Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 25, 2005, 297–311

FIXED POINT APPROACHES TO THE SOLUTION OF INTEGRAL INCLUSIONS

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Abstract. Solutions to generalizations of the Volterra and Hammerstein integral inclusions are found by using the fixed point theorems of Covitz–Nadler and Bohnenblust–Karlin. Several illustrative examples are presented. Some conditions are given which also allow Lipschitz solutions to be obtained.

1. Introduction

Numerous studies have considered the Hammerstein integral inclusion

$$x(t) \in \int_a^b k(t,s)F(s,x(s))\,ds + g(t) \tag{1.1}$$

or the Volterra integral inclusion

$$x(t) \in \int_a^t k(t,s)F(s,x(s))\,ds + g(t). \tag{1.2}$$

We note that, as pointed out in [21], if $k(t,s)=0$ for $0 \le t < s \le T$, (1.1) becomes (1.2), and hence existence theorems for (1.1) also apply to (1.2). However, stronger theorems for (1.2) might be proven by considering it directly. Existence theorems for continuous solutions of (1.1) were studied in [2], [7], [9], [10], [19], [21] and [22]. Existence for (1.2) was proven in [6], [17], [19], [21] and [22]. The results in [10], [20]–[22] were established by applying a set-valued version

2000 *Mathematics Subject Classification.* 34A60, 47H10.
Key words and phrases. Integral inclusions, fixed points.

of the fixed point theorem due to Mönch found in [18]. Results found in [20] were proven using a nonlinear alternative of the Leray–Schauder type. Existence of solutions for integral inclusions in abstract spaces include [1], [7], [9], [10], [13], [20]–[22]. Solutions which are in L^p rather than continuous were studied in [10] and [22]. The structure of the solution set to integral inclusions was considered in [6] and in [17]. Applications of integral inclusions can be found in [2], [7], [8], [14], [15] and [23].

In this article we study more general forms of (1.1) and (1.2), namely,

$$x(t) \in \int_a^b K(t,s,x(s))\,ds + g(t), \tag{1.3}$$

$$x(t) \in \int_a^t K(t,s,x(s))\,ds + g(t). \tag{1.4}$$

Inclusions (1.3) and (1.4) were studied by Petrusel in [24].

Theorems 3.1, 3.3, and 3.7 cover cases not covered by Theorem 3.1 of [24], and Theorem 3.8 covers examples not covered by Theorem 3.3 in [24].

2. Preliminaries

In this paper, $|\cdot|$ is used to denote either absolute value or the vector norm in $\mathbb{R}^n$, which of the two of these being evident from the context. The notation $\|\cdot\|$ is reserved for the sup norm in an appropriate function space. In a metric space (X,d), we use $B_r(x)$ to denote the open ball $\{y : d(x,y) < r\}$.

We next list several definitions which play an important role in this paper.

Definitions 2.1.

(1) If (X,d) is a metric space and $p(X)$ is the power set of X, then

$$p_{\mathrm{cl,cv}}(X) = \{Y \in p(X) : Y \text{is closed and convex}\}.$$

(2) A metric space (X,d) is *ε-chainable* (where $\varepsilon > 0$) if for each $x,y \in X$ there is a finite set of points $z_0 = x, z_1, \ldots, z_{n-1}, z_n = y$ such that $d(z_{i-1}, z_i) < \varepsilon$ for all $i = 1, \ldots, n$.

(3) A mapping T from a metric space (X,d) into its nonempty, closed subsets is *$(\varepsilon - \lambda)$-uniformly locally contractive* (where $\varepsilon > 0$ and $0 \le \lambda < 1$) if $H(T(x),T(y)) \le \lambda d(x,y)$ for all $x,y \in X$ such that $d(x,y) < \varepsilon$, where H represents the Hausdorff distance.

(4) Let $F\colon X \to p(Y)$ be given, where X is a measurable space and Y is a topological space. F is *measurable* if $F^{-1}(A) \equiv \{x \in X : F(x) \cap A \ne \emptyset\}$ is measurable for each open subset A of Y.

(5) Let $F\colon X \to p(Y)\setminus\{\emptyset\}$, where X and Y are Hausdorff topological spaces. We say that F is *upper semicontinuous* (u.s.c.) at $x \in X$ if for any open set V such that $F(x) \subseteq V$, there exists an open set U such that $x \in U$ and $F(U) \subseteq V$.

We say that F is *upper semicontinuous* if it is upper semicontinous at every $x \in X$.

(6) Let $F\colon X \to p(Y) \setminus \{\emptyset\}$, where X and Y are Hausdorff topological spaces. We say that F is *lower semicontinuous* (l.s.c.) at $x \in X$ if for any $y \in F(x)$ and any open set $V(y)$ containing y, there exists an open set U containing x such that $F(x') \cap V(y) \neq \emptyset$ for all $x' \in U$. We say that F is *lower semicontinuous* if it is lower semicontinous at every $x \in X$.

(7) Let T be a measure space, and let X and Y be metric spaces. A map $\varphi\colon T \times X \to p(Y)$ is said to be *Carathéodory* if

(a) for every $x \in X$, $\varphi(\,\cdot\,, x)$ is measurable, and
(b) for every $t \in T$, $\varphi(t, \,\cdot\,)$ is continuous.

(8) Let X be a metric space and for every $t \in [a,b]$, let $C(t) \subseteq X$ be a given set. A map $F\colon [a,b] \times X \to p(\mathbb{R}^n)$ is said to be *measurable/Lipschitz* on $\{C(t)\}_{t\in[a,b]}$ if for every $t \in [a,b]$, there exists $k(t) \geq 0$ such that

(a) for all $x \in X$, $F(\,\cdot\,, x)$ is measurable, and
(b) for all $t \in [a,b]$, $F(t, \,\cdot\,)$ is $k(t)$-Lipschitz on $C(t)$.

(9) Let X be a Banach space, suppose $\emptyset \neq D \subseteq X$, and let $F\colon D \to p(D)\setminus\{\emptyset\}$. A point $x \in X$ is called a *fixed point* of F if $x \in Fix(F) = \{y \in D : y \in F(y)\}$.

We now state, as lemmas, some important results from the literature which are used in the proofs of the theorems in this paper. In particular, Lemmas 2.2 and 2.7 are, respectively, the Covitz–Nadler and Bohnenblust–Karlin fixed point theorems.

Lemma 2.2 ([11, Corollary 4]). *Let (X,d) be a complete ε-chainable metric space and let $T\colon X \to p(X)$ be nonempty, closed valued and $(\varepsilon - \lambda)$-uniformly locally contractive. Then T has a fixed point.*

We note that $C[a,b]$ is a complete ε-chainable metric space.

Lemma 2.3 ([25, Theorem 2]). *Let S be a complete measurable space, X be a Polish space* (*a complete, separable metric space*) *and Y be a separable Banach space. Suppose that $F(t,x)\colon S \times X \to p_{\mathrm{cl,cv}}(Y) \setminus \{\emptyset\}$ is jointly measurable and lower semicontinuous in x for each fixed $t \in S$. Then, there exists $f\colon S \times X \to Y$ such that $f(t,x) \in F(t,x)$ for every $(t,x) \in S \times X$, f is jointly measurable and f is continuous in x for each fixed $t \in S$.*

Lemma 2.4 ([4, Theorem 8.1.3]). *Let X be a measurable space, Y a complete separable metric space, and F a measurable set-valued map from X to closed nonempty subsets of Y. Then there exists a measurable function $f\colon X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$.* (*We shall refer to such an f as a measurable selection of F.*)

Lemma 2.5 ([4, Theorem 9.5.3]). *Let X be a metric space, let $F: [a,b] \times X \to p_{\mathrm{cl,cv}}(\mathbb{R}^n) \setminus \{\emptyset\}$, let $z: [a,b] \to X$ be a single-valued map, and let $w: [a,b] \to \mathbb{R}^n$ be measurable. For every $t \in [a,b]$, let $C(t) \subseteq X$ be a given set. Suppose that F is Carathéodory and measurable/Lipschitz on $\{C(t)\}_{t\in[a,b]}$, and let $\alpha(t)$, $t \in [a,b]$, denote the corresponding Lipschitz constants. Suppose further that $w(t) \in F(t, z(t))$ almost everywhere in $[a,b]$. Then, there exists a measurable/Lipschitz selection f of F on $\{C(t)\}_{t\in[a,b]}$ such that $f(t,\,\cdot\,)$ is $5n\alpha(t)$-Lipschitz on $C(t)$ and for almost every $t \in [a,b]$, $w(t) = f(t, z(t))$.*

Lemma 2.6 ([16, Proposition 15.6]). *Let X and Y be metric spaces. Let $\phi: X \to p(Y)$ be lower semicontinuous and let $f: X \to Y$, $\lambda: X \to (0,\infty)$ be continuous. Define $\psi: X \to p(Y)$ by $\psi(x) = \overline{B_{\lambda(x)}(f(x))}$. Assume also for all $x \in X$ that $\phi(x) \cap B_{\lambda(x)}(f(x)) \neq \emptyset$. Then, $\phi \cap \psi$ is lower semicontinuous.*

Lemma 2.7 ([12, Corollary 11.3(e)], [5]). *Let X be a real Banach space, $\emptyset \neq D \subseteq X$ be closed bounded convex and $F: D \to p_{\mathrm{cl,cv}}(X) \setminus \{\emptyset\}$ be upper semicontinuous. If $F(D) \subseteq D$ and $\overline{F(D)}$ is compact, then F has a fixed point.*

3. Main results

Our first theorem is as follows.

Theorem 3.1. *Let $K: [a,b] \times [a,b] \times \mathbb{R}^n \to p_{\mathrm{cl,cv}}(\mathbb{R}^n) \setminus \{\emptyset\}$ satisfy the following conditions:*

(a) *For all $t \in [a,b]$, $x \in C[a,b]$, there exists $M: [a,b] \to \mathbb{R}$ such that M is integrable and nonnegative and $K(t,\,\cdot\,, x(\,\cdot\,)) \subseteq M(\,\cdot\,)B_1(0)$ a.e. on $[a,b]$.*
(b) *For all $x \in C[a,b]$, $K(t,s,x(s)): [a,b] \times [a,b] \to p(\mathbb{R}^n)$ is jointly measurable.*
(c) *For all $(s,u) \in [a,b] \times \mathbb{R}^n$, $K(\,\cdot\,, s, u): [a,b] \to p(\mathbb{R}^n)$ is lower semicontinuous.*
(d) *There exists an $\varepsilon > 0$ such that for all $(t,s) \in [a,b] \times [a,b]$, $u, v \in \mathbb{R}^n$ with $|u - v| < \varepsilon$, $H(K(t,s,u), K(t,s,v)) \leq l(t,s)|u - v|$, where l is continuous in t and jointly measurable in (t,s) and*

$$\sup_{t\in[a,b]} \int_a^b l(t,s)\,ds < 1.$$

(e) *$g: [a,b] \to \mathbb{R}^n$ is continuous.*

Then, there exists a continuous solution to the integral inclusion

$$x(t) \in \int_a^b K(t,s,x(s))\,ds + g(t) \quad \text{for } t \in [a,b].$$

Proof. Define $T: C[a,b] \to p(C[a,b])$ by

$$T(x) = \left\{ v \in C[a,b] : v(t) \in \int_a^b K(t,s,x(s))\,ds + g(t),\ t \in [a,b] \right\}.$$

Claim 1. For all $x \in C[a,b]$, there exists $k: [a,b] \times [a,b] \to \mathbb{R}^n$ such that, for all t, $k(t,s)$ is an integrable selection of $K(t,s,x(s))$ and $k(t,s)$ is continuous in t.

Note that by assumptions (a)–(c) we may apply Lemma 2.3 to $K(t,s,x(s))$: $[a,b] \times [a,b] \to p_{\mathrm{cl,cv}}(\mathbb{R}^n) \setminus \{\emptyset\}$ to obtain the desired k.

Claim 2. $T(x) \neq \emptyset$ for each $x \in C[a,b]$.

This follows easily from Claim 1 and assumption (e).

Claim 3. $T(x)$ is closed for each $x \in C[a,b]$.

Assumptions (a) and (b) allow us to use part 3 of Theorem 8.6.4 in [4] to verify this claim.

Claim 4. For each $x_1, x_2 \in C[a,b]$ with $\|x_1 - x_2\| < \varepsilon$, for each $\gamma > 0$, for each k_1 that satisfies Claim 1 for $x = x_1$, there exists a k_2 that satisfies Claim 1 for $x = x_2$ such that for all $t \in [a,b]$, for almost all $s \in [a,b]$, we have $|k_1(t,s) - k_2(t,s)| < l(t,s)\|x_1 - x_2\| + \gamma$.

To prove Claim 4, we proceed as follows. Define $F: [a,b] \times [a,b] \to p(\mathbb{R}^n)$ by $F(t,s) = \overline{B_C(k_1(t,s))} \cap K(t,s,x_2(s))$, where $C = l(t,s)\|x_1 - x_2\| + \gamma$.

Subclaim 4a. F is nonempty, closed, and convex valued.

Fix (t,s). Recall that $k_1(t,s) \in K(t,s,x_1(s))$. We have by assumption (d) that there exists $k_2^* \in K(t,s,x_2(s))$ such that

$$|k_1(t,s) - k_2^*| < l(t,s)|x_1(s) - x_2(s)| + \gamma \le l(t,s)\|x_1 - x_2\| + \gamma.$$

Thus, k_2^* is also in $B_C(k_1(t,s)) \subseteq \overline{B_C(k_1(t,s))}$. Since both $\overline{B_C(k_1(t,s))}$ and $K(t,s,x_2(s))$ are closed and convex valued, we have Subclaim 4a.

Subclaim 4b. F is jointly measurable in (t,s).

Clearly, $\overline{B_C(k_1(t,s))}$ is jointly measurable in (t,s) since k_1 is. Using assumption (b), we may apply Proposition 3.4a, p. 25 of [12], to get Subclaim 4b.

Subclaim 4c. F is lower semicontinuous in t for fixed $s \in [a,b]$.

This is a direct application of Lemma 2.6.

We may now apply Lemma 2.3 to F to obtain Claim 4.

Claim 5. T is $(\varepsilon - \lambda)$-uniformly locally contractive, where

$$\lambda = \sup_{t\in[a,b]} \int_a^b l(t,s)\,ds$$

and ε is from (d).

To prove Claim 5, we proceed as follows. Let $x_1, x_2 \in C[a,b]$ with $\|x_1 - x_2\| < \varepsilon$ and let $\delta > 0$. Let $v_1 \in T(x_1)$. Then there exists a k_1 satisfying Claim 1 for $x = x_1$ such that $v_1(t) = \int_a^b k_1(t,s)\,ds + g(t)$. Apply Claim 4 to obtain k_2, for $\gamma = \delta/(b-a)$. Note that

$$v_2(t) = \int_a^b k_2(t,s)\,ds + g(t) \in T(x_2).$$

Also,

$$\begin{aligned}
\sup_{t\in[a,b]} |v_1(t) - v_2(t)| &= \sup_{t\in[a,b]} \left| \int_a^b [k_1(t,s) - k_2(t,s)]\,ds \right| \\
&\leq \sup_{t\in[a,b]} \int_a^b |k_1(t,s) - k_2(t,s)|\,ds \\
&\leq \sup_{t\in[a,b]} \int_a^b \left[l(t,s)\|x_1 - x_2\| + \frac{\delta}{b-a} \right] ds \quad \text{(by Claim 4)} \\
&= \lambda\|x_1 - x_2\| + \delta.
\end{aligned}$$

Since v_1 was arbitrary, we have shown that

$$\sup_{w_1\in T(x_1)} \inf_{w_2\in T(x_2)} \|w_1 - w_2\| \leq \lambda\|x_1 - x_2\|.$$

Similarly, it can be shown that

$$\sup_{w_2\in T(x_2)} \inf_{w_1\in T(x_1)} \|w_1 - w_2 t\| \leq \lambda\|x_1 - x_2\|,$$

establishing Claim 5.

Using Claims 2, 3 and 5, we may apply Lemma 2.2 to finish the proof of the theorem. $\square$

Example 3.2. Let $K\colon [-1,1] \times [-1,1] \times \mathbb{R} \to p(\mathbb{R})$ be defined by

$$K(t,s,u) = F(t) + \chi_{\mathbb{Q}}(s) + \frac{u}{4},$$

where

$$F(t) = \begin{cases} \{0\} & \text{for } t = 0, \\ [0,1] & \text{for } t \neq 0, \end{cases}$$

$\chi_{\mathbb{Q}}$ represents the characteristic function of the rationals, and let $g(t) = \sin t$. The hypotheses of Theorem 3.1 can be easily verified and hence there exists a continuous solution x to the integral inclusion

$$x(t) \in \int_{-1}^{1} \left[F(t) + \chi_{\mathbb{Q}}(s) + \frac{x(s)}{4}\right] ds + g(t).$$

We note that K does not satisfy all of the hypotheses of Theorem 3.1 in [24].

Theorem 3.3. *Let $K\colon [a,b] \times [a,b] \times \mathbb{R}^n \to p_{\mathrm{cl,cv}}(\mathbb{R}^n) \setminus \{\emptyset\}$ satisfy the following conditions:*

(a) *$K(t,\cdot,u)$ is measurable.*
(b) *There exists $M\colon [a,b] \to \mathbb{R}$ nonnegative and integrable such that for all $(t,u) \in [a,b] \times \mathbb{R}^n$, $K(t,s,u) \subseteq M(s)B_1(0)$ a.e. $s \in [a,b]$.*
(c) *For all $\varepsilon > 0$, $t \in [a,b]$, there exists $\eta(t,\varepsilon)$ such that for all $u,v \in \mathbb{R}^n$ and all $s \in [a,b]$, $|u-v| < \eta(t,\varepsilon) \Rightarrow H(K(t,s,u),K(t,s,v)) < \varepsilon$.*
(d) *there exists $T \in L^1[a,b]$, $T > 0$, such that for all $t,t_0 \in [a,b]$ and $u \in \mathbb{R}^n$,*

$$H(K(t,s,u),K(t_0,s,u)) \leq \frac{T(s)}{5n}|t-t_0|.$$

(e) *g is Lipschitz on $[a,b]$.*

Then, the inclusion $x(t) \in \int_a^b K(t,s,x(s))\,ds + g(t)$ has a Lipschitz solution.

Remark 3.4. Note that assumptions (d) and (e) allow us to obtain existence of a Lipschitz, rather than simply continuous, solution.

Proof. Let

$$K_0 = \int_a^b M(s)\,ds + \|g\| \quad \text{and} \quad \tau = \int_a^b T(s)\,ds + L,$$

where L is the Lipschitz constant of g. Define $D_{\tau K_0}$ by

$$D_{\tau K_0} = \{v \in C[a,b] : \|v\| \leq K_0,\ v \text{ has Lipschitz constant } \tau\}.$$

Note that $D_{\tau K_0}$ is nonempty, closed, convex and bounded in $C[a,b]$. Define $\Phi\colon D_{\tau K_0} \to p(D_{\tau K_0})$ by

$$\Phi(x) = \left\{v \in D_{\tau K_0} : v(t) \in \int_a^b K(t,s,x(s))\,ds + g(t),\ t \in [a,b]\right\}.$$

Claim 1. $\Phi(x) \neq \emptyset$ for all $x \in D_{\tau K_0}$.

Let $G(s,t) = K(t,s,x(s))$. Since K is closed and convex valued and for t fixed, $K(t,s,u)$ is Carathéodory in s and u, Theorem 8.2.8 in [4] implies that $G(\cdot,t) = K(t,\cdot,x(\cdot))$ is measurable. Also, we know from condition (d) that $G(s,\cdot)$ is Lipschitz. Since $G(\cdot,t)$ is measurable and $G(s,\cdot)$ is Lipschitz, it follows that $G(s,t)$ is Carathéodory in s and t, and hence (again using Theorem

8.2.8 in [4]) $G(s,s)$ is measurable. It then follows from Lemma 2.4 that $G(s,s)$ has a measurable selection $w(s)$. From Lemma 2.5 we may conclude that $G(s,t)$ has a measurable/Lipschitz selection $K_1(t,s)$, having Lipschitz constant $T(s)$. Now, let

$$v_1(t) = \int_a^b K_1(t,s)\,ds + g(t).$$

Then

$$\begin{aligned} |v_1(t) - v_1(t_0)| &\leq \int_a^b |K_1(t,s) - K_1(t_0,s)|\,ds + |g(t) - g(t_0)| \\ &\leq \bigg(\int_a^b T(s)\,ds\bigg)|t-t_0| + L|t-t_0| = \tau|t-t_0| \end{aligned}$$

and

$$\|v_1\| \leq \sup_{t\in[a,b]} \int_a^b |K_1(t,s)|\,ds + \|g\| \leq \int_a^b M(s)\,ds + \|g\| = K_0.$$

It follows that $v_1 \in D_{\tau K_0}$, completing the proof of the claim.

Note that $D_{\tau K_0}$ is equicontinuous since it is defined with a uniform Lipschitz constant. Since $\Phi(D_{\tau K_0}) \subseteq D_{\tau K_0}$, it follows that $\Phi(D_{\tau K_0})$ is equicontinuous and bounded. It follows from the Arzela–Ascoli Theorem that $\overline{\Phi(D_{\tau K_0})}$ is compact in $C[a,b]$.

Claim 2. Φ is upper semicontinuous.

To show this, we will show that Φ has a closed graph. It then follows that Φ is upper semicontinuous. The argument to show that Φ has closed graph is as follows.

Suppose $x_n \to x$ and $y_n \to y$ in $C[a,b]$, with $y_n \in \Phi(x_n)$ for all n. We need to show that $y \in \Phi(x)$. Fix $t \in [a,b]$. Let $m \in \mathbb{Z}^+$ be arbitrary. Since $x_n \to x$, there exists some N_m such that if $n > N_m$ then $\|x_n - x\| < \eta(t,1/m)$. Now, $|x_n(s) - x(s)| < \eta(t,1/m)$ implies that $H(K(t,s,x_n(s)), K(t,s,x(s))) < 1/m$, so $K(t,s,x_n(s)) \subseteq B_{1/m}(K(t,s,x(s)))$. Since $y_n \in \Phi(x_n)$,

$$y_n(t) \in \int_a^b K(t,s,x_n(s))\,ds + g(t),$$

and therefore

$$y_n(t) = \int_a^b K_n(t,s)\,ds + g(t),$$

where $K_n(t,s) \in K(t,s,x_n(s))$ for all s. From this, it follows that $K_n(t,s) \in B_{1/m}(K(t,s,x(s)))$ for all s. Now, consider the set valued function

$$G(s) = \overline{B_{1/m}(K_n(t,s))} \cap K(t,s,x(s))$$

for t fixed. As in the proof of Claim 1, $K(t,s,x(s))$ is measurable in s; also $\overline{B_{1/m}(K_n(t,s))}$ is measurable with closed images. Hence, $G(s)$ is measurable with closed images. Thus, by Lemma 2.4 we can choose $K^{(m)}(t,s) \in G(s)$ measurable in s such that $|K_n(t,s) - K^{(m)}(t,s)| < 1/m$ for all s. Therefore,

$$\begin{aligned}
&\left|y(t) - \left(\int_a^b K^{(m)}(t,s)\,ds + g(t)\right)\right| \\
&\qquad \le \|y_n - y\| + \left|y_n(t) - \left(\int_a^b K^{(m)}(t,s)\,ds + g(t)\right)\right| \\
&\qquad = \|y_n - y\| + \left|\int_a^b (K^{(m)}(t,s) - K_n(t,s))\,ds\right| \\
&\qquad \le \|y_n - y\| + \int_a^b |K^{(m)}(t,s) - K_n(t,s)|\,ds \\
&\qquad \le \|y_n - y\| + \frac{b-a}{m}
\end{aligned}$$

for $n > N_m$.

Now, let $\varepsilon > 0$ be arbitrary and choose m such that $(b-a)/m < \varepsilon/2$ and choose $N > N_m$ such that $\|y_n - y\| < \varepsilon/2$ for all $n > N$. Then, if $n > N$ we have

$$\left|y(t) - \left(\int_a^b K^{(m)}(t,s)\,ds + g(t)\right)\right| < \varepsilon$$

for fixed t. Since K is convex valued and integrably bounded, $\int_a^b K(t,s,x(s))\,ds + g(t)$ is closed (by Theorem 8.6.4 of [4]). Since $y(t)$ is arbitrarily close to the set $\int_a^b K(t,s,x(s))\,ds + g(t)$, it must be a limit point and hence (since the set is closed), $y(t) \in \int_a^b K(t,s,x(s))\,ds + g(t)$. Since $t \in [a,b]$ was arbitrary, this shows that $y \in \Phi(x)$. We have shown that Φ has closed graph, and therefore Φ is upper semicontinuous. This completes the proof of Claim 2.

We use Claims 1 and 2 and apply the Bohnenblust–Karlin fixed point theorem (Lemma 2.7) to Φ to yield a solution to the integral inclusion which is Lipschitz.□

Before considering examples, we list several conditions which would imply some of the conditions in Theorem 3.3. First, if K is bounded (there exists K_0 such that for all $u \in \mathbb{R}^n$, $t \in [a,b]$, $|K(t,s,u)| \le K_0$ a.e. $s \in [a,b]$), then K is clearly integrably bounded (condition (b)). Condition (c) is satisfied if, for τ fixed, $\{K_s(u) = K(\tau,s,u),\ s \in [a,b]\}$ is equicontinuous. Also, conditions (c) and (d) are satisfied if

$$H(K(t,s,u), K(t_0,s,v)) \le \frac{T(s)}{5n}|t - t_0| + L|u - v|$$

or if

$$H(K(t,s,u), K(t_0,s,v)) \le \frac{L}{5n}|(t,u) - (t_0,v)|$$

(Lipschitz in t, u with constant independent of s).

Example 3.5. We now exhibit a single-valued function K which satisfies the conditions of Theorem 3.3. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be any uniformly continuous bounded function. Define $K(t,s,u)$ by $K(t,s,u) = stf(u)$. We make the following observations.

- $K\colon [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in s (condition (a)).
- $|K(t,s,u)| = |stf(u)| \le c^2M \in L^1[a,b]$, where $c = \max\{|a|,|b|\}$ and $M = \sup_{u\in\mathbb{R}^n} |f(u)|$ (condition (b)).
- Clearly $|K(t,s,u) - K(t,s,v)| \le c^2|f(u) - f(v)|$. Given any $\varepsilon > 0$, choose $\eta > 0$ such that if $|u - v| < \eta$ then $|f(u) - f(v)| < \varepsilon/c^2$. Then $|K(t,s,u) - K(t,s,v)| < \varepsilon$ whenever $|u - v| < \eta$ (condition (c)).
- $H(K(t,s,u), K(t_0,s,u)) = |K(t,s,u) - K(t_0,s,u)| \le Mc|t - t_0|$ (which gives us condition (d)).

We note that K does not satisfy all of the hypotheses of Theorem 3.1 in [24].

Example 3.6. Example 3.5 may be converted into a set-valued example as follows. Define $K(t,s,u) = \overline{B_1(stf(u))}$, where f is as given above. Then

- K is measurable in s (condition (a)).
- $|K(t,s,u)| \le c^2M + 1 \in L^1[a,b]$, where c and M are as given in Example 3.5 (giving us condition (b)).
- Since the Hausdorff distance between balls of the same radius is the same as the distance between their centers, we can proceed as in Example 3.5 to conclude that
  (a) for any $\varepsilon > 0$, there exists $\eta > 0$ such that
  $$H(K(t,s,u), K(t,s,v)) < \varepsilon$$
  whenever $|u - vt| < \eta$ (condition (c)), and also
  (b) $H(K(t,s,u), K(t_0,s,u)) \le Mc|t - t_0|$ (giving us condition (d)).

We now consider an alternate theorem, in which we assume K is bounded and we replace the Lipschitz condition on K with a continuity condition to obtain a continuous solution.

Theorem 3.7. *Let* $K\colon [a,b] \times [a,b] \times \mathbb{R}^n \to p_{\mathrm{cl,cv}}(\mathbb{R}^n) \setminus \{\emptyset\}$ *satisfy the following conditions:*

(a) $K(t,\cdot\,,u)$ *is measurable.*
(b) K *is bounded.*
(c) *For all* $\varepsilon > 0$, $t \in [a,b]$, *there exists* $\eta(t,\varepsilon)$ *such that if for all* $u,v \in \mathbb{R}^n$ *and all* $s \in [a,b]$, $|u - v| < \eta(t,\varepsilon)$ *then* $H(K(t,s,u), K(t,s,v)) < \varepsilon$.
(d) *For all* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that if* $|t - t_0| < \delta(\varepsilon)$ *then* $H(K(t,s,u), K(t_0,s,u)) < \varepsilon$ *for all* $(s,u) \in [a,b] \times \mathbb{R}^n$.
(e) $g \in C[a,b]$.

Then, the inclusion $x(t) \in \int_a^b K(t,s,x(s))\,ds + g(t)$ *has a continuous solution.*

Proof. Let $x \in C[a,b]$. Define $G(s,t) = K(t,s,x(s))$; note that G is Carathéodory in s and t. From Theorem 8.2.8 in [4] we know that $G(s,s)$ is measurable, and hence (from Lemma 2.4) it follows that $G(s,s)$ has a measurable selection. Let $w(s)$ be such a selection. Let $f(s,t)$ be that element of $G(s,t)$ which is closest to $w(s)$. This element, called the projection, satisfies the conditions that $f(s,t) \in G(s,t)$ and $d(w(s), G(s,t)) = |f(s,t) - w(s)|$.

We now show continuity of f in t by considering two cases.

Let $\varepsilon > 0$ be arbitrary, and let $\delta = \delta(\min\{\varepsilon, \varepsilon/(2K_0)\})$, where $K_0 \geq \max\{\|K\|, (b-a)\|K\|\}$. Let $s^*, t^* \in [a,b]$ be arbitrary.

Case 1. Suppose $w(s^*) = f(s^*, t^*)$. Then $w(s^*) \in G(s^*, t^*)$ and therefore (from (c)) there exists some $y \in G(s^*, t)$ such that $|w(s^*) - y| < \varepsilon$ whenever $|t - t^*| < \delta$. Hence $|w(s^*) - f(s^*, t)| < \varepsilon$, because $f(s^*, t)$ is the projection, and therefore $|f(s^*, t^*) - f(s^*, t)| = |w(s^*) - f(s^*, t)| < \varepsilon$. This completes Case 1.

Case 2. Suppose that $|w(s^*) - f(s^*, t^*)| > 0$. Let $|t - t^*| < \delta(\min\{\varepsilon, \varepsilon/(2K_0)\})$.

Claim 1. If $H(G(s^*, t), G(s^*, t^*)) < \varepsilon$ then

$$|f(s^*, t) - w(s^*)| < |f(s^*, t^*) - w(s^*)| + \varepsilon.$$

If $H(G(s^*, t), G(s^*, t^*)) < \varepsilon$ then there exists $y \in G(s^*, t)$ such that $|y - f(s^*, t^*)| < \varepsilon$. Hence,

$$|y - w(s^*)| \leq |y - f(s^*, t^*)| + |f(s^*, t^*) - w(s^*)| < |f(s^*, t^*) - w(s^*)| + \varepsilon.$$

Therefore,

$$|f(s^*, t) - w(s^*)| = d(w(s^*), G(s^*, t)) \leq |y - w(s^*)| < |f(s^*, t^*) - w(s^*)| + \varepsilon,$$

which proves Claim 1.

Claim 2. Let $H(G(s^*, t), G(s^*, t^*)) < \varepsilon$. If $\|G\| \leq K_0$ then

$$|f(s^*, t) - w(s^*)|^2 < |f(s^*, t^*) - w(s^*)|^2 + 4K_0\varepsilon + \varepsilon^2.$$

Using Claim 1, we can conclude that

$$\begin{aligned} |f(s^*, t) - w(s^*)|^2 &< |f(s^*, t^*) - w(s^*)|^2 + 2\varepsilon|f(s^*, t^*) - w(s^*)| + \varepsilon^2 \\ &\leq |f(s^*, t^*) - w(s^*)|^2 + 4K_0\varepsilon + \varepsilon^2. \end{aligned}$$

This completes the proof of Claim 2.

Then, since (by (d)) $H(G(s^*, t), G(s^*, t^*)) < \varepsilon/(2K_0)$, it follows that there exists $y_t \in G(s^*, t^*)$ such that

$$|f(s^*, t) - y_t| < \frac{\varepsilon}{2K_0} \leq \frac{\varepsilon}{|f(s^*, t^*) - w(s^*)|},$$

which implies that

$$\begin{aligned}(3.1)\qquad &\langle f(s^*,t^*) - w(s^*), f(s^*,t)\rangle \\ &= \langle f(s^*,t^*) - w(s^*), f(s^*,t) - y_t\rangle + \langle f(s^*,t^*) - w(s^*), y_t\rangle \\ &> \langle f(s^*,t^*) - w(s^*), y_t\rangle - \varepsilon.\end{aligned}$$

Corollary 1 of [3, p. 23] implies that, for any $y \in G(s^*,t^*)$,

$$\langle f(s^*,t^*) - w(s^*), f(s^*,t^*) - y\rangle \le 0,$$

from which it follows that

$$(3.2)\qquad \langle f(s^*,t^*) - w(s^*), y\rangle \ge \langle f(s^*,t^*) - w(s^*), f(s^*,t^*)\rangle.$$

Using (3.1) and (3.2) with $y = y_t$, we obtain the inequality

$$(3.3)\qquad \langle f(s^*,t^*) - w(s^*), f(s^*,t) - f(s^*,t^*)\rangle > -\varepsilon.$$

It can easily be shown that, in any Hilbert space over $\mathbb{R}$,

$$\|x\|^2 = \|y\|^2 + \|x-y\|^2 + 2\langle y, x-y\rangle.$$

Applying this inequality to the present situation, we have

$$\begin{aligned}(3.4)\quad |f(s^*,t) - w(s^*)|^2 &= |f(s^*,t^*) - w(s^*)|^2 + |f(s^*,t) - f(s^*,t^*)|^2 \\ &\quad + 2\langle f(s^*,t^*) - w(s^*), f(s^*,t) - f(s^*,t^*)\rangle.\end{aligned}$$

Using (3.3), (3.4) and Claim 2, we obtain

$$\begin{aligned}|f(s^*,t^*)-& f(s^*,t)|^2 - 2\varepsilon \\ &< |f(s^*,t^*) - f(s^*,t)|^2 + 2\langle f(s^*,t^*) - w(s^*), f(s^*,t) - f(s^*,t^*)\rangle \\ &= |f(s^*,t) - w(s^*)|^2 - |f(s^*,t^*) - w(s^*)|^2 < 4K_0\varepsilon + \varepsilon^2.\end{aligned}$$

Therefore,

$$|f(s^*,t^*) - f(s^*,t)|^2 < (4K_0 + 2)\varepsilon + \varepsilon^2.$$

Let $\varepsilon > 0$ and let $\tau(\varepsilon) = -(1+2K_0) + \sqrt{(1+2K_0)^2 + \varepsilon^2}$. Choose $\delta > 0$ such that if $|t - t_0| < \delta$ then $H(G(s,t), G(s,t_0)) < \min\{\varepsilon, \varepsilon/(2K_0), \tau(\varepsilon)\}$. It is straightforward to show that $(4K_0+2)\tau(\varepsilon) + [\tau(\varepsilon)]^2 = \varepsilon^2$. Using this equation, it can be shown that if $|t^* - t| < \delta$ then

$$H(G(s^*,t), G(s^*,t^*)) < \min\{\varepsilon, \varepsilon/(2K_0), \tau(\varepsilon)\} = \widehat{\varepsilon}.$$

This implies

$$|f(s^*,t^*) - f(s^*,t)|^2 < (4K_0+2)\widehat{\varepsilon} + \widehat{\varepsilon}^2 \le (4K_0+2)\tau(\varepsilon) + [\tau(\varepsilon)]^2 = \varepsilon^2$$

and therefore

$$|f(s^*,t^*) - f(s^*,t)| < \varepsilon \quad \text{for all } s^* \in [a,b],$$

which completes Case 2.

Replacing ε in either case by $\varepsilon/(b-a)$ tells us that if δ is chosen so that

$$H(G(s^*,t),G(s^*,t^*)) < \min\left\{\frac{\varepsilon}{b-a}, \frac{\varepsilon}{2K_0(b-a)}, \tau\left(\frac{\varepsilon}{b-a}\right)\right\},$$

then

$$|f(s^*,t^*) - f(s^*,t)| < \frac{\varepsilon}{b-a} \quad \text{for all } s^* \in [a,b]. \tag{3.5}$$

Let $\beta = \min\{\varepsilon/(b-a), \varepsilon/(2K_0), \tau(\varepsilon/(b-a))\}$. Define $D \subseteq C[a,b]$ by

$$\begin{aligned} D = \{v \in C[a,b] : \|v\| \le K_0 + \|g\| \text{ and, for all } \varepsilon > 0,\\ \text{if } |t-t_0| < \delta(\beta) \text{ then } |v(t)-v(t_0)| \le \varepsilon + |g(t)-g(t_0)|\}. \end{aligned}$$

Note that $D \neq \emptyset$ since the zero function is an element of D. Since D is equicontinuous and bounded, $\overline{D}$ is compact. D is closed, convex, and bounded. Define $T\colon D \to p(D)$ by

$$T(x) = \left\{v \in D : v(t) \in \int_a^b K(t,s,x(s))\,ds + g(t),\ t \in [a,b]\right\}.$$

Claim 3. $T(x) \neq \emptyset$ for all $x \in D$.

Define $G(s,t) = K(t,s,x(s))$. Note that G is Carathéodory because $G(\,\cdot\,,t)$ is measurable and $G(s,\,\cdot\,)$ is continuous. Let $w(s)$ be a measurable selection of $G(s,s)$. Let $\varepsilon > 0$. From condition (d) it follows that we can choose δ such that if $|t-t_0| < \delta(\beta)$ then $H(G(s,t),G(s,t_0)) < \beta$. Now, let

$$v(t) = g(t) + \int_a^b f(s,t)\,ds.$$

Then, using (3.5), we have

$$|v(t)-v(t_0)| \le \int_a^b |f(s,t)-f(s,t_0)|\,ds + |g(t)-g(t_0)| < |g(t)-g(t_0)| + \varepsilon.$$

Therefore, $v \in T(x)$. This completes the proof of Claim 3.

To complete the proof of Theorem 3.7 we then proceed as in the proof of Theorem 3.3, beginning with the proof of Claim 2 in the proof of that theorem, using Bohnenblust–Karlin to show that T has a fixed point. That the fixed point is continuous follows immediately since $D \subseteq C[a,b]$. $\square$

All of the theorems thus far have dealt with the inclusion (1.3), but inclusion (1.4) can be examined as well. For example, Theorem 3.3 can be altered to accommodate this case, as shown in the following theorem.

Theorem 3.8. *Let* $K\colon [a,b] \times [a,b] \times \mathbb{R}^n \to p_{\mathrm{cl,cv}}(\mathbb{R}^n) \setminus \{\emptyset\}$ *satisfy the following conditions:*

(a) $K(t, \cdot\,, u)$ *is measurable.*
(b) *There exists* $M : [a,b] \to \mathbb{R}$ *nonnegative and integrable such that for all* $(t,u) \in [a,b] \times \mathbb{R}^n$, $K(t,s,u) \subseteq M(s)B_1(0)$ *a.e.* $s \in [a,b]$.
(c) *For all* $\varepsilon > 0$, $t \in [a,b]$, *there exists* $\eta(t,\varepsilon)$ *such that for all* $u, v \in \mathbb{R}^n$ *and all* $s \in [a,b]$, *if* $|u - v| < \eta(t,\varepsilon)$ *then* $H(K(t,s,u), K(t,s,v)) < \varepsilon$.
(d) *There exists* $T \in L^1[a,b]$, $T > 0$, *such that for all* $t, t_0 \in [a,b]$ *and* $u \in \mathbb{R}^n$,

$$H(K(t,s,u), K(t_0,s,u)) \le \frac{T(s)}{5n}|t - t_0|.$$

(e) $g \in C[a,b]$. *Then, the inclusion* $x(t) \in \int_a^t K(t,s,x(s))\,ds + g(t)$ *has a continuous solution.*

Proof. The proof follows along the lines of that of Theorem 3.3, now with $K_0 = \int_a^b M(s)\,ds + \|g\|$, with $D_{\tau K_0}$ being replaced by

$$D_{TK_0} = \bigg\{ v \in C[a,b] : \|v\| \le K_0 \text{ and for all } t, t_0 \in [a,b],$$
$$|v(t) - v(t_0)| \le \bigg(\int_a^b T(s)\,ds\bigg)|t - t_0| + \bigg|\int_{t_0}^t M(s)\,ds\bigg| + |g(t) - g(t_0)|\bigg\},$$

and with $\Phi\colon D_{\tau K_0} \to p(D_{\tau K_0})$ being replaced by $\Psi\colon D_{TK_0} \to p(D_{TK_0})$, defined by

$$\Psi(x) = \bigg\{ v \in D_{TK_0} : v(t) \in \int_a^t K(t,s,x(s))\,ds + g(t),\ t \in [a,b]\bigg\}. \qquad \square$$

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Manuscript received July 5, 2004

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