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A SHARKOVSKII-TYPE THEOREM FOR MINIMALLY FORCED INTERVAL MAPS

Roberta Fabbri — Tobias Jäger
Russel Johnson — Gerhard Keller

Abstract. We state and prove a version of Sharkovskiĭ's theorem for forced interval maps in which the forcing flow is minimal (Birkhoff recurrent). This setup includes quasiperiodically forced interval maps as a special case. We find that it is natural to substitute the concept of "fixed point" with that of "core strip." Core strips are frequently of almost automorphic type.

1. Introduction

A well-known theorem of Sharkovskiĭ regarding continuous maps $f: I \to I$ of an interval into itself states that, if f admits a periodic point x of minimal period p, then f admits a periodic point of minimal period q if q lies below p in the Sharkovskiĭ ordering of the natural numbers:

$$3 > 5 > \ldots > 2n+1 > \ldots > 6 > 10 > \ldots > 2\cdot(2n+1) \\ > \ldots > 2^m \cdot 3 > 2^m \cdot 5 > \ldots > 2^m \cdot (2n+1) > \ldots > 2^n > \ldots > 2 > 1.$$

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In particular, if f admits a periodic point of period 3, then it admits periodic points of all integer periods.

This theorem can be proved by a simple-looking but subtle analysis of the f-images of those subintervals of I whose endpoints are elements of the orbit of the periodic point x; see ([7], [14], [4]). Our purpose in this note is to extend the Sharkovskiĭ theorem to the case of certain mappings of skew-product form defined on a product space $\Theta \times I$. Precisely, let Θ be a compact metric space, and let $R: \Theta \to \Theta$ be a minimal homeomorphism with the property that every power R^l $(l = 1, 2, \ldots)$ is minimal. Let $I \subset \mathbb{R}$ be a compact interval, and let $T: \Theta \times I \to \Theta \times I$ be a continuous map with the property that, if $\pi: \Theta \times I \to \Theta$ is the projection onto the first factor, then $\pi(T(\theta, x)) = R(\theta)$ for all $\theta \in \Theta$, $x \in I$. We propose to generalize the statement and the proof of the Sharkovskiĭ theorem in the context of such mappings T.

We were motivated to study this question by recent work on "forced" interval maps ([9]–[11], [13]). Many such maps are of the form we consider here. Numerical studies of such maps indicate that they often give rise to so-called non-chaotic strange attractors. It has recently been emphasized that these attractors appear to have a topological structure of almost automorphic type ([9], [10]). While we do not address directly the properties of attractors for maps of the form T, we do find a strong connection between phenomena of Sharkovskiĭ type and the presence of almost automorphic subsets of $\Theta \times I$ which have periodicity properties with respect to T.

The connection arises as follows. To realize a generalization of Sharkovskiĭ's theorem, it is necessary to determine an appropriate analogue of the concept of "periodic point" for a skew-product mapping of the form T. It turns out that the notion of measurable section $\phi: \Theta \to \Theta \times I$ of the trivial fiber bundle $\Theta \times I \to \Theta$ does not provide a useful analogue of the concept of periodic point. We will see, however, that a version of Sharkovskiĭ's theorem for skew-product maps T can be stated and proved in which periodic points are substituted by "periodic core strips". Here a strip is a certain type of compact subset $A \subset \Theta \times I$ which covers Θ in the sense that $\pi(A) = \Theta$, and a core strip satisfies further conditions to be discussed in Section 3. A particular type of core strip is defined by a continuous section ϕ if one sets $A = \operatorname{Im} \phi$; however, in developing our theory, we will need to consider core strips A which are not necessarily images of continuous sections. Indeed we will be led in a natural way to core strips of almost automorphic type. It should be noted that, even when each $T_\theta: I \to I: x \to \pi_2 T(\theta, x)$ is strictly monotone, the map T may admit an invariant set which is of almost automorphic type but is not a section (here $\pi_2: \Theta \times I \to I$ is the projection onto the second factor). For concrete examples illustrating this phenomenon see [17] and [12].

We were also motivated by the work of Andres and his collaborators ([1]–[3]) on Sharkovskiĭ-type results for differential inclusions. These authors work with points which are periodic in the sense of the theory of differential inclusions.

The paper is organized as follows. In Section 2 we give all definitions necessary to state abridged versions of the main results. In Section 3 we first introduce cores and strips and prove some of their basic properties, and at the end of that section we give an example which discourages the consideration of measurable sections in the context of a Sharkovskiĭ-type theory for skew-product maps T. Section 4 contains a detailed analysis of strips and their semicontinuous bounding sections. Finally, in Section 5 we state and prove our Sharkovskiĭ-type theorem.

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2. The main concepts and results

In this preliminary section, we let Θ and I be as above, and set $\Omega = \Theta \times I$. (In Section 3 we will sometimes impose weaker conditions on Θ and Ω.) Let R be a homomorphism of Θ onto itself, let $\pi : \Omega \to \Theta$ be the natural projection, and let $T : \Omega \to \Omega$ be a continuous mapping such that $\pi(T(\theta, x)) = R(\theta)$ for all $(\theta, x) \in \Omega$.

As usual, we say that a subset $G \subseteq \Theta$ is *residual* if it contains the intersection of a countable family of open dense subsets of Θ. Let $\mathcal{G}$ be the family of all such subsets of Θ. We introduce the following notions:

Cores. A set $M \subseteq \Omega$ is a *core*, if

$$M = \bigcap_{G \in \mathcal{G}} \overline{M \cap \pi^{-1}(G)}.$$

(Solid) strips, pinched sets.

(a) A closed subset $A \subseteq \Omega$ is called a *strip*, if $\{\theta \in \Theta : A^\theta \text{ is an interval}\}$ is residual. Here A^θ is the fiber $\{x \in I \mid (\theta, x) \in \Omega\}$.

(b) A strip A is *solid*, if *each* fiber of A is an interval and if $\delta(A) := \inf\{|A^\theta| : \theta \in \Theta\} > 0$.

(c) A closed subset $A \subseteq \Omega$ is called *pinched*, if $P_A := \{\theta \in \Theta : \operatorname{card} A^\theta = 1\}$ is dense in Θ. (In this case, P_A is residual; that is, each pinched set is a strip.)

(Strongly) T-invariant, minimal. A subset $M \subseteq \Omega$ is said to be T-*invariant* if $T(M) \subseteq M$. It is said to be *strongly T-invariant*, if $T(M) = M$. The set $M \subseteq \Omega$ is said to be *minimal* if it is nonempty, T-invariant, closed, and does not strictly contain any other non-empty, T-invariant, closed subset of Ω.

Almost automorphic. Let A be a core strip; say that A is T-*almost automorphic* if it is pinched and minimal with respect to T. (Our usage of this notion is a bit more general than that in the literature, where it is also required that the base homeomorphism $R\colon \Theta \to \Theta$ is almost periodic. See [15] for general properties of almost automorphic dynamical systems.)

The following theorem, which is a corollary to Theorem 4.11, provides a structure dichotomy for strongly invariant core strips.

Theorem 2.1 (Structure dichotomy for core strips). *Let R be a minimal homeomorphism of Θ, and let A be a strongly invariant core strip. Then A is either almost automorphic, or it is solid.*

To formulate our main result we need two more more definitions.

Ordered strips. Say that two strips A and B satisfy $A < B$ if there is a residual set G such that for all $\theta \in G$, $x \in A^\theta$, and $y \in B^\theta$ there holds $x < y$. We say that the strips are *ordered*, if either $A < B$ or $B < A$.

Periodic strips. Let $p > 1$ be an integer. A strip $A \subseteq \Omega$ is called *p-periodic* if $T^p(A) = A$ and if the image sets $A, T(A), \ldots, T^{p-1}(A)$ are pairwise disjoint and pairwise ordered.

Now we can state our main result, which is a corollary to Theorem 5.6.

Theorem 2.2 (Sharkovskiĭ for strips). *Suppose that T admits a p-periodic strip B and that $p > q$ in the Sharkovskiĭ ordering. Then T admits a q-periodic core strip C. This strip C is either T^q-almost automorphic or solid. In the latter case it is "bounded" above and below by a pair of T^{2q}-almost automorphic strips.*

The difficulty of the proof is to replace the intermediate value theorem – the only piece of real analysis in the proof of the classical Sharkovskiĭ theorem, used there to guarantee the existence of fixed points – by a (constructive) procedure that provides, under suitable assumptions, invariant core strips. The combinatorial part of the proof is – modulo certain details – essentially the same as for the classical theorem.

3. Preliminaries on strips and cores

In this section we will collect some basic definitions and results, and give an example which indicates that it is pointless to try to formulate an analogue of the Sharkovskiĭ theorem in which the concept of periodic point is substituted by that of measurable section of the bundle $\Theta \times I \to \Theta$.

We begin with some rather general considerations. Let Θ and Ω be complete separable metric spaces, and let $\pi\colon \Omega \to \Theta$ be a continuous surjective map. If $M \subseteq \Omega$, let $M^\theta := M \cap \pi^{-1}\{\theta\}$ the fiber of M over θ. Say that a subset $G \subseteq \Theta$ is *residual* if it contains the intersection of a countable family of open dense subsets of Θ. Let $\mathcal{G}$ be the family of all such subsets of Θ.

Definition 3.1 (Core). If $M \subseteq \Omega$, the *core* of M (relative to π) is defined to be

$$M^C = \bigcap_{G \in \mathcal{G}} \overline{M \cap \pi^{-1}(G)}.$$

If $M = M^C$, then we say that M is a core.

Lemma 3.2. *For $M \subseteq \Omega$, define $\mathcal{G}_M := \{G \in \mathcal{G} : M^C = \overline{M \cap \pi^{-1}(G)}\}$.*

(a) *For each $M \subseteq \Omega$ there is a $G \in \mathcal{G}$ such that $M^C = \overline{M \cap \pi^{-1}(G)}$. That is, $\mathcal{G}_M$ is not empty.*
(b) *If $G \in \mathcal{G}_M$ and $G_0 \in \mathcal{G}$, then $G \cap G_0 \in \mathcal{G}_M$. In particular $\mathcal{G}_M \cap \mathcal{G}_N \neq \emptyset$ if M and N are subsets of Ω.*
(c) *If $\pi(M) = \Theta$ and if M is compact, then $\pi(M^C) = \Theta$.*
(d) *If $M \subseteq \Omega$ is closed and $G \in \mathcal{G}_M$, then $M \cap \pi^{-1}(G) = M^C \cap \pi^{-1}(G)$.*
(e) *Let $M, N \subseteq \Omega$. If there exists $G \in \mathcal{G}$ such that $M \cap \pi^{-1}(G) = N \cap \pi^{-1}(G)$, then $M^C = N^C$.*
(f) *If $M, N \subseteq \Omega$ are closed, then $M^C = N^C$ if and only if there exists $G \in \mathcal{G}$ such that $M \cap \pi^{-1}(G) = N \cap \pi^{-1}(G)$.*

Proof. (a) Let $(U_j)_{j \in \mathbb{N}}$ be a basis for the topology of Ω. If $G \in \mathcal{G}$, let $J_G := \{j \in \mathbb{N} : U_j \cap \overline{M \cap \pi^{-1}(G)} = \emptyset\}$. Let $J := \bigcup_{G \in \mathcal{G}} J_G$. Then $M^C = \bigcap_{j \in J}(\Omega \setminus U_j)$. Use the axiom of choice to choose, for each $j \in J$, a set $G_j \in \mathcal{G}$ such that $j \in J_{G_j}$. Then $J = \bigcup_{j=1}^{\infty} J_{G_j}$, and for the residual set $G := \bigcap_{j \in J} G_j$ one has $\overline{M \cap \pi^{-1}(G)} \subseteq \bigcap_{j \in J} \overline{M \cap \pi^{-1}(G_j)} \subseteq \bigcap_{j \in J}(\Omega \setminus U_j) = M^C$. Hence $M^C = \overline{M \cap \pi^{-1}(G)}$.

(b) The proof is quite simple.

(c) Let $G \in \mathcal{G}_M$ and $\theta \in \Theta$. Choose sequences $\theta_n \in G$ and $x_n \in M$ such that $\theta_n \to \theta$ and $\pi(x_n) = \theta_n$. Let x be a limit point of (x_n). Then $x \in \overline{M \cap \pi^{-1}(G)} = M^C$ and $\pi(x) = \theta$.

(d) Observe that $M \cap \pi^{-1}(G) \subseteq \overline{M \cap \pi^{-1}(G)} \subset \overline{M} = M$, from which it follows that $M \cap \pi^{-1}(G) = \overline{M \cap \pi^{-1}(G)} \cap \pi^{-1}(G) = M^C \cap \pi^{-1}(G)$.

(e) Set $G_1 = G \cap G_M \cap G_N$, $G_M \in \mathcal{G}_M, G_N \in \mathcal{G}_N$. Then $G_1 \in \mathcal{G}_M \cap \mathcal{G}_N$ so that $M^C = \overline{M \cap \pi^{-1}(G_1)} = \overline{N \cap \pi^{-1}(G_1)} = N^C$.

(f) Let $G_M \in \mathcal{G}_M$, $G_N \in \mathcal{G}_N$, and set $G_0 := G_M \cap G_N$. Suppose first that $M^C = N^C$. Then using (d), one has $M \cap \pi^{-1}(G_0) = M^C \cap \pi^{-1}(G_0) = N^C \cap \pi^{-1}(G_0) = N \cap \pi^{-1}(G_0)$. The reverse implication follows from (e). □

Remark 3.3. Let A be a compact subset of Ω such that $\pi(A) = \Theta$. Let d be a metric on Ω, and let 2^Ω be the family of nonempty compact subsets $Y \subseteq \Omega$, endowed with the Hausdorff metric ρ:

$$\rho(Y,Z) = \max\{\max_{y\in Y}\min_{z\in Z} d(y,z), \max_{z\in Z}\min_{y\in Y} d(y,z)\}.$$

It is well-known that the set $G \subseteq \Theta$ consisting of those points θ' such that the map $\theta \mapsto A^\theta\colon \Theta \to 2^\Omega$ is ρ-continuous at θ' is residual in Θ.[1] It follows that A is a core if and only if $\overline{A \cap \pi^{-1}(G)} = A$.

Suppose now that $T\colon \Omega \to \Omega$ is a continuous map, and that $R\colon \Theta \to \Theta$ is a homeomorphism such that $\pi \circ T = R \circ \pi$. Since π is surjective we have

$$\pi \circ T(\pi^{-1}(U)) = RU \quad \text{for each } U \subseteq \Theta.$$

It follows that $T(\pi^{-1}(U)) \subseteq \pi^{-1}(RU)$ and that

$$TM \cap \pi^{-1}(RU) = T(M \cap \pi^{-1}(U)) \quad \text{whenever } U \subseteq \Theta \text{ and } M \subseteq \Omega. \tag{3.1}$$

While the "$\supseteq$" inclusion is trivial, we show the other direction: For $y \in T(M) \cap \pi^{-1}(R(U))$ there exists $x \in M$ such that $y = T(x)$ and $\pi(y) \in R(U)$. Hence $R(\pi(x)) = \pi(T(x)) = \pi(y) \in R(U)$ so that $\pi(x) \in U$ and thus $x \in M \cap \pi^{-1}(U)$.

Lemma 3.4. *Let M and N be subsets of Ω.*

(a) *If $T(M) \subseteq N$, then $T(M^C) \subseteq N^C$.*
(b) *If $T(M) \supseteq N$ and if M^C is compact, then $T(M^C) \supseteq N^C$.*
(c) *If M^C is compact, then $(T(M))^C = T(M^C)$.*

Proof. (a) Let $G_M \in \mathcal{G}_M$, $G_N \in \mathcal{G}_N$, and set $G = G_M \cap R^{-1}(G_N)$. Then $G \in \mathcal{G}_M$, $R(G) \in \mathcal{G}_N$, and $T(M \cap \pi^{-1}(G)) \subseteq T(M) \cap T(\pi^{-1}(G)) \subseteq N \cap \pi^{-1}(R(G)) \subseteq N^C$. Therefore $T(M^C) = T(\overline{M \cap \pi^{-1}(G)}) \subseteq \overline{N^C} = N^C$.

(b) In a similar way, $N \cap \pi^{-1}(R(G)) \subseteq T(M) \cap \pi^{-1}(R(G)) \subseteq T(M^C)$. Since M^C is compact this implies that $N^C = \overline{N \cap \pi^{-1}(RG)} \subseteq \overline{T(M^C)} = T(M^C)$.

(c) This follows from (a) and (b) when applied to $N = TM$. □

Corollary 3.5. *If M is a compact core, then $T(M)$ is a compact core.*

Proof. It follows from Lemma 3.4(c) that $(T(M))^C = T(M^C) = T(M)$. □

Definition 3.6 ((Strongly) T-invariant, minimal). A subset $M \subseteq \Omega$ is said to be *T-invariant* if $T(M) \subseteq M$. It is said to be *strongly T-invariant*, if $T(M) = M$. The set $M \subseteq \Omega$ is said to be *minimal* if it is nonempty, T-invariant, closed, and does not strictly contain any other non-empty, T-invariant, closed subset of Ω.

It is easy to see that, if $M \subseteq \Omega$ is compact and minimal, then it is strongly invariant: if this were not so, then $\bigcap_{n=1}^\infty T^n(M)$ would be a nonempty, T-invariant, compact subset of Ω which is strictly contained in M.

[1] See e.g. [6, Theorem 7.10] and note that the same proof given there works for upper semicontinuous functions as well as for lower semicontinuous ones.

For later use we note some simple consequences of Lemmas 3.2 and 3.4.

Corollary 3.7. *If A is a minimal compact T-invariant set, then either $A^C = A$ or $A^C = \emptyset$.*

Proof. This follows from Lemma 3.4(c). □

Corollary 3.8. *If M and N are cores and if $T(M \cap \pi^{-1}(G)) = N \cap \pi^{-1}(R(G))$ for some $G \in \mathcal{G}$, then $T(M) \subseteq N$. If M^C is also compact, then $T(M) = N$.*

Proof. Observe first that $T(M \cap \pi^{-1}(G)) = T(M) \cap \pi^{-1}(R(G))$ by equation (3.1). Without loss of generality we can assume that $R(G) \in \mathcal{G}_{T(M)} \cap \mathcal{G}_N$, see Lemma 3.2. Hence the assumption implies that $T(M)^C = N^C$. Therefore $T(M) = T(M^C) \subseteq (T(M))^C = N^C = N$ by Lemma 3.4, and if $M = M^C$ is compact the same lemma also yields the converse inclusion. □

General assumption. From now on, let $I \subseteq \mathbb{R}$ be a compact interval, and set $\Omega = \Theta \times I$. Let $\pi: \Theta \times I \to \Theta$ be the natural projection.

Definitions 3.9 ((Solid) strip, pinched set).

(a) A closed subset $A \subseteq \Omega$ is called a *strip*, if $\{\theta \in \Theta : A^\theta \text{ is an interval}\}$ is residual. (In particular there exists $G \in \mathcal{G}_A$ such that A^θ is an interval for all $\theta \in G$.) We denote

$$\widetilde{\mathcal{G}}_A := \{G \in \mathcal{G}_A : A^\theta \text{ is an interval for all } \theta \in G\}.$$

(Observe that $\pi(A) = \Theta$ if A is a strip.)

(b) A strip A is called *solid*, if *each* fiber of the strip is an interval and if

$$\delta(A) := \inf\{|A^\theta| : \theta \in \Theta\} > 0.$$

(c) A closed subset $A \subseteq \Omega$ is called *pinched*, if $P_A := \{\theta \in \Theta : \operatorname{card} A^\theta = 1\}$ is dense in Θ. (In this case, $P_A \in \mathcal{G}$; that is, each pinched set is a strip.)

Lemma 3.10. *Let $A \subseteq \Omega$ be a strip.*

(a) *A^C is a strip.*

(b) *A minimal T-invariant strip is a core.*

Proof. (a) Since $A^C \subseteq A$ we have $A \cap \pi^{-1}(G) \subseteq A^C \cap \pi^{-1}(G) \subseteq A \cap \pi^{-1}(G)$ for $G \in \widetilde{\mathcal{G}}_A$ and therefore A^C is a strip. (b) The statement follows from part (a) and Lemma 3.4. □

General assumption. From now on we assume that Θ is a compact metric space, so that $\Omega = \Theta \times I$ is compact as well.

Lemma 3.11. *Every T-invariant strip contains at least one minimal T-invariant core strip. Each minimal T-invariant core strip is strongly invariant.*

Proof. We prove only the first statement; the second statement follows from a remark made earlier.

Let A be a T-invariant strip, and let $\{A_i : i \in I\}$ be a nested family of T-invariant strips contained in A – thus if $i, j \in I$ then either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Let $A_\infty = \bigcap_{i \in I} A_i$. Then A_∞ is compact, non-empty, T-invariant, and $\pi(A_\infty) = \Theta$. There is a countable directed subset $i_1 < \ldots < i_n < \ldots$ of I such that $A_\infty = \bigcap_{n=1}^{\infty} A_{i_n}$, and it follows that A_∞ is a strip.

By Zorn's lemma there exists a minimal T-invariant strip $B \subseteq A$. By Lemma 3.10(b) B is a core. This completes the proof. □

Let us now recall that a homeomorphism $R: \Theta \to \Theta$ of a compact metric space Θ is called *minimal* if there is no proper nonempty closed subset $\Theta_1 \subset \Theta$ such that $R(\Theta_1) \subseteq \Theta_1$. It is easy to see that R is minimal if and only if, for each $\theta \in \Theta$, the forward orbit of $\{R^k(\theta) : k = 1, 2, \ldots\}$ is dense in Θ. This condition is equivalent to the seemingly less restrictive one that, for each $\theta \in \Theta$, the full orbit $\{R^k(\theta) : k = 0, \pm 1, \pm 2, \ldots\}$ is dense in Θ. A homeomorphism $R: \Theta \to \Theta$ is called *totally minimal* if each power $R^l: \Theta \to \Theta$ is minimal ($l = 1, 2, \ldots$).

As an example of a totally minimal homeomorphism, let $\Theta = \mathbb{R}/\mathbb{Z}$ be the circle, and let $R: \Theta \to \Theta$ be the rotation $\theta \mapsto \Theta + \gamma$ where γ is an irrational number. More generally, let $\Theta = \mathbb{R}^m/\mathbb{Z}^m$ be the m-torus with angular coordinates $(\theta_1, \ldots, \theta_m) = \theta$, and let $R: \Theta \to \Theta$, $\theta \mapsto \theta + \gamma$ where $(\gamma_1, \ldots, \gamma_m)$ is a vector of real numbers with the property that the components $\gamma_1, \ldots, \gamma_m$ are independent over the rational field $\mathbb{Q}$.

Lemma 3.12. *Suppose that R is minimal. Then the intersection of two T-invariant strips is either empty or is a T-invariant strip.*

Proof. If A and B are two T-invariant strips such that $A \cap B \neq \emptyset$, then $\pi(A \cap B)$ is compact and non-empty. It is also R-invariant because A and B are T-invariant. Since R is a minimal homeomorphism of Θ we must have $\pi(A \cap B) = \Theta$. It is now clear that, for a generic set of $\theta \in \Theta$, the fiber $(A \cap B)^\theta$ is a compact non-empty subinterval of I. We conclude that, if $A \cap B \neq \emptyset$, then $A \cap B$ is a T-invariant strip. □

Definition 3.13 (Ordered strips). Say that two strips A and B satisfy $A < B$ if there is a set $G \in \mathcal{G}$ such that for all $\theta \in G$, $x \in A^\theta$, and $y \in B^\theta$ there holds $x < y$. We say that the strips are *ordered*, if either $A < B$ or $B < A$.

While two disjoint core strips need not be ordered even if Θ is connected and locally connected, we do have the following

LEMMA 3.14. *Suppose that Θ is connected, and that A and B are disjoint full strips, i.e. A^θ and B^θ are intervals for all $\theta \in \Theta$. Then either $A > B$ or $B > A$.*

PROOF. Fix $\theta \in \Theta$. Then either $A^\theta > B^\theta$ in the sense that $x > y$ whenever $x \in A^\theta$ and $y \in B^\theta$, or $B^\theta > A^\theta$ in the same sense. Let $V = \{\theta \in \Theta : A^\theta > B^\theta\}$ so that $\Theta \setminus V = \{\theta \in \Theta : B^\theta > A^\theta\}$. By the compactness of A and B, both these sets are open in Θ. So one of them is empty. □

DEFINITION 3.15 (Periodic strip). Let $p > 1$ be an integer. A strip $A \subseteq \Omega$ is called *p-periodic* if $T^p(A) = A$ and if the image sets $A, T(A), \dots, T^{p-1}(A)$ are pairwise disjoint and pairwise ordered.

If A happens to be a full strip and if Θ is connected, then A is p-periodic if and only if $T^p(A) = A$ and the image sets $A, T(A), \dots, T^{p-1}(A)$ are pairwise disjoint (Lemma 3.14).

In Section 5 we will state and prove a generalization of the Sharkovskiĭ theorem for skew-product maps, where the concept of periodic point is replaced by that of periodic strip. We will see that periodic strips of almost automorphic type (i.e. those which are pinched cores) arise naturally in this context. We finish this section by giving an example which clearly indicates that another possible analogue of "periodic point" – namely, the concept of measurable section $\phi : \Theta \to \Theta \times I$ – cannot be fruitfully used to generalize the Sharkovskiĭ theorem for such maps.

EXAMPLE 3.16. Let $\Theta = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be the circle, and let $R(\theta) = \theta + \gamma$ where $\gamma \in \mathbb{R}$ is irrational. Let $I = [0,1]$, and let $T_\theta : I \to I$ be the full tent map for each $\theta \in \Theta$. Thus

$$T_\theta(x) = f(x) = 1 - |2x - 1| \quad (0 \le x \le 1).$$

The map $T\colon \Theta \times I \to \Theta \times I$, $(\theta, x) \mapsto (\theta + \gamma, T_\theta(x)) = (\theta + \gamma, f(x))$ satisfies all the conditions imposed so far.

Write $I_0 = [0, 1/2]$, $I_1 = [1/2, 1]$. To each infinite sequence $a_0, a_1, \dots$ of binary digits in $\{0, 1\}$ we associate the unique point $x = x(a_0, a_1, \dots) \in [0,1]$ such that $f^n(x) \in I_{a_n}$ for all $n = 0, 1, \dots$. Let $B \subset [0,1]$ be any measurable set, and set

$$\phi_B(\theta) = x(\mathbf{1}_B(\theta), \mathbf{1}_B(R(\theta)), \mathbf{1}_B(R^2(\theta)), \dots)$$

where $\mathbf{1}_B$ is the indicator function of B. Then ϕ_B defines a measurable section of the trivial bundle $\Theta \times I \xrightarrow{\pi} \Theta$ which is invariant in the sense that $f(\phi_B(\theta)) = \phi_B(R(\theta))$ for all $\theta \in \Theta$.

Next let $B, C \subset [0,1]$ be measurable sets whose symmetric difference has positive Lebesgue measure: $|B \triangle C| > 0$. Then for Lebesgue-a.e. θ the two sequences $(\mathbf{1}_B(\theta), \mathbf{1}_B(R(\theta)), \mathbf{1}_B(R^2(\theta)), \dots)$ and $(\mathbf{1}_C(\theta), \mathbf{1}_C(R\theta), \mathbf{1}_C(R^2\theta), \dots)$

differ in infinitely many places. Fix such a θ; if it were true that $x := \phi_B(\theta) = \phi_C(\theta)$, then the point $f^n(x)$ would be in $I_0 \cap I_1 = \{1/2\}$ for infinitely many n, which is impossible because $f^k(1/2) = 0$ for $k \geq 1$. Thus $\phi_B(\theta) \neq \phi_C(\theta)$ a.e. and we must conclude that our map T admits uncountably many measurable invariant sections.

The fact that this phenomenon occurs motivates our search for another analogue of the concept of fixed point and for the concept of periodic point.

4. Results on strips and their bounding sections

In this section, we state and prove basic results concerning strips and core strips. Throughout the section, Θ denotes a compact metric space; $R : \Theta \to \Theta$ is a homeomorphism of Θ onto itself; $\Omega = \Theta \times I$; and $T : \Omega \to \Omega$ is a continuous map satisfying $\pi \circ T = R \circ \pi$ where $\pi : \Omega \to \Theta$ is the natural projection.

Definition 4.1 (Sections). Let $\phi : \Theta \to \Theta \times I$ be a section (which a priori need not even be measurable). If $\pi_2 : \Theta \times I \to I$ is the projection onto I, then $\pi_2 \circ \phi$ is a map from Θ to I which we also denote by ϕ and which we also call a section.

(a) Let $A \subseteq \Omega$ be compact with $\pi(A) = \Theta$. The *upper* and *lower bounding sections* of A are given by $\upsilon_A(\theta) = \sup A^\theta$ and $\lambda_A(\theta) = \inf A^\theta$, respectively. Observe that υ_A is upper semicontinuous (u.s.c.) and λ_A lower semicontinuous (l.s.c.).

(b) For a section $\phi : \Theta \to \Theta \times I$, let $\Phi = \{(\theta, \phi(\theta)) : \theta \in \Theta\}$ be the image of ϕ, let $\overline{\Phi}$ be the topological closure of Φ in $\Theta \times I$, and let ϕ^+, resp. ϕ^- be the upper and lower bounding sections of $\overline{\Phi}$. Instead of $(\phi^+)^-$ we write ϕ^{+-} etc. If λ and υ are sections we write Λ and Υ for their images, etc. If ϕ is a section we set $(T\phi)(R(\theta)) = T_\theta(\phi(\theta))$.

(c) If ϕ, ψ are sections then $\phi \leq \psi$ has the natural meaning $\phi(\theta) \leq \psi(\theta)$ for all $\theta \in \Theta$.

If ϕ, ψ are sections, then we will use the notation $\{\phi \leq \psi\}$ for the set $\{\theta \in \Theta : \phi(\theta) \leq \psi(\theta)\}$ and analogously for other order relations.

Remark 4.2. If $\phi \leq \upsilon$ are two sections and if υ is u.s.c., then $\phi^+ \leq \upsilon$. An analogous remark is true for l.s.c. sections. We will use this remark repeatedly without further comment.

Lemma 4.3. *Let ϕ be a semicontinuous section.*

(a) *$\overline{\Phi}$ is pinched.*

(b) $\Phi^C = \overline{\Phi}^C$.

(c) *If ϕ is u.s.c. then $\phi^- = \lambda_{\overline{\Phi}} = \lambda_{\Phi^C}$; in particular, $\{\phi = \phi^-\} = P_{\overline{\Phi}} \in \mathcal{G}$ (is residual in Θ).*

(d) *If ϕ is l.s.c. then $\phi^+ = \upsilon_{\overline{\Phi}} = \upsilon_{\Phi^C}$; in particular, $\{\phi = \phi^+\} = P_{\overline{\Phi}} \in \mathcal{G}$.*

PROOF. (a) Suppose that ϕ is u.s.c. We need only show that $P_{\overline{\Phi}}$ is dense in Θ. For this let $\theta \in \Theta$. There is a sequence $\theta_n \to \theta$ such that $\lim_{n\to\infty} \phi(\theta_n) = \phi^-(\theta)$. Since ϕ^- is l.s.c., we also have $\liminf_{n\to\infty} \phi^-(\theta_n) \geq \phi^-(\theta)$. This shows that $\lim_{n\to\infty} |\phi(\theta_n) - \phi^-(\theta_n)| = 0$. Since θ is an arbitrary point of Θ, we have that, for each $\varepsilon > 0$, the open set $\{\phi - \phi^- < \varepsilon\}$ is dense in Θ. So $P_{\overline{\Phi}} = \bigcap_{k=1}^{\infty} \{\phi - \phi^- < k^{-1}\}$ is residual in Θ. The proof is similar if ϕ is l.s.c.

(b) We now know that $P_{\overline{\Phi}}$ is residual, so there exists $G \subseteq P_{\overline{\Phi}}$, $G \in \mathcal{G}_{\overline{\Phi}}$. Since $(\theta, \phi(\theta)) \in \overline{\Phi}_\theta$ we have $\overline{\Phi}_\theta = \{(\theta, \phi(\theta))\} = \Phi_\theta$ for all $\theta \in G$. Hence $\Phi \cap \pi^{-1}(G) = \overline{\Phi} \cap \pi^{-1}(G)$. By (the proof of) Lemma 3.2(f) we have $\Phi^C = \overline{\Phi}^C$.

(c) We have $\lambda_{\overline{\Phi}} \leq \lambda_{\Phi^C}$ because $\Phi^C = \overline{\Phi}^C \subseteq \overline{\Phi}$. On the other hand, let $\theta \in \Theta$ and $G \subseteq P_{\overline{\Phi}}$, $G \in \mathcal{G}_{\overline{\Phi}}$. Since ϕ is u.s.c. there is a sequence $\theta_n \in G$ such that $\theta_n \to \theta$ and $\lambda_{\overline{\Phi}}(\theta) = \lim_{n\to\infty} \phi(\theta_n)$. Therefore $(\theta, \lambda_{\overline{\Phi}}(\theta)) \in \Phi^C$ so that $\lambda_{\overline{\Phi}} \geq \lambda_{\Phi^C}$.

(d) The proof is analogous to the previous one. □

LEMMA 4.4.

(a) *If A is a pinched subset of Ω, then $\Lambda_A^C = \Upsilon_A^C = A^C$.*
(b) *If A is a pinched core, then $\overline{\Lambda}_A = \overline{\Upsilon}_A = A$. In particular, $\lambda_A^+ = \upsilon_A$ and $\upsilon_A^- = \lambda_A$.*

PROOF. (a) Since A is pinched we have $(\overline{\Lambda}_A)^\theta = (\overline{\Upsilon}_A)^\theta = A^\theta$ for a residual set of θ. Hence $\overline{\Lambda}_A^C = \overline{\Upsilon}_A^C = A^C$ by Lemma 3.2(f). The statement now follows from Lemma 4.3(b).

(b) If A is a pinched core, then $A = A^C = \Lambda_A^C \subseteq \overline{\Lambda}_A \subseteq A$. Arguing similarly one proves $\overline{\Upsilon}_A = A$. □

LEMMA 4.5. *Let ϕ be a semicontinuous section.*

(a) *If ϕ is u.s.c. then $\phi^{-+} = \upsilon_{\Phi^C}$.*
(b) *If ϕ is l.s.c. then $\phi^{+-} = \lambda_{\Phi^C}$.*

PROOF. By Lemma 4.3(c) we have $\phi^{-+} = \lambda_{\Phi^C}^+$. By Lemma 4.4(b) applied to the pinched core Φ^C we have $\lambda_{\Phi^C}^+ = \upsilon_{\Phi^C}$. This proves part (a); part (b) is proved in a similar way. □

LEMMA 4.6. *Let A be a strip.*

(a) $\upsilon_A^{-+} = \upsilon_{A^C} = \upsilon_{\Upsilon_A^C}$.
(b) $\lambda_A^{+-} = \lambda_{A^C} = \lambda_{\Lambda_A^C}$.

PROOF. It suffices to prove part (a). Let $G \in \mathcal{G}_A$. In view of Lemma 4.3(c) also $G' := G \cap \{\upsilon_A = \upsilon_A^-\} \in \mathcal{G}_A$. For each $(\theta, x) \in A^C$, there is a sequence $(\theta_n, x_n) \in A \cap \pi^{-1}(G')$ such that $(\theta_n, x_n) \to (\theta, x)$. As $x_n \leq \upsilon_A(\theta_n) = \upsilon_A^-(\theta_n)$

we see that $x = \lim_{n\to\infty} x_n \le \upsilon_A^{-+}(\theta)$. But this holds for each $(\theta, x) \in A^C$, whence $\upsilon_{A^C} \le \upsilon_A^{-+}$. The identity $\upsilon_A^{-+} = \upsilon_{\Upsilon_A^C}$ follows from Lemma 4.5(a), and $\upsilon_{\Upsilon_A^C} \le \upsilon_{A^C}$ follows from the observation that $\Upsilon_A^C \subseteq A^C$. □

We have the following immediate corollary to this lemma:

Corollary 4.7. *Let A be a core strip, and let υ_A resp. λ_A be the upper resp. lower bounding section of A.*

(a) *υ_A is also the upper bounding section of Υ_A^C.*
(b) *λ_A is also the lower bounding section of Λ_A^C.*

Combining Lemmas 4.3, 4.5, and Corollary 4.7 we obtain

Proposition 4.8. *Let A be a core strip. Then $\upsilon_A^{-+} = \upsilon_A$ and $\lambda_A^{+-} = \lambda_A$. In particular one has*

(a) *$A^\theta = [\lambda_A^+(\theta), \upsilon_A^-(\theta)]$ for θ in a residual $G \subseteq \Theta$.*
(b) *$\overline{\Lambda}_A = \Lambda_A^C$ and $\overline{\Upsilon}_A = \Upsilon_A^C$.*

Lemma 4.9. *Let A be a core strip, and denote $\widetilde{A} = \{(\theta, x) : \lambda_A(\theta) \le x \le \upsilon_A(\theta)\}$ the corresponding "filled in" strip. Then $A = \widetilde{A}^C$.*

Proof. A^θ is an interval for θ in a residual $G \subseteq \Theta$. Hence $A^\theta = [\lambda_A(\theta), \upsilon_A(\theta)] = \widetilde{A}^\theta$ for $\theta \in G$. Now Lemma 3.2(f) implies $A = A^C = \widetilde{A}^C$. □

Before stating the next result we introduce some terminology.

Definition 4.10 (Almost automorphic). Let A be a core strip; say that A is T-*almost automorphic* if it is pinched and minimal with respect to T. (Our usage of this notion is a bit more general than that in the literature, where it is also required that the base homeomorphism $R: \Theta \to \Theta$ is almost periodic. See [15] for general properties of almost automorphic dynamical systems.)

Theorem 4.11. *Let R be a minimal homeomorphism of Θ, and let A be a strongly invariant core strip. Define $\Theta_A := \{\theta \in \Theta : \lambda_A^+(\theta) < \upsilon_A^-(\theta)\}$. Then Θ_A is open and either*

(a) *Θ_A is empty and A is almost automorphic, or*
(b) *Θ_A is dense in Θ and A is solid.*

Proof. The set Θ_A is open because $\upsilon_A^- - \lambda_A^+$ is lower semicontinuous.

Suppose first that $\overline{\Theta}_A \ne \Theta$. Then $\upsilon_A^- \le \lambda_A^+$ on some open set $U \subseteq \Theta \setminus \Theta_A$. By Proposition 4.8, $\upsilon_A = \upsilon_A^{-+} \le \lambda_A^+ \le \upsilon_A$ on U. Using Proposition 4.8 again, one has that $A^\theta = [\upsilon_A(\theta), \upsilon_A^-(\theta)]$ consists of exactly one point for a residual subset of $\theta \in U$. In particular, the pinching set P_A is non-empty. Since A is strongly invariant, we have $R(P_A) \subseteq P_A$ and so, by minimality of R, P_A is dense in Θ; that is A is pinched. Since Θ_A is open and $P_A \cap \Theta_A = \emptyset$ we must have $\Theta_A = \emptyset$.

We must still show that A is minimal invariant if $\Theta_A = \emptyset$. Let $B \subseteq A$ be a closed invariant set. Then $\lambda_A \leq \upsilon_B \leq \upsilon_A$, hence $\lambda_A^+ \leq \upsilon_B \leq \upsilon_A$. By Lemma 4.4(a) one has $\upsilon_A = \lambda_A^+$, hence $\upsilon_B = \upsilon_A$. Therefore $A = \overline{\Upsilon}_A = \overline{\Upsilon}_B \subseteq B$, where we have used Lemma 4.4(b).

Let us now consider the case where $\overline{\Theta}_A = \Theta$. We claim that $A^\theta = [\lambda_A(\theta), \upsilon_A(\theta)]$ for each $\theta \in \Theta_A$. To see this, let $G \in \widetilde{\mathcal{G}}_A$. Let $\theta \in \Theta_A$, and choose a sequence (θ_n, x_n) in $A \cap \pi^{-1}(G)$ which converges to $(\theta, \upsilon_A(\theta))$. Then the interval $[\lambda_A(\theta_n), x_n] \subseteq A^{\theta_n}$ for each n, and $\limsup_{n\to\infty} \lambda_A(\theta_n) \leq \lambda_A^+(\theta)$. Hence $[\lambda_A^+(\theta), \upsilon_A(\theta)] \subseteq A^\theta$. In a similar way one proves that $[\lambda_A(\theta), \upsilon_A^-(\theta)] \subseteq A^\theta$. Since $\theta \in \Theta_A$, these two intervals overlap, and therefore $[\lambda_A(\theta), \upsilon_A(\theta)] \subseteq A^\theta$. The reverse inclusion is trivial, so indeed $A^\theta = [\lambda_A(\theta), \upsilon_A(\theta)]$ if $\theta \in \Theta_A$. This shows that the set Θ' of those θ for which A^θ is an interval contains the open set Θ_A. As A is strongly invariant under T, thet set Θ' is forward invariant under R. Now the minimality of R^{-1} implies that $\Theta' = \Theta$, i.e. all A^θ are intervals.

Observe now that $\Theta_A = \bigcup_{k=1}^\infty \Theta_A^k$ where $\Theta_A^k := \{\theta \in \Theta_A : \lambda_A^+(\theta) < \upsilon_A^-(\theta) - 1/k\}$ are open sets. As $\Theta_A \neq \emptyset$, there is some k such that $\Theta_A^k \neq \emptyset$. Because R is minimal (and Θ compact), there is some $N > 0$ such that $\Theta = \bigcup_{n=0}^N R^{-n}\Theta_A^k$, and in view of the uniform continuity of T there is for each n some $\varepsilon = \varepsilon(k,n) > 0$ such that $R^{-n}\Theta_A^k \subseteq \{\theta \in \Theta : \lambda_A(\theta) < \upsilon_A(\theta) - \varepsilon\}$. It follows that A is a solid strip. □

REMARK 4.12. With reference to the preceeding proof: if A is a solid strip then $\overline{\Theta}_A = \Theta$. If Λ_A^C and Υ_A^C are T-invariant, we can say more: Suppose that $\Lambda_A^C \cap \Upsilon_A^C$, which is also a T-invariant set, is nonempty. Then $\pi(\Lambda_A^C \cap \Upsilon_A^C)$ is a nonempty, closed R-invariant set. Therefore $\pi(\Lambda_A^C \cap \Upsilon_A^C) = \Theta$ by minimality of the homeomorphism R. In view of Lemma 4.3(c) and (d) this implies $\lambda_A^+(\theta) \geq \upsilon_A^-(\theta)$ for all $\theta \in \Theta$ which contradicts the assumption $\overline{\Theta}_A = \Theta$.

It follows that, if Λ_A^C and Υ_A^C both are invariant, then $\lambda_A^+ < \upsilon_A^-$ everywhere. Thus the sets Λ_A^C and Υ_A^C have strictly positive distance; i.e. they are "separated by an open tube." By Proposition 4.8 the same is true for $\overline{\Lambda}_A$ and $\overline{\Upsilon}_A$.

In the next theorem we will see that the same conclusion holds also if Λ_A^C and Υ_A^C are not invariant provided T satisfies some nondegeneracy condition.

THEOREM 4.13. *Suppose that in the situation of Theorem* 4.11 *the map T has the following additional property: For each $\theta \in \Theta$ and each nondegenerate interval $J \subseteq I$ the interval $T_\theta J$ is nondegenerate. Suppose that A is a solid invariant core strip. Then the sets Λ_A^C and Υ_A^C have strictly positive distance and are separated by an open "tube." By Proposition* 4.8 *the same holds for $\overline{\Lambda_A}$ and $\overline{\Upsilon_A}$.*

PROOF. Since A is solid the open set Θ_A is nonempty. Hence we find a compact set $K \subseteq \Theta$ with nonempty interior and a nondegenerate interval $[a,b] \subseteq$

I such that $W := K \times [a,b] \subseteq A$. Let $N > 0$ be such that $\bigcup_{n=0}^{N} R^n(\operatorname{int} K) = \Theta$. In view of the skew product structure of T each $T^n(W)$ is a compact set with $\pi(T^n(W)) = R^n K$ and fibers which are nondegenerate intervals. We will show below that $\lambda_{T^n(W)}$ and $\upsilon_{T^n(W)}$ are continuous functions from $R^n(K)$ to I. Given this fact it follows that each $\theta \in \Theta$ has a neighbourhood on which the sets Λ_A^C and Υ_A^C have strictly positive distance, and the compactness of Θ concludes the argument.

It remains to show that $\lambda_{T^n(W)}$ and $\upsilon_{T^n(W)}$ are continuous functions from $R^n(K)$ to I. We carry out the argument for $\lambda_{T^n(W)}$, that for $\upsilon_{T^n(W)}$ is the same. Since $T^n(W)$ is compact, $\lambda_{T^n(W)}$ is l.s.c. Now we fix $R^n(\theta) \in R^n(K)$ and consider any sequence $\theta_j \in K$ that converges to θ. Let $(\theta, x) \in W$ be a preimage of $(R^n(\theta), \lambda_{T^n(W)}(R^n(\theta)))$ under T^n. Because of the product structure of W all (θ_j, x) are in W so that

$$\begin{aligned}(R^n(\theta), \lambda_{T^n(W)}(R^n(\theta))) &= T^n(\theta, x) = \lim_{j\to\infty} T^n(\theta_j, x) \\ &= (R^n(\theta), \lim_{j\to\infty} \pi_2(T^n(\theta_j, x))).\end{aligned}$$

It follows that

$$\limsup_{j\to\infty} \lambda_{T^n(W)}(R^n(\theta_j)) \leq \lambda_{T^n(W)}(R^n(\theta)).$$

Since $(R^n(\theta_j))_j$ is an arbitrary sequence converging to $R^n(\theta)$, this proves the upper semicontinuity of $\lambda_{T^n(W)}$. □

Now we turn to the construction of invariant core strips in a situation which will arise in Section 5.

Definition 4.14 (Strips mapped over another). Let A_0 and A be core strips.

(a) We say that T maps A *upward over* A_0 if

$$T(\lambda_A) \leq \lambda_{A_0}^+ \quad \text{and} \quad T(\upsilon_A) \geq \upsilon_{A_0}^-. \tag{4.1}$$

In this case we write $A \xrightarrow{\text{u.o.}} A_0$, or $A \xrightarrow{\text{u.o.}} A_0$ with respect to (w.r.t.) T.

(b) We say that T maps A *downward over* A_0 if

$$T(\lambda_A) \geq \upsilon_{A_0}^- \quad \text{and} \quad T(\upsilon_A) \leq \lambda_{A_0}^+. \tag{4.2}$$

In this case we write $A \xrightarrow{\text{d.o.}} A_0$, or $A \xrightarrow{\text{d.o.}} A_0$ with respect to (w.r.t.) T.

(c) We say that T maps A *over* A_0 if either $A \xrightarrow{\text{u.o.}} A_0$ or $A \xrightarrow{\text{d.o.}} A_0$ w.r.t. T. In this case we write $A \xrightarrow{\text{o.}} A_0$ or $A \xrightarrow{\text{o.}} A_0$ w.r.t. T.

LEMMA 4.15. *Let A and A_0 be core strips. If $A \xrightarrow{\text{o.}} A_0$, then $T(A) \supseteq A_0$.* T*hus the terminology "mapped over" is justified.*)

PROOF. Let $G \in \widetilde{\mathcal{G}}_A$. For all $\theta \in G$ the set $T_\theta(A^\theta)$ is an interval. Also, by Proposition 4.8, $(A_0)^\theta = [\lambda^+_{A_0}(\theta), \upsilon^-_{A_0}(\theta)]$ for a residual subset of $G_0 \subseteq R(G)$. Hence $A_0 \cap \pi^{-1}(G_0) \subseteq T(A)$, and so $A_0 = A_0^C \subseteq \overline{T(A)} = T(A)$. □

In Definition 3.13 we introduced a strict order relation $A < B$ between strips. With the notation introduced in this section we can characterize that relation as follows:

$$A < B \quad \text{if } \{\upsilon_A < \lambda_B\} \in \mathcal{G}.$$

This motivates the following definition.

DEFINITION 4.16 (Weakly ordered strips). Let A and B be strips. Then $A \prec B$ if $\{\upsilon^-_A \leq \lambda^+_B\} \in \mathcal{G}$, and $A \succ B$ if $\{\lambda^+_A \geq \upsilon^-_B\} \in \mathcal{G}$. We say that A and B are *weakly ordered.*

(If A is a core strip, then $A \prec A$ if and only if A is pinched.)

A closer look at this definition reveals that the weak order (in contrast to the strict order) is not really a notion depending on the residual subsets of Θ.

LEMMA 4.17. *If $A \prec B$, then $\upsilon^-_A(\theta) \leq \lambda^+_B(\theta)$ for all $\theta \in \Theta$.*

PROOF. The set $\{\upsilon^-_A \leq \lambda^+_B\}$ is closed because of the semicontinuity properties of υ^-_A and λ^+_B. At the same time it is residual because $A \prec B$. Hence it is all of Θ. □

Obviously, $A < B$ implies $A \prec B$. Here is a kind of reverse implication:

LEMMA 4.18. *If A and B are disjoint strips and if $A \prec B$, then $A < B$.*

PROOF. As A and B are disjoint strips, the set $\{\upsilon_A < \lambda_B\} \cup \{\upsilon_B < \lambda_A\}$ is residual. By Lemma 4.17, $\{\upsilon_B < \lambda_A\} \subseteq \{\lambda^+_B < \upsilon^-_A\} = \emptyset$. Hence $\{\upsilon_A < \lambda_B\}$ is residual, i.e. $A < B$. □

We now formulate and prove a key result.

LEMMA 4.19. *Suppose that the core strip A is mapped upwards* (*downwards*) *over the core strip A_0.*

(a) *There is a core strip $A_1 \subseteq A$ with $T(A_1) = A_0$ which is mapped upwards* (*downwards*) *over A_0. If $A_1 \xrightarrow{\text{u.o.}} A_0$, then $T(\Lambda^C_{A_1}) \subseteq \Lambda^C_{A_0}$ and $T(\Upsilon^C_{A_1}) \subseteq \Upsilon^C_{A_0}$; if, on the other hand, $A_1 \xrightarrow{\text{d.o.}} A_0$, then $T(\Lambda^C_{A_1}) \subseteq \Upsilon^C_{A_0}$ and $T(\Upsilon^C_{A_1}) \subseteq \Lambda^C_{A_0}$.*

(b) *Let A_0^* be another core strip and suppose that A is mapped upwards* (*downwards*) *over A_0 and A_0^*. If A_0 and A_0^* are weakly ordered, then the core strips A_1 and A_1^* which are mapped over A_0 and A_0^* as in* (a) *can* *be chosen weakly ordered as well. More precisely, if* $A_0 \prec A_0^*$ ($A_0^* \prec A_0$), *then* $A_1 \prec A_1^*$ ($A_1^* \prec A_1$).

Proof. Assume first that $A \xrightarrow{\text{u.o.}} A_0$. By Lemma 4.15, we have $\Upsilon_{A_0}^C \subseteq A_0 \subseteq T(A)$. Therefore $\pi(A \cap T^{-1}(\Upsilon_{A_0}^C)) = \Theta$. Set

$$\phi(\theta) = \inf(A \cap T^{-1}(\Upsilon_{A_0}^C))^\theta.$$

As the lower bounding section of a compact set, ϕ is l.s.c.

Let $\widetilde{A} := \{(\theta, x) \in A : x \le \phi(\theta)\}$ and $A' := \widetilde{A}^C$. A' is a core by definition, and we show now that it is a strip. To this end let $G \in \mathcal{G}_A \cap \mathcal{G}_{\widetilde{A}}$ be such that $A^\theta = [\lambda_A^+(\theta), \upsilon_A^-(\theta)]$ for all $\theta \in G$, see Proposition 4.8. It suffices to show that ${A'}^\theta = [\lambda_A(\theta), \upsilon_{A'}(\theta)]$ for $\theta \in G$. As $A' = \widetilde{A}^C \subseteq A^C = A$, the inclusion "$\subseteq$" is obvious. For the other direction we must show that each x with $\lambda_A(\theta) < x < \upsilon_{A'}(\theta)$ belongs to ${A'}^\theta$. Now $x < \upsilon_{A'}(\theta)$ implies that there are $\theta_n \in G$ converging to θ with $x < \phi(\theta_n) \le \upsilon_A(\theta_n) = \upsilon_A^-(\theta_n)$. As $\lambda_A(\theta) = \lambda_A^+(\theta)$ for $\theta \in G$, the inequality $x > \lambda_A(\theta)$ implies that $x > \lambda_A^+(\theta_n)$ for sufficiently large n. Hence $(\theta_n, x) \in \widetilde{A}$ for large n, and it follows that $(\theta, x) \in \overline{\widetilde{A} \cap \pi^{-1} G} = \widetilde{A}^C = A'$.

Let us show that

$$\upsilon_{A'} = \upsilon_{\Phi^C} = \phi^+ \tag{4.3}$$

First of all, $\Phi^C \subseteq A'$ by definition of ϕ and $\widetilde{A}$, hence $\upsilon_{\Phi^C} \le \upsilon_{A'}$. On the other hand, let $(\theta, x) \in A'$. There is a sequence (θ_n, x_n) in A with $x_n \le \phi(\theta_n)$ such that $(\theta_n, x_n) \to (\theta, x)$. Hence $x \le \limsup_{n\to\infty} \phi(\theta_n) \le \phi^+(\theta)$. Therefore $\upsilon_{A'} \le \phi^+(\theta)$. By Lemma 4.3(d) we have $\phi^+ = \upsilon_{\Phi^C}$, which finishes the proof of (4.3).

Next, Lemma 4.4(a) implies that $\Upsilon_{A'}^C = \Upsilon_{\Phi^C}^C = \Phi^C$, hence $T(\Upsilon_{A'}^C) = T(\Phi^C) \subseteq \Upsilon_{A_0}^C$ where we have observed that $T\Phi \subseteq \Upsilon_{A_0}^C$ by the definition of ϕ. By Corollary 4.7(a) we have $\Upsilon_{A'} \subseteq \Upsilon_{A'}^C$, hence $T(\upsilon_{A'}) \ge \lambda_{\Upsilon_{A_0}^C} = \upsilon_{A_0}^-$. In order to see that also $T(\lambda_{A'}) = T(\lambda_A) \le \lambda_{A_0}^+$ we show

$$\lambda_A = \lambda_{A'}. \tag{4.4}$$

As $A' \subseteq A^C = A$ by definition, $\lambda_A \le \lambda_{A'}$ is obvious. For the converse inequality observe first that $\Lambda_A \subseteq \widetilde{A}$ by definition. So $\Lambda_A^C \subseteq A'$, and it follows from Corollary 4.7(b) that $\lambda_A = \lambda_{\Lambda_A^C} \ge \lambda_{A'}$.

We now apply the above procedure to the "lower boundary" of A' rather than to the "upper boundary" of A_0. Specifically, define

$$\psi(\theta) = \sup(A' \cap T^{-1}(\Lambda_{A_0}^C)) \ .$$

Then ψ is u.s.c. and $\psi \le \upsilon_{A'}$. Set $\widetilde{A}_1 = \{(\theta, x) \in A' : x \ge \psi(\theta)\}$ and $A_1 = \widetilde{A}_1^C$. As in the first part of the proof one checks that A_1 is a core strip, that $T(\Lambda_{A_1}^C) \subseteq$

$\Lambda^C_{A_0}$, that $T(\lambda_{A_1}) \le \lambda^+_{A_0}$, and that

$$\lambda_{A_1} = \lambda_{\Psi^C} = \psi^-. \tag{4.5}$$

As in (4.4) one shows

$$\upsilon_{A_1} = \upsilon_{A'}. \tag{4.6}$$

This is all one needs to check that A_1 has all the properties required in Lemma 4.19(a) in the case $A \xrightarrow{\text{u.o.}} A_0$, except for the property $T(A_1) = A_0$.

In order to prove this we observe that the above construction (and further choices as in Proposition 4.8 furnish a set $G \in \mathcal{G}$ such that $(A_1)^\theta = [\lambda_{A_1}(\theta), \upsilon_{A_1}(\theta)]$, $\lambda_{A_0}(R(\theta)) = \lambda^+_{A_0}(R(\theta))$ and $\upsilon_{A_0}(R(\theta)) = \upsilon^+_{A_0}(R(\theta))$ for all $\theta \in G$. Hence $T(A_1)^\theta \supseteq (A_0)^{R(\theta)}$ for each $\theta \in G$ in such a way that the endpoints of $(A_1)^\theta$ are mapped onto the corresponding endpoints of $(A_0)^{R(\theta)}$. Without loss of generality we can also assume that $\phi(\theta) = \phi^+(\theta)$ for $\theta \in G$ (see Lemma 4.3(d)), so that indeed $\phi(\theta) = \upsilon_{A'}(\theta)$ in view of equation (4.3). But this excludes the possibility that there is $x \in A'^\theta$ for which $T_\theta x > \upsilon_{A_0}(R(\theta))$. With an analoguous argument on the "lower boundaries" one finally shows that indeed $T(A_1^\theta) = (A_0)^{R(\theta)}$ for all $\theta \in G$. Now $T(A_1) = A_0$ follows from Corollary 3.8.

There remains to reduce the "downward over" case to the "upward over" one. Suppose without loss of generality that I is symmetric about $x = 0$, and let $\tau(x) = -x$ be the symmetry. Set $\widetilde{A}_0 := \tau(A_0)$ and $\widetilde{T} := \tau \circ T$. Then $A \xrightarrow{\text{u.o.}} \widetilde{A}_0$ w.r.t. $\widetilde{T}$, so there exists a core strip $A_1 \subseteq A$ with $\widetilde{T}(A_1) = A_0$ and $A_1 \xrightarrow{\text{u.o.}} \widetilde{A}_0$ w.r.t. $\widetilde{T}$. But then $T(A_1) = A_0$ and $A_1 \xrightarrow{\text{d.o.}} A_0$ w.r.t. T. The other properties required in Lemma 4.19(a) can be checked immediately.

We turn to the proof of the second part of Lemma 4.19. We denote the auxiliary objects in the above construction applied to A_0^* by ϕ^*, ψ^*, etc. Then $T\Psi^* \subseteq \Lambda^C_{A_0^*}$ by definition of ψ^*. Hence, observing (4.5) and Lemma 3.4(a), $T\Lambda_{A_1^*} \subseteq T(\Psi^{*C}) \subseteq \Lambda^C_{A_0^*}$. Now, by assumption, $\upsilon^-_{A_0} \le \lambda^+_{A_0^*}$. As A_0^* is a core strip, this implies $\lambda_{A_0^*} = \lambda^{+-}_{A_0^*} \ge \upsilon^-_{A_0}$, see Proposition 4.8. Hence $T\lambda^+_{A_1^*} \ge \upsilon^-_{A_0}$ so that $\{\lambda^+_{A_1^*} \ge \phi\} \in \mathcal{G}$ by definition of ϕ. (Observe that generically A^θ is an interval and $\lambda^+_{A_0}(R\theta) \le \upsilon^-_{A_0}(R\theta)$, see Proposition 4.8.) Hence $\lambda^+_{A_1^*} \ge \phi^+ = \upsilon_{A'} = \upsilon_{A_1} \ge \upsilon^-_{A_1}$, see also (4.3) and (4.4). This finishes the proof of Lemma 4.19. □

Remark 4.20. Let A and A_0 be core strips such that either

(a) $A \xrightarrow{\text{u.o.}} A_0$ w.r.t. T, $T(\Lambda^C_A) \subseteq \Lambda^C_{A_0}$, and $T(\Upsilon^C_A) \subseteq \Upsilon^C_{A_0}$; or

(b) $A \xrightarrow{\text{d.o.}} A_0$ w.r.t. T, $T(\Lambda^C_A) \subseteq \Upsilon^C_{A_0}$ and $T(\Upsilon^C_A) \subseteq \Lambda^C_{A_0}$.

Then we write $A \twoheadrightarrow A_0$, or $A \twoheadrightarrow A_0$ w.r.t. T. Note that $\twoheadrightarrow$ is a transitive relation: if $A \twoheadrightarrow A_0$ w.r.t. T and $A_0 \twoheadrightarrow A_1$ w.r.t. T_1, then $A \twoheadrightarrow A_1$ w.r.t. $T_1 \circ T$.

We note also that one can replace the inclusions in (a) and (b) above by equalities without changing anything. This follows from Corollary 3.8. (For the first inclusion, for example, apply this corollary to the residual set $G = P_{\Lambda^C_A} \cap R^{-1} P_{\Lambda^C_{A_0}}$.)

Now we state and prove the main result of this section.

Theorem 4.21. *Let R be a homeomorphism of Θ. Suppose that the core strip A is mapped upwards* (*downwards*) *over itself by T. Then there is a core strip $A_\infty \subseteq A$ with $T(A_\infty) = A_\infty$ which is mapped upwards (downwards) over itself. In fact, $A_\infty \twoheadrightarrow A_\infty$ w.r.t. T.*

Proof. In view of the fact that $A \overset{\text{o.}}{\longrightarrow} A$, we can apply Lemma 4.19 to find a core strip $A_1 \subseteq A$ with $A_1 \twoheadrightarrow A$ w.r.t. T. In particular, $A_1 \overset{\text{o.}}{\longrightarrow} A_1$. Applying Lemma 4.19 to A_1 we find a core strip $A_2 \subseteq A_1$ such that $A_2 \twoheadrightarrow A_1$, and inductively we construct a sequence of core strips $A = A_0 \supseteq A_1 \supseteq \dots$ such that $A_i \twoheadrightarrow A_{i-1}$ w.r.t. T $(i = 0, 1, \dots)$. Let $\widetilde{A}_\infty := \bigcap_{i=0}^\infty A_i$, and set $A_\infty := \widetilde{A}^C_\infty$. Then $T(\widetilde{A}_\infty) = \widetilde{A}_\infty$ and hence $T(A_\infty) = A_\infty$ by Lemma 3.4(c). As a countable decreasing intersection of strips, the set $\widetilde{A}_\infty$ is a strip, hence A_∞ is a core strip by Lemma 3.10(a).

Let us show that, if $A \overset{\text{u.o.}}{\longrightarrow} A$, then $T(\Upsilon^C_{A_\infty}) \subseteq \Upsilon^C_{A_\infty}$ and that $T(\Lambda^C_{A_\infty}) \subseteq \Lambda^C_{A_\infty}$. Observe first that, for each $\theta \in \Theta$,

$$(T(\upsilon_{\widetilde{A}_\infty}))(\theta) = \lim_{i\to\infty} (T(\upsilon_{A_i}))(\theta) \geq \limsup_{i\to\infty} \upsilon^-_{A_{i-1}}(R(\theta)) \geq \upsilon^-_{\widetilde{A}_\infty}(R(\theta))$$

i.e. $T(\upsilon_{\widetilde{A}_\infty}) \geq \upsilon^-_{\widetilde{A}_\infty}$ so that also $T(\upsilon^{-+}_{\widetilde{A}_\infty}) \geq \upsilon^-_{\widetilde{A}_\infty}$. Next, Lemma 4.5 implies that $\upsilon^{-+}_{\widetilde{A}_\infty} = \upsilon_{\widetilde{A}^C_\infty} = \upsilon_{A_\infty}$. Also $\upsilon^-_{\widetilde{A}_\infty} \geq \upsilon^-_{\widetilde{A}^C_\infty} = \upsilon^-_{A_\infty}$ (because $\widetilde{A}_\infty \supseteq \widetilde{A}^C_\infty$ which implies that $\upsilon_{\widetilde{A}_\infty} \geq \upsilon_{\widetilde{A}^C_\infty}$). Therefore $T(\upsilon_{A_\infty}) \geq \upsilon^-_{A_\infty}$. Since $T(A_\infty) = A_\infty$, this means that

$$\upsilon^-_{A_\infty} \leq T(\upsilon_{A_\infty}) \leq \upsilon_{A_\infty}. \tag{4.7}$$

But $\{\upsilon^-_{A^\infty} = \upsilon_{A^\infty}\} \in \mathcal{G}$ by Lemma 4.3(c), so also $\{T(\upsilon_{A^\infty}) = \upsilon_{A^\infty}\} \in \mathcal{G}$. Hence $T(\Upsilon^C_{A_\infty}) = (T(\Upsilon_{A_\infty}))^C = \Upsilon^C_{A_\infty}$ by Lemmas 3.4(c) and 3.2(e). In the same way one proves $T\Lambda^C_{A_\infty} = \Lambda^C_{A_\infty}$.

The "downward" case can be reduced to the "upward" one as in the proof of Lemma 4.19. □

Corollary 4.22. *Let R be a minimal homeomorphism of Θ.*

(a) *If the core strip A is mapped upward over itself, then A contains a core strip which is pinched and minimal w.r.t. T; that is, which is T-almost automorphic.*
(b) *If A is mapped downwards over itself and if A does not contain a T-almost automorphic core strip, then it contains a solid T-invariant core*

strip A_∞ for which $\Upsilon^C_{A_\infty}$ and $\Lambda^C_{A_\infty}$ are permuted under the action of T. If R^2 is a minimal homeomorphism of Θ, then both sets are almost automorphic under T^2.

Proof. We need only to note that, if $A \xrightarrow{\text{u.o.}} A$, then $\Upsilon^C_{A_\infty}$ and $\Lambda^C_{A_\infty}$ are T-almost automorphic; they may coincide. $\square$

We do not know of any example for the second case of this corollary. When the map $\theta \mapsto T_\theta$ is only required to be measurable – so also the sections need only to be measurable – it is known that such situations can occur (see [13]).

5. A Sharkovskiĭ type theorem

In this section Θ denotes a compact metric space and $R\colon \Theta \to \Theta$ is a totally minimal homeomorphism of Θ. Also $\Omega = \Theta \times I$, and $T\colon \Omega \to \Omega$ is a continuous map such that $\pi \circ T = R \circ \pi$ where $\pi\colon \Omega \to \Theta$ is the projection.

Let $B \subset \Omega$ be a strip, and let $p > 1$ be an integer. Recall (Definition 3.15) that B is *p-periodic* if $T^p(B) = B$ and if the image sets $B, T(B), \dots, T^{p-1}(B)$ are pairwise disjoint and pairwise ordered. Suppose that q is an integer which is below p in the Sharkovskiĭ ordering. Thus if $p = 3$, then q can be any positive integer. Our goal is to determine a strip C which is q-periodic for T; that is, $T^q(C) = C$ and the images $C, T(C), \dots, T^{q-1}(C)$ are pairwise disjoint and pairwise ordered.

We begin the analysis. By Lemma 3.11, we can assume that B is a minimal strongly T^p-invariant core strip. We order the core strips $B, T(B), \dots, T^{p-1}(B)$ in the natural way:

$$B_0 < B_1 < \ldots < B_{p-1}$$

where $B_j = T^{k_j}(B)$ for a unique integer $k_j \in \{0, \dots, p-1\}$ $(0 \le j \le p-1)$. Let λ_j resp. υ_j be the lower resp. upper bounding section of B_j. Observe that $B_{j-1} < B_j$ implies $B_{j-1} \prec B_j$ so that $\upsilon^-_{j-1} \le \lambda^+_j$, see Lemmas 4.17 and 4.18.

Set $[\upsilon^-_{j-1}, \lambda^+_j] = \{(\theta, x) \in \Omega \mid \upsilon^-_{j-1}(\theta) \le x \le \lambda^+_j(\theta)\}$, then define $I_j = [\upsilon^-_{j-1}, \lambda^+_j]^C$. By Lemma 3.10(a), I_j is a core strip. Using Lemma 4.6 and Remark 4.2, one checks that, for $1 \le j \le p-1$,

$$\upsilon_{I_j} = \upsilon^{-+}_{[\upsilon^-_{j-1}, \lambda^+_j]} = \lambda^{+-+}_j = \lambda^+_j, \tag{5.1}$$

$$\lambda_{I_j} = \lambda^{+-}_{[\upsilon^-_{j-1}, \lambda^+_j]} = \upsilon^{-+-}_{j-1} = \upsilon^-_{j-1}, \tag{5.2}$$

$$\upsilon^-_{I_j} = \lambda^{+-}_j \le \upsilon^{-+}_j = \lambda^+_{I_{j+1}}. \tag{5.3}$$

This implies that the strips I_j are weakly ordered, see Definition 4.16. For later use we note that if $I_i \cap I_j$ contains a strip for some $i \ne j$, then $|i - j| = 1$.

Comparing with Definition 4.14(a), one now sees that, if $0 \le j < p-1$ and if $T(B_{j-1}) = B_r$ and $T(B_j) = B_s$ with $B_s > B_r$, then T maps I_j upwards over each strip $I_{r+1}, \dots, I_s$. Similarly one can show that, if $B_s < B_r$, then T maps I_j downward over each strip $I_{s+1}, \dots, I_r$.

We will apply the results of Section 4 together with the arguments exposed in [4], [7], [14] in proving our version of Sharkovskiĭ's theorem. Set $[B_j, B_{j+1}] = [\upsilon_j^-, \lambda_{j+1}^+] \cup \{(\theta, x) \in \Omega \mid x \in B_j^\theta \cup B_{j+1}^\theta\}$ for $0 \le j \le p-1$. Motivated by a standard construction in the theory of interval maps, we introduce the directed graph (digraph) of B whose vertices are the strips $I_1, \dots, I_{p-1}$ and whose edges $I_j \to I_k$ are stipulated as follows: $I_j \to I_k$ just when $T[B_j, B_{j+1}]$ contains $[B_k, B_{k+1}]$ in the set-theoretic sense.

Let us compare this use of the symbol "$\to$" with that of the symbol "$\xrightarrow{\text{o.}}$" given in Definition 4.14(c). According to the preceding discussion, if $T(B_j) = B_r$ and $T(B_{j+1}) = B_s$, then $I_j \xrightarrow{\text{o.}} I_k$ in the sense of Definition 4.14(c) whenever I_k is "between" I_r and I_s in the obvious sense. However the digraph may contain edges which are defined neither by the upward over nor by the downward over relation. Thus the sense of the symbol "$\to$" in the context of the digraph of B is more inclusive than the sense attached to the symbol "$\xrightarrow{\text{o.}}$" in Definition 4.14(c).

In the developments below we follow [7] though we could just as well read in [4, pp. 22–25]. We introduce some standard terminology, following Coppel [7]. First, we construct the standard p-cycle. Let us view B_{j-1} and B_j as the endstrips of I_j $(1 \le j \le p-1)$. We define vertices $J_0, \dots, J_{p-1}, J_p = J_0$ in the following way. Put $J_0 = I_1$; let J_1 be that vertex I_j contained in the strip $\{T(B_0), T(B_1)\}$, such that (with slight imprecision of language) $T(B_0)$ is an endstrip of I_j, etc. Here and below we use the brackets to indicate the strip defined by the appropriate boundary sections of $T(B_0)$ and $T(B_1)$. We obtain a cycle $J_0 \to J_1 \to \dots \to J_{p-1} \to J_0$ of length p in the digraph of B. This is the standard p-cycle; it is characterized uniquely as that p-cycle $J_0 \to J_1 \to \dots \to J_{p-1} \to J_0$ in the digraph of B having the property that J_0 (now not necessarily assumed to be I_1) admits an endstrip C such that $T^k(C)$ is an endstrip of J_k for $1 \le k < p$. If J_0 is chosen as I_1, then $C = B_0$. Note that each arrow "$\to$" in the standard p-cycle satisfies the condition of Definition 41.4(c); thus we actually have $J_0 \xrightarrow{\text{o.}} J_1 \xrightarrow{\text{o.}} \cdots \xrightarrow{\text{o.}} J_0$.

We say that a cycle in the digraph of B is *primitive* if it does not consist entirely of a cycle of smaller length repeated several times.

Lemma 5.1. *Let q an integer. Suppose that the digraph of B contains a primitive cycle $J_0 \xrightarrow{\text{o.}} J_1 \xrightarrow{\text{o.}} \cdots \xrightarrow{\text{o.}} J_{q-1} \xrightarrow{\text{o.}} J_0$ of length q where all arrows are as in Definition* 4.14(c). *Then there exists a core strip C such that $T^k(C) \subset J_k$ $(0 \le k \le q-1)$ and such that either C is q-periodic, or $C = B_i$ for some i and q is an integer multiple of p. Moreover, $C \twoheadrightarrow C$ w.r.t. T^q (see Remark* 4.20).

PROOF. Suppose first that $q = 1$. Then the primitive cycle $J_0 \xrightarrow{\text{o.}} J_0$ is a loop. We can apply Theorem 4.21 strongly T-invariant core strip C such that $C \twoheadrightarrow C$. Suppose from now on that $q \geq 2$.

Let $J_k = J_{k \bmod q}$ for $k \geq q$. We use Lemma 4.19(a) to define recursively core strips $J_i^\ell \subseteq J_i$ ($\ell = 0, 1, 2, \ldots$):

$$J_i^0 = J_i, \quad J_i^{\ell+1} \subseteq J_i^\ell, J_i^{\ell+1} \twoheadrightarrow J_{i+1}^\ell \text{ w.r.t. } T \quad (\ell \geq 0). \tag{5.4}$$

We claim that, for all $\ell, i, j \geq 0$ either $J_i^\ell = J_j^\ell$ or these two strips are weakly ordered. The proof is by induction on ℓ: for $\ell = 0$ this is obvious, because all J_i^0 are among the intervals $I_1, \ldots, I_{p-1}$. So suppose that the claim holds true for ℓ and consider $J_i^{\ell+1}$ and $J_j^{\ell+1}$. If $J_i^\ell \neq J_j^\ell$, then J_i^ℓ and J_j^ℓ are weakly ordered by the inductive assumption, and as $J_i^{\ell+1} \subseteq J_i^\ell$, $J_j^{\ell+1} \subseteq J_j^\ell$, also $J_i^{\ell+1}$ and $J_j^{\ell+1}$ are weakly ordered. We turn to the case where $J_i^\ell = J_j^\ell$. Suppose first that $J_{i+1}^\ell = J_{j+1}^\ell$. Then both, $J_i^{\ell+1}$ and $J_j^{\ell+1}$, are constructed by Lemma 4.19(a) with the same "ingredients", and so they coincide. It remains to treat the case where $J_i^\ell = J_j^\ell$ but $J_{i+1}^\ell \neq J_{j+1}^\ell$. In this case J_{i+1}^ℓ and J_{j+1}^ℓ are weakly ordered by the inductive assumption, and Lemma 4.19(b) tells us that also $J_i^{\ell+1}$ and $J_j^{\ell+1}$ are weakly ordered.

A first consequence of this construction is that $J_0^{2q} \twoheadrightarrow J_0^q \twoheadrightarrow J_0$ w.r.t. T^q by Remark 4.20 and $J_0^{2q} \subseteq J_0^q \subseteq J_0$. So we can apply Theorem 4.21 to $A = J_0^{2q}$ to find a core strip $C \subset J_0^{2q}$ such that $T^q(C) = C$ and also $C \twoheadrightarrow C$ w.r.t. T^q. The previous construction yields also $T^k(C) \subseteq J_k^{2q-k} \subseteq J_k^\ell \subseteq J_k$ for $0 \leq \ell \leq 2q-k$. Because of Corollary 4.22 we may also assume that either C is T^q-almost automorphic or C is solid and contains no T^q-almost automorphic substrip.

Let $0 \leq i < j < q$. Suppose for a contradiction that $J_i^\ell = J_j^\ell$ for $\ell = 0, \ldots, q$. As

$$J_i^\ell \twoheadrightarrow J_{i+1}^{\ell-1} \twoheadrightarrow J_{i+2}^{\ell-2} \twoheadrightarrow \ldots \twoheadrightarrow J_{i+\ell}^0 = J_{i+\ell},$$

we conclude that $J_{i+\ell} = J_{j+\ell}$ for $\ell = 0, \ldots, q$. But, as we assumed that the J_i form a primitive cycle, this leads to the contradiction $i = j$.

Hence there exists $\ell \in \{0, \ldots, q\}$ such that $J_i^\ell \neq J_j^\ell$. We argued above that this implies that J_i^ℓ and J_j^ℓ are weakly ordered. As $T^i(C) \subseteq J_i^\ell$ and $T^j(C) \subseteq J_j^\ell$, it follows that $T^i(C)$ and $T^j(C)$ are weakly ordered.

Suppose first that for all $0 \leq i < j < q$ the two strips $T^i(C)$ and $T^j(C)$ are disjoint. Since they are weakly ordered, they are then indeed (strictly) ordered in view of Lemma 4.18. Hence the core strip C is q-periodic in this case.

Now suppose that $D := T^i(C) \cap T^j(C) \neq \emptyset$ for some $0 \leq i < j < q$. As intersection of two T^q-invariant strips D is a T^q-invariant strip, see Lemma 3.12. Then $\widetilde{D} := T^\ell(D) \subseteq J_{i+\ell} \cap J_{j+\ell}$ for all $\ell \geq 0$. As the J_i form a primitive cycle, there is some $\ell \geq 0$ such that $J_{i+\ell} \neq J_{j+\ell}$. Let $J_{i+\ell} = I_r$, $J_{j+\ell} = I_s$. Then $I_r \cap I_s \neq \emptyset$ so that $|r - s| = 1$.

Without loss of generality $s = r + 1$. Then $\widetilde{D} \subseteq I_r \cap I_s \subseteq [\upsilon_r^-, \lambda_r^+]$ so that $\emptyset \neq \widetilde{D}^C \subseteq [\upsilon_r^-, \lambda_r^+]^C$. In view of Theorem 4.11 this implies that B_r is T^p-almost automorphic, and it follows from Lemmas 4.4(b) and 4.9 that $\widetilde{D}^C \subseteq [\lambda_r, \upsilon_r]^C = B_r$. Hence $\widetilde{D}^C = B_r$ and the T^q-almost automorphic strip $T^{2q-i-\ell}(B_r)$ is contained in $T^{2q}(C) = C$. This excludes the possibility that C is a solid strip. Hence C is T^q-automorphic, and we can conclude that $T^i(C) = T^j(C) = B_r$. It follows that C coincides with some B_i and q is an integer multiple of p.

This completes the proof of Lemma 5.1. □

Remark 5.2. According to Corollary 4.22, either C is T^q-almost automorphic or it contains a core strip which is T^{2q}-almost automorphic.

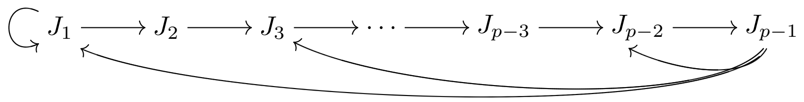
Our goal now is to determine primitive cycles with arrows $\xrightarrow{\text{o.}}$ whose lengths correspond to the numbers q which are below p in the Sharkovskiĭ ordering. We proceed using the arguments of [7]. As a warm-up exercise (strictly speaking not needed in what follows), we show that there is a vertex $\widetilde{I}$ such that $\widetilde{I} \xrightarrow{\text{o.}} \widetilde{I}$. To see this, write again the strips $B, T(B), \ldots, T^{p-1}(B)$ in their natural order:

$$B_0 < B_1 < \ldots < B_{p-1}$$

where, as before, $B_j = T^{k_j}(B)$ for a unique integer $k_j \in \{0, \ldots, p-1\}$ $(0 \leq j \leq p-1)$. Let $u = \max\{i : T(B_i) > B_i\}$. Then u is well-defined by assumption. Set $\widetilde{I} := I_{u+1} = [\upsilon_u^-, \lambda_{u+1}^+]^C$. Since $T(B_u) > B_u$ and $T(B_{u+1}) \leq B_u$, $\widetilde{I}$ is mapped directed over $\widetilde{I}$ by T; i.e. $\widetilde{I} \xrightarrow{\text{o.}} \widetilde{I}$. Indeed, as $\lambda_{\widetilde{I}} = \upsilon_u^-$ by (5.2), we have $T(\lambda_{\widetilde{I}}) = T(\upsilon_u^-) \geq \lambda_{u+1}$, and $\lambda_{u+1} = \lambda_{u+1}^{+-} = \upsilon_{\widetilde{I}}^-$ in view of Lemma 4.6(b) and equation (5.1). Similarly, $T(\upsilon_{\widetilde{I}}) \leq \lambda_{\widetilde{I}}^+$.

We now formulate a version of the key lemma of [7] (see [7, Proposition 3]).

Lemma 5.3. *Suppose that B is a p-periodic strip with p odd, $p > 1$. Suppose that T admits no periodic strip of odd period q strictly between* 1 *and p. Then the vertices of the digraph of B admit a labelling $J_1, \ldots, J_{p-1}$ with respect to which the digraph has the following form:*



All the arrows in the digraph are of the "directed over" type (*Definition* 4.14(c)). *The digraph admits the following paths:*

(a) $J_1 \xrightarrow{\text{o.}} J_2 \xrightarrow{\text{o.}} \cdots \xrightarrow{\text{o.}} J_{p-1} \xrightarrow{\text{o.}} J_1 \xrightarrow{\text{o.}} J_1 \xrightarrow{\text{o.}} \cdots \xrightarrow{\text{o.}} J_1$

(b) $J_{p-1} \xrightarrow{\text{o.}} J_{2i}$ whenever $2i + 1 < p$.

Proof. We follow the arguments of ([7, pp. 8–10]). Consider the standard p-cycle $J_0 \xrightarrow{\text{o.}} J_1 \xrightarrow{\text{o.}} \cdots \xrightarrow{\text{o.}} J_{p-1} \xrightarrow{\text{o.}} J_0$ introduced earlier. It contains some vertex $\widetilde{I}$ at least twice because there are only $p-1$ vertices. On the other hand, any vertex can occur at most two times because a vertex has only two end-strips. If the standard p-cycle contains a vertex twice, then it can be decomposed into two cycles of smaller length, each of which contains $\widetilde{I}$ just once and is hence primitive.

In the case at hand, the standard p-cycle decomposes into two smaller primitive cycles, one of which must have length 1 because there is no periodic strip with period q if $q \in \{3, \ldots, p-1\}$ (use Lemma 5.1). We can thus re-label the standard p-cycle and write it in the form

$$J_1 \xrightarrow{\text{o.}} J_1 \xrightarrow{\text{o.}} J_2 \xrightarrow{\text{o.}} \cdots \xrightarrow{\text{o.}} J_{p-1} \xrightarrow{\text{o.}} J_1$$

where $J_1 = \widetilde{I}$ defines the 1-cycle and $J_i \neq J_1$ if $1 < i < p$. Suppose for contradiction that $J_i = J_k$ for some $1 < i < k < p$. Then by omitting the intermediate vertices one obtains a shorter primitive cycle with arrows $\xrightarrow{\text{o.}}$, and by omitting the loop at J_1 if necessary one obtains a primitive cycle of odd length strictly between 1 and p with arrows $\xrightarrow{\text{o.}}$. This together with Lemma 5.1 leads to a contradiction with the hypothesis of the present lemma. So we conclude that $J_1, \ldots, J_{p-1}$ is a permutation of $I_1, \ldots, I_{p-1}$.

If $k > i+1$ we cannot have $J_i \xrightarrow{\text{o.}} J_k$ because if we did we could construct a primitive cycle of odd length strictly between 1 and p. For the same reason we cannot have $J_i \xrightarrow{\text{o.}} J_k$ if $k = 1$ and $i \neq 1$, $i \neq p-1$.

Now let C be the middle strip among the strips $B_0 < \ldots < B_{p-1}$. We claim that $J_1 = \{C, T(C)\}$ where we use the brackets to indicate the core strip determined by C and $T(C)$. We also claim that $J_k = \{T^{k-2}(C), T^k(C)\}$ for $2 \leq k \leq p-1$. These statements can be proved by basically following word-for-word the arguments given in ([7, pp. 9–10]). For the reader's convenience we give them here.

Write $J_1 = I_h = [\mathcal{A}, \mathcal{B}]$ where $\mathcal{A}, \mathcal{B} \in \{B_0, \ldots, B_{p-1}\}$ and where we commit an obvious abuse of notation. We know that J_1 is $\xrightarrow{\text{o.}}$ -connected only to J_1 and J_2 in the digraph of B. It follows that J_2 is adjacent to J_1 in the natural sense, and T must map one end strip of J_1 to the other end strip of J_1, while it maps the other end strip of J_1 to an end strip of J_2. Thus there are only two possibilities:

$$(*) \qquad B_{h-1} = \mathcal{A}, \quad B_h = T(\mathcal{A}), \quad B_{h-2} = T^2(\mathcal{A})$$

or

$$(**) \qquad B_h = \mathcal{B}, \quad B_{h-1} = T(\mathcal{B}).$$

Consider the first possibility. If $p = 3$ then it is easily seen that $T^2(C) < C < T(C)$ and that $J_1 = [C, T(C)]$, $J_2 = [T^2(C), C]$. If $p > 3$ we argue as follows. If $T^3(\mathcal{A}) < T^2(\mathcal{A})$ then we must have $J_2 \xrightarrow{\text{o.}} J_1$, which does not happen. Hence

$T^3(\mathcal{A}) > T^2(\mathcal{A})$. Since J_2 is not mapped over J_k for $k > 3$, we must have that $J_3 = [T(\mathcal{A}), T^3(\mathcal{A})]$ is adjacent to J_1 on the right. If $T^4(\mathcal{A}) > T^3(\mathcal{A})$ then we must have $J_3 \xrightarrow{\text{o.}} J_1$, which does not happen. Hence $T^4(\mathcal{A}) < T^3(\mathcal{A})$. Since J_3 is not mapped over J_k for $k > 4$ we must have $J_4 = [T^4(\mathcal{A}), T^3(\mathcal{A})]$ is adjacent to J_2 on the left. Continuing in this way we obtain

$$\begin{aligned} J_{p-1} &= [T^{p-1}(\mathcal{A}), T^{p-3}(\mathcal{A})] < \ldots < J_4 = [T^4(\mathcal{A}), T^2(\mathcal{A})] < J_2 \\ &= [T^2(\mathcal{A}), \mathcal{A}] < J_1 = [\mathcal{A}, T(\mathcal{A})] < \ldots < J_{p-2} = [T^{p-4}(\mathcal{A}), T^{p-2}(\mathcal{A})]. \end{aligned}$$

This shows that $\mathcal{A} = C$ and that $J_k = \{T^{k-2}(C), T^k(C)\}$ $(2 \le k \le p-1)$.

We see now that $J_{p-1} \to J_k$ if and only if k is odd. We also see that there are no arcs in the digraph other than those already found. Moreover, all the arrows in the digraph are of the directed over type. This completes the proof of the Lemma if $(*)$ holds.

If $(**)$ holds, one argues analogously, and finds that, if C is the middle strip among $\{B_0, \ldots, B_{p-1}\}$, then

$$T^{p-2}(C) < T^{p-4}(C) < \ldots < T(C) < C < T^2(C) < \ldots < T^{p-3}(C) < T^{p-1}(C).$$

Setting $J_1 = \{C, T(C)\}$, one also finds that $J_k = \{T^{k-2}(C), T^k(C)\}$, and that Lemma 5.3 holds in this case. □

Proposition 5.4. *Let $p > 1$ be an odd integer, and let B be a p-periodic strip for T. Assume that T admits no q-periodic strip if $q \in \{2, \ldots, p-1\}$ is odd. Then T admits a q-periodic strip whenever $q > p$ (in the natural ordering on the positive integers) and whenever $q \in \{2, \ldots, p-1\}$ is even.*

Proof. It is sufficient to recognize that the paths of (a) and (b) in Lemma 5.3 are primitive, and apply Lemma 5.1. □

We continue to follow the arguments of [7].

Lemma 5.5. *Let B be a periodic strip for T of period p. Then for each positive integer h, B is a periodic strip of T^h of period $p/(h,p)$, where (h,p) is the greatest common divisor of h and p. Conversely, if B is a periodic strip of T^h of period m, then B is a periodic strip of T of period mh/d, where d divides h and is relatively prime to m.*

Proof. Consider the first statement. Suppose B has period p for T and that $m = p/(h,p)$. Then $T^{mh}(B) = B$. If $T^{kh}(B) = B$ then p divides kh and so m divides k.

Passing to the second statement, suppose B has period m for T^h. Then B has period p for T where p divides mh. Write $p = mh/d$. Then by the previous statement, $p/(h,p) = pd/h$, and therefore $(h,p) = h/d$. Hence we can write $h = de$ where $(de, me) = e$; that is, d is relatively prime to m. □

Theorem 5.6 (Sharkovskiĭ for strips). *Suppose that T admits a p-periodic strip B and that $p > q$ in the Sharkovskiĭ ordering. Then T admits a q-periodic core strip C such that $C \twoheadrightarrow C$ w.r.t. T^q. In addition, either C is T^q-almost automorphic or it contains a core strip which is T^{2q}-almost automorphic.*

Proof. The last statement follows from the preceding ones and Corollary 4.22.

Note that the existence of a strongly T-invariant core strip C such that $C \twoheadrightarrow C$ w.r.t. T follows from Lemma 5.1 together with the existence of a loop $\widetilde{I} \to \widetilde{I}$ in the digraph of B satisfying $\widetilde{I}$ do $\widetilde{I}$.

Next we show that T admits a 2-periodic strip. If $p = 2$ the standard p-cycle contains a primitive cycle of length 2. By Lemma 5.1 we obtain a 2-periodic core strip C satisfying $C \twoheadrightarrow C$ w.r.t. T^2. Suppose that B is a periodic strip of least period $p > 2$. Then the standard p-cycle decomposes into two primitive cycles, at least one of which has length strictly between 1 and p, and by Lemma 5.1 we obtain a periodic core strip with period less than p (natural ordering). This shows that in fact T admits a 2-periodic core strip C such that $C \twoheadrightarrow C$ w.r.t. T^2.

Next write $p = 2^d \cdot s$ where s is odd. Suppose first that $s = 1$ and that $q = 2^e$ where $0 \le e < d$. We can assume that $e > 1$ by what has already been proved. Consider the map $S = T^{q/2}$. By Lemma 5.5, S admits a periodic strip of period 2^{d-e+1}. Therefore S admits a periodic core strip C of period 2 such that $C \twoheadrightarrow C$ w.r.t. S^2. Using Lemma 5.5 again, we see that this last strip is q-periodic for T, and moreover $C \twoheadrightarrow C$ w.r.t. T^q.

Now suppose that $s > 1$. We write $q = 2^d r$ and consider the following cases: (a) r is even; (b) r is odd and $r > s$. Consider the map $S = T^{2^d}$. Tt admits a periodic strip of period s, and hence also a periodic strip C of period r (Proposition 5.4); one has $C \twoheadrightarrow C$ w.r.t. S. In the case (a) this strip has period $q = 2^d r$ for T (Lemma 5.5), and one checks that $C \twoheadrightarrow C$ w.r.t. T^q. In the case (b) it has period $2^e r$ for T, for some $e \le d$. If $e < d$, replace p by $2^e r$. Since $q = 2^e \cdot 2^{d-e} r$, we can use case (a) to conclude that T admits a periodic strip C such that $C \twoheadrightarrow C$ w.r.t. T^q. This completes the proof of Theorem 5.6. □

Remark 5.7. If R is not minimal but is simply a homeomorphism of Θ, we still have a version of Sharkovskiĭ's theorem, as follows. If T admits a p-periodic strip, and if $p > q$ in the Sharkovskiĭ ordering, then T admits a q-periodic core strip C such that $C \twoheadrightarrow C$. It can no longer be stated that C has the property of almost automorphicity. On the other hand, if R is minimal and totally minimal, then we have shown that C does exhibit this property.

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Roberta Fabbri and Russel Johnson
Dipartimento di Sistemi e Informatica
Università di Firenze
50139 Firenze, ITALY

E-mail address: fabbri@dsi.unifi.it, johnson@dsi.unifi.it

Tobias Jäger and Gerhard Keller
Mathematisches Institut
Universität Erlangen–Nürnberg
D-91054 Erlangen, GERMANY

E-mail address: jager@mi.uni-erlangen.de, keller@mi.uni-erlangen.de