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TOPOLOGICAL INDEX FOR CONDENSING MAPS ON FINSLER MANIFOLDS WITH APPLICATIONS TO FUNCTIONAL-DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. The topological index for maps of infinite-dimensional Finsler manifolds, condensing with respect to internal Kuratowski's measure of non-compactness, is constructed under the hypothesis that the manifold can be embedded into a certain Banach linear space as a neighbourhood retract so that the Finsler norm in tangent spaces and the restriction of the norm from enveloping space on the tangent spaces are equivalent. It is shown that the index is an internal topological characteristic, i.e. it does not depend on the choice of enveloping space, embedding, etc. The total index (Lefschetz number) and the Nielsen number are also introduced. The developed machinery is applied to investigation of functional-differential equations of neutral type on Riemannian manifolds. A certain existence and uniqueness theorem is proved. It is shown that the shift operator, acting in the manifold of C^1-curves, is condensing, its total index is calculated to be equal to the Euler characteristic of (compact) finite-dimensional Riemannian manifold where the equation is given. Some examples of calculating the Nielsen number are also considered.

1. Introduction

This paper has two main goals. One of them is to describe and investigate functional differential equations of neutral type on Riemannian manifolds. This

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is a natural development of the theory yielding interesting invariant geometric and topological structures and interrelations but also requiring new complicated methods and constructions. Unlike the theory of ordinary functional differential equation (whose investigation on manifolds was started probably in Oliva's paper [12]) the equations of neutral type on manifolds practically were not considered. Our particular purpose here is to find interrelations between the topological properties of the manifold and those of solutions of the equation.

Among our tools we should first mention the so-called shift operator along trajectories of the equation that acts in the infinite-dimensional functional manifold of C^1-curves. It is a well-known fact that in the case of linear spaces this operator is condensing (k-set contraction), see e.g. [11], so that it is a demand to develop the topological theory of condensing maps on Banach manifolds. The main difficulty here is that formulations of many facts from this theory in linear spaces sound reasonably on manifolds but cannot be proved directly since the topological theory of condensing maps is essentially based on the notion of convex closure that is absolutely ill-posed on nonlinear manifolds.

In a series of previous works (see, e.g. [5], [7], [3] and references there) such a theory was constructed for the case of Finsler manifolds that can be embedded isometrically into a certain Banach linear space as a neighbourhood retract. But it turns out that the latter condition is not satisfied for the manifold of C^1-curves where the shift operator of neutral type functional differential equation acts. Thus, our another main goal is to generalize the topological theory of condensing maps in order to cover the case under consideration.

Here we modify the previous approach and construct the topological index for some sort of condensing maps of Finsler manifolds that can be embedded into a linear Banach space as a neighbourhood retract so that the Finsler norm in tangent spaces and the restriction of the norm from the enveloping Banach space onto those tangent spaces are equivalent. It is shown that the construction does not depend on the choice of enveloping space, embedding and other details. Thus the index is an internal topological characteristic in spite of the fact that in its construction the enveloping space is involved.

It is shown that under some natural hypotheses the shift operator of a neutral type equation is condensing with respect to internal Kuratowski's measure of non-compactness of a certain Finsler metric on the manifold of C^1-curves in a Riemannian manifold and that the manifold of C^1-curves satisfies the above condition so that the topological index for the shift operator is well-posed. The total index (Lefschetz number) and the Nielsen number for this operator are investigated and calculated. In particular, it is shown that if the Riemannian manifold is compact, the total index of shift operator is equal to its Euler characteristic. This leads to an existence theorem of periodic solutions of the neutral type equation on a manifold with non-zero Euler characteristic.

The structure of the paper is as follows. In Section 2 we present the construction of topological index for condensing maps of Finsler manifolds under the above-mentioned conditions. The constructions of total index (Lefschetz number) and Nielsen number are given. In Section 3 we illustrate the general construction on a model example of the manifold of C^1-curves in a Riemannian manifold. This material is the basis for further investigation of the shift operator.

Section 4 is devoted to general theory of functional-differential equations of neutral type on Riemannian manifolds. In particular we prove a certain existence and uniqueness theorem that is in use below. In the fourth section we investigate the topological properties of the shift operator of a neutral type equation and calculate its total index (Lefschetz number). In Appendix we consider a model example of condensing operator whose Lefschetz number is zero while the Nielsen number is equal to 2.

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2. Topological characteristics of condensing maps of Finsler manifolds

Let $\mathcal{M}$ be a Finsler manifold and $\mathcal{M}$ be embedded (possibly not isometrically) into a Banach space $\mathcal{E}$ with the norm $\|\cdot\|$ as a neighbourhood retract. Denote by $\|\cdot\|_I$ the internal (Finsler) norm in tangent spaces $T_m\mathcal{M}$, and by $\|\cdot\|_E$ the restriction of the norm in $\mathcal{E}$ onto T_mM. Below we suppose that the norms $\|\cdot\|_I$ and $\|\cdot\|_E$ are equivalent, i.e. that there exist constants $0 < c(m) \leq C(m)$ continuously depending on $m \in \mathcal{M}$ such that for any $Y \in T_m\mathcal{M}$ the relation

$$c(m)\|Y\|_I \leq \|Y\|_E \leq C(m)\|Y\|_I \tag{2.1}$$

takes place.

Starting from the norms in tangent spaces to $\mathcal{M}$, one can find the corresponding lengths of piece-wise smooth curves in $\mathcal{M}$ as integrals of norms of velocity (derivative) vectors and then define the distance functions on $\mathcal{M}$ as infimums of the lengths of curves connecting the points (standard constructions of Riemannian and Finsler geometry). Denote by ρ_I the distance generated with $\|\cdot\|_I$, and by ρ_E — the distance generated with $\|\cdot\|_E$. Besides, the distance can be measured directly in $\mathcal{E}$, as the norm of difference. Note that the latter two distance functions are related by obvious estimate

$$\rho_E(m^0, m^1) \geq \|m^0 - m^1\|, \tag{2.2}$$

for all couples m^0 and $m^1 \in \mathcal{M}$ since the lengths with respect to $\|\cdot\|_E$ and $\|\cdot\|$ coincide but ρ_E is the infimum of lengths of curves on M while $\|\cdot\|$ – in $\mathcal{E}$.

Recall (see details, e.g. in [1]) the notions of Kuratowski's measure of non-compactness in a metric space E. Let $\Omega \subset E$ be a bounded subset.

Definition 2.1. $\alpha(\Omega) = \inf\{d > 0 \mid \Omega$ permits its partition in E into a finite number of subsets with diameters less than d with respect to the metric in $E\}$ is called the Kuratowski's measure of non-compactness of Ω.

Having defined the distances in $\mathcal{M}$, denote by α_I the Kuratowski's measure of noncompactness with respect to ρ_I, by α_E — the Kuratowski's measure of noncompactness with respect to ρ_E and by $\alpha_{\|\cdot\|}$ — the Kuratowski's measure of noncompactness with respect to $\|\cdot\|$.

Let ψ be a certain measure of noncompactness.

Definition 2.2. A continuous operator $F\colon \mathcal{M} \to \mathcal{M}$ is called condensing with respect to a measure of non-compactness ψ with constant $q < 1$ if for any bounded set $\Omega \subset \mathcal{M}$ the inequality

$$\psi(F\Omega) < q\psi(\Omega) \tag{2.3}$$

holds.

Definition 2.3. A continuous operator $F\colon \mathcal{M} \to \mathcal{M}$ is called locally condensing with respect to ψ, if any point $x \in \mathcal{M}$ has a neighbourhood U_x such that for any bounded set $\Omega \subset U_x$ the inequality

$$\psi(F\Omega) < q\psi(\Omega), \quad q < 1 \tag{2.4}$$

is satisfied.

Let the operator $F : \mathcal{M} \to \mathcal{M}$ be condensing with respect to $\psi = \alpha_I$ with a constant $q < 1$ and $\Omega \subset \mathcal{M}$ be a bounded domain. Consider the set $F^\infty\Omega = \bigcap_{k=1}^{\infty} F^k\Omega$, where F^k is the k-th iteration of F. Sometimes we shall introduce the additional assumption that F sends the entire $\mathcal{M}$ into a domain having finite diameter with respect to the distance ρ_I. In this case we can consider the set $F^\infty\mathcal{M} = \bigcap_{k=1}^{\infty} F^k\mathcal{M}$.

Lemma 2.4. *The set $F^\infty\Omega$ is compact. If F sends the entire $\mathcal{M}$ into a domain having finite diameter with respect to the distance ρ_I, then the set $F^\infty\mathcal{M}$ is compact.*

The proof can be found, e.g. in [5], [7]. Notice that the set $f^\infty\mathcal{M}$ contains all fixed points of F from $\mathcal{M}$ and $F^\infty\Omega$ contains all fixed points of F from Ω.

Since the functions $C(m)$ and $c(m)$ are continuous and the sets $F^\infty\Omega$ and $F^\infty\mathcal{M}$ from Lemma 6 are compact, there exist constants $C > c > 0$ and a neighbourhood $\mathcal{A}$ of $F^\infty\Omega$ or of $F^\infty\mathcal{M}$ such that for any $m \in \mathcal{A}$, $Y \in T_m\mathcal{M}$

$$c\|Y\|_I \leq \|Y\|_E \leq C\|Y\|_I. \tag{2.5}$$

Let $V \subset \mathcal{A}$ be bounded. Then from (2.5) and (2.3) we obtain the following inequalities:

$$\alpha_I(F(V)) \leq q\alpha_I(V) \leq \frac{1}{c}q\alpha_E(V). \tag{2.6}$$

But on the other hand, from (2.3) and (2.6) it follows that

$$\alpha_I(F(V)) \geq \frac{1}{C}\alpha_E(F(V)). \tag{2.7}$$

Then from (2.4), (2.6) and (2.7) we get

$$\alpha_E(F(V)) \leq \frac{C}{c}q\alpha_E(V). \tag{2.8}$$

Denote by $R\colon \overline{U} \to \mathcal{M}$ the smooth retraction of a certain tubular neighbourhood $\overline{U} \subset \mathcal{E}$ of $\mathcal{M}$ and by $TR\colon T\overline{U} \to T\mathcal{M}$ the tangent map to R. Recall that the tangent map sends the vector $X \in T_xU$ into $TRX = d_xRX \in T_{Rm}\mathcal{M}$, where the linear operator $d_xR\colon T_x\overline{U} \to T_{Rx}\mathcal{M}$ is the Frechet derivative of R at the point $x \in \overline{U}$.

Theorem 2.5 (cf. [7]). *For any $m \in \mathcal{M} \subset \overline{U}$ there exists a ball $B_m \subset \overline{U}$ centered at m and having finite radius, where the retraction R is Lipschitz continuous with respect to ρ and $\|\cdot\|$ with a certain constant $Q > 0$, i.e.:*

$$\rho_E(R(u_0), R(u_1)) \leq Q\|u_0 - u_1\|, \tag{2.9}$$

for any u_0 and $u_1 \in B_m$.

Proof. Let $\|d_mR\|$ be the norm of Frechet derivative of R at m. Since R is smooth, its Frechet derivative d_xR is continuous in $x \in \overline{U}$ and so for a certain $Q > \|d_mR\|$ there exists an open neighbourhood $V_m \subset \overline{U}$ of m such that $\|d_xR\| < Q$ for any $x \in V_m$.

Since V_m is open, it contains a certain ball B_m with a finite radius centered at m. Consider two points u_0 and u_1 in B and denote by $u(s)$, $s^0 \leq s \leq s^1$, the line interval connecting u_0 and u_1. Since B is convex, the complete interval $u(s)$ belongs to B and thus the map R is well-posed at its points. Consider the curve $m(s) = R(u(s))$ in M connecting $m_0 = R(u_0) \in M$ with $m_1 = R(u_1) \in M$. Let s be the natural parameter (the length) on $u(s)$ (if it is not so, change the parameter). Thus $\|\dot{u}(s)\| = 1$ for $s \in [s^0, s^1]$ where $\dot{u} = du/ds$. Evidently the length $\int_{s^0}^{s^1} \|\dot{u}(s)\|\,ds$ of $u(s)$ from u_0 to u_1 is equal to the distance $\|u_0 - u_1\|$ between those points in E and is equal to $|s^1 - s^0|$ since $\|\dot{u}(s)\| = 1$. The length of $m(s)$, $s^0 \le s \le s^1$, between m_0 and m_1 is equal to $\int_{s^0}^{s^1} \|\dot m(s)\|\,ds$ and is not shorter than the intrinsic distance $d(m_0, m_1)$ between m_0 and m_1.

Notice in addition that by definition $\dot m(s) = TR\dot u(s)$. Then applying the above arguments we get the following estimates:

$$\begin{aligned} d(m_0, m_1) &\le \int_{s^0}^{s_1} \|\dot m(s)\|\,ds = \int_{s^0}^{s^1} \|d_{u(s)}R\dot u(s)\|\,ds \le \int_{s^0}^{s^1} \|d_{u(s)}R\|\|\dot u(s)\|\,ds \\ &\le \int_{s^0}^{s^1} Q\|\dot u(s)\|\,ds = Q\int_{s^0}^{s^1} ds = Q(s^1 - s^2) = Q\|u_0 - u_1\|. \end{aligned}$$

The theorem follows. □

Introduce $\overline{F}\colon \overline{U} \to \mathcal{M} \subset \overline{U}$, by the formula $\overline{F} = F \circ R$. From (2.2) it follows that for any $u_0, u_1 \in B_m$

$$\|\overline{F}(u_0) - \overline{F}(u_1)\| = \|FR(u_0) - FR(u_1)\| \le \rho_E(FR(u_0), FR(u_1)).$$

From this and from (2.1), (2.3), (2.8) and (2.9) we get that for a bounded set $V \subset B_m$

$$\alpha_{\|\,\cdot\,\|}(\overline{F}(V)) \le \alpha_E(FR(V)) \le \frac{C}{c} q\alpha_E(R(V)) \le Q\frac{C}{c} q\alpha_{\|\,\cdot\,\|}(V).$$

Consider a bounded domain $\Omega \subset \mathcal{M}$ such that F has no fixed points on its boundary $\dot\Omega$, or the entire $\mathcal{M}$ under the assumption that $F\mathcal{M}$ is bounded (see above). For any $x^* \in F^\infty\Omega$ ($x^* \in F^\infty\mathcal{M}$, respectively) take the ball $B_{x^*} \subset \overline{U}$ as above. Cover the set $F^\infty\Omega \bigcap \Omega$ ($F^\infty\mathcal{M}$, respectively) with the balls B_{x^*}. This is an open covering of the compact set $F^\infty\Omega \bigcap \Omega$ ($F^\infty\mathcal{M}$) and so there exists its finite subcovering B_{x_i}. Denote by Q_i the Lipschitz constant for R in B_{x_i} and set $Q = \max_i Q_i$. Let $V_B = \bigcup_i B_{x_i}$. For any $V \subset V_B$. there exists a finite number of sets $V_i = V \bigcap B_{x_i}$ and from above arguments we get

$$\alpha_{\|\,\cdot\,\|}(\overline{F}(V_i)) \le Q_i \frac{C}{c} q\alpha_{\|\cdot\|}(V_i).$$

Then

$$\alpha_{\|\,\cdot\,\|}(\overline{F}(V)) = \max_i \alpha_{\|\,\cdot\,\|}(\overline{F}(V_i)) \le \max_i Q_i \frac{C}{c} q\alpha_{\|\,\cdot\,\|}(V_i) = Q\frac{C}{c} q\alpha_{\|\,\cdot\,\|}(V). \tag{2.10}$$

By the construction, on the boundary of V_B there are no fixed points of $\overline{F}$.

Recall the following

Definition 2.6. A set $S \subset \mathcal{E}$ is called fundamental for an operator $F\colon \overline{U} \to \mathcal{E}$, if:

(a) $S \ne \emptyset$ is convex and compact;
(b) $F(\overline{U} \cap S) \subset S$;
(c) if $x_0 \in \overline{U} \setminus S$, then $x_0 \notin \overline{\operatorname{co}}[\{F(x_0)\} \cup S]$.

In the standard theory of topological index for condensing maps in Banach spaces (see, e.g. [1]) the index is defined as that for the contraction of the operator to a certain fundamental set, containing the set of fixed points of this operator. The key fact here is that for an operator, condensing with a constant less than 1, such a fundamental set exists. We shall show that for the above-mentioned operator $\overline{F}$ a fundamental set, containing fixed points of $\overline{F}$, does exist in spite of the fact that the constant for $\overline{F}$ in (2.10) is greater than 1.

Lemma 2.7. *There exists a fundamental set for operator $\overline{F}$, constructed above, that contains all fixed points of $\overline{F}$ in Ω (in $\mathcal{M}$, respectively).*[1]

Proof. We shall deal here with the bounded domain Ω, the case of $\mathcal{M}$ with $F(\mathcal{M})$ bounded is absolutely analogous. Since F is condensing with the constant $q < 1$, F^k is obviously condensing with the constant q^k. Notice that from the properties of the retraction it follows that $\overline{F}^k = (F \circ R)^k = F^k \circ R$. Then with the same scheme of arguments as above we can show that $\overline{F}^k$ is condensing with respect to $\alpha_{\|\cdot\|}$ with the constant $\overline{q} = QCq^k/c$. Since $q < 1$, for k large enough $q^k < 1/(QC/c)$. For such k we get $\overline{q} < 1$.

Choose such k. Denote by $\aleph$ the collection of all closed sets containing $F^\infty(\Omega)$ and satisfying all conditions from the definition of fundamental set for $\overline{F}$ and $\overline{F}^k$ together except maybe compactness.

The collection $\aleph$ is not empty since at least the set $T_0 = \overline{\mathrm{co}}[F^\infty(\Omega) \cup F(\Omega)] = \overline{\mathrm{co}}(F(\Omega))$ belongs to $\aleph$. Indeed, since $T_0 = \overline{\mathrm{co}}(F(\Omega))$, $F(T_0 \cap \Omega) \subset F(\Omega) \subset T_0$ and analogously $F^k(T_0 \cap \Omega) \subset F(\Omega) \subset T_0$. Let $x_0 \in \Omega \setminus T_0$, then, since $F(x_0) \in F(\Omega) \subset T_0$ and $F^k(x_0) \in F(\Omega) \subset T_0$, $x_0 \notin \overline{\mathrm{co}}[F(x_0) \cup T_0]$ means that $x_0 \notin T_0$, and $x_0 \notin \overline{\mathrm{co}}[F^k(x_0) \cup T_0]$ also means that $x_0 \notin T_0$. But these two conditions are satisfied by the hypothesis $x_0 \in \Omega \setminus T_0$.

Let a set $T \in \aleph$. This means that $F^\infty(\Omega) \subset T$, if $x_0 \in \Omega \cap T$, then $F(x_0)$ and $F^k(x_0)$ belongs to the set T and that if $x_0 \in \Omega \setminus T$, then $x_0 \notin \overline{\mathrm{co}}[F(x_0) \cup T]$ and $x_0 \notin \overline{\mathrm{co}}[F^k(x_0) \cup T]$.

Consider the set $T_1 = \overline{\mathrm{co}}[F^\infty(\Omega) \cup F(\Omega \cap T)]$. By the construction $T \supset T_1 = \overline{\mathrm{co}}[F(\Omega \cap T)] \supset \overline{\mathrm{co}}[F^k(\Omega \cap T)]$. Hence $F(\Omega \cap T_1) \subset F(\Omega \cap T) \subset \overline{\mathrm{co}}[F(\Omega \cap T)] = T_1$, and consequently $F^k(\Omega \cap T_1) \subset T_1$.

Let $x_0 \in \Omega \setminus T_1$. Consider two cases:

Case 1. $x_0 \notin T$, then $x_0 \notin \overline{\mathrm{co}}[F(x_0) \cup T]$, and so $x_0 \notin \overline{\mathrm{co}}[F(x_0) \cup T_1]$ and from $x_0 \notin \overline{\mathrm{co}}[F^k(x_0) \cup T]$, it follow that $x_0 \notin \overline{\mathrm{co}}[F^k(x_0) \cup T_1]$;

Case 2. $x_0 \notin T_1$ and $x_0 \in T$, hence $x_0 \in \Omega \cap T$. Thus $F(x_0) \in T$ and $F^k(x_0) \in T$. From this it follows that $F(x_0) \in F(\Omega \cap T) \subset \overline{\mathrm{co}}[F(\Omega \cap T)] \subset T_1$ and $F^k(x_0) \in F^k(\Omega \cap T) \subset \overline{\mathrm{co}}[F^k(\Omega \cap T)] \subset T_1$. Then since $x_0 \notin T_1$ and

[1] This result for F condensing with respect to internal Kuratowski's and Hausdorff's measures of noncompactness, was announced in [4].

$F(x_0) \in T_1$, we get $x_0 \notin \overline{\text{co}}[F(x_0) \cup T_1]$ since $\overline{\text{co}}[F(x_0) \cup T_1] \subset T_1$. Analogously $x_0 \notin \overline{\text{co}}[F^k(x_0) \cup T_1] \subset T_1$.

Thus conditions (a) and (b) of Definition 2.6 are fulfilled both for $\overline{F}$ and $\overline{F}^k$, i.e. $T_1 \in \aleph$.

Determine the set S as $S = \bigcap_{T \in \aleph} T$ that belongs to $\aleph$. Hence, as it is proved above, the set $S_1 = \overline{\text{co}}[F^\infty(\Omega) \cup F^k(\Omega \cap S)]$ also belongs to $\aleph$. Show that S is fundamental for F. Conditions (a) and (b) of the definition are fulfilled both for the F and for F^k by the construction. The set $S \in \aleph$ is minimal in $\aleph$. Hence $S_1 = \overline{\text{co}}[F^\infty(\Omega) \cup F^k(\Omega \cap S)]$ coincides with S. Then since $\overline{F}^k$ is condensing with a constant less than 1, from the equality $S = \overline{\text{co}}[F^\infty(\Omega) \cup F^k(\Omega \cap S)]$ it follows that S is compact. □

Thus the index of vector field $I - \overline{F}$ on the boundary of V_B is well-posed. For the case of entire $\mathcal{M}$ we call this index *the Lefschetz number* Λ_F of F on $\mathcal{M}$ and for the case of bounded $\Omega \subset \mathcal{M}$ we call it *the index* $\text{ind}_F(\dot{\Omega})$ of F on $\dot{\Omega}$.

Notice that $\text{ind}_F(\dot{\Omega})$ is well posed for a condensing F under the above assumptions for a bounded domain Ω. In particular, the index of an isolated fixed point is also well-posed. The Lefschetz number Λ_F is well-posed under additional assumption that $F\mathcal{M}$ has finite diameter with respect to ρ_I. In particular, this is true for a condensing F if $\mathcal{M}$ has finite diameter.

Our definition of Lefschetz number is compatible with the usual terminology since in the finite-dimensional case the Lefschetz number (in the sense of usual homological definition) is equal to the total index of fixed points. Suppose that F has only isolated fixed points and denote by j_i the index of F in a neighbourhood of the fixed point x_i. One can easily see that Λ_F is equal to the sum of indices, i.e. $\Lambda_F = \sum_i j_i$.

The following statements are proved in complete analogy with [7].

Lemma 2.8. *$\text{ind}_F(\dot{\Omega})$ and Λ_f do not depend on the choice of $\mathcal{E}$ and embedding, $\overline{U}$ and retraction R and of the choice of V_B.*

Lemma 2.9. *Let $\mathcal{M}_i$ be a submanifold in $\mathcal{M}$, such that $F: \mathcal{M} \to \mathcal{M}_i$. Then $\text{ind}_F(\dot{\Omega}) = \text{ind}_{F|\mathcal{M}_i}(\Omega \dot{\cap} \mathcal{M}_i)$ and $\Lambda_F = \Lambda_{F|\mathcal{M}_i}$, where $F|_{\mathcal{M}_i}$ is the restriction of F on $\mathcal{M}_i$.*

The next statements follow from the construction and routine facts of the topological fixed point theory for condensing maps of Banach linear spaces.

Theorem 2.10. *The Lefschetz number is invariant under homotopies in the class of condensing maps. The index is invariant under homotopies in the class of condensing maps having no fixed points on $\dot{\Omega}$.*

Theorem 2.11. *If $\Lambda_F \neq 0$, then F has a fixed point in $\mathcal{M}$. If $\operatorname{ind}_F(\dot{\Omega}) \neq 0$, then F has a fixed point in Ω.*

For the manifolds and maps, considered above, the general scheme of constructing Nielsen number is well-posed.

Let $\mathcal{M}$ be connected but not simply connected. A continuous map from $[0,1] \subset R$ into $\mathcal{M}$ is called a path.

Definition 2.12. Fixed points x_1 and x_2 of F belong to the same Nielsen's equivalence class if there exists a path w, connecting them, such that $w \circ f(w)^{-1} = 0$ in $\pi_1(\mathcal{M})$.

Definition 2.13. A Nielsen's equivalence class of fixed points is called essential if ind $\neq 0$ and not essential otherwise.

Definition 2.14. The number of essential classes of F is called Nielsen number N_f.

Lemma 2.15. *The Nielsen number is constant under homotopies in the class of condensing maps.*

Indeed, under condensing homotopies the indices remain constant and it is easy to see that an equivalence class of fixed points transforms into an equivalence class of fixed points. Thus an essential class turns into an essential class and vice versa.

It should be pointed out that the construction of index, Lefschetz and Nielsen numbers, described above, can be obviously generalized for locally condensing maps F of Finsler manifolds of the same sort under the assumption that for a certain integer l, $0 < l \leq \infty$, the iteration F^l sends a bounded domain Ω or the entire $\mathcal{M}$, respectively, into a compact set.

3. The manifold of C^1-curves on a Riemannian manifold

Consider a basic example of the situation, described above: the manifold of C^1-curves on a compact Riemannian manifold. This manifold is a natural phase space for functional-differential equations of neutral type, investigated below. Here we present some basis for further developments.

Let M be a compact Riemannian manifold. By Nash's theorem it can be isometrically embedded into Euclidean space R^N, where N is large enough, as a neighbourhood retract. Denote by $i\colon M \to R^N$ this embedding and by Ti its tangent map. Notice that all tangent spaces to R^N are canonically isomorphic to R^N itself and that is why we consider Ti as a map sending TM into R^N.

Denote by $C^1([-h,0],M)$ the Banach manifold of C^1-curves in M, given on the interval $[-h,0]$ and by $C^0([-h,0],M)$ the Banach manifold of continuous curves defined on the same interval. For a curve $x(\cdot)$ from $C^1([-h,0],M)$ the tangent space $T_{x(\,\cdot\,)}C^1([-h,0],M)$ is the set of C^1 vector fields along $x(\,\cdot\,)$. The tangent space $T_{x(\,\cdot\,)}C^0([-h,0],M)$ at $x(\,\cdot\,)\in C^0([-h,0],M)$ is the set of all continuous vector fields along $x(\,\cdot\,)$.

Define the internal Finsler metric on $C^1([-h,0],M)$ by constructing the norm in $T_{x(\,\cdot\,)}C^1([-h,0],M)$ of the form:

$$\|Y(\,\cdot\,)\|_I^{C^1}=\sup_{t\in[-h,0]}\|Y(t)\|+\sup_{t\in[-h,0]}\bigg\|\frac{D}{dt}Y(t)\bigg\|, \tag{3.1}$$

where $(D/dt)Y(t)$ is the covariant derivative of Levi–Civitá connection (see, e.g. [6]) on M of the vector field $Y(t)$ along $x(\,\cdot\,)$ (emphasize that norm (3.1) is given in intrinsic terms); The internal Finsler metric on $C^0([-h,0],M)$ is introduced analogously via the norm

$$\|Y(\,\cdot\,)\|_I^{C^0}=\sup_{t\in[-h,0]}\|Y(t)\|. \tag{3.2}$$

The map i generates the embedding of the manifold $C^1([-h,0],M)$ into Banach space $C^1([-h,0],R^N)$ as a neighbourhood retract. Introduce the following norm in $T_{x(\,\cdot\,)}C^1([-h,0],M)$:

$$\|Y(\,\cdot\,)\|_E^{C^1}=\sup_{t\in[-h,0]}\|Y(t)\|+\sup_{t\in[-h,0]}\|(TiY(t))'\| \tag{3.3}$$

where $(TiY(t))'$ is the derivative of curve $TiY(t)$ in R^N.

Introduce the norm, analogous to (3.3) in $C^1([-h,0],R^N)$. Then (3.3) is its restriction onto tangent spaces to $C^1([-h,0],M)$.

As well as in Section 2 construct the distance functions in $C^1([-h,0],M)$, corresponding to norms (3.1) and (3.3) and denote them by ρ_I and ρ_E, respectively.

Denote by P the orthogonal projection of R^N onto T_mM. It is well-known that $(D/dt)Y(t)=P(TiY(t))'$, thus

$$\|Y(\,\cdot\,)\|_I^{C^1}\le\|Y(\,\cdot\,)\|_E^{C^1}.$$

One can easily see that

$$(TiY(t))'=\frac{D}{dt}Y(t)+(I-P)\bigg(T^2i\bigg(\frac{d}{dt}y(t),Y(t)\bigg)\bigg),$$

where T^2i is the bilinear operator of second derivative of embedding i.

For $x(\,\cdot\,)\in C^1([-h,0],M)$ its velocity vector field $x'(t)$ is continuous and so its norm $\|x'(t)\|$ is bounded of the compact interval $[-h,0]$ by a certain constant $k(x(\,\cdot\,))>0$. The operator norm of $(I-P)T^2i$ on M is bounded as a continuous function on the compact manifold M, i.e. $\|(I-P)Ti^2\|\le\Xi$ for some $\Xi>0$.

Hence, using the above estimates and the obvious fact that $\sup_{t\in[-h,0]}\|Y(t)\| \le \|Y(t)\|_I^{C^1}$, we see that

$$\begin{aligned}\|Y(\,\cdot\,)\|_E^{C^1} &= \sup_{t\in[-h,0]}\|Y(t)\| + \sup_{t\in[-h,0]}\left\|\frac{D}{dt}Y(t) + \left((I-P)Ti^2\left(\frac{d}{dt}x(t), Y(t)\right)\right)\right\| \\ &\le \sup_{t\in[-h,0]}\|Y(t)\| + \sup_{t\in[-h,0]}\left\|\frac{D}{dt}Y(t)\right\| + \Xi \sup_{t\in[-h,0]}\left(\left\|\frac{d}{dt}x(t)\right\|, \|Y(t)\|\right) \\ &\le \|Y(t)\|_I^{C^1}(1+\Xi k(x(\,\cdot\,))).\end{aligned}$$

So, we obtain the following estimate for the norms:

$$\|Y(\,\cdot\,)\|_I^{C^1} \le \|Y(\,\cdot\,)\|_E^{C^1} \le \|Y(\,\cdot\,)\|_I^{C^1}(1+\Xi k(x(\,\cdot\,))). \tag{3.4}$$

Inequality (3.4) means that the norms $\|\cdot\|_I^{C^1}$ and $\|\cdot\|_E^{C^1}$ are equivalent. Evidently (3.4) can be transformed to the form (2.1). In particular, consider a set $\overline{\Omega} \in C^1([-h,0],M)$ having finite diameter with respect to ρ_I. Then for all curves $x(\,\cdot\,) \in \overline{\Omega}$ the velocity vector filed is bounded: $\|x'(t)\| \le k$ for some $k \ge 0$ independent of $x(\,\cdot\,)$. Hence from (3.4) we obtain on $\overline{\Omega}$

$$\|Y(\,\cdot\,)\|_I^{C^1} \le \|Y(\,\cdot\,)\|_E^{C^1} \le \|Y(\,\cdot\,)\|_I^{C^1}(1+\Xi k). \tag{3.5}$$

Thus we can apply the constructions of Section 2 to condensing maps of $C^1([-h,0],M)$.

4. Functional-differential equations of neutral type on a Riemannian manifold

The functional-differential equations of neutral type describe processes whose first derivative at a time instant t explicitly depends on the values of process and its first derivatives at previous time instants. Applied problems, where such equations arise (as well as the bibliography up to 1982 for the case of linear spaces), are presented, e.g. in survey [2]. The basic definitions and facts from this theory in linear spaces can be found, e.g. in [2], [10].

In this section we extend the theory of functional-differential equations of neutral type to the case of smooth manifolds. For this purpose we first introduce the geometrically invariant notion of functional vector field of neutral type (FVFN).

Let M be a complete Riemannian finite-dimensional manifold and TM be its tangent bundle. By $\pi\colon TM \to M$ we denote the natural projection. Consider the Banach manifolds $C^0([-h,0],M)$ and $C^0([-h,0],TM)$ of continuous curves in M and TM, respectively, given for $s \in [-h,0]$. Notice that the norms of type (3.2) are well-posed in tangent spaces to those manifolds as well as Finsler metrics on the manifolds, generated by the norms (see Section 3).

Definition 4.1. The map $f\colon I \times C^0([-h,0],TM) \to TM$ such that for any $y(\,\cdot\,) \in C^0([-h,0],TM)$ the condition $f(t,y(\,\cdot\,)) \in T_{\pi(y(0))}M$ holds, where I is a certain interval of real line (in particular, it may be the entire real line), is called a functional vector field of neutral type (FVFN).

Below, if the contrary is not postulated, for the sake of simplicity we assume that I is the entire real line. In various concrete problems we also suppose f to satisfy some additional conditions from the following list:

Condition 4.2. FVFN f *is bounded, i.e. for a certain constant* $C > 0$ *and any* $y(\,\cdot\,) \in C^0([-h,0],TM)$, $t \in I$ *the inequality* $\|f(t,y(\,\cdot\,))\| \le C$ *holds, where the norm is generated by the Riemannian metric on* M.

It is clear that any curve $y(\,\cdot\,) \in C^0([-h,0],TM)$ can be represented as the couple $y(t) = (x(t),X(t))$, where $x(t) = \pi y(t)$ is a continuous curve in M and $X(t)$ is a continuous vector field along $x(t)$. Note that for the vector field $X(\,\cdot\,)$ the uniform norm $\|X(\,\cdot\,)\|^{C^0} = \sup_{s\in[-h,0]} \|X(s)\|$ is well-posed, where $\|\cdot\|$ (as well as above) is the norm in tangent spaces to M, generated by the Riemannian metric.

Condition 4.3. *For any continuous curve* $x(\,\cdot\,)$ FVFN f *is Lipschitz continuous in the third argument, i.e. for some constant* $C_1 > 0$ *the inequality* $\|f(t,x(\,\cdot\,),X_0(\,\cdot\,)) - f(t,x(\,\cdot\,),X_1(\,\cdot\,))\| < C_1\|X_0(\,\cdot\,) - X_1(\,\cdot\,)\|^{C^0}$ *holds, where* C_1 *does not depend on* t *and* $x(\,\cdot\,)$.

Condition 4.4. *The map* $f\colon I \times C^0([-h,0],TM) \to TM$ *is* C^1*-smooth as a mapping of manifolds.*

In particular from Condition 4.4 it follows that f is continuous.

For a continuous curve $y(t)$ in TM, $t \in [-h,T]$ for some $T > 0$ we introduce the family of curves from $C^0([-h,0],TM)$ by the standard construction $y_t(\theta) = y(t+\theta)$, $\theta \in [-h;0]$, $t \in [0,T]$. Obviously the family $y_t(\theta)$ forms a curve in $C^0([-h,0],TM)$ that is continuous in $t \in [0,T]$. Then $f(t,y_t)$, $t \in [0,T]$ is a continuous curve in TM.

Condition 4.5. *For a bounded set* $\mathbf{Y} \subset C^0([-h,T],TM)$ *such that* $\pi\mathbf{Y}$ *is compact in* $C^0([-h,T],M)$, *the set* $\{f(t,y_t) | y \in \mathbf{Y}\}$ *is compact in* $C^0([0,T],TM)$.

Remark 4.6. For the case, where M is a Euclidean space R^n, Condition 4.5 follows from a certain condition from [11]. However the latter condition is specific for linear spaces and for manifolds it can be formulated only in charts.

Condition 4.7. FVFN f *is* ω*-periodical, i.e. for a certain* $\omega > 0$ *for any* $y(\,\cdot\,) \in C^0([-h,0],TM)$ *and any* $t \in R$ *the equality* $f(t+\omega,y(\,\cdot\,)) = f(t,y(\,\cdot\,))$ *holds.*

A particular case of continuous curve in TM of the form $y(t) = (x(s), x'(s))$, where $x(s)$ is a C^1-curve in M and $x'(s)$ is its derivative, plays an important role below. We emphasize that the vector $f(t, x(\,\cdot\,), x'(\,\cdot\,)) \in T_{x(0)}M$ is well-posed. Sometimes we shall suppose that a curve $x(\,\cdot\,) \in C^1([-h, 0], M)$ satisfies the following condition:

Condition 4.8 (Condition of compatibility).

$$x'_-(0) = f(0, x(t), x'(t)),$$

where $x'_-(0)$ is the backward derivative at zero point.

As above, for a C^1-curve $x(t)$, $t \in [-h, T]$, $T \geq 0$ we introduce the standard notations $x_t(\theta) = x(t + \theta)$, $x'_t(\theta) = x'(t + \theta)$, where $\theta \in [-h; 0]$. Consider the following problem:

$$x'(t) = f(t, x_t, x'_t), \quad t \geq 0, \tag{4.1}$$

Definition 4.9. Equation (4.1) is called functional-differential equation of neutral type (FDEN) on the Riemannian manifold M, generated by FVFN f. A C^1-curve $x^\varphi(t)$, $t \in [-h, T)$ for a certain $T > 0$ is called a solution of (4.1) with initial condition φ, satisfying Condition 4.8, if $x^\varphi(t) = \varphi(t)$ for $t \in [-h, 0]$ and $x^\varphi(t)$ satisfies (4.1) for $t \in [0, T)$.

We emphasize that the above definition of solution is consistent if the initial curve $\varphi(t)$ satisfies Condition 4.8. In order to get a solution for any initial curve, we modify (4.1) and consider the following equation

$$x'(t) = f(t, x_t, x'_t) + \|\nu_\eta(t)[x'_-(0) - f(0, x_0, x'_0)], \quad t \geq 0, \tag{4.2}$$

where $\eta > 0$ is a small enough real number, $\nu_\eta(t) = 1 - \eta^{-1}t$ if $t \in [0, \eta]$ and $\nu_\eta(t) = 0$ if $t > \eta$, $\|$ denotes the Riemannian parallel translation of the vector $\nu_\eta(t)[x'_-(0) - f(0, x_0, x'_0)] \in T_{x_0(0)}M$ along the solution $x(\,\cdot\,)$ from $x(0)$ to $x(t)$. This is a natural modification of a standard method.

As well as above embed M isometrically into a Euclidean space R^N of a dimension large enough. Denote by $r\colon U \to M$ the retraction of tubular neighbourhood U in R^N onto M. One can easily see that $r\colon U \to M$ generates the smooth map from $C^0([-h, 0], U)$ onto $C^0([-h, 0], M)$. In particular, for $\varphi(t) \in C^0([-h, 0], U)$ we get that

$$r\varphi(t) \in C^0([-h, 0], M).$$

Over any chart V_α in M the tubular neighbourhood U is represented in the form:

$$r^{-1}(V_\alpha) = V_\alpha \times W,$$

where W is an open ball in R^{N-n}, n is the dimension of M. Any $\psi(t) \in C^0([-h,0],U)$ is represented in the form:

$$\psi(t) = (\varphi(t), Z(t)),$$

where $\varphi(t) \subset V_\alpha$, $Z(t) \subset W$. For any vector field $Z(t)$ along $\psi(t)$ we obtain the decomposition $Z(t) = (Z^T(t), Z^N(t))$, where $Z^T(t)$ is tangent to V_α, and $Z^N(t)$ is tangent to W. Evidently $TrZ(t) = TrZ^T(t)$. Introduce the function $\overline{f}(t,\psi(t)) = \overline{f}(t,\varphi(t),Z(t)) = \overline{f}(t,\varphi(t),Z^T(t))$ by the formulae

$$\overline{f}^T(t,\varphi(t),Z(t)) = f(t,r\varphi(t),TrZ(t)), \quad \overline{f}^N(t,\varphi(t),Z(t)) = 0.$$

The map $\overline{f}$ is well-posed on the domain $C^0([-h,0],U) \times C^0([-h,0],R^N)$. Consider the open ball $B_{C+\varepsilon}(0)$ centered at the origin in the space $C^0([-h,0],R^N)$ and having radius $C+\varepsilon$ where C is introduced in Condition 4.2 and $\varepsilon > 0$ is a certain number. Below we consider $\overline{f}$ defined on $C^0([-h,0],U) \times B_{C+\varepsilon}$, i.e. on a bounded domain in the Banach space $C^0([-h,0],U) \times C^0([-h,0],R^N)$.

Theorem 4.10. *Under Conditions* 4.2–4.4 *above:*

(a) *for any curve* $\varphi(t) \in C^1([-h,0],M)$, *satisfying Condition* 4.8, *and any* $T > 0$ *there exists a unique solution* $x^\varphi(t)$, $t \in [-h,T]$ *of* (4.1) *on* M *with initial condition* $x_0^\varphi(s) = \varphi(s)$;

(b) *for any curve* $\varphi(t) \in C^1([-h,0],M)$ *any* $T > 0$ *there exists a unique solution* $x^\varphi(t)$, $t \in [-h,T]$ *of* (4.2) *on* M *with initial condition* $x_0^\varphi(s) = \varphi(s)$.

Proof. Consider the following equation of type (4.1) in R^N

$$x'(t) = \overline{f}(t,x_t,x'_t) \tag{4.3}$$

with initial condition $\varphi(t)$. It is easy to see that for (4.3) all the hypotheses of Theorem 4.3.6 [1] (see also [10]) are fulfilled and so for $\varphi(t) \in C^0([-h,0],U) \times B_{C+\varepsilon}(0)$ there exists a unique solution for $t \in [0,T]$. Moreover, by the construction $x'(t) = \overline{f}^T(t,x_t,x'_t)$ and $\overline{f}^N(t,x_t,x'_t) = 0$, hence (4.3) is decomposed into the system

$$\begin{cases} x'(t) = \overline{f}^T(t,x_t,x'_t), \\ y'(t) = 0. \end{cases}$$

Thus $x(t) \subset M$ for all $t \in [0,\varepsilon]$. Since M is complete, f is bounded and the embedding of M into R^N is isometric we obtain that $x(t)$ does exist for $t \in [0,T]$.

The proof of the second part is quite analogous. □

5. Shift operator along the trajectories of FDEN and its Lefschetz number

In this section M is a compact Riemannian manifold. Suppose that Conditions 4.2–4.4 of Section 4 are satisfied. Then by Theorem 4.10 for any initial curve φ there exists unique solution of (4.2) that is continuous in φ. Specify $\omega > 0$.

Definition 5.1. Operator $u_\omega: C^1([-h,0],M) \to C^1([-h,0],M)$ sending the curve φ into the curve x_ω^φ is called the shift operator along the solution of FDEN (4.1).

One can easily show that under Condition 4.7 of Section 4 the fixed points of u_ω and only they are ω-periodic solutions of (4.1). Here we show that if in addition Condition 4.5 is satisfied, u_ω is a condensing map with respect to internal Kuratowski's measure of noncompactness of a certain special Finsler metric on $C^1([-h,0],M)$.

Below in this section we suppose Conditions 4.2–4.5 and 4.7 to be fulfilled.

Modify metric (3.1) on the manifold $C^1([-h,0],M)$ as follows:

$$\|Y(\,\cdot\,)\|_I^{\mathbf{C}^1} = \sup_{t\in[-h,0]} e^t\|Y(t)\| + \sup_{t\in[-h,0]} e^t\left\|\frac{D}{dt}Y(t)\right\|. \tag{5.1}$$

Analogous modification of (3.3) has the form

$$\|Y(\,\cdot\,)\|_E^{\mathbf{C}^1} = \sup_{t\in[-h,0]} e^t\|Y(t)\| + \sup_{t\in[-h,0]} e^t\|(TiY(t))'\| \tag{5.2}$$

and of (3.2) has the form

$$\|Y(\,\cdot\,)\|_I^{\mathbf{C}^0} = \sup_{t\in[-h,0]} e^t\|Y(t)\|. \tag{5.3}$$

The trivial modification of constructions of Section 3 where (3.1) and (3.3) replaced with (5.1) and (5.2), respectively, makes obvious the fact that the index and Lefschetz number are well-posed for an operator on $C^1([-h,0],M)$ that is condensing with respect to Kuratowski's measure of noncompactness, corresponding to norm (5.1). We shall show that u_ω is of this sort and calculate its Lefschetz number.

Let $x_0(t)$ and $x_1(t)$ be curves in M, $t\in[-h,0]$, i.e. points in $C^1([-h,0],M)$. Connect them by a C^1-curve $\xi(s) = x(t,s)$ where $t\in[-h,0]$ and $s\in[0,1]$, $\xi(0)=x_0$ and $\xi(1)=x_1$, in the manifold $C^1([-h,0],M)$. Denote $(d/dt)x(t,s)$ by $W(t,s)$. Then the length of $\xi(s)$ with respect to $\|\cdot\|_I^{\mathbf{C}^1}$ is calculated by the formula:

$$l(\xi)_I^{\mathbf{C}^1} = \int_0^1 \left(\sup_{t\in[-h,0]} e^t\|W(t,s)\| + \sup_{t\in[-h,0]} e^t\left\|\frac{D}{dt}W(t,s)\right\|\right) ds. \tag{5.4}$$

The distance in $C^1([-h,0],M)$ between $x_0(\,\cdot\,)$ and $x_1(\,\cdot\,)$, corresponding to norm (5.1), is the infimum of lengths of curves, connecting $\eta(\,\cdot\,)$ and $\theta(\,\cdot\,)$, where the length is calculated by formula (5.4). Denote by $\overline{\alpha}_I$ the internal Kuratowski's measure of non-compactness generated by this distance.

Let Ω be a set in $C^1([-h,0],M)$, bounded with respect to the above distance. First suppose that $\omega \geq h$.

Lemma 5.2. *If $\omega \geq h$, for any bounded set Ω in $C^1([-h,0],M)$ the set $u_\omega(\Omega)$ is compact in $C^1([-h,0],M)$.*

Proof. Since M is compact, all curves from Ω belong to a compact set and so are uniformly bounded. Moreover, boundedness of Ω in $C^1([-h,0],M)$ means that for all $\varphi(\,\cdot\,) \in \Omega$ the derivatives $(d/dt)\varphi(t)$ are uniformly bounded and so the curves from Ω are equicontinuous. Recall that by definition for $t \in [0,\omega]$ we have $(d/dt)x^\varphi(t) = f(t, x^\varphi_t, (d/dt)x^\varphi_t)$. Then from Condition 4.2 it follows that the curves $\{x^\varphi(t) | \varphi \in \Omega\}$ have uniformly bounded derivatives on the entire interval $[-h,\omega]$. Thus they are uniformly bounded and equicontinuous and so they form a compact set in $C^0([-h,\omega],M)$. From Condition 4.5 we obtain that the curves $(d/dt)x^\varphi(t) = f(t, x^\varphi_t, (d/dt)x^\varphi_t)$, $t \in [0,\omega]$ form a compact set in $C^0([0,\omega],TM)$. Hence the set $u_\omega(\Omega)$ is compact in $C^1([-h,0],M)$ since the derivatives of those curves form a compact set in $C^0([-h,0],TM)$. □

Now consider the case $\omega < h$.

Introduce the Kuratowski's measure of noncompactness $\overline{\alpha}_I$ in $C^1([-h,0],M)$ with respect to the distance function, defined as inf of lengths of curves in $C^1([-h,0],M)$ with respect to norm (5.1). For any point $-h+\omega \in [-h,0]$ consider the restrictions of curves from $C^1([-h,0],M)$ on the intervals $[-h,-h+\omega]$ and $[-h+\omega,0]$. On the manifolds of curves, defined on those subintervals, one can define the Finsler metrics, the distances and Kuratowski's measures of noncompactness as above. Denote those measures of noncompactness by $\overline{\alpha}_{I,[-h,-h+\omega]}(\Omega)$ and $\overline{\alpha}_{I,[-h+\omega,0]}(\Omega)$.

Lemma 5.3. *For a bounded set $\Theta \subset C^1([-h,0],M)$ the equality*

$$\overline{\alpha}_I(\Theta) = \max(\overline{\alpha}_{I,[-h,-h+\omega]}(\Theta), \overline{\alpha}_{I,[-h+\omega,0]}(\Theta)),$$

holds.

Lemma 5.3 is a well-known property of Kuratowski's measure of non-compactness (see, e.g. [11]).

Lemma 5.4. *Let $\omega < h$ and Ω be a bounded set in $C^1([-h,0],M)$. Then $\overline{\alpha}_I u_\omega(\Omega) < e^{-\omega}\overline{\alpha}_I(\Omega)$.*

Proof. By Lemma 5.3 we have $\overline{\alpha}_I(u_\omega(\Omega)) = \max(\overline{\alpha}_{[-h,-\omega]}(u_\omega(\Omega)), \overline{\alpha}_{[-\omega,0]}(u_\omega(\Omega)))$ and $\overline{\alpha}_{[-h,0]}(\Omega) = \max(\overline{\alpha}_{[-h,-h+\omega]}(\Omega), \overline{\alpha}_{[-h+\omega,0]}(\Omega))$.

Taking into account the arguments used in proof of Lemma 5.2 one can easily see that on $[-\omega,0]$ the curves $u_\omega(\Omega)$ form a compact set and so $\overline{\alpha}_{[-\omega,0]}u_\omega(\Omega) = 0$. Thus $\overline{\alpha}_I(u_\omega(\Omega)) = \overline{\alpha}_{[-h,-\omega]}(u_\omega(\Omega))$. By the construction of norm (5.1), length (5.4) and corresponding distance we obtain $\overline{\alpha}_{[-h,-\omega]}(u_\omega(\Omega)) = e^{-\omega}\overline{\alpha}_{[-h+\omega,0]}(\Omega)$.

Hence $\overline{\alpha}_I(u_\omega(\Omega)) = \overline{\alpha}_{[-h,-\omega]}(u_\omega(\Omega)) = e^{-\omega}\overline{\alpha}_{[-h+\omega,0]}(\Omega) < e^{-\omega}\max(\overline{\alpha}_{[-h,-h+\omega]}(\Omega), \overline{\alpha}_{[-h+\omega,0]}(\Omega)) = e^{-\omega}\overline{\alpha}_I(\Omega)$. □

Theorem 5.5. *u_ω is condensing with respect to $\overline{\alpha}_I$ with a constant less than* 1 *and so the index, constructed above, for it is well-posed.*

Theorem 5.5 follows from Lemmas 5.2 and 5.4.

Specify $\varepsilon > 0$. Notice that from Condition 4.2 it follows that the operator u_ω sends $C^1([-h,0],M)$ into the domain $B_{C+\varepsilon} \subset C^1([-h,0],M)$ such that for any $x(t) \in B_{C+\varepsilon}$ we have $\|x'(t)\| < C+\varepsilon$. By the construction u_ω has no fixed points on the boundary of $B_{C+\varepsilon}$ and $B_{C+\varepsilon}$ has finite diameter. Thus the construction of Lefschetz number of Section 2 is well-posed for u_ω.

Theorem 5.6. *The Lefschetz number Λ_{u_ω} is equal to Euler characteristic $\chi(M)$.*

Proof. Let $s \in [0,1]$. Introduce the curve $\kappa^s(x(\,\cdot\,)) \in C^1([-h,0],M)$ by the formula

$$\kappa^s(x(t)) = x(st).$$

Consider the homotopy $\overline{u}_s(x(\,\cdot\,)) = u_{s\omega}(\kappa^s(x(\,\cdot\,)))$. As well as above one can easily prove that this homotopy lives in the class of condensing operators. By the construction $u_{\omega s}$ at $s = 1$ coincides with u_ω, and at $s = 0$ the operator u_0 sends the curve $x(\,\cdot\,)$ into the constant curve $x(0)$. By Lemma 2.9 we can restrict u_0 from $C^1([-h,0],M)$ on the submanifold consisting of constant curves that is isomorfic to M. But on M the operator u_0 is the identical map and so its total index (Lefschetz number) is equal to Euler characteristic $\chi(M)$. □

Corollary 5.7. *Let $\chi(M) \neq 0$ and f satisfy Conditions* 4.2–4.5 *and* 4.7. *Then* (4.1) *has an ω-periodic solution.*

6. Appendix. An example of calculation of Nielsen number

Now consider a model example where an operator, constructed from u_ω, has zero Lefschetz number while its Nielsen number is not equal to zero. Here we involve a special manifold M and a certain continuous map $\widehat{h}: M \to M$, considered in [8], [9].

Introduce the equivalence relation in $\mathbb{R}^3$ of the form

$$(x,y,z) \sim (x+a, (-1)^a y + b, (-1)^a z + c), \tag{6.1}$$

where a, b, c are integers. Let M be the manifold obtained from $\mathbb{R}^3$ by factorization with respect to this equivalence.

Consider in $\mathbb{R}^3$ the map $h: \mathbb{R}^3 \to \mathbb{R}^3$, given by the formula:

$$h(x,y,z) = (-x, y+z, y). \tag{6.2}$$

This map induces the continuous map $\widehat{h}: M \to M$ having two fixed points:

$$x_1 = (0,0,0) \quad \text{and} \quad x_2 = (1/2,0,0).$$

It is shown in [8] and [9] that $\mathrm{ind}_{x_1} = -1$ and $\mathrm{ind}_{x_2} = 1$. Hence the Lefschetz number $\Lambda_{\widehat{h}} = 0$. The fixed points x_1 and x_2 belong to different essential Nielsen equivalence classes and so the Nielsen number $N_{\widehat{h}} = 2$.

Consider the shift operator $u_\omega: C^1([-h,0],M) \to C^1([-h,0],M)$ from Section 5 and introduce the operator $\widehat{h}u_\omega$. From the fact that u_ω is condensing it follows that $\widehat{h}u_\omega$ is also condensing with respect to Kuratowski's measure of noncompactness of the same metric and so the Lefschetz and Nielsen numbers are well-posed for it.

Theorem 6.1. *$\Lambda_{\widehat{h}u_\omega} = 0$ and $N_{\widehat{h}u_\omega} = 2$.*

Proof. The proof is a modification of that for Theorem 5.6. Consider the homotopy $\widehat{u}_s = \widehat{h}\overline{u}_s$ where $\overline{u}_s$ was introduced in the proof of Theorem 5.6. By the same arguments as in that proof one can easily show that $\widehat{u}_1 = \widehat{h}u_\omega$ and that the homotopy $\widehat{u}_s$ lives in the class of condensing maps. The operator $\widehat{u}_0$ can be restricted to the manifold of constant curves, homeomorfic to M, and coincides there with $\widehat{h}$. Since $\Lambda_{\widehat{h}} = 0$ and $N_{\widehat{h}} = 2$, this completes the proof. □

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