in H^1(\mathbb{R}^{M+1})$. We get

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\{(x,y)\in\Omega:0<x\le\varepsilon C_1\}} u^2\,dx\,dy \\
&\le \frac{2}{\varepsilon}\int_{\{(x,y)\in\Omega:0<x\le\varepsilon C_1\}} |u(x,y)-\widetilde{u}(0,y)|^2\,dx\,dy \\
&\quad + \frac{2}{\varepsilon}\int_{\{(x,y)\in\Omega:0<x\le\varepsilon C_1\}} \widetilde{u}^2(0,y)\,dx\,dy \\
&\le \frac{2}{\varepsilon}\int_0^\varepsilon \|\widetilde{u}(x,y)-\widetilde{u}(0,y)\|^2_{L^2(\mathbb{R}^M)}\,dx + 2C_1\|\widetilde{u}(0,\,\cdot\,)\|^2_{L^2(\mathbb{R}^M)} \\
&\le \frac{C_2}{\varepsilon}\int_0^\varepsilon x\|\widetilde{u}\|^2_{H^1(\mathbb{R}^{M+1})}\,dx + C_3\|\widetilde{u}\|^2_{H^1(\mathbb{R}^{M+1})} \le C_4\|u\|^2_{H^1(\Omega)},
\end{aligned}$$

where we used Theorem 6.2.29 in [7]. □

Lemma 3.7. *There is a constant $C > 0$ such that for all $[u] \in H_\varepsilon$, $\varepsilon > 0$, and $d > 1/2$*

$$\sum_{j=K_E+1}^{K_E+K_N} \|u_j\|^2_{L^2(G_{\varepsilon,j})} \le C|[u]|^2_{\varepsilon,d}. \tag{3.5}$$

Proof. Fix $j_0 \in \{K_E+1,\dots,K_E+K_N\}$ and let $[u] \in H_\varepsilon$, $0<\varepsilon$. If G_{ε,j_0} has empty interior, nothing has to be shown. If this is not the case, by (C8) there are open, bounded, connected, Lipschitz $G_{j_0,k} \subset \mathbb{R}^{M+1}$, $i_0 \in \{1,\dots,K_E\}$, $U_{j_0,k} \subset {\Psi_{\varepsilon,j_0}}^{-1}(\Omega_{\varepsilon,j_0} \cap \Omega_{\varepsilon,i_0})$, $k = 1,\dots,N_{j_0}$, $r, C_1 > 0$, all independent of ε, C^1-diffeomorphisms $\Psi_{\varepsilon,j_0,k}$, $z_{\varepsilon,j_0,k} \in \mathbb{R}^{M+1}$ such that

$$\begin{aligned}
\Bigl|G_{\varepsilon,j_0} \setminus \bigcup_{k=1}^{N_{j_0}} \Psi_{\varepsilon,j_0,k}(G_{j_0,k})\Bigr| &= 0, \\
B_r(z_{\varepsilon,j_0,k}) &\subset U_{j_0,k} \cap {\Psi_{\varepsilon,j_0}}^{-1}(\Omega_{\varepsilon,i_0}), \\
|{\Psi_{\varepsilon,j_0,k}}^{-1}(U_{j_0,k} \cap \Psi_{\varepsilon,j_0,k}(G_{j_0,k}))| &\ge \frac{1}{C_1}, \\
{\Psi_{\varepsilon,i_0}}^{-1} \circ \Psi_{\varepsilon,j_0}(B_r(z_{\varepsilon,j_0,k})) &\subset \,]0,\varepsilon C_1[\, \times \mathbb{R}^M \quad \text{or} \\
{\Psi_{\varepsilon,i_0}}^{-1} \circ \Psi_{\varepsilon,j_0}(B_r(z_{\varepsilon,j_0,k})) &\subset \,]1-\varepsilon C_1, 1[\, \times \mathbb{R}^M,
\end{aligned}$$

if the edge Ω_{ε,i_0} begins at the node Ω_{ε,j_0}, or if Ω_{ε,i_0} ends at Ω_{ε,j_0}, respectively.

Without loss of generality we assume Ω_{ε,i_0} begins at Ω_{ε,j_0}. Define

$$\begin{aligned}
u_{j_0,k,U} &:= u_i \circ {\Psi_{\varepsilon,i}}^{-1} \circ \Psi_{\varepsilon,j_0} && \text{on } G_i, \text{ resp. } G_{\varepsilon,i}, \\
u_{j_0,k} &:= u_{j_0} \circ \Psi_{\varepsilon,j_0,k} && \text{on } G_{j_0,k},
\end{aligned}$$

for all possible j, k. Then $u_{j_0,k,U} \in H^1(U_{j_0,k})$, $u_{j_0,k} \in H^1(G_{j_0,k})$. Note that these functions may depend on ε.

Now, if $V \subset U \subset \mathbb{R}^{M+1}$ are given, U is open, bounded, connected, Lipschitz, $|V| \geq C_2 > 0$ and $v \in H^1(U)$, we set $c_v := |U|^{-1}(v,1)_{L^2(U)}$, $w := v - c_v$. Then $(w,1)_{L^2(U)} = 0$ and by the generalized Poincaré-inequality (see e.g. [1, 5.15]) there is a $C_3 = C_3(U)$ such that

$$\|w\|_{L^2(U)} \leq C_3 \|Dw\|_{L^2(U)} = C_3 \|Dv\|_{L^2(U)}.$$

Also

$$\begin{aligned} |c_v| &\leq \frac{1}{C_2} \int_V |c_v| \, dz \leq \frac{1}{C_2} \bigg(\int_V |v| \, dx + \int_U |w| \, dz \bigg) \\ &\leq \frac{|U|^{1/2}}{C_2} (\|v\|_{L^2(V)} + C_3 \|Dv\|_{L^2(U)}), \end{aligned}$$

$$\|v\|_{L^2(U)} \leq |U|^{1/2} |c_v| + C_3 \|Dv\|_{L^2(U)} \leq C_4 (\|v\|_{L^2(V)} + \|Dv\|_{L^2(U)}), \tag{3.6}$$

with some constant $C_4 = C_4(|U|, C_2)$. Apply this first for

$$U = G_{j_0,k}, \quad V = V(\varepsilon) = \Psi_{\varepsilon,j_0,k}{}^{-1}(U_{j_0,k} \cap \Psi_{\varepsilon,j_0,k}(G_{j_0,k})),$$

then

$$\|u_{j_0,k}\|_{L^2(G_{j_0,k})} \leq C_5(\|u_{j_0,k}\|_{L^2(\Psi_{\varepsilon,j_0,k}{}^{-1}(U_{j_0,k} \cap \Psi_{\varepsilon,j_0,k}(G_{j_0,k})))} + \|Du_{j_0,k}\|_{L^2(G_{j_0,k})}).$$

Apply to this inequality the transformation $\Psi_{\varepsilon,j_0,k}$, then by (C8) there is a constant C_6 such that

$$\|u_{j_0}\|_{L^2(G_{\varepsilon,j_0} \cap \Psi_{\varepsilon,j_0,k}(G_{j_0,k}))} \leq C_6(\|u_{j_0,k,U}\|_{L^2(U_{j_0,k})} + \|Du_{j_0}\|_{L^2(G_{\varepsilon,j_0})}). \tag{3.7}$$

Now apply inequality (3.6) a second time, with $U = U_{j_0,k}$, $V = V(\varepsilon) = B_r(z_{\varepsilon,j_0,k})$. There is a C_7 such that

$$\begin{aligned} \|u_{j_0,k,U}\|_{L^2(U_{j_0,k})} &\leq C_7(\|u_{j_0,k,U}\|_{L^2(B_r(z_{\varepsilon,j_0,k}))} + \|Du_{j_0,k,U}\|_{L^2(U_{j_0,k})}) \\ &\leq C_8(\varepsilon^{-1/2} \|u_{i_0}\|_{L^2(\{(x,y) \in G_{i_0} : 0 < x \leq \varepsilon C_1\})} \\ &\quad + \sum_{i=1}^{K_E} \varepsilon^{1/2} \|D_x u_i\|_{L^2(\Psi_{\varepsilon,i}{}^{-1}(\Psi_{\varepsilon,j_0}(U_{j_0,k}) \cap \Omega_{\varepsilon,i}))} \\ &\quad + \varepsilon^{-1/2} \|D_y u_i\|_{L^2(\Psi_{\varepsilon,i}{}^{-1}(\Psi_{\varepsilon,j_0}(U_{j_0,k}) \cap \Omega_{\varepsilon,i}))} \\ &\quad + \sum_{i=K_E+1}^{K_E+K_N} \|Du_i\|_{L^2(\Psi_{\varepsilon,i}{}^{-1}(\Psi_{\varepsilon,j_0}(U_{j_0,k}) \cap \Omega_{\varepsilon,i}))}) \end{aligned}$$

making the transformations onto G_i and $G_{\varepsilon,i}$, respectively, and using the boundedness of $\mathcal{A}_{\varepsilon,i}$.

By Lemma 3.6 there is a C_9 such that, for $d > 1/2$,

$$\|u_{j_0,k,U}\|_{L^2(U_{j_0,k})} \leq C_9(\|u_{i_0}\|_{H^1(G_{i_0})} + |[u]|_\varepsilon, d) \leq C_{10} |[u]|_{\varepsilon,d}.$$

Using inequality (3.7) and summing over all k, there is a C_{11} such that

$$\|u_{j_0}\|_{L^2(G_{\varepsilon,j_0})} \leq C_{11}|[u]|_{\varepsilon,d}$$

which implies (3.5). □

Lemma 3.8. *There is a constant $C > 0$ such that $\mathcal{A}_\varepsilon \subset B_C(0) \subset H_\varepsilon$ for all $\varepsilon \geq 0$, i.e. the attractors are bounded uniformly in $\|\cdot\|_{H_\varepsilon}$ for $\varepsilon \geq 0$.*

Proof. Let $[u_\varepsilon]: \mathbb{R} \to H_\varepsilon$ be a full bounded solution of equation (2.9). Then $\mathcal{G}_{\varepsilon,H}([u_\varepsilon(t)])$ is bounded, by say $C_1(\varepsilon)$. There is a sequence $t_n \to -\infty$ such that $(t \mapsto \mathcal{G}_{\varepsilon,H}([u_\varepsilon(t)]))'|_{t=t_n} = -\|\partial_t[u_\varepsilon(t_n)]\|^2_{L_\varepsilon} \to 0$.

Let $C_2 = C(1)$ the constant of Lemma 3.5. For n big enough $\|\partial_t[u_\varepsilon(t_n)]\|_{L_\varepsilon} < 1$ and this lemma implies $\|[u_\varepsilon(t_n)]\|_{H_\varepsilon} \leq C_2$. Since $\mathcal{G}_{\varepsilon,H}([u_\varepsilon(t)])$ is non increasing,

$$\mathcal{G}_{\varepsilon,H}([u_\varepsilon(t)]) \leq C_3\|u_\varepsilon(t_n)\|^2_{H_\varepsilon} \leq C_2C_3$$

for $t \geq t_n$ and thus for all $t \in \mathbb{R}$. (3.4) now proves the lemma. □

Lemma 3.9. *Let $\varepsilon_n \to 0$ and assume $\sigma_n: \mathbb{R} \to H_{\varepsilon_n}$ are solutions of equation (2.9) with $\|\sigma_n(t)\|_{H_{\varepsilon_n}} \leq C$ for all n and t. Then there are a constant $C_1 = C_1(C) > 0$ and a solution $\sigma_0: \mathbb{R} \to H_0$ of (2.9) with $\|\sigma_0(t)\|_{H_0} \leq C_1$. $C_1(C) \to 0$ as $C \to 0$, and there is a subsequence, called ε_n too, such that $|\sigma_n(t) - \Phi^H_{\varepsilon_n}\sigma_0(t)|_{\varepsilon_n,d} \to 0$, as $n \to \infty$, for all $0 \leq d < 1$, $t \in \mathbb{R}$.*

Proof. Fix $0 \leq d < 1$. For each $k \in \mathbb{N}$ fixed, $\|\sigma_n(-k)\|_{H_n}$ is bounded, hence taking a subsequence, called ε_n again, $(\sigma_n(-k))_j$ converges weakly in $H^1(G_j)$, $j = 1, \ldots, K_E$. By conditions (C9) and (C10) there is a $[u_{0,k}] \in H_0$ with $\|\sigma_n(-k) - \Phi^L_n[u_{0,k}]\|_{L_n} \to 0$, as $n \to \infty$. We can apply Theorem 2.2 to get

$$|\sigma_n(-k+t) - \Phi^H_n[u_{0,k}]\pi_0 t|_{n,d} \to 0 \quad \text{as } n \to \infty, \tag{3.8}$$

for each $t > 0$. Using the Cantor diagonal procedure there is a subsequence and $[u_{0,k}] \in H_0$ such that (3.8) holds for all $k \in \mathbb{N}$, $t > 0$.

Since for $k > l$ and $t > 0$

$$\sigma_n(-l+t) = \sigma_n(-k+t+k-l)$$

we have $[u_{0,l}]\pi_0 t = [u_{0,k}]\pi_0(t+k-l)$ and we can define $\sigma_0: \mathbb{R} \to H_0$, $\sigma_0(t) := [u_{0,k}]\pi_0(t+k)$ if $t > -k$. σ_0 is a solution of equation (2.9) (for $\varepsilon = 0$). (3.8) implies $\|\sigma_0(t)\|_{H_0} \leq C_1$ for all $t \in \mathbb{R}$, and $C_1 = C_1(C) \to 0$ as $C \to 0$. □

We want to prove the convergence of eigenvalues and eigenvectors if the linear operator is A_ε plus a potential V_ε. We assume that the given potentials $V_\varepsilon: L_\varepsilon \to L_\varepsilon$, $\varepsilon \geq 0$, satisfy the following conditions:

(V1) There is a constant $C_v > 0$, independent of $\varepsilon \geq 0$, such that $\|V_\varepsilon[u]\|_{L_\varepsilon} \leq C_v\|[u]\|_{L_\varepsilon}$ for all $[u] \in L_\varepsilon$.

(V2) For all $\varepsilon \geq 0$ is V_ε symmetric.

(V3) If $[u_\varepsilon] \in L_\varepsilon$, $[u_0] \in L_0$, $\|[u_\varepsilon] - \Phi^L_\varepsilon[u_0]\|_{L_\varepsilon} \to 0$, as $\varepsilon \to 0$, then $\|V_\varepsilon[u_\varepsilon] - \Phi^L_\varepsilon V_0[u_0]\|_{L_\varepsilon} \to 0$, as $\varepsilon \to 0$.

Note that (V3) implies

(V3') If $[u_\varepsilon], [w_\varepsilon] \in L_\varepsilon$, $[u_0], [w_0] \in L_0$ and $\lim_{\varepsilon\to 0} \|[u_\varepsilon] - \Phi^L_\varepsilon[u_0]\|_{L_\varepsilon} = 0 = \lim_{\varepsilon\to 0} \|[w_\varepsilon] - \Phi^L_\varepsilon[w_0]\|_{L_\varepsilon}$ then

$$(V_\varepsilon[u_\varepsilon], w_\varepsilon)_{L_\varepsilon} \to (V_0[u_0], [w_0])_{L_0}, \quad \text{as } \varepsilon \to 0.$$

For $\varepsilon \geq 0$ define a bilinear form $b_\varepsilon: H_\varepsilon \times H_\varepsilon \to H_\varepsilon$ by $b_\varepsilon([u],[v]) = a_\varepsilon([u],[v]) + (V_\varepsilon[u],[v])_{L_\varepsilon}$. In the same way as a_ε this bilinear form b_ε defines an operator $B_\varepsilon: D(B_\varepsilon) \subset L_\varepsilon \to L_\varepsilon$, B_ε is selfadjoint, sectorial, and has compact resolvent. There is a complete ONS $([u^b_{\varepsilon,j}])_j$ of L_ε consisting of eigenvectors $[u^b_{\varepsilon,j}]$ of B_ε with corresponding eigenvalues $\lambda^b_{\varepsilon,j}$. Without loss of generality we can assume these eigenvalues to be ordered $\lambda^b_{\varepsilon,1} \leq \lambda^b_{\varepsilon,2} \leq \dots$

Note that $B_\varepsilon = A_\varepsilon + V_\varepsilon$, $D(B_\varepsilon) = D(A_\varepsilon)$, H_ε is still the fractional power space $X^{1/2}_\varepsilon$, and $dist(\sigma(A_\varepsilon), \sigma(B_\varepsilon)) \leq \|V_\varepsilon\|_{L_\varepsilon} \leq C_v$, C_v as in (V1) (for the inequality see e.g. Theorem 4.10 in [11, Chapter V]).

Lemma 3.10. *Theorem* 2.1 *holds for* B_ε. *I.e. if* $\varepsilon_n \to 0$, *then* $\lambda^b_{\varepsilon_n,l} \to \lambda^b_{0,l}$, *for all* $l \in \mathbb{N}$. *There is a subsequence, called* ε_n *too, and a complete* ONS $([u^b_l])_l$ *of* L_0 *consisting of eigenvectors belonging to* $\lambda^b_{0,l}$ *such that* $|[u^b_{\varepsilon_n,l}] - \Phi^H_{\varepsilon_n}[u^b_l]|_{\varepsilon_n,d} \to 0$, *as* $n \to \infty$, *for all* $0 \leq d < 1$.

Proof. Let $\varepsilon_n \to 0$ and fix $0 \leq d < 1$. Since $\lambda_{n,l} \to \lambda_{0,l}$, as $n \to \infty$, the remark above implies that $(\lambda^b_{n,l})_n$ is bounded, for all $l \in \mathbb{N}$. Thus for l fixed, we can take a subsequence, called ε_n again, such that $\lambda^b_{n,l} \to \mu_l$.

We have

$$\|[u^b_{n,l}]\|^2_{H_n} = a_n([u^b_{n,l}],[u^b_{n,l}]) + \|[u^b_{n,l}\|^2_{L_n} \leq |\lambda^b_{n,l}| + C_v + 1.$$

Thus (C10) (recall (2.4) to bound $|\cdot|_{\varepsilon,1}$) shows

$$\varepsilon_n \sum_{j=K_E+1}^{K_E+K_N} \|[u^b_{n,l,j}]\|^2_{L^2(G_{n,j})} \to 0.$$

Also $[u^b_{n,l,j}]$ is bounded in $\|\cdot\|_{H^1(G_j)}$, for all $j = 1, \dots, K_E$. This in turn implies — taking again a subsequence — there are $u_{l,j} \in H^1(G_j)$ and $u_{n,l,j} \rightharpoonup u_{l,j}$ weakly in H^1 and strongly in L^2. Thus $[u_l] = [u_{l,1}, \dots, u_{l,K_E}] \in H_0$ (see condition (C9)), $\|[u^b_{n,l}] - \Phi^L_n[u_l]\|_{L_n} \to 0$, and $1 = \|[u^b_{n,l}]\|_{L_n} \to \|[u_l]\|_{L_0}$.

Using Lemma 3.4 and (V3') we find for all $[u] \in H_0$, as $n \to \infty$

$$\begin{aligned}\mu_l([u_l],[u])_{L_0} \leftarrow \lambda^b_{n,l}([u^b_{n,l}], \Phi^H_n[u])_{L_n} &= a_n([u^b_{n,l}], \Phi^H_n[u]) + (V_n[u^b_{n,l}], \Phi^H_n[u])_{L_n} \\ &\to a_0([u_l],[u]) + (V_0[u_l],[u])_{L_0} = b_0([u_l],[u])\end{aligned}$$

and $(\mu_l, [u_l])$ is an eigenvalue, vector pair of B_0. Also

$$\begin{aligned} a_n([u^b_{n,l}],[u^b_{n,l}]) &= b_n([u^b_{n,l}],[u^b_{n,l}]) - (V_n[u^b_{n,l}],[u^b_{n,l}])_{L_n} \to \mu_l - (V_0[u_l],[u_l])_{L_0} \\ &= b_0([u_l],[u_l]) - (V_0[u_l],[u_l])_{L_0} = a_0([u_l],[u_l]) \end{aligned}$$

and we can apply Lemma 3.4 getting $|[u^b_{n,l}] - \Phi^H_n[u_l]|_{n,d} \to 0$, as $n \to \infty$.

With the Cantor diagonal procedure there is a subsequence such that we have above results not only for one l but for all $l \in \mathbb{N}$. That is we can assume

$$\lim_{n\to\infty} \lambda^b_{n,l} = \mu_l, \quad \lim_{n\to\infty} |[u^b_{n,l}] - \Phi^H_n[u_l]|_{n,d} = 0, \quad \|[u_l]\|_{L_0} = 1$$

and $(\mu_l, [u_l])$ is an eigenvector, value pair for B_0, for all $l \in \mathbb{N}$.

If $l \neq k$, then as $n \to \infty$,

$$0 = ([u^b_{n,l}],[u^b_{n,k}])_{L_n} \to ([u_l],[u_k])_{L_0}$$

and $([u_l])_l$ is an ONS of L_0. Assume it is not complete, then there is a $0 \neq [u] \in L_0$ such that $([u_l],[u])_{L_0} = 0$ for all l. Since there is a complete ONS of L_0 consisting entirely of eigenvectors of B_0, we can without loss of generality assume $[u]$ to be such an eigenvector, and in particular $[u] \in H_0$.

Write $\Phi^H_n[u] = \sum_{l\geq 1} \alpha_{n,l}[u_{n,l}]$ (recall that $[u_{n,l}]$ is an eigenvector for the eigenvalue $\lambda_{n,l}$ of A_n). Then for all l

$$\alpha_{n,l} = (\Phi^H_n[u],[u_{n,l}])_{L_n} \to ([u],[u_l])_{L_0} = 0, \quad \text{as } n \to \infty,$$

and by (2.1) there is a constant C_1 such that

$$\sum_{l\geq 1} \lambda_{n,l}\alpha^2_{n,l} = a_n(\Phi^H_n[u], \Phi^H_n[u]) \leq C_1.$$

For every $\delta > 0$ there are $n_1, l_1 \in \mathbb{N}$ such that for $n \geq n_1, l \geq l_1$ we have $\lambda_{n,l} \geq 1/\delta$. For $n \geq n_1$ we get

$$\sum_{l>l_1} \alpha^2_{n,l} \leq \frac{C_1}{\lambda_{n,l_1}} \leq \delta C_1$$

and thus, as $n \to \infty$

$$\|[u]\|^2_{L_0} \leftarrow \|\Phi^H_n[u]\|^2_{L_n} \leq \underbrace{\sum_{l=1}^{l_1} \alpha^2_{n,l}}_{\to 0} + C_1\delta.$$

Hence $[u] = 0$, which cannot be, and $([u_l])_l$ has to be complete.

The only thing we still have to show is $\lambda^b_{n,l} \to \lambda^b_{0,l}$ for all l for the original sequence ε_n. For this it is sufficient to show $\mu_l = \lambda^b_{0,l}$ for all l.

Assume this to be false. Then there is a $l_2 \in \mathbb{N}$ such that $\mu_l = \lambda^b_{0,l}$, $l = 1, \ldots, l_2 - 1$, $\mu_{l_2} \neq \lambda^b_{0,l_2}$. Now if $\mu_{l_2} < \lambda^b_{0,l_2}$, then $\mu_{l_2} = \lambda^b_{0,l}$ for some $l \in \{1, \ldots, l_2 - 1\}$ and $[u_{l_2}]$ is a linear combination of the first $l_2 - 1$ eigenvectors of b_0, i.e. of $[u_1], \ldots, [u_{l_2-1}]$, which contradicts the orthogonality of $([u_l])_l$. If $\mu_{l_2} > \lambda^b_{0,l_2}$, then there is an eigenvector of B_0 for the eigenvalue λ_{0,l_2} which is orthogonal to all $[u_l]$, which contradicts the completeness of $([u_l])_l$.

Hence the assumption is false and $\mu_l = \lambda^b_{0,l}$ for all l. □

Lemma 3.11. *Assume $0 \notin \sigma(B_0)$. Then there are $\varepsilon_0, C > 0$ such that for all $0 \le \varepsilon \le \varepsilon_0$ we have $0 \notin \sigma(B_\varepsilon)$ and $\|B_\varepsilon{}^{-1}[u]\|_{H_\varepsilon} \le C\|[u]\|_{L_\varepsilon}$, for all $[u] \in L_\varepsilon$. Also, if $[u_\varepsilon] \in L_\varepsilon$, $[u_0] \in L_0$, $\lim_{\varepsilon\to 0} \|[u_\varepsilon] - \Phi^L_\varepsilon[u_0]\|_{L_\varepsilon} = 0$, then as $\varepsilon \to 0$, for all $0 \le d < 1$,*

$$|B_\varepsilon{}^{-1}[u_\varepsilon] - \Phi^H_\varepsilon B_0{}^{-1}[u_0]|_{\varepsilon,d} \to 0.$$

Proof. Fix $0 \le d < 1$. $0 \notin \sigma(B_\varepsilon)$ for $0 \le \varepsilon \le \varepsilon_0$, for some $\varepsilon_0 > 0$ follows directly from the convergence of the eigenvalues of B_ε to those of B_0 (Lemma 3.10).

For given $[u] \in L_\varepsilon$ we can use the ONS $(u^b_{\varepsilon,l})_l$ to find $\|B_\varepsilon{}^{-1}[u]\|_{L_\varepsilon} \le C_1\|[u]\|_{L_\varepsilon}$, where $C_1 = C_1(\varepsilon_0)$ is independent of ε. Thus

$$\begin{aligned}\|B_\varepsilon{}^{-1}[u]\|^2_{H_\varepsilon} &= (A_\varepsilon B_\varepsilon{}^{-1}[u], B_\varepsilon{}^{-1}[u])_{L_\varepsilon} + \|B_\varepsilon{}^{-1}[u]\|^2_{L_\varepsilon} \\ &= ([u], B_\varepsilon{}^{-1}[u])_{L_\varepsilon} - (V_\varepsilon B_\varepsilon{}^{-1}[u], B_\varepsilon{}^{-1}[u])_{L_\varepsilon} + \|B_\varepsilon{}^{-1}[u]\|^2_{L_\varepsilon} \le C_2\|[u]\|^2_{L_\varepsilon},\end{aligned}$$

where C_2 is independent of ε (see (V1)).

Assume the convergence of the resolvents is not true. Then there is a sequence $\varepsilon_n \to 0$, $\delta_1 > 0$, $[u_n] \in L_n$, $[u_0] \in L_0$ such that $|B_n{}^{-1}[u_n] - \Phi^H_n B_0{}^{-1}[u_0]|_{n,d} \ge \delta_1$ for all n and $\|[u_n] - \Phi^L_n[u_0]\|_{L_n} \to 0$.

Taking a subsequence, called ε_n too, by Lemma 3.10 we can assume $|[u^b_{n,l}] - \Phi^H_n[u^b_{0,l}]|_{n,d} \to 0$, $n \to \infty$, for all $l \in \mathbb{N}$.

Setting $[w_n] := B_n{}^{-1}[u_n]$, $[w_0] := B_0{}^{-1}[u_0]$, we see $\|[u_n]\|_{L_n}$, $\|[w_n]\|_{H_n}$ and $\|A_n[w_n]\|_{L_n}$ are bounded. If $[u_n] = \sum_{l\ge 1}\alpha_{n,l}[u^b_{n,l}]$, $[u_0] = \sum_{l\ge 1}\alpha_{0,l}[u^b_{0,l}]$, then $\alpha_{n,l} \to \alpha_{0,l}$, for all l, as $n \to \infty$. Since for all $C > 0$ there is a $l_1 = l_1(C)$ such that $\lambda_{n,l} \ge C$ for all $l \ge l_1$, $n \in \mathbb{N}$, we see as $n \to \infty$

$$\begin{aligned}\|[w_n] - \Phi^L_n[w_0]\|_{L_n} \le{}& \underbrace{\bigg\| \sum_{l=1}^{l_1} \frac{\alpha_{n,l}}{\lambda^b_{n,l}}[u^b_{n,l}] - \frac{\alpha_{0,l}}{\lambda^b_{0,l}}\Phi^L_n[u^b_{0,l}]\bigg\|_{L_n}}_{\to 0} \\ &+ \underbrace{\bigg\| \sum_{l>l_1} \frac{\alpha_{n,l}}{\lambda^b_{n,l}}[u^b_{n,l}]\bigg\|_{L_n}}_{\le \|[u_n]\|_{L_n}/\lambda^b_{n,l_1}} + \underbrace{\bigg\| \Phi^L_n \sum_{l>l_1} \frac{\alpha_{0,l}}{\lambda^b_{0,l}}[u^b_{0,l}]\bigg\|_{L_n}}_{\le C_3\|[u_0]\|_{L_0}/\lambda^b_{0,l_1}} \to 0,\end{aligned}$$

$$b_n([w_n],[w_n]) = ([u_n],[w_n])_{L_n} = \sum_{l=1}^{l_1} \frac{\alpha_{n,l}^2}{\lambda_{n,l}^b} + \underbrace{\sum_{l>l_1} \frac{\alpha_{n,l}^2}{\lambda_{n,l}^b}}_{\leq \|[u_n]\|_{L_n}^2/\lambda_{n,l_1}^b}$$

$$\to \sum_{l\geq 1} \frac{\alpha_{0,l}^2}{\lambda_{0,l}^b} = ([u_0],[w_0])_{L_0} = b_0([w_0],[w_0]).$$

By (V3')

$$(V_n[w_n],[w_n])_{L_n} \to (V_0[w_0],[w_0])_{L_0}, \quad a_n([w_n],[w_n]) \to a_0([w_n],[w_n]),$$

as $n \to \infty$. We can apply Lemma 3.4 and get $|[w_n] - \Phi_n^H[w_0]|_{n,d} \to 0$. This is a contradiction, and the proof is complete. □

4. Continuity

In this section we shall prove Theorem 1.1. That is we shall show that the family of attractors $\mathcal{A}_\varepsilon$ is lower-semi-continuous at $\varepsilon = 0$. Theorem 2.2 then implies the continuity, i.e. Theorem 1.1.

Assume f satisfies (H2). We know already that there is a $C_A > 0$ such that $\|[u]\|_{L_\varepsilon^\infty} \leq C_A$ for all $[u] \in \mathcal{A}_\varepsilon$ and $\varepsilon \geq 0$ (see Proposition 3.3). Eventually increasing C_A we can assume

$$\frac{f(s)}{s} \leq -\frac{3}{4}\xi, \quad |s| \geq C_A,$$

where ξ is as in condition (H2).

There is a C^2-function $g: \mathbb{R} \to \mathbb{R}$ which coincides with f on $|s| \leq 2C_A$, satisfies

$$\frac{g(s)}{s} \leq -\frac{1}{2}\xi \quad \text{for } |s| \geq C_A, \quad g''(s) = 0 \quad \text{for } |s| \geq 3C_A,$$

and g still satisfies conditions (H1) and (H2'). Denote by g_ε the Nemitsky operator of g on H_ε, $\varepsilon \geq 0$. The differential equations

$$(4.1) \qquad [u_t] = -A_\varepsilon[u] + g_\varepsilon([u]), \quad t > 0$$

define semiflows $\widetilde{\pi}_\varepsilon$ on H_ε, $\varepsilon \geq 0$. Theorem 2.2 still holds, thus all $\widetilde{\pi}_\varepsilon$ are global semiflows and $\widetilde{\pi}_\varepsilon$ converges to $\widetilde{\pi}_0$ in the sense of this theorem.

On the attractor $\mathcal{A}_\varepsilon$ the semiflows $\widetilde{\pi}_\varepsilon$ and π_ε coincide. Also, for all $[u] \in \mathcal{A}_\varepsilon$, $[v] \in L_\varepsilon$, we have $Df_\varepsilon([u])[v] = Dg_\varepsilon([u])[v]$, $\varepsilon \geq 0$. By Lemma 3.2(c) any $[u]$ which is a point of equilibrium of $\widetilde{\pi}_\varepsilon$, $\varepsilon > 0$, satisfies $\|[u]\|_{L_\varepsilon^\infty} \leq C_A$, that is $[u] \in \mathcal{A}_\varepsilon$. In Theorem 1.1 the condition on the spectrum of $Df_\varepsilon([u])$, $[u]$ a point of equilibrium for π_ε, becomes simply the following:

The semiflow $\widetilde{\pi}_0$ has only finitely many points of equilibrium $\{[\widetilde{u}_1^0], \dots, [\widetilde{u}_{M_0}^0]\}$ and 0 is not in the spectrum of the linear operators $A_0 - Dg_0([\widetilde{u}_j^0])\mathrm{id}: D(A_0) \to L_0$ for all $j = 1, \dots, M_0$.

In this section we will consider equation (4.1) and assume the condition above holds.

To simplify notation we will drop the tilde "$\widetilde{\ }$" in the notation of the semiflows and their points of equilibrium. That is we shall write π_ε and $[u_m^0]$ for $\widetilde{\pi}_\varepsilon$ and $[\widetilde{u}_m^0]$.

As a first step we show that each $[u_m^0]$ is the limit of a point of equilibrium of π_ε. Before we do this, we need some technical lemmas.

Lemma 4.1. *Let $p > 2$, $G \subset \mathbb{R}^{M+1}$ be open, bounded, and g_G the Nemitsky operator of g on G. Let $C_1 \geq |G|$. Then $g_G: L^p(G) \to L^2(G)$ is C^1, $Dg_G(u)v(z) = g'(u(z))v(z)$, $z \in G$, and there are $\beta > 1 > \gamma > 0$, $C_2 > 0$, β, γ, C_2 independent of G, $C_2 = C_2(C_1)$, such that for all $u, v, w \in L^p(G)$*

$$\|g_G(u+v) - g_G(u) - Dg_G(u)v\|_{L^2(G)} \leq C_2 \|v\|^\beta_{L^p(G)}, \tag{4.2}$$

$$\begin{aligned} &\|g_G(u+v) - g_G(u+w) - Dg_G(u)(v-w)\|_{L^2(G)} \\ &\qquad\qquad \leq C_2 \|v-w\|_{L^p(G)} (\|v\|^2_{L^2(G)} + \|w\|^2_{L^2(G)})^\gamma, \end{aligned} \tag{4.3}$$

$$\|g_G(u+v) - g_G(u)\|_{L^2(G)} \leq C_2 \|v\|_{L^2(G)} \quad \textit{for all } u, v \in L^2(G). \tag{4.4}$$

Proof. Note that $g''(s) = 0$ for $|s|$ big enough shows that g' and g'' are bounded. Thus indeed $g_G: L^2(G) \to L^2(G)$ and (4.4) holds.

Let $u, v, w \in L^p(G)$. Then

$$\begin{aligned} &\int_G (g_G(u+v) - g_G(u+w) - Dg_G(u)(v-w))^2 \, dz \\ &\qquad \leq \|v-w\|^2_{L^p(G)} \bigg(\int_G (g'(u+w+\xi) - g'(u))^{2p/(p-2)} \, dz \bigg)^{(p-2)/p} \\ &\qquad \leq C_2 \|v-w\|^2_{L^p(G)} \bigg(\int_G |g''(\zeta)|^2 |w+\xi|^2 \, dz \bigg)^{(p-2)/p} \\ &\qquad \leq C_3 \|v-w\|^2_{L^p(G)} \|w+\xi\|^{2(p-2)/p}_{L^2(G)}, \end{aligned}$$

where C_2, C_3 depend only on g and p, $\xi = \xi(z)$ is between 0 and $v(z) - w(z)$, $\zeta = \zeta(z)$.

Choose $w = 0$, then $\|\xi\|_{L^2(G)} \leq \|v\|_{L^2(G)}$. Thus $g_G: L^p(G) \to L^2(G)$ is Frechét-differentiable and (4.2) holds with $\beta = 2 - 2/p$.

If w is arbitrary, then $\|w+\xi\|^2_{L^2(G)} \leq \|v\|^2_{L^2(G)} + \|w\|^2_{L^2(G)}$ and (4.3) holds.

Now

$$\begin{aligned} &\int_G ((Dg_G(u+v) - Dg_G(u))w)^2 \, dz \\ &\qquad\qquad \leq \|w\|^2_{L^p(G)} \bigg(\int_G |g'(u+v) - g'(u)|^{2p/(p-2)} \, dz \bigg)^{(p-2)/p}. \end{aligned}$$

As $\|v\|_{L^p(G)} \to 0$, $g'(u(z)+v(z))-g'(u(z)) \to 0$ for a.a. $z \in G$. With the Lebesgue dominated convergence the integral on the right-hand-side above tends to 0 and $Dg_G\colon L^p(G) \times L^p(G) \to L^2(G)$ is continuous. □

As a direct consequence of condition (C10) of the second section we have

Lemma 4.2. *If*

$$C(\varepsilon) := \sup\left(\frac{\varepsilon\sum_{j=K_E+1}^{K_E+K_N}\|u_j\|^2_{L^2(G_{\varepsilon,j})}}{\|[u]\|^2_{H_\varepsilon}} : 0 \neq [u] \in H_\varepsilon\right),$$

then $C(\varepsilon) \to 0$, *as* $\varepsilon \to 0$.

Lemma 4.3. $g_\varepsilon\colon H_\varepsilon \to L_\varepsilon$ *is* C^1, $Dg_\varepsilon([u])[v](z) = g'(u_j(z))v_j(z)$ *for* $z \in G_j$ *or* $z \in G_{\varepsilon,j}$, *resp. and all possible* j, $\varepsilon \geq 0$. *Also the following hold:*

(a) *Let* $\varepsilon \geq 0$. $g_\varepsilon\colon L_\varepsilon \to L_\varepsilon$ *and* $Dg_\varepsilon\colon L_\varepsilon \times L_\varepsilon \to L_\varepsilon$ *are well defined. For each* $[u] \in L_\varepsilon$ *is* $Dg_\varepsilon([u])\colon L_\varepsilon \to L_\varepsilon$ *a symmetric operator. There is a* $C_1 > 0$, *independent of* ε, *such that*

$$\|Dg_\varepsilon([u])[v]\|_{L_\varepsilon} \leq C_1\|[v]\|_{L_\varepsilon} \quad \text{for all } [u],[v] \in L_\varepsilon,$$
$$\|g_\varepsilon([u]+[v]) - g_\varepsilon([u])\|_{L_\varepsilon} \leq C_1\|[v]\|_{L_\varepsilon} \quad \text{for all } [u],[v] \in L_\varepsilon.$$

(b) *Let* $0 \leq d < 1$. *There are* $\beta > 1$, γ, $C_2 = C_2(d) > 0$, *all independent of* $\varepsilon \geq 0$, *and* $C_3(\varepsilon) > 0$, $C_3(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, $C_3(0) = 0$, *such that, for all* $[u],[v],[w] \in H_\varepsilon$,

$$\|g_\varepsilon([u]+[v]) - g_\varepsilon([u]) - Dg_\varepsilon([u])[v]\|_{L_\varepsilon} \leq C_2|[v]|^\beta_{\varepsilon,d} + C_3(\varepsilon)\|[v]\|_{H_\varepsilon},$$

$$\|g_\varepsilon([u]+[v]) - g_\varepsilon([u]+[w]) - Dg_\varepsilon([u])([v]-[w])\|_{L_\varepsilon}$$
$$\leq C_2|[v]-[w]|_{\varepsilon,d}(\|[v]\|^2_{L_\varepsilon} + \|[w]\|^2_{L_\varepsilon})^\gamma + C_3(\varepsilon)\|[v]-[w]\|_{H_\varepsilon}.$$

(c) *Let* $\varepsilon \geq 0$. *For all* $\widetilde{C} > 0$ *there is a* $C_4(\varepsilon) = C_4(\varepsilon,\widetilde{C}) > 0$ *such that*

$$\|g_\varepsilon([u]+[v]) - g_\varepsilon([u]+[w]) - Dg_\varepsilon([u])([v]-[w])\|_{L_\varepsilon} \leq \widetilde{C}\|[v]-[w]\|_{H_\varepsilon}$$

for all $[u],[v],[w] \in H_\varepsilon$, $\|[v]\|_{H_\varepsilon}, \|[w]\|_{H_\varepsilon} \leq C_4(\varepsilon)$.

(d) *If* $[u_\varepsilon],[v_\varepsilon] \in L_\varepsilon$, $[u_0],[v_0] \in L_0$ *and*

$$\lim_{\varepsilon\to 0}\|[u_\varepsilon] - \Phi^L_\varepsilon[u_0]\|_{L_\varepsilon} = \lim_{\varepsilon\to 0}\|[v_\varepsilon] - \Phi^L_\varepsilon[v_0]\|_{L_\varepsilon} = 0,$$

then

$$\lim_{\varepsilon\to 0}\|g_\varepsilon([u_\varepsilon]) - \Phi^L_\varepsilon g_0([u_0])\|_{L_\varepsilon} = \lim_{\varepsilon\to 0}\|Dg_\varepsilon([u_\varepsilon])[v_\varepsilon] - \Phi^L_\varepsilon Dg_0([u_0])[v_0]\|_{L_\varepsilon} = 0.$$

Proof. Let $2 < p \leq p^* = 2(M+1)/(M-1)$. All G_j and Ω_ε are Lipschitz, hence $H^1(G_j) \subset L^p(G_j)$, $H^1(\Omega_\varepsilon) \subset L^p(\Omega_\varepsilon)$.

Thus for $\varepsilon > 0$ Lemma 4.1 implies directly $g_\varepsilon \colon H_\varepsilon \to L_\varepsilon$ is C^1. For $\varepsilon = 0$ we use the same argument for each G_j separately to get the same conclusion.

The formula for Dg_ε obviously holds. The boundedness of g', Ω_ε and all G_j imply that $g_\varepsilon \colon L_\varepsilon \to L_\varepsilon$, $Dg_\varepsilon \colon L_\varepsilon \times L_\varepsilon \to L_\varepsilon$ are well defined for all $\varepsilon \geq 0$ and (a) is true.

Now assume the situation in (d). For $j \in \{1, \dots, K_E\}$ we have $u_{\varepsilon,j} \to u_{0,j}$, $v_{\varepsilon,j} \to v_{0,j}$ in $L^2(G_j)$. Hence with Lemma 4.1 follow

$$\sum_{j=1}^{K_E} \int_{G_j} (g(u_{\varepsilon,j}) - g(u_{0,j}))^2 d\lambda_{\varepsilon,j} \to 0,$$

$$\begin{aligned}\sum_{j=1}^{K_E} \int_{G_j} (g'(u_{\varepsilon,j})v_{\varepsilon,j} - g'(u_{0,j})v_{0,j})^2 d\lambda_{\varepsilon,j} \leq C_5 \sum_{j=1}^{K_E} &\|g'(u_{\varepsilon,j})(v_{\varepsilon,j} - v_{0,j})\|^2_{L^2(G_j)} \\ &+ \|(g'(u_{\varepsilon,j}) - g'(u_{0,j}))v_{0,j}\|^2_{L^2(G_j)} \to 0.\end{aligned}$$

For $j > K_E$ and $\varepsilon > 0$ we have $(\Phi^L_\varepsilon[u])_j = 0$ for all $[u] \in L_\varepsilon$. Hence

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \|u_{\varepsilon,j}\|^2_{L^2(G_{\varepsilon,j})} = \lim_{\varepsilon \to 0} \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \|v_{\varepsilon,j}\|^2_{L^2(G_{\varepsilon,j})} = 0$$

and by inequality (4.4) there is a C_6 such that

$$\begin{aligned}\varepsilon &\sum_{j=K_E+1}^{K_E+K_N} \|g(u_{\varepsilon,j})\|^2_{L^2(G_{\varepsilon,j})} \\ &\qquad \leq 2\varepsilon \sum_{j=K_E+1}^{K_E+K_N} (C_6 \|u_{\varepsilon,j}\|^2_{L^2(G_{\varepsilon,j})} + \|g(0)\|^2_{L^2(G_{\varepsilon,j})}) \to 0,\end{aligned}$$

$$\varepsilon \sum_{j=K_E+1}^{K_E+K_N} \|Dg(u_{\varepsilon,j})v_{\varepsilon,j}\|^2_{L^2(G_{\varepsilon,j})} \leq C_6 \varepsilon \sum_{j=K_E+1}^{K_E+K_N} \|v_{\varepsilon,j}\|^2_{L^2(G_{\varepsilon,j})} \to 0,$$

as $\varepsilon \to 0$. This proves (d).

To prove (b) assume for a moment $j \in \{1, \dots, K_E\}$. By Lemma 4.1 there are $\beta > 1$, γ, $C_7 > 0$, all independent of $\varepsilon \geq 0$, such that for all $[u], [v], [w] \in H_\varepsilon$, $\varepsilon \geq 0$,

$$\begin{aligned}\|g(u_j + v_j) - g(u_j) - Dg(u_j)v_j\|^2_{L^2(G_j)} &\leq C_7 \|v_j\|^{2\beta}_{L^p(G_j)} \\ &\leq C_8 \|v_j\|^{2\beta}_{H^1(G_j)} \leq C_9 |[v]|^{2\beta}_{\varepsilon,d}.\end{aligned}$$

Analogously

$$\|g(u_j+v_j)-g(u_j+w_j)-Dg(u_j)(v_j-w_j)\|^2_{L^2(G_j)} \\ \leq C_{10}|[v]-[w]|^2_{\varepsilon,d}(\|[v]\|^2_{L_\varepsilon}+\|[u]\|^2_{L_\varepsilon})^{2\gamma}.$$

If $j>K_E$, $\varepsilon>0$, let $C_{11}(\varepsilon)\to 0$ be the constant from Lemma 4.2. Then

$$\varepsilon\|g(u_j+v_j)-g(u_j+w_j)-Dg(u_j)(v_j-w_j)\|^2_{L^2(G_{\varepsilon,j})} \\ \leq \varepsilon C_{12}\|v_j-w_j\|^2_{L^2(G_{\varepsilon,j})}\leq C_{11}(\varepsilon)C_{12}\|[v]-[w]\|^2_{H_\varepsilon}.$$

Thus the second inequality in (b) holds, and choosing $[w]=0$ the first one too.

Analogously, either using Lemma 4.1 directly for Ω_ε, $\varepsilon>0$, or for each G_j separately ($\varepsilon=0$), one proves (c). □

Now we prove the continuity — in a certain sense — of the equilibrium points of π_ε. Recall that $\{[u_1^0],\dots,[u_{M_0}^0]\}$ are the points of equilibrium of π_0.

Lemma 4.4. *Fix $m\in\{1,\dots,M_0\}$ and $0\leq d<1$. There are $\varepsilon_0,C>0$ and for all $0\leq\varepsilon\leq\varepsilon_0$ there are $[u_m^\varepsilon]\in D(A_\varepsilon)$ such that $\|[u_m^\varepsilon]\|_{L_\varepsilon^\infty}\leq C$, $A_\varepsilon[u_m^\varepsilon]=g_\varepsilon([u_m^\varepsilon])$ and $|[u_m^\varepsilon]-\Phi_\varepsilon^H[u_m^0]|_{\varepsilon,d}\to 0$, as $\varepsilon\to 0$.*

Proof. Recall that $\Phi_0^H=\mathrm{id}$ on H_0. For $\varepsilon\geq 0$ set $V_\varepsilon=V_\varepsilon(m)\colon L_\varepsilon\to L_\varepsilon$ by

$$V_\varepsilon[u]:=Dg_\varepsilon(\Phi_\varepsilon^H[u_m^0])[u].$$

The potentials $-V_\varepsilon$ satisfy conditions (V1)–(V3) of section three. In particular there is a linear operator $B_\varepsilon:=A_\varepsilon-V_\varepsilon\colon D(A_\varepsilon)\to L_\varepsilon$ which is selfadjoint, sectorial, has compact resolvent, and there are complete ONS $([u^b_{\varepsilon,l}])_l$ of L_ε consisting of eigenvectors with corresponding eigenvalues $\lambda^b_{\varepsilon,1}\leq\lambda^b_{\varepsilon_2}\leq\dots$

By Lemma 3.10 the $\lambda^b_{\varepsilon,l}\to\lambda^b_{0,l}$, $\varepsilon\to 0$, for all l, and by Lemma 3.11 the assumption $0\notin\sigma(B_0)$ shows $0\notin\sigma(B_\varepsilon)$, $0\leq\varepsilon\leq\varepsilon_0$, for some $\varepsilon_0>0$.

For $\varepsilon\geq 0$ define $T_\varepsilon=T_{\varepsilon,m}\colon H_\varepsilon\to H_\varepsilon$ by

$$T_\varepsilon[u]:={B_\varepsilon}^{-1}(g_\varepsilon([u])-V_\varepsilon[u]).$$

We shall show that T_ε has a fixed point, which will be $[u_m^\varepsilon]$.

Using Lemmas 3.11 and 4.3(b) there are constants $C_1,\gamma>0$, independent of ε, and $C_2(\varepsilon)\to 0=C_2(0)$ such that for $[v],[w]\in L_\varepsilon$

$$\|T_\varepsilon[v]-T_\varepsilon[w]\|_{H_\varepsilon}\leq C_1|[v]-[w]|_{\varepsilon,d}(\|[v]-\Phi_\varepsilon^H[u_m^0]\|^2_{L_\varepsilon}+\|[w]-\Phi_\varepsilon^H[u_m^0]\|^2_{L_\varepsilon})^\gamma \\ +C_2(\varepsilon)\|[v]-[w]\|_{H_\varepsilon}.$$

Since there is a C_3 such that $\|\cdot\|_{L_\varepsilon}\leq C_3|\cdot|_{\varepsilon,d}\leq(C_3)^2\|\cdot\|_{H_\varepsilon}$, there is a C_4 such that for $[v],[w]\in\{[u]\in H_\varepsilon:|[u]-\Phi_\varepsilon^H[u_m^0]|_{\varepsilon,d}\leq C_4\}$

$$\begin{aligned}|T_\varepsilon[v]-T_\varepsilon[w]|_{\varepsilon,d}&\leq\frac12|[v]-[w]|_{\varepsilon,d}+C_2(\varepsilon)C_3\|[v]-[w]\|_{H_\varepsilon},\\ \|T_\varepsilon[v]-T_\varepsilon[w]\|_{H_\varepsilon}&\leq\frac12|[v]-[w]|_{\varepsilon,d}+C_2(\varepsilon)\|[v]-[w]\|_{H_\varepsilon}.\end{aligned}\tag{4.5}$$

Recall that $\Phi_\varepsilon^H: H_0 \to H_\varepsilon$ is a bounded operator, the bound being independent of ε. Thus by Lemma 4.3(a) $\|g_\varepsilon(\Phi_\varepsilon^H[u_m^0])\|_{L_\varepsilon}$ is bounded and by Lemma 3.11 there is a $C_5 > 4C_4$ such that $\|\Phi_\varepsilon^H[u_m^0]\|_{H_\varepsilon}, \|T_\varepsilon\Phi_\varepsilon^H[u_m^0]\|_{H_\varepsilon} \le 1/2C_5$.

Eventually decreasing $\varepsilon_0 > 0$ so that $C_2(\varepsilon) \le (1/4)\min(1, C_4/C_5)$, we get for $[v]$ as above

$$\|T_\varepsilon[v]\|_{H_\varepsilon} \le \frac{3}{4}C_4 + \frac{1}{2}C_5 + \frac{\|[v]\|_{H_\varepsilon}}{4} < \frac{3}{4}C_5 + \frac{1}{4}\|[v]\|_{H_\varepsilon}.$$

Now $T_0[u_m^0] = [u_m^0]$ and

$$\|g_\varepsilon(\Phi_\varepsilon^H[u_m^0]) - Dg_\varepsilon(\Phi_\varepsilon^H[u_m^0])\Phi_\varepsilon^H[u_m^0] - \Phi_\varepsilon^L(g_0([u_m^0]) - Dg_0([u_m^0])[u_m^0])\|_{L_\varepsilon} \to 0$$

by Lemma 4.3(d), thus $|T_\varepsilon\Phi_\varepsilon^H[u_m^0] - \Phi_\varepsilon^H T_0[u_m^0]|_{\varepsilon,d} \to 0$ by Lemma 3.11, as $\varepsilon \to 0$.

Putting all together, and eventually decreasing $\varepsilon_0 > 0$ further, for $0 < \varepsilon \le \varepsilon_0$

$$\begin{aligned} T_\varepsilon: \{[u] \in H_\varepsilon : \|u\|_{H_\varepsilon} \le C_5,\ |[u] - \Phi_\varepsilon^H[u_m^0]|_{\varepsilon,d} \le C_4\} \\ \to \{[u] \in H_\varepsilon : \|u\|_{H_\varepsilon} \le C_5,\ |[u] - \Phi_\varepsilon^H[u_m^0]|_{\varepsilon,d} \le C_4\}. \end{aligned}$$

A_ε has compact resolvent, hence $D(A_\varepsilon) \subset H_\varepsilon$ compactly and $T_\varepsilon: H_\varepsilon \to H_\varepsilon$ is completely continuous. By the Schauder fixed-point theorem T_ε has a fixed-point, say $[u_m^\varepsilon]$, and $|[u_m^\varepsilon] - \Phi_\varepsilon^H[u_m^0]|_{\varepsilon,d} \le C_4$. Since we can choose C_4 arbitrarily small (and decrease ε_0 with C_4), we can assume $|[u_m^\varepsilon] - \Phi_\varepsilon^H[u_m^0]|_{\varepsilon,d} \to 0$, as $\varepsilon \to 0$.

Obviously $[u_m^\varepsilon] \in D(A_\varepsilon)$ and $g([u_m^\varepsilon]) = A_\varepsilon[u_m^\varepsilon]$. By Lemma 3.2(c)

$$\|[u_m^\varepsilon]\|_{L_\varepsilon^\infty} \le C_6. \qquad \square$$

Now we can show that the points of equilibrium depend continuously on ε at $\varepsilon = 0$.

Lemma 4.5. *The family of points of equilibrium*

$$\mathcal{E}_\varepsilon := \{[u] \in D(A_\varepsilon) : A_\varepsilon[u] = g_\varepsilon([u])\}$$

is continuous at $\varepsilon = 0$, i.e. for $0 \le d < 1$

$$\lim_{\varepsilon \to 0} \operatorname{dist}_{\varepsilon,d}(\mathcal{E}_\varepsilon, \mathcal{E}_0) = 0,$$

where $\operatorname{dist}_{\varepsilon,d}$ *is defined in Theorem* 1.1.

Proof. Fix $0 \le d < 1$. We have to show

$$\text{(4.6)} \qquad \lim_{\varepsilon \to 0} \sup_{[u] \in \mathcal{E}_\varepsilon} \inf_{[v] \in \mathcal{E}_0} |[u] - \Phi_\varepsilon^H[v]|_{\varepsilon,d} = \lim_{\varepsilon \to 0} \sup_{[v] \in \mathcal{E}_0} \inf_{[u] \in \mathcal{E}_\varepsilon} |[u] - \Phi_\varepsilon^H[v]|_{\varepsilon,d} = 0.$$

Assume the first limit is not 0. Then there is a sequence $\varepsilon_n \to 0$, $\delta > 0$, $[u_n] \in \mathcal{E}_n$ such that

$$\inf_{[v] \in \mathcal{E}_0} |[u_n] - \Phi_n^H[v]|_{n,d} \ge \delta \quad \text{for all } n.$$

By Lemma 3.8 $\mathcal{A}_\varepsilon$ are bounded in $\|\cdot\|_{H_\varepsilon}$ uniformly in ε, hence, taking a subsequence, by (C9) there is a $[u_0] \in H_0$ and by (C10) $\|[u_n] - \Phi_n^L[u_0]\|_{L_n} \to 0$ as $n \to \infty$.

By Theorem 2.2, for all $t > 0$,

$$|[u_n] - \Phi_n^H[u_0]\pi_0 t|_{n,d} = |[u_n]\pi_n t - \Phi_n^H[u_0]\pi_0 t|_{n,d} \to 0$$

as $n \to \infty$. Hence $[u_0]$ is a point of equilibrium for π_0, that is $[u_0] \in \mathcal{E}_0$, and we have a contradiction.

Assume now the second limit in (4.6) is not 0. Then there is a sequence $\varepsilon_n \to 0$, $[v_n] \in \mathcal{E}_0$, and $\delta > 0$ such that

$$\inf_{[u]\in\mathcal{E}_n} |[u] - \Phi_n^H[v_n]|_{n,d} \geq \delta$$

for all n. By assumption $\mathcal{E}_0$ is finite, hence taking a subsequence we can without loss of generality assume $[v_n] = [u_m^0]$ for all n.

By Lemma 4.4 for ε sufficiently small there are $[u_m^\varepsilon] \in \mathcal{E}_\varepsilon$ and $|[u_m^\varepsilon] - \Phi_\varepsilon^H[u_m^0]|_{\varepsilon,d} \to 0$, as $\varepsilon \to 0$. This is a contradiction, and the lemma has been proven. □

Now we prove that the family of attractors is lower semicontinuous at $\varepsilon = 0$. We do this by essentially proving the continuity of the unstable manifolds for the points of equilibrium $[u_m^\varepsilon]$ of π_ε. Since all the semiflows are gradient-like, any $[u_0] \in \mathcal{A}_0 \setminus \mathcal{E}_0$ has to be in the unstable manifold of some $[u_m^0] \in [E]_0$. Thus the convergence of the unstable manifolds allows to get $[u_\varepsilon] \in [A]_\varepsilon$ which converge to the given $[u_0]$.

We look first at what happens around a given point of equilibrium $[u_m^0] \in \mathcal{E}_0$, $m \in \{1, \dots, M_0\}$. To simplify notations, we drop in what follows the index "m".

Set $[u^0] := [u_m^0]$ and let $0 \leq d < 1$, $\varepsilon_n \to 0$. By Lemma 4.5 there are $[u^n] = [u_m^n] \in \mathcal{E}_{\varepsilon_n}$ and $|[u^n] - \Phi_{\varepsilon_n}^H[u^0]|_{\varepsilon_n,d} \to 0$, as $n \to \infty$. Set $B_n \colon D(A_{\varepsilon_n}) \to L_{\varepsilon_n}$, $B_n[u] := A_{\varepsilon_n}[u] - Dg_{\varepsilon_n}([u^n])[u]$. Define $B_0 \colon D(A_0) \to L_0$ by $B_0[u] := A_0[u] - Dg_0([u^0])[u]$. (Note that B_n is not the operator B_ε of the proof of Lemma 4.4.)

We shall in what follows abuse notation and include the limit case $\varepsilon = 0$ by writing something is defined (or holds) for all $n \geq 0$. E.g. we would say $B_n[u] := A_n[u] - Dg_n([u^n])[u]$, $n \geq 0$, to define the operators B_n, B_0 above.

Set $V_n[u] := -Dg_n([u^n])[u]$ for $n \geq 0$. (Again note that V_n is not the V_ε of the proof of Lemma 4.4.) Then by Lemma 4.3(d) the potentials V_n satisfy conditions (V1)–(V3) of Section 3, and Lemmas 3.10 and 3.11 hold. In particular the operators B_n have all the properties stated in Section 3.

Thus, eventually taking a subsequence, we have eigenvalue, eigenvector pairs $(\lambda^b_{n,l},[u^b_{n,l}])$ of B_n, $\lambda^b_{n,1}\le\lambda^b_{n,2}\le\ldots$, $([u^b_{n,l}])_l$ is a complete ONS of L_n, and

$$\lim_{n\to\infty}|\lambda^b_{n,l}-\lambda^b_{0,l}|=\lim_{n\to\infty}|[u^b_{n,l}]-\Phi^H_{\varepsilon_n}[u^b_{0,l}]|_{\varepsilon_n,d}=0,$$

for all $0\le d<1$, $l\in\mathbb{N}$.

Assume π_0 has an unstable manifold at $[u^0]$. Then by Lemma 3.11 there is a $l_1\ge 1$ such that (eventually taking again a subsequence) $\lambda^b_{n,l_1}<0<\lambda^b_{n,l_1+1}$, for all $n\ge 0$. Fix $C_B>0$ such that

$$\frac{1}{2}\lambda^b_{n,l_1}<-C_B\quad\text{for all } n\ge 0.$$

Define for $n\ge 0$

$$\begin{aligned}W_n &:=\Big\{[u]\in H_n:[u]=\sum_{l=1}^{l_1}\alpha_l[u^b_{n,l}],\ \alpha_l\in\mathbb{R}\Big\},\\ W_n^{\perp} &:=\{[u]\in L_n:([u],[w])_{L_n}=0\text{ for all }[w]\in W_n\},\end{aligned}$$

$$\begin{gathered}P_n\colon L_n\to W_n\text{ orthogonal projection},\quad Q_n:=\mathrm{id}-P_n\colon L_n\to W_n^{\perp},\\ h_n\colon H_n\to L_n,\quad h_n([u]):=g_n([u]+[u^n])-g_n([u^n])-Dg_n([u^n])[u],\\ B_{1,n}:=B_n|_{W_n}\colon W_n\to W_n,\quad B_{2,n}:=B_n|_{D(A_n)\cap W_n^{\perp}}\colon D(A_n)\cap W_n^{\perp}\to W_n^{\perp}.\end{gathered}$$

We shall need the space of functions on $]-\infty,0]$ which decrease at least with e^{C_Bt} as $t\to-\infty$. For $\sigma\colon]-\infty,0]\to H_n$, $n\ge 0$, define

$$\begin{aligned}\|\sigma\|_{H_n} &:=\sup_{t\le 0}(e^{-C_Bt}\|\sigma(t)\|_{H_n}),\\ BH_n &:=\{\sigma\colon]-\infty,0]\to H_n|:\|\sigma\|_{H_n}<\infty\}.\end{aligned}$$

BH_n with $\|\cdot\|_{H_n}$ is a Banach space for all $n\ge 0$.

Note that $[u(t)]$ satisfies equation (4.1) (with $\varepsilon=\varepsilon_n$ or $\varepsilon=0$) if and only if $[v]=[v(t)]:=[u(t)]-[u^n]$ satisfies

$$[v_t]=-B_n[v]+h_n([v]). \tag{4.7}$$

We construct the unstable manifold via a contraction map on the space of functions with exponential growth (as $t\to-\infty$; see e.g. [16], [5], [15]). For this we need the following well known result we state without proof.

Lemma 4.6. *Let $n\ge 0$ and $\sigma\colon]-\infty,0]\to H_n$. σ is a solution of equation (4.7) and $\|\sigma(t)\|_{H_n}\to 0$ as $t\to-\infty$ if and only if $\sigma\in BH_n$ and*

$$\begin{aligned}\sigma(t)=e^{-B_{1,n}t}P_n\sigma(0)+\int_0^t e^{-B_{1,n}(t-s)}P_nh_n(\sigma(s))\,ds\\ +\int_{-\infty}^t e^{-B_{2,n}(t-s)}Q_nh_n(\sigma(s))\,ds.\end{aligned}$$

A list of some properties of h_n follows, the proof is a simple application of Lemma 4.3.

Lemma 4.7.

(a) $h_n(0) = 0 = Dh_n(0)$ *and* h_n *is* C^1.

(b) *There is a constant* $C_1 > 0$, *independent of* n, *such that*

$$\|h_n([u]+[v]) - h_n([u])\|_{L_{\varepsilon_n}} \le C_1\|[v]\|_{L_{\varepsilon_n}} \quad \text{for all } [u],[v] \in L_{\varepsilon_n},\ n \ge 0.$$

(c) *For all* $\widetilde{C} > 0$ *there are* $C_2 = C_2(\widetilde{C}) > 0$, *independent of* n, *and* $C_3(n) > 0$, *independent of* $\widetilde{C}$, $C_3(n) \to 0$ *as* $n \to \infty$, $C_3(0) = 0$, *such that for all* $n \ge 0$, $[u],[v] \in H_{\varepsilon_n}$, $|[u]|_{\varepsilon_n,d}, |[v]|_{\varepsilon_n,d} \le C_2$

$$\|h_n([u]+[v]) - h_n([u])\|_{L_{\varepsilon_n}} \le \widetilde{C}|[v]|_{\varepsilon_n,d} + C_3(n)\|[v]\|_{H_{\varepsilon_n}}.$$

(d) *If* $[u_n] \in L_{\varepsilon_n}$, $[u_0] \in L_0$, $\lim_{n\to\infty} \|[u_n] - \Phi^L_{\varepsilon_n}[u_0]\|_{L_{\varepsilon_n}} = 0$, *then*

$$\|h_n([u_n]) - \Phi^L_{\varepsilon_n} h_0([u_0])\|_{L_{\varepsilon_n}} \to 0 \quad n \to \infty.$$

The fixed points of the maps Ψ_n we define in the following lemma define the unstable manifold near to a point of equilibrium $[u^n]$.

Lemma 4.8. *Recall that* l_1 *is the index of the last negative eigenvalue of* B_n. *For* $\xi \in \mathbb{R}^{l_1}$, $\sigma \in BH_n$, $t \le 0$ *define*

$$\begin{aligned}\Psi_n(\xi,\sigma)(t) := e^{-B_{1,n}t}\sum_{l=1}^{l_1}\xi_l[u^b_{n,l}] + \int_0^t e^{-B_{1,n}(t-s)}P_n h_n(\sigma(s))\,ds \\ + \int_{-\infty}^t e^{-B_{2,n}(t-s)}Q_n h_n(\sigma(s))\,ds.\end{aligned}$$

Then $\Psi_n\colon \mathbb{R}^{l_1} \times BH_n \to BH_n$ *is continuous and* $\Psi_n(\xi,\cdot)\colon BH_n \to BH_n$ *is completely continuous for each* $\xi \in \mathbb{R}^{l_1}$, $n \ge 0$.

Proof. Set

$$\begin{aligned}\|\sigma\|_{L_n} &:= \sup_{t\le 0}(e^{-C_B t}\|\sigma(t)\|_{L_n}), \\ BL_n &:= \{\sigma\colon]-\infty,0] \to L_n : \|\sigma\|_{L_n} < \infty\}.\end{aligned}$$

By Lemma 4.7(b) for $\sigma \in BL_n$ and $s \le 0$

$$\|h_n(\sigma(s))\|_{L_n} \le C_1\|\sigma(s)\|_{L_n} \le C_1 e^{C_B s}\|\sigma\|_{L_n}$$

for some constant $C_1 > 0$. There is a $C_2 > 0$ such that $\|[u]\|^2_{H_n} \le b_n([u],[u]) + C_2\|[u]\|^2_{L_n}$. We get for $t \le 0$

$$(4.8)\quad \begin{aligned} e^{-C_B t}\|\Psi_n(\xi,\sigma)(t)\|_{H_n} &\le C_3(\|\xi\| + \|\sigma\|_{L_n}) \\ &\quad + \int_{t-(\lambda^b_{n,l_1+1}+C_2)^{-1}/2}^{t} \frac{e^{-C_B t}\|h_n(\sigma(s))\|_{L_n}}{\sqrt{2(t-s)}}\,ds \\ &\quad + \int_{-\infty}^{t-(\lambda^b_{n,l_1+1}+C_2)^{-1}/2} (\lambda^b_{n,l_1+1}+C_2)^{1/2} e^{-C_B t-\lambda^b_{n,l_1+1}(t-s)}\|h_n(\sigma(s))\|_{L_n}\,ds \\ &\le C_4(\|\xi\| + \|\sigma\|_{L_n}). \end{aligned}$$

Hence $\Psi_n(\xi,\,\cdot\,)\colon BL_n \to BH_n$ maps bounded sets into bounded sets. In a completely analogous way one shows that if $\|\cdot\|_{\alpha,n}$ is the norm of the fractional power space X^α_n of B_n, $1/2 < \alpha < 1$, then

$$(4.9)\qquad e^{-C_B t}\|\Psi_n(\xi,\sigma)(t)\|_{\alpha,n} \le C_5(\|\xi\| + \|\sigma\|_{L_n}).$$

Ψ_n is obviously continuous with respect to ξ. To prove continuity in σ let $\sigma, \sigma_1 \in BH_n$, $t \le 0$ and assume $\|\sigma_1\|_{H_n} \to 0$. Then

$$\begin{aligned} \|\Psi_n(\xi,\sigma+\sigma_1)(t) - \Psi_n(\xi,\sigma)(t)\|_{H_n} &\le C_6\bigg(\int_t^0 e^{-\lambda^b_{n,l_1}(t-s)+C_B s}\|\sigma_1\|_{H_n}\,ds \\ &\quad + \int_{t-(\lambda^b_{n,l_1+1}+C_2)^{-1}/2}^{t} \frac{e^{C_B s}}{\sqrt{t-s}}\|\sigma_1\|_{H_n}\,ds + \int_{-\infty}^{t} e^{-\lambda^b_{n,l_1+1}(t-s)+C_B s}\|\sigma_1\|_{H_n}\,ds\bigg) \\ &\le C_7 e^{C_B t}\|\sigma_1\|_{H_n} \to 0 \end{aligned}$$

and Ψ_n is indeed continuous.

If we show $\{\Psi_n(\xi,\sigma) : \|\sigma\|_{H_n} \le C\}$ is compact for all $C > 0$, then $\Psi_n(\xi,\,\cdot\,)$ is completely continuous.

So let $\sigma_j \in BH_n$ be a sequence for which $\sup_{t\le 0}(e^{-C_B t}\|\sigma_j\|_{H_n}) \le C$. Let $\bigcup_{i\ge 1} t_i = \mathbb{Q} \cap {]-\infty, 0]}$. For each i fixed, $(\Psi_n(\xi,\sigma_j)(t_i))_j$ is in a compact set by (4.9), hence taking a subsequence, called σ_j too,

$$(4.10)\qquad \|\Psi_n(\xi,\sigma_j)(t_i) - \mu(t_i)\|_{H_n} \to 0,$$

as $j \to \infty$, for some $\mu(t_i) \in H_n$. With the Cantor diagonal procedure there is a subsequence, called σ_j again, such that for all i (4.10) holds.

$t \mapsto \Psi_n(\xi,\sigma)(t) \in H_n$ is continuous: by Lemma 4.6 it is a solution of equation (4.7), as such it is continuous (see e.g. [10, Theorem 3.5.2]). But we need more, namely bounds independent of j. To get them let $\sigma \in B_C(0) \subset BH_n$, $\tau_1 < \tau_2 \le 0$,

$\tau_2-\tau_1<1$. Arguing as in (4.8)

$$\begin{aligned}
\|\Psi_n(\xi,\sigma)(\tau_1) &- \Psi_n(\xi,\sigma)(\tau_2)\|_{H_n}\\
&\le \|(e^{-B_{1,n}(\tau_1-\tau_2)}-\mathrm{id})e^{-B_{1,n}\tau_2}\sum_{l=1}^{l_1}\xi_l[u^b_{n,l}]\|_{H_n}\\
&\quad+\int_{\tau_2}^0 \|(e^{-B_{1,n}(\tau_1-\tau_2)}-\mathrm{id})e^{-B_{1,n}(\tau_2-s)}P_nh_n(\sigma(s))\|_{H_n}\,ds\\
&\quad+\int_{\tau_1}^{\tau_2} \|e^{-B_{1,n}(\tau_1-s)}P_nh_n(\sigma(s))\|_{H_n}\,ds\\
&\quad+\int_{\tau_1}^{\tau_2} \|e^{-B_{2,n}(\tau_2-s)}Q_nh_n(\sigma(s))\|_{H_n}\,ds\\
&\quad+\int_{-\infty}^{\tau_1} \|(e^{-B_{2,n}(\tau_2-\tau_1)}-\mathrm{id})e^{-B_{2,n}(\tau_1-s)}Q_nh_n(\sigma(s))\|_{H_n}\,ds\\
&\le C_8\bigg(\|\xi\|(\tau_2-\tau_1)+((\tau_2-\tau_1)\int_{\tau_2}^0 e^{-\lambda^b_{n,l_1}(\tau_2-s)+C_Bs}\,ds\\
&\quad+(\tau_2-\tau_1)+\sqrt{\tau_2-\tau_1}+(\tau_2-\tau_1)^{\widetilde{\alpha}})\|\sigma\|_{L_n}\bigg)\\
&\le C_9(\tau_2-\tau_1)^{\widetilde{\alpha}}(\|\xi\|+\|\sigma\|_{L_n}),
\end{aligned}$$

where the constants C_8, C_9 are independent of τ_1, τ_2, and $0<\widetilde{\alpha}<1/2$. Thus for $\delta_1>0$, i, $\widetilde{i}$, $|t_i-t_{\widetilde{i}}|\le\min(1,(\delta_1/4((\|\xi\|+C)C_9))^{1/\widetilde{\alpha}})$, there is a $j=j(\delta_1,i,\widetilde{i})$, such that

$$\begin{aligned}
\|\mu(t_i)-\mu(t_{\widetilde{i}})\|_{H_n} &\le \|\mu(t_i)-\Psi_n(\xi,\sigma_j)(t_i)\|_{H_n}\\
&+\|\Psi_n(\xi,\sigma_j)(t_i)-\Psi_n(\xi,\sigma_j(t_{\widetilde{i}})\|_{H_n}+\|\Psi_n(\xi,\sigma_j)(t_{\widetilde{i}})-\mu(t_{\widetilde{i}})\|_{H_n}\le\delta_1.
\end{aligned}$$

Hence we can define $\mu(t)$ for all $t\le0$ by continuously extending $\mu(t_i)$. Then for $t\le0$ and $t_i\in\mathbb{Q}$ near to t

$$\begin{aligned}
e^{-C_Bt}\|\Psi_n(\xi,\sigma_j)(t)-\mu(t)\|_{H_n}&\le e^{-C_Bt}(\|\Psi_n(\xi,\sigma_j)(t)-\Psi_n(\xi,\sigma_j)(t_i)\|_{H_n}\\
&+\|\Psi_n(\xi,\sigma_j)(t_i)-\mu(t_i)\|_{H_n}+\|\mu(t_i)-\mu(t)\|_{H_n})
\end{aligned}$$

shows $\|\Psi_n(\xi,\sigma_j)-\mu\|_{H_n}\to0$ as $j\to\infty$. I.e. we have found a convergent subsequence, and $\{\Psi_n(\xi,\sigma):\|\sigma\|_{H_n}\le C\}$ is indeed compact in BH_n. □

Lemma 4.9. *There is a subsequence, called ε_n too, constants $C_1,C_2,C_3>0$, maps $\sigma^*_n\colon B_{C_1}(0)\subset\mathbb{R}^{l_1}\to BH_n$, such that*

$$\begin{aligned}
\sigma^*_n(\xi)(t)&=e^{-B_{1,n}t}\sum_{l=1}^{l_1}\xi_l[u^b_{n,l}]+\int_0^t e^{-B_{1,n}(t-s)}P_nh_n(\sigma^*_n(\xi)(s))\,ds\\
&\quad+\int_{-\infty}^t e^{-B_{2,n}(t-s)}Q_nh_n(\sigma^*_n(\xi)(s))\,ds,
\end{aligned}$$

for all $t \le 0$, $n \ge 0$. $\sigma_n^*(\xi)(\,\cdot\,)$ *can be extended to a function on* $\mathbb{R}$ *in such a way that it is a solution of equation* (4.7). *It is the only solution* σ *in* $\{\sigma \in BH_n : \|\sigma\|_{H_n} \le C_2,\ |\sigma|_{H_n} \le C_3\}$ *with* $P_n\sigma(0) = \sum_{l=1}^{l_1} \xi_l [u_{0,l}^b]$. *Moreover,*

$$|\sigma_n^*(\xi)(t) - \Phi_{\varepsilon_n}^H \sigma_0^*(\xi)(t)|_{\varepsilon_n,d} \to 0$$

as $n \to \infty$, *for all* $\xi \in B_{C_1}(0) \subset \mathbb{R}^{l_1}$, $t \in \mathbb{R}$.

Proof. Let Ψ_n be as in Lemma 4.8. We know already that for each fixed $\xi \in \mathbb{R}^{l_1}$ is $\Psi_n(\xi, \cdot\,) \colon BH_n \to BH_n$ a completely continuous map. We claim that, with some restrictions on ξ and σ, this map is a contraction.

Given $\widetilde{C} > 0$, by Lemma 4.7(c) there are $C_1 = C_1(\widetilde{C}) > 0$, $C_2(n) > 0$, $C_2(n) \to 0$ as $n \to \infty$, $C_2(0) = 0$, such that

$$\|h_n(\sigma(s) + \sigma_1(s)) - h_n(\sigma(s))\|_{L_n} \le \widetilde{C}|\sigma_1(s)|_{n,d} + C_2(n)\|\sigma_1(s)\|_{H_n},$$

whenever $|\sigma(s)|_{n,d}, |\sigma_1(s)|_{n,d} \le C_1$. Thus, if

$$|\sigma|_{n,d} := \sup_{t\le 0}(e^{-C_B t}|\sigma(t)|_{n,d})$$

we find for $|\sigma|_{n,d}, |\sigma_1|_{n,d} \le C_1$, $t \le 0$

$$\begin{aligned}(4.11)\quad & e^{-C_B t}\|\Psi_n(\xi, \sigma + \sigma_1)(t) - \Psi_n(\xi, \sigma)(t)\|_{H_n} \\ &\le C_3(\widetilde{C}|\sigma_1|_{n,d} + C_2(n)\|\sigma_1\|_{H_n}) \le C_3(C_4\widetilde{C} + C_2(n))\|\sigma_1\|_{H_n},\end{aligned}$$

where C_3, C_4 do not depend on $\widetilde{C}$, and C_4 is such that $|\cdot|_{n,d} \le C_4\|\cdot\|_{H_n}$ for all $n \ge 0$.

Let C_5 denote the constant in (4.8). Choose $\widetilde{C} \le 1/(4C_3C_4)$, $C_6 = C_6(\widetilde{C}) \le C_1/C_4$, $C_7 = C_7(\widetilde{C}) \le C_6/(2C_5)$ and, taking a subsequence, assume $C_2(n) \le 1/(4C_3)$. Then inequalities (4.8) and (4.11) show

$$\begin{aligned}(4.12)\qquad \|\Psi_n(\xi, \sigma)\|_{H_n} &\le \|\Psi_n(\xi, \sigma) - \Psi_n(\xi, 0)\|_{H_n} + \|\Psi_n(\xi, 0)\|_{H_n} \\ &\le (C_3C_4\widetilde{C} + C_3C_2(n))\|\sigma\|_{H_n} + C_5\|\xi\| \le C_6, \\ |\Psi_n(\xi, \sigma)|_{n,d} &\le C_4C_6 \le C_1,\end{aligned}$$

for all $n \ge 0$, $\|\xi\| \le C_7$, $\|\sigma\|_{H_n} \le C_6$, $|\sigma|_{n,d} \le C_1$. If additionally $|\sigma_1|_{n,d} \le C_1$ we get

$$(4.13)\qquad \|\Psi_n(\xi, \sigma + \sigma_1) - \Psi_n(\xi, \sigma)\|_{H_n} \le \frac{1}{2}\|\sigma_1\|_{H_n}.$$

Thus, for these ξ,

$$\begin{aligned}\Psi_n(\xi, \cdot\,) \colon \{\sigma \in BH_n : \|\sigma\|_{H_n} \le C_2,\ |\sigma|_{H_n} \le C_3\} \\ \to \{\sigma \in BH_n : \|\sigma\|_{H_n} \le C_2,\ |\sigma|_{H_n} \le C_3\}\end{aligned}$$

is contracting. Hence for each $\xi \in B_{C_7}(0) \subset \mathbb{R}^{l_1}$ there is a unique $\sigma_n^*(\xi)$ in above set with $\sigma_n^*(\xi) = \Psi_n(\xi, \sigma_n^*(\xi))$. By Lemma 4.6 the map $t \mapsto \sigma_n^*(\xi)(t)$ is a solution of equation (4.7) for all $n \geq 0$. Since this equation is just the original (4.1) under the substitution $[u(t)] \to [u(t)] + [u^n]$, and the semiflows π_ε are global ones, all $\sigma_n^*(\xi)(t)$ can be extended to full solutions $\sigma_n^*(\xi)\colon \mathbb{R} \to H_n$ of (4.7).

Note that $\sigma_n^*(\xi)(t) \in \mathcal{A}_n$ for all $t \in \mathbb{R}$, $n \geq 0$.

The only thing we have not shown already is the convergence $|\sigma_n^*(\xi) - \Phi_n^H \sigma_0^*(\xi)|_{n,d} \to 0$, as $n \to \infty$.

The solutions $\sigma_n^*(\xi)(t) + [u^n]$ are in the respective attractors, thus by Lemma 3.8 there is a constant C_8 such that

$$\|\sigma_n^*(\xi)(t) + [u^n]\|_{H_n} \leq C_8,$$

for all $n \geq 0$, $t \in \mathbb{R}$ and ξ. But then Lemma 3.9 shows the existence of a solution $\sigma^*(\xi)(t) + [u^0]$ with $\|\sigma^*(\xi)(t) + [u^0]\|_{H_0} \leq C_9$, some constant $C_9 > 0$, and, taking again a subsequence,

$$|\sigma_n^*(\xi)(t) - \Phi_n^H \sigma^*(\xi)(t)|_{n,d} \to 0, \quad \text{as } n \to \infty, \text{ for all } t \in \mathbb{R}.$$

We have

$$P_0(\sigma^*(\xi)(0)) \leftarrow P_n(\sigma_n^*(\xi)(0)) = \sum_{l=1}^{l_1} \xi_l [u_{n,l}^b] \to \sum_{l=1}^{l_1} \xi_l [u_{0,l}^b] \quad \text{as } n \to \infty.$$

If $\|\xi\|$ is small, we can choose C_7 and C_6 small, and by Lemma 3.9 $\|\sigma^*(\xi)\|_{H_0}$ is small too. By Lemma 4.6 $\sigma^*(\xi) = \Psi_0(\xi, \sigma^*(\xi))$ and the uniqueness of σ_0^* on $\overline{B_{C_6}(0)} \subset BH_0$ yields $\sigma^*(\xi) = \sigma_0^*(\xi)$. Thus

$$|\sigma_n^*(\xi)(t) - \Phi_n^H \sigma_0^*(\xi)(t)|_{n,d} \to 0,$$

as $n \to \infty$, for each $t \in \mathbb{R}$ fixed. □

Now we can prove our main theorem:

Proof of Theorem 1.1. Fix $0 \leq d < 1$. We shall show that for given $\delta > 0$, $[u_0] \in \mathcal{A}_0$ there is a $0 < \varepsilon_0$, and for all $0 < \varepsilon \leq \varepsilon_0$ there are $[u_\varepsilon] \in \mathcal{A}_\varepsilon$ such that $|[u_\varepsilon] - \Phi_\varepsilon^H [u_0]|_{\varepsilon,d} \leq \delta$. Together with Theorem 2.2 this proves Theorem 1.1.

Assume that these $[u_\varepsilon]$ do not exist, then there are $[u_0] \in \mathcal{A}_0$, $\varepsilon_n \to 0$, $\delta > 0$ such that for all $n \in \mathbb{N}$

$$\inf_{[u] \in \mathcal{A}_n} |[u] - \Phi_n^H [u_0]|_{n,d} \geq \delta.$$

There is a full solution $\sigma_0\colon \mathbb{R} \to H_0$ of (4.1) such that $[u_0] = \sigma_0(0)$. By Lemma 4.5 $[u_0]$ is no point of equilibrium. We have already shown in Section 3 that π_0 is gradient like, thus $\sigma_0(t) \to [u_m^0]$, as $t \to -\infty$, for some $m \in \{1, \ldots, M_0\}$. This implies $[u_m^0]$ has an unstable manifold and we can use Lemma 4.9.

Let C_1, C_2 be as in this lemma. Note that $\sigma_0(t + t_0) - [u_m^0] \in BH_0$ for all t_0, and $\|\sigma_0(\,\cdot\, + t_0) - [u_m^0]\|_{H_0} \leq C_3 e^{C_B t_0}$ by Lemma 4.6.

Setting $\sigma_0^*(t) := \sigma_0(t+t_0) - [u_m^0]$ and choosing $t_0 < 0$ small enough, we have $\|\sigma_0^*\|_{H_0} < C_2$ and if ξ is defined by

$$P_0\sigma_0^*(0) = \sum_{l=1}^{l_1} \xi_l [u_{0,l}^b],$$

then also $\|\xi\| < C_1/2$. σ_0^* solves (4.7), and Lemma 4.9 shows $\sigma_0^*(t) = \sigma_0^*(\xi)(t)$, $\sigma_0^*(\xi)$ as in this lemma.

Now $\sigma_n^*(\xi)(-t_0) + [u_m^n] \in \mathcal{A}_n$, and since $|[u_m^n] - \Phi_n^H[u_m^0]|_{n,d} \to 0$

$$\begin{aligned} |\sigma_n^*(\xi)(-t_0) + [u_m^n] - \Phi_n^H[u_0]|_{n,d} &\leq |\sigma_n^*(\xi)(-t_0) - \Phi_n^H\sigma_0^*(\xi)(-t_0)|_{n,d} \\ &+ |\Psi_n^H(\sigma_0^*(\xi)(-t_0) - [u_0] + [u_m^0])|_{n,d} + |[u_m^n] - \Phi_n^H[u_m^0]|_{n,d} \to 0, \end{aligned}$$

as $n \to \infty$. This contradicts our assumption, and the proof is complete. □

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