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SYMMETRY BREAKING SOLUTIONS OF NONLINEAR ELLIPTIC SYSTEMS

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Abstract. We consider nonlinear elliptic systems with Dirichlet boundary condition on a bounded domain in $\mathbb{R}^N$ which is invariant with respect to the action of some group G of orthogonal transformations. For every subgroup K of G we give a simple criterion for the existence of infinitely many solutions which are K-invariant but not G-invariant. We include a detailed discussion of the case $N = 3$.

1. Introduction

Consider the elliptic system

$$(\wp) \qquad \begin{cases} -\Delta u = F_u(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, and $F : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ is a C^1-function. A solution to this problem is a vector-valued function $u = (u_1, \ldots, u_d) : \overline{\Omega} \to \mathbb{R}^d$ which satisfies $(\wp)$.

If Ω is a ball or an annulus and F is radial in x one may ask whether this problem has infinitely many nonradial solutions. This question has been

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extensively studied, see for example [1], [4], [5], [8]–[10], [17], [18], [21], [22], [25], [28], [29]. One can also ask the question whether the problem has infinitely many solutions which are nonradial but which possess some other specific type of symmetry. For a single equation and some special nonlinearities Kajikiya recently gave a characterization of those symmetries for which this question has a positive answer [19], [20].

Here we address the following question: Assume that Ω is invariant with respect to the action of some closed subgroup G of the group $O(N)$ of orthogonal transformations of $\mathbb{R}^N$, that is, $gx \in \Omega$ for every $x \in \Omega$, $g \in G$. Moreover, assume that

(S^G) $F(gx,u) = F(x,u)$ for every $g \in G$, $x \in \Omega$, $u \in \mathbb{R}^d$.

Given a subgroup K of G, are there infinitely many solutions u of ($\wp$) which are K-invariant but not G-invariant? In other words, are there infinitely many solutions u which satisfy $u(gx) = u(x)$ for every $g \in K$, $x \in \Omega$, but for which there exist $g_0 \in G$, $x_0 \in \Omega$ with $u(g_0x_0) \neq u(x_0)$?

If G is not the whole orthogonal group $O(N)$ this turns out to be, in general, a harder question than the one about existence of K-invariant nonradial solutions. The existence of infinitely many nonradial solutions requires careful estimates on the growth of the energy levels of the radial solutions, which are obtained using ODE-methods (see for example [1], [19]). These methods do not apply to other symmetry groups G.

Here we give some positive answers to this question based merely on group theoretical methods. We exploit an interplay between the given orthogonal action of G on the domain $\Omega \subset \mathbb{R}^N$ and some properly chosen orthogonal representation structure $\rho\colon G \to O(d)$ on the space $\mathbb{R}^d$ of values of the function u. Our results provide criteria for the existence of infinitely many K-invariant solutions which are not G-invariant, in terms of the groups themselves, under the usual growth conditions on the nonlinearity F. These criteria are easy to check. We include a detailled discussion for the case $N = 3$.

Our methods apply also to other types of elliptic systems, including Hamiltonian systems. But since the goal of this paper is to study the symmetry breaking phenomenon, we have chosen to restrict ourselves to the simplest type, that of gradient systems ($\wp$), to avoid additional technicalities. Our approach should also be useful in the study of bifurcation of symmetry breaking solutions of nonlinear problems, as considered in [14], [26], [27], [28].

2. Statement of results

Before stating our main results we recall some basic facts about transformation groups and introduce some notation. Details may be found for example in [6], and [15].

Let G be a closed subgroup of the orthogonal group $O(N)$. The G-orbit of a point $x \in \mathbb{R}^N$ is the set $Gx = \{gx \in \mathbb{R}^N : g \in G\}$. It is G-homeomorphic to the homogeneous space G/G_x, where G_x is the isotropy group $G_x = \{g \in G : gx = x\}$ of x. The isotropy groups of two points in the same orbit are conjugate. The conjugacy class (G_x) of G_x is called an isotropy class. There exists a unique isotropy class (P_G) such that $\{x \in \mathbb{R}^N : (G_x) = (P_G)\}$ is open and dense in $\mathbb{R}^N$. Any other isotropy class satisfies $(P_G) \le (G_x)$, i.e. $P_G \subset gG_xg^{-1}$ for some $g \in G$. (P_G) is called the *principal isotropy class* and G/P_G is called the *principal orbit type* of G.

Given a closed subgroup K of G we denote by $N_GK = \{g \in G : gKg^{-1} = K\}$ the normalizer of K in G. The *Weyl group of K in G* is the quotient group

$$W_GK = N_GK/K.$$

For every subgroup Γ of W_GK we write

$$\widetilde{\Gamma} := q^{-1}(\Gamma)$$

where $q\colon N_GK \to W_GK$ is the natural epimorphism.

We assume the following standard conditions on F :

(F_1) If $N \ge 3$ there are constants $2 < p < 2^* := 2N/(N-2)$ and $c > 0$ such that for every $x \in \Omega$, $u \in \mathbb{R}^d$,

$$|F_u(x,u)| \le c(1 + |u|^{p-1}).$$

If $N = 2$ this assumption can be weakened, if $N = 1$ it can be omitted.

(F_2) There are constants $\mu > 2$ and $R > 0$ such that for every $x \in \Omega$, $u \in \mathbb{R}^d$, $|u| \ge R$,

$$0 < \mu F(x,u) \le u \cdot F_u(x,u).$$

(S) $F(x,-u) = F(x,u)$ for every $x \in \Omega$, $u \in \mathbb{R}^d$.

If $d = 2s$ we identify $\mathbb{R}^d \equiv \mathbb{C}^s$. We say that F is *toroidal* in $u = (z_1, \ldots, z_i, \ldots, z_s)$ if $F(x, z_1, \ldots, z_i, \ldots, z_s) = F(x, z_1, \ldots, |z_i|, \ldots, z_s)$ for every $x \in \Omega$, $z_i \in \mathbb{C}$, $i = 1, \ldots, s$.

The following result gives an easy criterion for the existence of symmetry breaking solutions.

Theorem 2.1. *Let Ω be G-invariant and let K be a closed subgroup of G. Assume that F satisfies* (F_1)*,* (F_2)*,* (S) *and* (S^G).

(a) *If W_GK contains a subgroup Γ of order 2 such that $P_{\widetilde{\Gamma}} \subset K$, then problem ($\wp$) has infinitely many solutions which are K-invariant but not G-invariant.*

(b) *If W_GK contains a nontrivial subgroup Γ such that $P_{\widetilde{\Gamma}} \subset K$, and if $d = 2s$ and F is toroidal in u, then problem ($\wp$) has infinitely many solutions which are K-invariant but not G-invariant.*

The criteria given by this theorem depend merely on the subgroups $K \subset G$ of $O(N)$ and are easy to check. We shall study the case $N = 3$ in detail and see which subgroups $K \subset G$ of $O(3)$ satisfy this criteria. Notice that a necessary condition for the existence of solutions which are K-invariant but not G-invariant is that the K-orbit and the G-orbit of some point in Ω do not coincide. We shall show that this condition is also sufficient if $G \neq O(3)$ and the Weyl group $W_G K$ contains an element of order 2. More precisely, we shall prove the following.

Theorem 2.2. *Let $K \subset G$ be proper closed subgroups of $O(3)$ such that $Kx \neq Gx$ for some $x \in \mathbb{R}^3$. Assume that $\Omega \subset \mathbb{R}^3$ is G-invariant and that F satisfies* (F_1), (F_2), (S) *and* (S^G).

(a) *If $W_G K$ contains an element of order 2, then problem $(\wp)$ has infinitely many solutions which are K-invariant but not G-invariant.*
(b) *If $W_G K$ is nontrivial, $d = 2s$ and F is toroidal in u, then problem $(\wp)$ has infinitely many solutions which are K-invariant but not G-invariant.*

For $G = O(3)$ we obtain the following.

Theorem 2.3. *Let K be a closed subgroup of $O(3)$ such that $K \neq O(3)$, $SO(3)$, $O(2) \times \mathbb{Z}_2^c$, $O(2)$, $I \times \mathbb{Z}_2^c$, $O \times \mathbb{Z}_2^c$. Let $\Omega \subset \mathbb{R}^3$ be a ball or an annulus, and assume that F satisfies* (F_1), (F_2), $(S^{N_{O(3)}K})$ *and* (S) *then problem $(\wp)$ has infinitely many nonradial solutions which are K-invariant.*

For $d = 1$ and $F(x,u) = |u|^p$ Kajikiya [19] showed that there are infinitely many nonradial K-invariant solutions of $(\wp)$ on a ball or an annulus in $\mathbb{R}^N$ if and only if K acts nontransitively on $\mathbb{S}^{N-1}$. That is, if $N = 3$, problem $(\wp)$ has infinitely many nonradial K-invariant solutions for every $K \neq O(3)$, $SO(3)$. Theorem 2.3 does not cover all of Kajikiya's cases: It does not include the groups $K = O(2) \times \mathbb{Z}_2^c$, $O(2)$, $I \times \mathbb{Z}_2^c$, $O \times \mathbb{Z}_2^c$. On the other hand, it applies to more general nonlinearities and, as we shall see below, the proof is quite elementary. Kajikiya's proof involves delicate arguments, including a careful analysis of the asymptotic growth of the radial critical values of the associated functional, which cannot be extended to groups other than $O(N)$.

Our method provides, in addition, precise information on how the symmetries are broken (see our remarks in the following section, after the proof of Theorem 2.1).

3. Intertwining solutions

In this section we prove Theorem 2.1. We shall obtain it as a consequence of a multiplicity result for solutions of $(\wp)$ having specific symmetries.

Let G be a closed subgroup of $O(N)$ and let ρ be a d-dimensional orthogonal representation of G, that is, a homomorphism $\rho: G \to O(d)$. Let $\Omega \subset \mathbb{R}^N$ be

G-invariant. A function $u\colon \Omega \to \mathbb{R}^d$ which satisfies

$$u(gx) = \rho(g)u(x) \quad \text{for all} \ \ g \in G, \ x \in \Omega,$$

will be called a *ρ-intertwining* function.

As in the previous section, we denote by (P_G) the principal isotropy class of G. We shall prove the following

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be G-invariant, and let $\rho\colon G \to O(d)$ be an orthogonal representation such that $P_G \subset \ker\rho$. Assume that F satisfies* (F_1)*,* (F_2)*,* (S) *and*

(S^G_ρ) *$F(gx, \rho(g)u) = F(x,u)$ for every $g \in G$, $x \in \Omega$, $u \in \mathbb{R}^d$.*

Then problem $(\wp)$ has infinitely many ρ-intertwining solutions.

Before giving the proof let us consider some easy consequences of this result. If ρ is the trivial representation $\rho \equiv 1 \in O(d)$, then a ρ-intertwining solution is just a G-invariant solution. The trivial representation obviously satisfies $P_G \subset \ker\rho = G$. So Theorem 3.1 supplies, in particular, G-invariant solutions:

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^N$ be G-invariant. Assume that F satisfies* (F_1)*,* (F_2)*,* (S) *and* (S^G)*. Then problem $(\wp)$ has infinitely many G-invariant solutions.*

Theorem 3.1 also supplies noninvariant solutions. For example, if Ω is symmetric with respect to the origin, that is, $x \in \Omega$ if and only if $-x \in \Omega$, we may take $G = \{\pm 1\}$ and $\rho(-1) = -1$. Then $1 = P_G = \ker\rho$. A ρ-intertwining solution is just an odd solution, so we obtain the following.

Corollary 3.3. *Assume that Ω is symmetric with respect to the origin, and that F satisfies* (F_1)*,* (F_2)*, and $F(-x,u) = F(x,u) = F(x,-u)$ for every $x \in \Omega$, $u \in \mathbb{R}^d$. Then problem $(\wp)$ has infinitely many even solutions and infinitely many odd solutions.*

If $d = 1$, looking for odd solutions is a convenient way for obtaining sign changing solutions, see for example [7].

We may also apply Theorem 3.1 to obtain solutions which are K-invariant but not G-invariant. Recall that a representation $\rho\colon G \to O(d)$ is said to be *fixed point free* if for every $0 \neq u \in \mathbb{R}^d$ there is a $g \in G$ such that $\rho(g)u \neq u$.

Corollary 3.4. *Let Ω be G-invariant and let K be a closed subgroup of G. Assume that there exists a fixed point free representation $\rho\colon G \to O(d)$ such that $P_G \subset \ker\rho$ whose restriction to K is the trivial representation. Assume further that F satisfies* (F_1)*,* (F_2)*,* (S) *and (S^G_ρ). Then problem $(\wp)$ has infinitely many solutions which are K-invariant but not G-invariant.*

Proof. By Theorem 3.1 there are infinitely many ρ-intertwining solutions. Let $u \neq 0$ be such a solution. Then, since $\rho(g) = 1$ for every $g \in K$, it follows that u is K-invariant. Moreover, since ρ is fixed point free, given $x_0 \in \Omega$ such that $u(x_0) \neq 0$ there is a $g_0 \in G$ such that $u(g_0x_0) = \rho(g_0)u(x_0) \neq u(x_0)$, that is, u is not G-invariant. $\square$

In order to apply Corollary 3.4 we need to look for fixed point free extensions of the trivial representation of K. An easy case where such an extension exists is when the K-fixed point space $(\mathbb{R}^N)^K = \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in K\}$ is nontrivial. Then, any nontrivial orthogonal involution τ of $V = (\mathbb{R}^N)^K$ (that is, any $\tau \in O(N)$ such that $\tau \neq 1$, $\tau^2 = 1$ and $\tau(x) = x$ for every $x \in V^{\perp}$) satisfies $g\tau = \tau g$ for every $g \in K$, and the group $G = K \cup \{\tau\}$ admits a representation $\rho: G \to O(d)$ which is trivial on K and $\rho(\tau) = -1$. Thus, if $(\mathbb{R}^N)^K \neq \{0\}$, Ω is a ball or an annulus, and F satisfies (F_1), (F_2), (S) and $(S^{O(N)})$ then problem $(\wp)$ has infinitely many nonradial K-invariant solutions.

In general, a natural way to look for fixed point free extensions of the trivial representation of K is by looking at fixed point free representations of the Weyl group of K in G. This is the main idea involved in the proof of Theorem 2.1. As in the previous section, for every subgroup Γ of W_GK we write $\widetilde{\Gamma} := q^{-1}(\Gamma)$, where $q: N_GK \to W_GK$ is the natural epimorphism of the normalizer onto the Weyl group.

Proof of Theorem 2.1. If there exists a fixed point free representation $\rho: \Gamma \to O(d)$ then $\widetilde{\rho} = \rho \circ q \to \widetilde{\Gamma} \to O(d)$ is a fixed point free representation whose restriction to K is trivial. Our hypotheses imply that the assumptions of Corollary 3.4 hold for $G = \widetilde{\Gamma}$. Hence we need only to show the existence of such a ρ. If $\Gamma = \{1, \tau\}$ the representation $\rho: \Gamma \to O(d)$ given by $\rho(\tau) = -1$ is fixed point free. If Γ contains an element ζ of order n, we may assume that Γ is the cyclic group $\{1, \zeta, \ldots, \zeta^{n-1}\}$ which admits a fixed point free unitary representation $\rho: \Gamma \to U(s)$. $\square$

Note that the solutions given by Theorem 2.1 are K-invariant but not (N_GK)-invariant. In fact, they are not $\widetilde{\Gamma}$-invariant, for some cyclic subgroup $\Gamma \cong \mathbb{Z}_n$ of the Weyl group W_GK. Moreover, they are $\widetilde{\rho}$-intertwining for $\widetilde{\rho} = \rho \circ q$ where $\rho: \mathbb{Z}_n \to O(d)$ may be chosen to be any sum of nontrivial irreducible representations of $\mathbb{Z}_n$.

Proof of Theorem 3.1. Assumptions (F_1), (F_2) and (S) guarantee that

$$\Phi(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} F(x, u) dx$$

is a well defined even functional of class C^1 on $H_0^1(\Omega, \mathbb{R}^d)$ and that it satisfies the Palais–Smale condition and all other conditions of the symmetric mountain pass theorem of Ambrosetti and Rabinowitz [2], [24]. The critical points of this functional are the solutions of problem $(\wp)$. As usual, $H_0^1(\Omega, \mathbb{R}^d)$ is endowed with the scalar product

$$\langle u, v\rangle = \sum_{i=1}^{d} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx.$$

We define an action of G on $H_0^1(\Omega, \mathbb{R}^d)$ as follows:

$$(gu)(x) = \rho(g)u(g^{-1}x), \quad u \in H_0^1(\Omega, \mathbb{R}^d). \tag{3.1}$$

This is an orthogonal action, that is, $\langle gu, gv\rangle = \langle u, v\rangle$ for every $g \in G$, $u, v \in H_0^1(\Omega, \mathbb{R}^d)$. The ρ-intertwining solutions of $(\wp)$ belong to the G-fixed point set

$$\begin{aligned} H_0^1(\Omega, \mathbb{R}^d)_\rho^G =& \{u \in H_0^1(\Omega, \mathbb{R}^d) : gu = u \text{ for every } g \in G\} \\ =& \{u \in H_0^1(\Omega, \mathbb{R}^d) : u(gx) = \rho(g)u(x) \text{ for every } g \in G\}. \end{aligned}$$

Assumption (S_ρ^G) guarantees that the functional Φ is G-invariant, that is, $\Phi(gu) = \Phi(u)$ for every $g \in G$, $u \in H_0^1(\Omega, \mathbb{R}^d)$. Therefore $\nabla\Phi(gu) = g\nabla\Phi(u)$. In particular, $\nabla\Phi(u) \in H_0^1(\Omega, \mathbb{R}^d)_\rho^G$ if $u \in H_0^1(\Omega, \mathbb{R}^d)_\rho^G$. This implies that the critical points of the restriction

$$\Phi\colon H_0^1(\Omega, \mathbb{R}^d)_\rho^G \to \mathbb{R}$$

of Φ to $H_0^1(\Omega, \mathbb{R}^d)_\rho^G$ are the ρ-intertwining solutions of $(\wp)$ (this property is usually refered to as the principle of symmetric criticallity, see [23]), and that $\Phi\colon H_0^1(\Omega, \mathbb{R}^d)_\rho^G \to \mathbb{R}$ satisfies the Palais–Smale condition. We shall show that $H_0^1(\Omega, \mathbb{R}^d)_\rho^G$ is infinite dimensional. Then the symmetric mountain pass theorem of Ambrosetti and Rabinowitz [24, Theorem 9.12] asserts the existence of an unbounded sequence of critical values of $\Phi\colon H_0^1(\Omega, \mathbb{R}^d)_\rho^G \to \mathbb{R}$.

Let $P = P_G$ be the principal isotropy class of G. The set $\Omega_{(P)} = \{x \in \Omega : (G_x) = (P)\}$ is open and dense in Ω. Moreover, the projection $\varphi\colon \Omega_{(P)} \to \widehat{\Omega}_{(P)}$ onto its orbit space $\widehat{\Omega}_{(P)} = \{Gx : x \in \Omega_{(P)}\}$ is a smooth fibre bundle with fibre G/P [15, Theorem I.5.14]. Let $\xi \in \widehat{\Omega}_{(P)}$ and let $\delta > 0$ be small enough so that this bundle is trivial over the open ball $B = B_\delta(\xi)$ of radius δ centered at ξ in $\widehat{\Omega}_{(P)}$. Since $P \subset \ker\rho$, the function

$$C_c^\infty(B, \mathbb{R}^d) \to C_c^\infty(B \times (G/P), \mathbb{R}^d)_\rho^G \cong C_c^\infty(\varphi^{-1}(B), \mathbb{R}^d)_\rho^G$$

given by $w \mapsto \widetilde{w}$, $\widetilde{w}(\zeta, gP) = \rho(g)w(\zeta)$, is well defined and a linear isomorphism. Therefore, $H_0^1(\varphi^{-1}(B), \mathbb{R}^d)_\rho^G$ is infinite dimensional and, since $H_0^1(\varphi^{-1}(B), \mathbb{R}^d)_\rho^G$ is a subspace of $H_0^1(\Omega, \mathbb{R}^d)_\rho^G$, this last space is also infinite dimensional. This concludes the proof of Theorem 3.1. $\square$

Note that the interplay between the given orthogonal action of G on the domain $\Omega \subset \mathbb{R}^N$ and the orthogonal representation structure $\rho\colon G \to O(d)$ on the space $\mathbb{R}^d$ is reflected by the infinite dimensional representation structure on $H_0^1(\Omega, \mathbb{R}^d)$ given by (3.1). As we have just seen, condition (S_ρ^G) guarantees that

Φ is G-invariant and allows us to reduce our problem to finding critical points of the restriction of Φ to the G-fixed point space $H^1_0(\Omega,\mathbb{R}^d)^G_\rho$. The crucial step in the proof of Theorem 3.1 is to show that this space is infinite dimensional. This is guaranteed by the assumption that $P_G \subset \ker \rho$. In fact, one can show that, if $H^1_0(\Omega,\mathbb{R}^d)^G_\rho \neq \{0\}$, then it is infinite dimensional. Conditions (F_1), (F_2) and (S) are the standard ones which are used to prove the existence of infinitely many solutions for the gradient system ($\wp$).

Thus, similar results may be obtained for other types of elliptic systems under appropriate growth conditions on the nonlinearity. For example, one may consider Hamiltonian systems (see for example [16] and the references therein), or systems where the symmetries are perturbed either by adding to the nonlinearity a noneven lower order term, or by a nonhomogeneous boundary condition (see for example [13], [12] and the references therein).

One may also replace condition (S) by a condition which involves the action of another group T on $\mathbb{R}^d$ as in [13] where F is required to be $\widetilde{\rho}$-invariant in u with respect to a representation $\widetilde{\rho}\colon T \to O(d)$ of a torus, a p-torus or a cyclic p-group T. In this case one should also assume that the representation $\rho\colon G \to O(d)$ commutes with $\widetilde{\rho}$, that is, $\rho(g)\widetilde{\rho}(t)x = \widetilde{\rho}(t)\rho(g)x$ for each $g \in G$, $t \in T$, $x \in \mathbb{R}^d$.

As was shown in the proof of Corollary 3.4, if ρ is fixed point free, the solutions provided by Theorem 3.1 are not G-invariant but they will be K-invariant for every subgroup K of G such that $K \subset \ker \rho$. It is worthwhile to emphazise that Theorem 3.1 may be used to obtain solutions of a special analytical form. We illustrate this with the following example.

Let $\Omega = \mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, and let $G = SO(2) \subset O(2)$ be the group of rotations of the plane $\mathbb{R}^2$. The action of the rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad 0 \le \theta < 2\pi,$$

on $\mathbb{R}^2$ is the same as multiplication with the unit complex number $e^{i\theta} \in \mathbb{S}^1 \equiv SO(2)$ on $\mathbb{C} \equiv \mathbb{R}^2$. This is a free action on $\mathbb{D}^2 \setminus \{0\}$, therefore, $P_{\mathbb{S}^1} = 1$. It is well known that every irreducible representation of $\mathbb{S}^1$ is either the one-dimensional trivial representation $\rho_0(e^{i\theta}) = 1$ or one of the countably many two-dimensional representations $\rho_n\colon \mathbb{S}^1 \to SO(2)$ given by the formula

$$\rho_n(e^{i\theta}) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}, \quad n \ge 1.$$

That is, $\rho_n(e^{i\theta})$ acts on $\mathbb{C} \equiv \mathbb{R}^2$ by multiplication with $e^{in\theta} \in \mathbb{S}^1$. For $d = 2s$, we denote again by $\rho_n\colon \mathbb{S}^1 \to SO(d)$ the direct sum of s copies of the representation ρ_n defined above. If we write $u \in H^1_0(\mathbb{D}^2,\mathbb{R}^d)$ in polar coordinates, $u(r,\varphi)$,

$0 \le r < 1$, $0 \le \varphi < 2\pi$, and expand it in Fourier series with respect to φ,

$$u(r,\varphi) = \sum_{k=0}^{\infty} a^k(r) e^{ik\varphi}, \quad \text{with } a^k(r) = \frac{1}{2\pi i} \int_0^{2\pi} u(r,\theta) e^{ik\theta} d\theta \in \mathbb{C}^s,$$

then, by definition, u is ρ_n-intertwining if and only if

$$\sum_{k=0}^{\infty} a^k(r) e^{ik(\varphi+\theta)} = u(r,\varphi+\theta) = \rho_n(e^{i\theta}) u(r,\varphi) = \sum_{k=0}^{\infty} a^k(r) e^{i(k\varphi+n\theta)}.$$

Therefore, u is ρ_n-intertwining if and only if $a^k \equiv 0$ for every $k \neq n$, that is, if and only if u is of the form $u(r,\varphi) = a^n(r)e^{in\varphi}$. From this discussion and Theorem 3.1 we conclude the following.

Proposition 3.5. *Let $\Omega = \mathbb{D}^2$ be the open unit disk in $\mathbb{R}^2$. Assume that $d = 2s$ and that F satisfies* (F_1), (F_2), *and* (S), *and is such that $F(e^{i\theta}z_0, e^{in\theta}z_1, \dots, e^{in\theta}z_s) = F(z_0, z_1, \dots, z_s)$ for every $0 \le \theta \le 2\pi$, $z_0 \in \Omega$, $z_i \in \mathbb{C}$, $i = 1, \dots, s$. Then problem ($\wp$) has infinitely many solutions of the form $u(r,\varphi) = a(r)e^{in\varphi}$, with $a \in H_0^1([0,1], \mathbb{R}^d)$. In particular, if F safisfies* (F_1), (F_2), (S^G) *and is toroidal then this assumption is satisfied for all n and consequently for each $n \in \mathbb{N}$ problem ($\wp$) has infinitely many solutions of the above form.*

4. The case $N = 3$

Now we shall investigate which subgroups $K \subset G$ of $O(3)$ satisfy the hypotheses of Theorem 2.1. We recall some well known facts on the subgroups of $O(3)$. Details may be found in [3, Section 8.2], [11], and the references therein.

The orthogonal group $O(3)$ consists of all 3×3 matrices g such that $g^t = g^{-1}$. The special orthogonal group $SO(3)$ is the subgroup of those matrices whose determinant is 1. The center of $O(3)$ is the group $\mathbb{Z}_2^c = \{\pm 1\}$. We identify $O(3)$ with $SO(3) \times \mathbb{Z}_2^c$ via the isomorphism

$$SO(3) \times \mathbb{Z}_2^c \cong O(3), \quad (g,t) \mapsto t \cdot g.$$

As usual, we think of $O(2)$ as being the subgroup of $SO(3)$ generated by the matrices

$$\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$\theta \in [0, 2\pi)$, and denote by $\mathbb{Z}_m$ the cyclic group with m elements and by D_m the dihedral group with $2m$ elements.

The subgroups of $O(3)$ are well known. Up to conjugacy they fall into three classes:

(I) *Subgroups of* $SO(3)$. The proper subgroups of $SO(3)$ are the planar groups $O(2)$, $SO(2)$, D_m ($m \ge 2$), and $\mathbb{Z}_m$ ($m \ge 1$), and the oriented symmetry groups I of the icosahedron, O of the octahedron and T of the tetrahedron. I is isomorphic to the alternating group A_5, O to the symmetric group S_4 and T to the alternating group A_4. Their normalizers and Weyl groups in $O(3)$ are as follows:

K	$N_{O(3)}K$	$W_{O(3)}K$
$SO(3)$	$O(3)$	$\mathbb{Z}_2$
$O(2)$	$O(2)\times\mathbb{Z}_2^c$	$\mathbb{Z}_2$
$SO(2)$	$O(2)\times\mathbb{Z}_2^c$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$D_m, m\geq 3$	$D_{2m}\times\mathbb{Z}_2^c$	$\mathbb{Z}_2\times\mathbb{Z}_2$
D_2	$O\times\mathbb{Z}_2^c$	$D_3\times\mathbb{Z}_2$
$\mathbb{Z}_m, m\geq 2$	$O(2)\times\mathbb{Z}_2^c$	$O(2)\times\mathbb{Z}_2$
I	$I\times\mathbb{Z}_2^c$	$\mathbb{Z}_2$
O	$O\times\mathbb{Z}_2^c$	$\mathbb{Z}_2$
T	$O\times\mathbb{Z}_2^c$	$\mathbb{Z}_2\times\mathbb{Z}_2$

(II) *Subgroups of $O(3)$ which contain -1.* These are of the form $K=H\times\mathbb{Z}_2^c$ where $\mathbb{Z}_2^c=\{\pm1\}$. Their normalizers are $N_{O(3)}K=N_{O(3)}H$.

(III) *Subgroups of $O(3)$ not contained in $SO(3)$ which do not contain -1.* These are determined by their intersection $K\cap SO(3)$ with $SO(3)$ and their projection $\psi(K)\subset SO(3)$ where $\psi\colon O(3)\to SO(3)$ is given by $\psi(\pm1\cdot g)=g$ for every $g\in SO(3)$. Their normalizers and Weyl groups in $O(3)$ are as follows.

K	$\psi(K)$	$K\cap SO(3)$	$N_{O(3)}K$	$W_{O(3)}K$
$O(2)^-$	$O(2)$	$SO(2)$	$O(2)\times\mathbb{Z}_2^c$	$\mathbb{Z}_2$
$D_{2m}^d(m\geq 2)$	D_{2m}	D_m	$D_{2m}\times\mathbb{Z}_2^c$	$\mathbb{Z}_2$
$D_m^z(m\geq 2)$	D_m	$\mathbb{Z}_m$	$D_{2m}\times\mathbb{Z}_2^c$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$\mathbb{Z}_{2m}^-(m\geq 1)$	$\mathbb{Z}_{2m}$	$\mathbb{Z}_m$	$O(2)\times\mathbb{Z}_2^c$	$O(2)$
O^-	O	T	$O\times\mathbb{Z}_2^c$	$\mathbb{Z}_2$

We shall need the following lemmas.

Lemma 4.1. *If G is a finite subgroup of $O(N)$ then its principal isotropy class is $P_G=1$.*

Proof. $\dim\ker(g-1)\leq N-1$ for every $g\in G$, $g\neq 1$. Since G is finite, $\bigcup\{\ker(g-1):1\neq g\in G\}\neq\mathbb{R}^N$. In other words, there is an $x\in\mathbb{R}^N$ such that $gx\neq x$ for every $g\in G\backslash\{1\}$. □

Lemma 4.2. *The principal isotropy class of a subgroup G of $O(3)$ is*

(a) $P_G=1$ *if* $G\neq O(3), SO(3), O(2)\times\mathbb{Z}_2^c$, *and* $O(2)^-$,

(b) $P_{O(2)\times\mathbb{Z}_2^c} = P_{O(2)^-} = \{1, -1\cdot\kappa\}$,
(c) $P_{SO(3)} = SO(2)$ *and*
(d) $P_{O(3)} = SO(2)\times\mathbb{Z}_2^c$.

Proof. For the finite subgroups of $O(3)$ the result follows from Lemma 4.1. The infinite ones are $O(3)$, $SO(3)$, $SO(2)$, $SO(2)\times\mathbb{Z}_2^c$, $O(2)$, $O(2)\times\mathbb{Z}_2^c$ and $O(2)^-$ for which the assertion can be easily verified. □

Proof of Theorem 2.3. By Theorem 2.1 it suffices to show that for every such K there is a subgroup Γ of $W_{O(3)}K$ of order two such that $P_{\widetilde{\Gamma}} \subset K$. If K is in class (I) then, by assumption, $K \neq SO(3), O(2)$. If $K = D_m, I, O$ or T its normalizer is finite and, by Lemma 4.1, any subgroup Γ of $W_{O(3)}K$ satisfies $P_{\widetilde{\Gamma}} = 1$. If $K = SO(2)$ the epimorphism

$$q\colon O(2)\times\mathbb{Z}_2^c \to \mathbb{Z}_2\times\mathbb{Z}_2$$

of the normalizer onto the Weyl group is given by $q(\theta) = (1,1)$, $q(\kappa) = (-1,1)$, $q(-1) = (1,-1)$. By Lemma 4.2 any subgroup Γ not containing $q(-1\cdot\kappa) = (-1,-1)$ satisfies $P_{\widetilde{\Gamma}} = 1$. If $K = \mathbb{Z}_m$ the epimorphism

$$q_m\colon O(2)\times\mathbb{Z}_2^c \to O(2)\times\mathbb{Z}_2$$

is given by $q_m(\theta) = (\theta^m, 1)$, $q_m(\kappa) = (\kappa, 1)$, $q_m(-1) = (1,-1)$. So any subgroup Γ not containing $q_m(-1\cdot\kappa) = (\kappa,-1)$ satisfies $P_{\widetilde{\Gamma}} = 1$. If K is in class (II) then, by assumption, $K \neq O(2)\times\mathbb{Z}_2^c, I\times\mathbb{Z}_2^c, O\times\mathbb{Z}_2^c$. The groups $K = D_m\times\mathbb{Z}_2^c$ and $T\times\mathbb{Z}_2^c$ have a finite normalizer so again any subgroup Γ of $W_{O(3)}K$ satisfies $P_{\widetilde{\Gamma}} = 1$. For $K = SO(2)\times\mathbb{Z}_2^c$ we take $\Gamma = \mathbb{Z}_2$, and for $K = \mathbb{Z}_m\times\mathbb{Z}_2^c$ we may take any subgroup Γ of order two of $O(2)$. In both cases $\widetilde{\Gamma} \subset O(2)$ and, hence, $P_{\widetilde{\Gamma}} = P_{O(2)} = 1$. If K is in class (III) and $K = D_{2m}^d, D_m^z$ or O^-, then it has a finite normalizer and any subgroup Γ of $W_{O(3)}K$ satisfies $P_{\widetilde{\Gamma}} = 1$. For $K = O(2)^-$ we take $\Gamma = \mathbb{Z}_2$. Then $\widetilde{\Gamma} = O(2)\times\mathbb{Z}_2^c$ and, by Lemma 4.2, $P_{\widetilde{\Gamma}} = \{1, -\kappa\} \subset O(2)^-$. Finally, for $K = \mathbb{Z}_{2m}^-$ the projection

$$q_m^-\colon O(2)\times\mathbb{Z}_2^c \to O(2)$$

is given by $q_m^-(\theta) = \theta^m$, $q_m^-(\kappa) = \kappa$, $q_m^-(-1) = \pi$. By Lemma 4.2 any subgroup Γ not containing $q_m^-(-1\cdot\kappa) = \pi\kappa$ will satisfy $P_{\widetilde{\Gamma}} = 1$. □

Note that the groups K excluded by Theorem 2.3 are precisely those which satisfy that $Kx = (N_{O(3)}K)x$ for every $x \in \mathbb{R}^3$. This is a necessary condition for applying Theorem 2.1. Indeed, recall that the solutions provided by Theorem 2.1, and hence those given by Theorem 2.3, are K invariant but not $(N_{O(3)}K)$-invariant.

Proof of Theorem 2.2. Let $K \subset G \subsetneq O(3)$. If $N_GK \neq SO(3), O(2)\times\mathbb{Z}_2^c, O(2)^-$ then, by Lemma 4.2, every subgroup Γ of W_GK satifies $P_{\widetilde{\Gamma}} = 1$. If

$W_G K$ contains an element τ of order 2 we take $\Gamma = \{1, \tau\}$. In any case $W_G K$ contains a nontrivial subgroup, so Theorem 2.1 gives the result. Observe that $N_G K = N_{O(3)} K \cap G$. So we are left with the cases $N_{O(3)} K = O(2) \times \mathbb{Z}_2^c$ and $G = O(2) \times \mathbb{Z}_2^c$ or $O(2)^-$. If $G = O(2) \times \mathbb{Z}_2^c = N_{O(3)} K$ then $N_G K = N_{O(3)} K = G$ and Theorem 2.3 gives the result. If $G = O(2)^-$ and $N_{O(3)} K = O(2) \times \mathbb{Z}_2^c$ then $N_{O(2)^-} K = O(2)^-$ and $K = SO(2)$ or $\mathbb{Z}_m$. Since $K = SO(2)$ has the same orbits as $G = O(2)^-$ this case should be excluded. For $K = \mathbb{Z}_m$ the quotient map $q_m: N_{O(2)^-} \mathbb{Z}_m = O(2)^- \to O(2)^- = W_{O(2)^-} \mathbb{Z}_m$ satisfies $q_m(-1 \cdot \kappa) = -1 \cdot \kappa$. So $P_{\widetilde{\Gamma}} = 1$ if $\Gamma = \{1, \pi\}$. $\square$

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