**T**opological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 25, 2005, 135–153

# SHARP SOBOLEV INEQUALITY INVOLVING A CRITICAL NONLINEARITY ON A BOUNDARY

Jan Chabrowski — Jianfu Yang

Abstract. We consider the solvability of the Neumann problem for the equation

$$-\Delta u + \lambda u = 0, \qquad \frac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u$$

on $\partial\Omega$, where $Q$ is a positive and continuous coefficient on $\partial\Omega$, $\lambda$ is a parameter and $q = 2(N-1)/(N-2)$ is a critical Sobolev exponent for the trace embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$. We investigate the joint effect of the mean curvature of $\partial\Omega$ and the shape of the graph of $Q$ on the existence of solutions. As a by product we establish a sharp Sobolev inequality for the trace embedding. In Section 6 we establish the existence of solutions when a parameter $\lambda$ interferes with the spectrum of $-\Delta$ with the Neumann boundary conditions. We apply a min-max principle based on the topological linking.

## 1. Introduction

In recent years, a number of sharp Sobolev inequalities have been established by applying the blow-up technique to nonlinear Neumann problems. The main purpose of this work is to prove a sharp Sobolev inequality involving the critical Sobolev exponent on a boundary of a bounded domain.

2000 *Mathematics Subject Classification.* 35B33, 35J20, 35J65.

*Key words and phrases.* Neumann problem, critical Sobolev exponent, topological linking.

The second named author was supported by the National Science Foundation of China, No. 10271118 and The National Key Program for Basic Research of China, No. 2002CCA03700.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with the smooth boundary $\partial\Omega$. We are mainly concerned with the nonlinear Neumann problem

$$
(1.1)\qquad \begin{cases} -\Delta u + \lambda u = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu} u(x) = Q(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $\nu$ is the outer normal on $\partial\Omega$ and the coefficient $Q$ is continuous and positive on $\partial\Omega$. $q = 2(N-1)/(N-2)$, $N \geq 3$, denotes the critical Sobolev exponent for the trace embedding of the space $H^1(\Omega)$ into $L^q(\partial\Omega)$. The embedding of $H^1(\Omega)$ into $L^q(\partial\Omega)$ is continuous, but not compact.

In Section 2 we establish a condition for the solvability of problem (1.1) which involves the best Sobolev constant $S_1$ for the trace embedding of the space $H^1(\mathbb{R}^N_+)$ into $L^q(\mathbb{R}^{N-1})$, where $\mathbb{R}^N_+ = \{x : x \in \mathbb{R}^N,\ x_N > 0\}$. The constant $S_1$ is defined by (see [12])

$$
S_1 = \inf\bigg\{ \int_{\mathbb{R}^N_+} |\nabla u|^2\,dx;\ u \in C^\infty(\mathbb{R}^N_+),\ \int_{\partial\mathbb{R}^N_+} |u(x',0)|^q\,dx' = 1 \bigg\}.
$$

For a point $x$ we use a notation $x = (x', x_N)$, $x' \in \mathbb{R}^{N-1}$. The constant $S_1$ is attained by the function

$$
W(x) = \frac{c_N}{[|x'|^2 + (x_N + (N-2))^2]^{(N-2)/2}},
$$

where $c_N > 0$ is a positive constant depending on $N$. The function $W$ satisfies

$$
\int_{\mathbb{R}^N_+} |\nabla W|^2\,dx = \int_{\mathbb{R}^{N-1}} W(x',0)^q\,dx' = S_1^{N-1}
$$

and moreover $W$ is a positive solution of the Neumann problem in the half-space

$$
(1.2)\qquad \begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u(x',0)}{\partial x_N} = |u(x',0)|^{q-1} & \text{on } \mathbb{R}^{N-1}. \end{cases}
$$

If $Q \equiv 1$ on $\Omega$, it is known that problem (1.1) has a solution for every $\lambda > 0$. This solution is obtained as a minimizer of the variational problem

$$
s_\lambda = \inf_{u \in H^1(\Omega) - \{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx}{(\int_{\partial\Omega} |u|^q\,dS_x)^{2/q}}.
$$

If $u$ is a minimizer for $s_\lambda$, then a multiple of $u$ given by $s_\lambda^{1/(q-2)} u$ is a solution of the problem (1.1). Minimizers for $s_\lambda$ are called least energy solutions of (1.1). It is not difficult to show that if

$$
(1.3)\qquad s_\lambda < S_1 \quad \text{for some } \lambda > 0,
$$

then problem (1.1) has a least energy solution, that is, there exists a minimizer for $s_\lambda$. The condition (1.3) can be verified by testing $s_\lambda$ with the instanton $W$

centered at a point on the boundary of $\Omega$ with a positive mean curvature. We set

$$W_{\varepsilon,y}(x) = \varepsilon^{-(N-2)/2} W\left(\frac{x-y}{\varepsilon}\right),$$

where $y \in \partial\Omega$ and the mean curvature $H(y)$ is positive. In the paper [28] it was noted that

$$\frac{\int_\Omega |\nabla W_{\varepsilon,y}|^2\,dx}{(\int_{\partial\Omega} W_{\varepsilon,y}^q\,dS_x)^{2/q}} = S_1 - \frac{N-2}{2} A_N H(y)\beta(\varepsilon) + o(1)\beta(\varepsilon), \tag{1.4}$$

where $A_N > 0$ is a constant and

$$\beta(t) = \begin{cases} t\log(1/t) & \text{for } N = 3, \\ t & \text{for } N \geq 4. \end{cases}$$

Thus for $\varepsilon > 0$ sufficiently small the right hand side of (1.4) is strictly less than $S_1$ and the condition (1.3) holds. The fact that problem (1.1) has a least energy solution for every $\lambda > 0$ implies that we cannot expect the following inequality

$$S_1\left(\int_{\partial\Omega} |u|^q\,dS_x\right)^{1/q} \leq \int_\Omega (|\nabla u|^2 + C(\Omega)u^2)\,dx \tag{1.5}$$

to hold for all $u \in H^1(\Omega)$ and some constant $C(\Omega) > 0$. In this paper we show that the situation changes if we consider problem (1.1) with a nonconstant weight function $Q$ on $\partial\Omega$. It is not difficult to show that problem (1.1) has a least energy solution for every $\lambda > 0$ if $Q_M = \max_{x\in\partial\Omega} Q(x)$ is attained at a point with positive mean curvature. However, if $Q_M$ is achieved only at points with negative mean curvature (or on a flat part of the boundary, if such part exists), then the least energy solution exists only for $\lambda$ in an interval $(0,\Lambda)$, $0 < \Lambda < \infty$ and there are no least energy solutions for $\lambda > \Lambda$. This obviously gives rise to the sharp Sobolev inequality of type (1.5) with a nonconstant weight function (see Remark 5.5 in Section 5).

The paper is organized as follows. In Section 2 we establish a criterion for the existence of least energy solutions of problem (1.1). Section 3 is devoted to the study of the asymptotic behaviour of least energy solutions of (1.1), when $\lambda \to \infty$. In Section 4 we give the energy estimates of instantons centered either on a flat part of the boundary or at a boundary point with negative curvature. The results of Sections 3 and 4 are used in Section 5 to establish the main theorem (Theorem 5.3) of this paper. In particular, Theorem 5.3 leads to a sharp Sobolev inequality (see Remark 1.5). Finally, in Section 6 we allow the parameter $\lambda$ to interfere with the spectrum of the operator "$-\Delta$" with the Neumann boundary conditions. To obtain the existence of a solution of problem (1.1) we apply the min-max principle argument based on the topological linking.

The Neumann problem involving a critical Soboev exponent in the equation and with zero boundary conditions has an extensive literature and we refer to papers [2]–[7], [13], [14], [17], [18], [20]–[26]. Our approach to problem (1.1) has been motivated by these papers.

Throughout this paper we denote strong convergence by " $\to$ " and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^q(\Omega)$ are denoted by $\|\cdot\|_q$. By $H^1(\Omega)$ we denote a standard Sobolev space on $\Omega$ equipped with norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\, dx.$$

## 2. Existence of least energy solutions

The least energy solutions of problem (1.1) with $Q \not\equiv$ constant are the minimizers of the following problem

$$s_{\lambda,Q} = \inf_{u\in H^1(\Omega)-\{0\}} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx}{(\int_{\partial\Omega} Q(x)|u|^q\, dS_x)^{2/q}}.$$

If $Q \equiv 1$ on $\Omega$ we write $s_{\lambda,1} = s_\lambda$. It follows from the Sobolev trace embedding that $0 < s_{\lambda,Q} < \infty$ for every $\lambda > 0$. It is easy to check that $s_{\lambda,Q}$ is continuous and nondecreasing for $\lambda > 0$. To show the existence of a minimizer for $s_{\lambda,Q}$, we use the P. L. Lions concentration-compactness principle [16]. Let $\{u_m\} \subset H^1(\Omega)$ be such that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and $u_m \rightharpoonup u$ in $L^q(\partial\Omega)$. Then there exist constants $\nu_j > 0$, $\mu_j > 0$, $j \in J$, and $\{x_j\} \subset \partial\Omega$ such that

$$|\nabla u_m|^2 \overset{*}{\rightharpoonup} d\mu \geq |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}, \tag{2.1}$$

$$|u_m|^q \overset{*}{\rightharpoonup} d\nu = |u|^q + \sum_{j\in J} \nu_j \delta_{x_j}, \tag{2.2}$$

in the space of measures and moreover,

$$S_1(\nu_j)^{2/q} \leq \mu_j \quad \text{for } j \in J. \tag{2.3}$$

The set $J$ of indices is at most countable.

Proposition 2.1. *If*

$$s_{\lambda,Q} < \frac{S_1}{Q_M^{(N-2)/(N-1)}} \tag{2.4}$$

*for some $\lambda > 0$, then problem* (1.1) *admits a solution.*

Proof. We follow the argument from the paper [10]. Let $\{u_m\}$ be a minimizing sequence for $s_{\lambda,Q}$ such that

$$\int_{\partial\Omega} Q(x)|u_m|^q\, dS_x = 1$$

for every $m$. Since $\{u_m\}$ is bounded in $H^1(\Omega)$ we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$ and in $L^q(\partial\Omega)$ and, moreover (2.1)–(2.3) hold. Thus

$$1 = \int_{\partial\Omega} Q(x)|u|^q \, dS_x + \sum_{j\in J} Q(x_j)\nu_j$$

and

$$\begin{aligned} s_{\lambda,Q} &= \int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx + \sum_{j\in J} \mu_j \\ &\geq s_{\lambda,Q}\biggl(\int_{\partial\Omega} Q(x)|u|^q \, dS_x\biggr)^{2/q} + \sum_{j\in J} S_1 \frac{(\nu_j Q(x_j))^{2/q}}{Q(x_j)^{2/q}} \\ &\geq s_{\lambda,Q}\biggl(\int_{\partial\Omega} Q(x)|u|^q \, dS_x\biggr)^{2/q} + \sum_{j\in J} S_1 \frac{(\nu_j Q(x_j))^{2/q}}{Q_M^{2/q}}. \end{aligned}$$

Since $s_{\lambda,Q} < S_1/Q_M^{2/q}$, we see that $\nu_j = 0$ for every $j \in J$ and the result follows.□

Proposition 2.1 combined with the asymptotic estimate (1.4) leads to the following result.

THEOREM 2.2. *Suppose that $Q(y) = Q_M$ for some $y \in \partial\Omega$ with $H(y) > 0$ and, moreover*

$$|Q(x) - Q(y)| = o(|x - y|) \tag{2.5}$$

*for $x \in \partial\Omega$ near $y$. Then problem* (1.1) *has a least energy solution for every $\lambda > 0$.*

PROPOSITION 2.3. *We always have $s_{\lambda,Q} \leq S_1/Q_M^{(N-2)/(N-1)}$ for every $\lambda > 0$ and, moreover $\lim_{\lambda\to\infty} s_{\lambda,Q} = S_1/Q_M^{(N-2)/(N-1)}$.*

The second assertion of this Proposition follows from the concentration-compactness principle.

From Proposition 2.3 we derive a weak form of the inequality (1.5).

LEMMA 2.4. *For every $\delta > 0$ small there exists a constant $C(\delta) > 0$ such that*

$$\biggl(\int_{\partial\Omega} Q(x)|u|^q \, dS_x\biggr)^{2/q} \leq \biggl(\frac{S_1}{Q_M^{(N-2)/(N-1)}} - \delta\biggr)^{-1} \int_\Omega |\nabla u|^2 \, dx + C(\delta)\int_\Omega u^2\, dx.$$

## 3. Behaviour of solutions when $\lambda \to \infty$

We commence by showing that for large $\lambda > 0$, least energy solutions of (1.1), up to a translation and dilation, are close to the instanton $W$.

Proposition 3.1. *Suppose that for every $\lambda > 0$ the inequality* (2.4) *is satisfied. Let $\{u_\lambda\}$, $\lambda > 0$, be the corresponding least energy solutions of* (1.1). *Then there exist sequences $\lambda_k \to \infty$, $\varepsilon_k \to 0$ and $\{y_k\} \subset \partial\Omega$, with $y_k \to x_0$ and $Q_M = Q(x_0)$ such that*

$$\lim_{k\to\infty} \int_\Omega \left| \nabla \left[ u_{\lambda_k}(\,\cdot\,) - \varepsilon_k^{-(N-2)/2} W\left( S_1 Q_M^{-1/(N-1)} \frac{\cdot - y_k}{\varepsilon_k} \right)\right]\right|^2 dx = 0. \tag{3.1}$$

Proof. We use some ideas from the papers [5] and [10]. Let

$$s_{\lambda,Q} = \int_\Omega (|\nabla u_\lambda|^2 + \lambda u_\lambda^2)\,dx \tag{3.2}$$

and $\int_{\partial\Omega} Q(x)|u_\lambda|^q\,dS_x = 1$ for every $\lambda > 0$. It is known (see [11]) that $u_\lambda$ are continuous up to the boundary and we set

$$u_\lambda(x_\lambda) = \max_{x\in\bar\Omega} u_\lambda(x), \quad x_\lambda \in \partial\Omega.$$

It follows from (3.2) that $\lim_{\lambda\to\infty} \int_\Omega u_\lambda^2\,dx = 0$. By Lemma 2.4 we have

$$\frac{S_1}{Q_M^{(N-2)/(N-1)}} - \delta \le \lim_{\lambda\to\infty} \int_\Omega |\nabla u_\lambda|^2\,dx \le \frac{S_1}{Q_M^{(N-2)/(N-1)}}.$$

Since $\delta > 0$ is arbitrary we have $\lim_{\lambda\to\infty} \int_\Omega |\nabla u_\lambda|^2\,dx = S_1/Q_M^{(N-2)/(N-1)}$ and necessarily $\lim_{\lambda\to\infty} \lambda \int_\Omega u_\lambda^2\,dx = 0$. We set $M_\lambda = u_\lambda(x_\lambda)$ and $\varepsilon_\lambda = M_\lambda^{(2-N)/2}$. We now rescale solutions $u_\lambda$ by setting

$$v_\lambda(x) = \varepsilon_\lambda^{(N-2)/2} u_\lambda(\varepsilon_\lambda x + x_\lambda) \quad \text{for } \Omega_\lambda = \frac{\Omega - x_\lambda}{\varepsilon_\lambda}.$$

Thus, since $0 \le v_\lambda(x) \le 1$, we have

$$\lambda \int_\Omega u_\lambda^2\,dx = \lambda\varepsilon_\lambda^2 \int_{\Omega_\lambda} v_\lambda^2\,dx \ge \lambda\varepsilon_\lambda^2 \int_{\Omega_\lambda} v_\lambda^{2^*}\,dx \ge C_1\lambda\varepsilon_\lambda^2 \tag{3.3}$$

for some $C_1 > 0$ as $\int_{\Omega_\lambda} v_\lambda^{2^*}\,dx$ is bounded away from 0. Indeed, if $\int_{\Omega_\lambda} v_\lambda^{2^*}\,dx \to 0$, then also $\int_\Omega u_\lambda^{2^*}\,dx \to 0$. It then follows from [1] that for every $\delta > 0$ there exists a constant $C(\delta) > 0$ such that

$$\left( \int_{\partial\Omega} |u_\lambda|^q\,dS_x \right)^{2/q} \le \delta \int_\Omega |\nabla u_\lambda|^2\,dx + C(\delta)\left( \int_\Omega |u_\lambda|^{2^*}\,dx \right)^{2/2^*}.$$

Letting $\lambda \to \infty$, since $\delta > 0$ is arbitrary, we get that $\lim_{\lambda\to\infty} \int_{\partial\Omega} |u_\lambda|^q\,dS_x = 0$, which is impossible. Therefore $\lim_{\lambda\to\infty} \varepsilon_\lambda = 0$. The rescaled solution $v_\lambda$ satisfies

$$\begin{cases} -\Delta v_\lambda + \varepsilon_\lambda^2 \lambda v_\lambda = 0 & \text{in } \Omega_\lambda, \\ \dfrac{\partial v_\lambda}{\partial \nu} = s_{\lambda,Q} Q(\varepsilon_\lambda x + x_\lambda) v_\lambda^{q-1} & \text{on } \partial\Omega_\lambda, \\ 0 \le v_\lambda(x) \le 1 & \text{on } \Omega_\lambda \text{ and } v_\lambda(0) = 1. \end{cases}$$

By the Schauder estimates, there exists a sequence $\lambda_k \to \infty$ such that $v_{\lambda_k} \to w$ in $C^2_{\rm loc}(\mathbb{R}^N_+)$. We may also assume that $x_{\lambda_k} \to x_0 \in \partial\Omega$. The limit function $w$ is a solution of the problem

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega_\infty, \\ \dfrac{\partial w}{\partial \nu} = \widetilde{S} Q(x_0) w^{q-1} & \text{for } 0 \le w \le 1,\ w(0) = 1, \end{cases}$$

where $\widetilde{S} = S_1/Q_M^{(N-2)/(N-1)}$. Since $\Omega_\infty$ is a half-space, we may assume that $\Omega_\infty = \mathbb{R}^N_+$. By the uniqueness result from [15] we know that $w(x) = W(\widetilde{S}Q(x_0)x)$. We now observe that by the Fatou lemma we have

$$\begin{aligned} \frac{S_1^{N-1}(\widetilde{S}Q(x_0))^2}{(\widetilde{S}Q(x_0))^N} &= \int_{\mathbb{R}^N_+} |\nabla w(x)|^2\,dx \le \lim_{\lambda_k\to\infty} \int_{\Omega_{\lambda_k}} |\nabla v_{\lambda_k}|^2\,dx \\ &= \lim_{\lambda_k\to\infty} \int_\Omega |\nabla u_{\lambda_k}|^2\,dx = \frac{S_1}{Q_M^{(N-2)/(N-1)}}. \end{aligned}$$

From this we deduce that $Q(x_0) = Q_M$ and the result follows. □

## 4. Estimates of the energy of $W_{\varepsilon,y}$

We let

$$J_\lambda(u) = \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx}{(\int_{\partial\Omega} |u|^q\,dS_x)^{2/q}}$$

for $u \in H^1(\Omega)$. First we consider the case where the boundary $\partial\Omega$ has a flat part. We let $D(0,\delta) = B(0,\delta) \cap (x_N = 0)$, where $B(0,\delta)$ is the open ball in $\mathbb{R}^N$ centered at 0 and of the radius $\delta$.

LEMMA 4.1. *Suppose that $D(0,\delta) \subset \partial\Omega$ for some $\delta > 0$ and let $y \in D(0,\delta)$. Then there exist constants $C_1 > 0$ and $\varepsilon_0 > 0$ such that*

$$J_\lambda(W_{\varepsilon,y}) \ge S_1 + \lambda C_1 \varepsilon^2 \tag{4.1}$$

*for $\lambda > 0$ and $0 < \varepsilon \le \varepsilon_0$.*

PROOF. For simplicity we assume that $y = 0$ and set $W_{\varepsilon,0} = W_\varepsilon$. We have

$$\begin{aligned} \int_\Omega |\nabla W_\varepsilon|^2\,dx &= \int_{\Omega\cap B(0,\delta)} |\nabla W_\varepsilon|^2\,dx + \int_{\Omega - B(0,\delta)} |\nabla W_\varepsilon|^2\,dx \\ &= \int_{\mathbb{R}^N_+} |\nabla W_\varepsilon|^2\,dx - \int_{\mathbb{R}^N_+ - (\Omega\cap B(0,\delta))} |\nabla W_\varepsilon|^2\,dx + O(\varepsilon^{N-2}) \\ &= K_1 + O(\varepsilon^{N-2}), \end{aligned}$$

where $K_1 = \int_{\mathbb{R}^N_+} |\nabla W(x)|^2\,dx$. We now estimate the surface integral $\int_{\partial\Omega} W_\varepsilon^q\,dS_x$. We have

$$\begin{aligned}\int_{\partial\Omega} W_\varepsilon^q\,dS_x &= \int_{D(0,\delta)} W_\varepsilon^q\,dS_x + \int_{\partial\Omega - D(0,\delta)} W_\varepsilon^q\,dS_x \\ &= \int_{\mathbb{R}^{N-1}} W_\varepsilon(x',0)^q\,dx' - \int_{|x'|>\delta} W_\varepsilon(x',0)^q\,dx' + O(\varepsilon^{N-1}) \\ &= K_2 + O(\varepsilon^{N-1}),\end{aligned}$$

where $K_2 = \int_{\mathbb{R}^{N-1}} W(x',0)^q\,dx'$. Since $S_1 = K_1/(K_2)^{(N-2)/(N-1)}$, the result follows. □

We now establish an analogue of (4.1) in the case where $y \in \partial\Omega$ has a negative curvature.

Lemma 4.2. *If $H(y) < 0$ for some $y \in \partial\Omega$, then there exist constants $\alpha > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that, for $0 < \varepsilon \le \varepsilon_0$,*

$$J_\lambda(W_{\varepsilon,y}) \ge S_1 - \alpha H(y)\varepsilon + \lambda C\varepsilon^2 + O(\varepsilon^2).$$

Proof. We follow some ideas from the paper [20]. Without loss of generality we may assume that $y = 0$ and that near 0 the boundary is represented, changing the coordinates if needed, by

$$x_N = h(x') = \frac{1}{2}\sum_{i=1}^{N-1} \alpha_i x_i^2 + O(|x'|^3)$$

for $x' \in D(0,a)$ for some $a > 0$, where $D(0,a) = B(0,a) \cap \partial\Omega$ and $a_i$, $i = 1,\ldots,N-1$, are principal curvatures of $\partial\Omega$ at 0. Then the mean curvature at 0 is given by $H(0) = (1/(N-1))\sum_{i=1}^{N-1} \alpha_i$. Let $g(x') = (1/2)\sum_{i=1}^{N-1} \alpha_i x_i^2$. Then

$$\begin{aligned}\int_\Omega |\nabla W_\varepsilon|^2\,dx = \int_{\mathbb{R}^N_+} |\nabla W_\varepsilon|^2\,dx &- \int_{D(0,a)\cap g(x')>0} dx' \int_0^{g(x')} |\nabla W_\varepsilon|^2\,dx_N \\ &+ \int_{D(0,a)\cap g(x')<0} dx' \int_{g(x')}^0 |\nabla W_\varepsilon|^2\,dx_N \\ &+ \int_{D(0,a)} dx' \int_{g(x')}^{h(x')} |\nabla W_\varepsilon|^2\,dx_N + O(\varepsilon^{N-2}).\end{aligned}$$

We now estimate the last integral on the right side of this relation. We can assume that $O(|y'|^3)$ is nonnegative and we obtain

$$\begin{aligned}&\int_{D(0,a)} dx' \int_{g(x')}^{h(x')} |\nabla W_\varepsilon|^2\,dx_N \\ &\qquad \le C(N) \int_{D(0,a/\varepsilon)} dy' \int_{\varepsilon g(y')}^{\varepsilon g(y') + \varepsilon^2 O(|y'|^3)} \frac{dy_N}{(|y'|^2 + (y_N + (N-2))^2)^{N-1}}\end{aligned}$$

$$
\begin{aligned}
&\le C(N)\int_{\mathbb{R}^{N-1}} dy' \int_{\varepsilon g(y')}^{\varepsilon g(y')+\varepsilon^2 O(|y'|^3)} \frac{dy_N}{(|y'|^2+(y_N+(N-2))^2)^{N-1}}\\
&= C(N)\int_{|y'|\le\rho} dy' \int_{\varepsilon g(y')}^{\varepsilon g(y')+\varepsilon^2 0(|y'|^3)} \frac{dy_N}{(|y'|^2+(y_N+(N-2))^2)^{N-1}}\\
&\quad + C(N)\int_{|y'|\ge\rho} dy' \int_{\varepsilon g(y')}^{\varepsilon g(y')+\varepsilon^2 O(|y'|^3)} \frac{dy_N}{(|y'|^2+(y_N+(N-2))^2)^{N-1}}\\
&= J_1 + J_2.
\end{aligned}
$$

To estimate $J_1$ we choose $\rho > 0$ so that

$$
-\frac{N-2}{2} \le \varepsilon g(y') + \varepsilon^2 O(|y'|^3), \quad \varepsilon g(y') \le \frac{N-2}{2}
$$

for every $0 < \varepsilon \le 1$ and $|y| \le \rho$. Thus

$$
J_1 \le C\varepsilon^2 \tag{4.2}
$$

for $0 < \varepsilon \le 1$. Let $\rho > 0$ be chosen so that (4.2) holds. Then

$$
|J_2| \le c_N \int_{|y'|\ge\rho} dy' \int_{\varepsilon g(y')}^{\varepsilon g(y')+\varepsilon^2 O(|y'|^3)} \frac{dy_N}{|y'|^{2(N-1)}} = C\varepsilon^2. \tag{4.3}
$$

We set

$$
I^-(\varepsilon) = \int_{D(0,a)\cap g(x')<0} dx' \int_{g(x')}^{0} |\nabla W_\varepsilon|^2 \, dx_N
$$

and

$$
I^+(\varepsilon) = \int_{D(0,a)\cap g(x')>0} dx' \int_{0}^{g(x')} |\nabla W_\varepsilon|^2 \, dx_N .
$$

We now observe that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}(I^-(\varepsilon) - I^+(\varepsilon))\\
&\quad = -\int_{\mathbb{R}^{N-1}\cap g(x')<0} g(x')|\nabla W(x',0)|^2\,dx' - \int_{\mathbb{R}^{N-1}\cap g(x')>0} g(x')|\nabla W(x',0)|^2\,dx'\\
&\quad = -\int_{\mathbb{R}^{N-1}} g(x')|\nabla W(x',0)|^2\,dx' = -\alpha_N H(0)
\end{aligned}
$$

for some constant $\alpha_N > 0$. Therefore we can write

$$
\int_\Omega |\nabla W_\varepsilon|^2\,dx \ge K_1 - C_1 H(0)\varepsilon + O(\varepsilon^2) \tag{4.4}
$$

for $0 < \varepsilon \leq \varepsilon^*$. We now estimate the surface integral

$$
\begin{aligned}
(4.5)\quad \int_{\partial\Omega} W_\varepsilon^q \, dS_x &= \int_{\partial\Omega\cap B(0,a)} W_\varepsilon^q \, dS_x + O(\varepsilon^{N-1}) \\
&= \int_{D(0,a)} W_\varepsilon(x', h(x'))^q \sqrt{1+|\nabla h(x')|^2}\, dx' + O(\varepsilon^{N-1}) \\
&= \int_{\mathbb{R}^{N-1}} W_\varepsilon(x',0)^q \, dx' - \int_{D(0,a)} W_\varepsilon(x',0)^q \, dx' \\
&\quad + \int_{D(0,a)} W_\varepsilon(x', h(x'))^q \sqrt{1+|\nabla h(x')|^2}\, dx' + O(\varepsilon^{N-1}) \\
&\leq K_2 - \int_{D(0,a)} W_\varepsilon(x',0)^q \, dx' \\
&\quad + \int_{D(0,a)} W(x', h(x'))^q (1+|\nabla h(x')|^2)\, dx' + O(\varepsilon^{N-1}) \\
&\leq K_2 + \int_{D(0,a)} W(x', h(x'))^q |\nabla h(x')|^2 \, dx' = K_2 + O(\varepsilon^2).
\end{aligned}
$$

Combining (4.4) and (4.5) the result follows. □

## 5. Existence results and sharp Sobolev inequalities

By rescaling we may assume that $Q_M = 1$. We define the following set

$$\mathcal{M} = \{CW_{\varepsilon,y} : C \in \mathbb{R},\ y \in \partial\Omega,\ \varepsilon > 0\}$$

and set for a function $\phi \in H^1(\Omega)$

$$d(\phi, \mathcal{M}) = \inf\{\|\nabla\phi - \nabla\psi\|_2^2 : \psi \in \mathcal{M}\}.$$

Lemma 5.1. *Let $\delta > 0$ and $\{z_m\} \subset H^1(\Omega)$ be such that $z_m \rightharpoonup 0$ in $H^1(\Omega)$ and $d(z_m, \mathcal{M})^2 \leq \|\nabla z_m\|^2 - 2\delta$. Then there exists $m_0 \geq 1$ such that for $m \geq m_0$, $d(z_m, \mathcal{M})$ is achieved by some function $C_m W_{\varepsilon_m, y_m} \in \mathcal{M}$. Moreover, if $w_m$ is defined by*

$$z_m = C_m W_{\varepsilon_m, y_m} + w_m$$

*then up to a subsequence*

(a) $\lim_{m\to\infty} \varepsilon_m = 0$,
(b) *if* $\lim_{m\to\infty} d(z_m, \mathcal{M}) = 0$, *then* $\lim_{m\to\infty} C_m = C_0 \neq 0$,
(c) *we also have*

$$\int_{\partial\Omega} w_m W_{\varepsilon, y_m}^{q-1} \, dS_x = \beta(\varepsilon_m)\|w_m\|.$$

For the proof we refer to the paper [5] (see also [28]). Also, using the Sobolev embedding theorem one can verify that for $N \geq 7$ we have (see a similar formula (2.32) in [5])

$$\int_\Omega W_{\varepsilon,y_m} w_m \, dx = O(\varepsilon^2 \|w_m\|). \tag{5.1}$$

Let $u_m = u_{\lambda_m}$ be a sequence of solutions from Proposition 3.1. Since we assume that $Q_M = 1$, after rescaling $v_m = S^{1/(q-2)}_{\lambda_m,Q} u_m$, we can rewrite the assertion of Proposition 3.1 in the form

$$\int_\Omega |\nabla(v_m - W_{\varepsilon_m,y_m})|^2 \, dx \to 0$$

as $m \to \infty$. It now follows from Lemma 5.1 that there exist sequences $\{\delta_m\} \subset (0,\infty)$ and $\{y_m\} \subset \partial\Omega$, with $\delta \to 0$, such that

$$v_m = C_m W_{\delta_m,y_m} + w_m. \tag{5.2}$$

As in Lemma 2.2 in [28] we check that $C_m \to 1$ and $\varepsilon_m/\delta_m \to 1$. Therefore we may assume that (5.2) holds with $\delta_m = \varepsilon_m$ and $y_m = x_m$. Lemma 5.2 below can be proved in the same way as Lemma 7.3 in [5] (see also Lemma 2.3 in [28]).

LEMMA 5.2. *There exists a constant $\alpha > 0$ such that*

$$\int_\Omega (|\nabla w_m|^2 + \lambda_m w_m^2) \, dx \geq (q-1+\alpha) \int_{\partial\Omega} Q(x) W^{q-2}_{\varepsilon_m,y_m} w_m^2 \, dx + O(\beta(\varepsilon_m)^2 \|w_m\|^2).$$

We are now in a position to establish our main result. We set

$$J_{\lambda,Q}(u) = \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx}{(\int_{\partial\Omega} |u|^q \, dS_x)^{2/q}}$$

for $u \in H^1(\Omega) - \{0\}$.

THEOREM 5.3. *Let $N \geq 7$.*

(a) *Suppose that $D(0,a) \subset \partial\Omega$ for some $a > 0$ and that $\{x;\, Q(x) = Q_M\} \subset D(0,a)$ and*

$$|Q(x) - Q(y)| = o(|x-y|^2) \tag{5.3}$$

*for some $y \in \partial\Omega$ with $Q(y) = Q_M$ and $x$ near $y$. Then there exists a $\Lambda_1 > 0$ such that problem* (1.1) *admits a least energy solution for every $\lambda \in (0, \Lambda_1)$ and no least energy solution for $\lambda > \Lambda_1$.*

(b) *Suppose that $H(y) < 0$ for some $y \in \partial\Omega$ and that $\{x : Q(x) = Q_M\} \subset \{y : H(y) < 0\}$. Moreover, we assume that*

$$|Q(x) - Q(y)| = o(|x-y|) \tag{5.4}$$

*for some* $y \in \{x : Q(x) = Q_M\}$ *and* $x$ *near* $y$. *Then there existsa* $\Lambda_2 > 0$ *such that problem* (1.1) *admits a least energy solution for every* $\lambda \in (0, \Lambda_2)$ *and no least energy solution for* $\lambda > \Lambda_2$.

Proof. (a) Arguing by contradiction, assume that problem (1.1) has a least energy solution $u_\lambda$ for every $\lambda > 0$. Then for a sequence $\lambda_m \to \infty$, we have decomposition (5.2). Then

$$\begin{aligned} J_{\lambda_m,Q}(v_m) = {} & \frac{1}{(\int_{\partial\Omega} Q|v_m|^q \, dS_x)^{2/q}} \\ & \cdot \bigg\{ C_m^2 \bigg( \int_\Omega |\nabla W_{\varepsilon_m,y_m}|^2 \, dx + \lambda_m \int_\Omega W^2_{\varepsilon_m,y_m} \, dx \bigg) \\ & + \|\nabla w_m\|_2^2 + \lambda_m \|w_m\|_2^2 + 2\lambda_m C_m \int_\Omega W_{\varepsilon_m,y_m} w_m \, dx \bigg\} \end{aligned}$$

and using (c) of Lemma 5.1 we obtain

$$\begin{aligned} \bigg( \int_{\partial\Omega} Q|v_m|^q \, dS_x \bigg)^{-2/q} & = C_m^2 \bigg( \int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x \bigg)^{-2/q} \\ & \cdot \bigg[ 1 + \frac{q(q-1) \int_{\partial\Omega} Q W^{q-2}_{\varepsilon_m,y_m} w_m^2 \, dS_x}{2C_m^2 \int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x} + O(\beta(\varepsilon_m)\|w_m\|) + \|w_m\|^r \bigg]^{-2/q} \\ & = C_m^{-2} \bigg( \int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x \bigg)^{-2/q} \\ & \cdot \bigg\{ 1 - \frac{(q-1)}{C_m^2} \frac{\int_{\partial\Omega} Q W^{q-2}_{\varepsilon_m,y_m} w_m^2 \, dS_x}{\int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x} + O(\beta(\varepsilon_m)\|w_m\|) + \|w_m\|^r \bigg\} \end{aligned}$$

for some $2 < r < q$. Combining the last two relations we get

$$\begin{aligned} & J_{\lambda_m,Q}(v_m) \\ & = \bigg\{ J_{\lambda_m,Q}(W_{\varepsilon_m,y_m}) + \frac{\|\nabla w_m\|_2^2 + \lambda_m \|w_m\|_2^2 + 2C_m \lambda_m \int_\Omega W_{\varepsilon_m,y_m} w_m \, dx}{C_m^2 (\int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x)^{2/q}} \bigg\} \\ & \quad \times \bigg\{ 1 - \frac{(q-1) \int_{\partial\Omega} Q W^{q-2}_{\varepsilon_m,y_m} w_m^2 \, dS_x}{C_m^2 \int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x} + O(\beta(\varepsilon_m)\|w_m\| + \|w_m\|^r) \bigg\}. \end{aligned}$$

Using (5.1) we derive from this

$$\begin{aligned} J_{\lambda_m,Q}(v_m) = {} & J_{\lambda_m,Q}(W_{\varepsilon_m,y_m}) \\ & - \frac{(q-1)}{C_m^2} \frac{\int_{\partial\Omega} Q W^{q-2}_{\varepsilon_m,y_m} w_m^2 \, dS_x}{\int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x} J_{\lambda_m,Q}(W_{\varepsilon_m,y_m}) \\ & + \frac{\|\nabla w_m\|_2^2 + \lambda_m \|w_m\|_2^2 + O(\lambda_m \varepsilon_m^2 \|w_m\|)}{C_m^2 (\int_{\partial\Omega} Q W^q_{\varepsilon_m,y_m} \, dS_x)^{2/q}} \\ & + O(\|w_m\|^2 + \beta(\varepsilon_m)\|w_m\| + \|w_m\|^r) \\ & \times (\|\nabla w_m\|_2^2 + \lambda_m \|w_m\|_2^2 + O(\lambda_m \varepsilon_m \|w_m\|)) \\ & + O(\beta(\varepsilon_m)\|w_m\| + \|w_m\|^r). \end{aligned}$$

According to Lemma 5.2 we can find $0 < \rho < 1$ and $\delta > 0$ such that

$$(1-\rho)\int_\Omega (|\nabla w_m|^2 + \lambda_m w_m^2)\,dx \geq (q-1+\delta)\int_{\partial\Omega} QW^{q-2}_{\varepsilon_m,y_m} w_m^2\,dS_x + O(\varepsilon_m^2\|w_m\|^2).$$

Thus,

$$\begin{aligned}\frac{(1-\rho)\int_\Omega(|\nabla w_m|^2+\lambda_m w_m^2)\,dx}{C_m^2(\int_{\partial\Omega} QW^q_{\varepsilon_m,y_m}\,dS_x)^{2/q}} &- \frac{q-1}{C_m^2} J_{\lambda_m,Q}(W_{\varepsilon_m,y_m})\frac{\int_{\partial\Omega} QW^{q-2}_{\varepsilon_m,y_m} w_m^2\,dS_x}{\int_{\partial\Omega} QW^q_{\varepsilon_m,y_m}\,dS_x} \\ \geq \int_{\partial\Omega} QW^{q-2}_{\varepsilon_m,y_m} w_m^2\,dS_x &\left[\frac{q-1+\delta}{C_m^2(\int_{\partial\Omega} QW^q_{\varepsilon_m,y_m}\,dS_x)^{2/q}} - \frac{(q-1)J_{\lambda_m,Q}(W_{\varepsilon_m,y_m})}{C_m^2\int_{\partial\Omega} QW^q_{\varepsilon_m,y_m}\,dS_x}\right] \\ &+ O(\varepsilon_m^2\|w_m\|^2) = D_m + O(\varepsilon_m^2\|w_m\|^2),\end{aligned}$$

where $D_m \geq 0$ for large $m$ (see also [28, p. 41–42]). Assuming that (5.3) holds and using Lemma 4.1 we see that

$$J_{\lambda_m,Q}(v_m) \geq S_1 + \lambda_m C_1\varepsilon_m^2 + D_m + \frac{\rho\int_\Omega(|\nabla w_m|^2+\lambda_m w_m^2)\,dx}{C_m^2(\int_{\partial\Omega} QW^q_{\varepsilon_m,y_m}\,dS_x)^{2/q}} + O(\varepsilon_m\|w_m\|).$$

Applying the Hölder inequality and taking $m$ sufficiently large we derive from this that

$$J_{\lambda_m,Q}(v_m) \geq S_1$$

which is impossible. The proof of part (b) is the same. □

Remark 5.4. Theorem 5.3 remains true for $N = 5$ and 6. In this case one can use the following modification of Lemma 5.10 in [23]. For every $q \in (N/(N-2),2)\cap(2N/(N+2),2)$ there exist constants $C(q) > 0$ and $a = a(q) \in [0,1)$ with

$$a(q) = \frac{Nq-2N+2q}{2q},$$

such that for every $\gamma > 1$

$$\left|\int_\Omega W_{\varepsilon,y} w\,dx\right| \leq \left(1-\frac{a}{2}\right)C(q)\gamma^{2/(2-a)}\varepsilon^2\|w\|_{2^*}^{2(1-a)/(2-a)} + \frac{a}{2}\frac{1}{\gamma^{2/a}}\|w\|_2^2$$

for every $w \in H^1(\Omega)$. Here $2/a = \infty$ if $a = 0$. This inequality replaces (5.1).

Remark 5.5. Theorem 1.2 yields that in both cases

$$s_{\lambda,Q} = \frac{S_1}{Q_M^{(N-2)/(N-1)}}$$

for $\lambda \geq \Lambda_1$ (or $\lambda \geq \Lambda_2$). This gives the rise to the sharp Sobolev inequality:

- under assumptions (a) or (b) of Theorem 5.3 there exists a constant $C > 0$ such that, for every $u \in H^1(\Omega)$,

$$\left(\int_{\partial\Omega} Q(x)|u|^q\,dS_x\right)^{2/q} \leq \frac{Q_M^{(N-2)/(N-1)}}{S_1}\int_\Omega |\nabla u|^2\,dx + C\int_\Omega u^2\,dx.$$

## 6. Application of the topological linking

We now consider problem (1.1) with parameter interfering with the spectrum of $-\Delta$. It is convenient to rewrite problem (1.1) as

$$\begin{cases} -\Delta u - \lambda u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = Q(x)|u|^{q-2}u & \text{in } \partial\Omega, \end{cases} \tag{6.1}$$

where $\lambda > 0$. By $\{\lambda_k\}$ we denote the sequence of eigenvalues for $-\Delta$ with the Neumann boundary conditions

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that $0 = \lambda_1 < \lambda_2 \leq \ldots$ and the eigenfunctions corresponding to $\lambda_1$ are constant functions. We assume that

$$\lambda_{k-1} \leq \lambda < \lambda_k \quad \text{for some } k. \tag{6.2}$$

Let $I_\lambda$ be a variational functional for (6.1) given by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx - \frac{1}{q}\int_{\partial\Omega} Q(x)|u|^q\,dS_x.$$

Lemma 6.1. *Let $\{u_n\} \subset H^1(\Omega)$ be a sequence satisfying*

$$I_\lambda(u_n) \to c < \frac{S_1^{N-1}}{2(N-1)Q_M^{N-2}} \tag{6.3}$$

*and*

$$I'_\lambda(u_n) \to 0 \quad \text{in } H^{-1}(\Omega). \tag{6.4}$$

*Then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.*

Proof. We commence by showing that $\{u_n\}$ is bounded in $H^1(\Omega)$. The relations (6.3) and (6.4) imply that

$$\int_{\partial\Omega} Q(x)|u_n|^q\,dS_x, \qquad \bigg|\int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\,dx\bigg| \leq C + o(\|u_n\|) \tag{6.5}$$

for some constant $C > 0$ and every $n$. Arguing by contradiction assume that $\|u_n\| \to \infty$. We set $v_n = u_n/\|u_n\|$. We may assume that $v_n \rightharpoonup v$ in $H^1(\Omega)$. Thus for every $\phi \in H^1(\Omega)$ we have

$$\int_\Omega (\nabla v_n \nabla\phi - \lambda v_n\phi)\,dx = \|u_n\|^{-1}\int_{\partial\Omega} Q|u_n|^{q-2}u_n\phi\,dS_x. \tag{6.6}$$

Since

$$\bigg|\int_{\partial\Omega} Q|u_n|^{q-2}u_n\phi\,dS_x\bigg| \leq Q_M\bigg(\int_{\partial\Omega}|u_n|^q\,dS_x\bigg)^{(q-1)/q}\bigg(\int_{\partial\Omega}|\phi|^q\,dS_x\bigg)^{1/q},$$

letting $n \to \infty$, we derive from (6.5) and (6.6) that

$$\int_\Omega (\nabla v \nabla \phi - \lambda v \phi)\, dx = 0$$

for every $\phi \in H^1(\Omega)$. Since $\lambda$ is not an eigenvalue we see that $v \equiv 0$ on $\Omega$. Furthermore, we may assume that $v_n \to 0$ in $L^2(\Omega)$. This allows us to deduce from (6.3) and (6.4) that

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2\, dx = \frac{\|u_n\|^{q-2}}{q}\int_{\partial\Omega} Q|v_n|^q\, dS_x + o(1)$$

and

$$\int_\Omega |\nabla v_n|^2\, dx = \|u_n\|^{q-2}\int_{\partial\Omega} Q|v_n|^q\, dS_x + o(1).$$

These two relations imply that $\nabla v_n \to 0$ in $L^2(\Omega)$, which is impossible. Consequently $\{u_n\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. By the concentration-compactness principle we have

$$|\nabla u_n|^2 \overset{*}{\rightharpoonup} d\nu \geq |\nabla u|^2 + \sum_{j\in J}\mu_j \delta_{x_j}$$

and

$$|u_n|^q \overset{*}{\rightharpoonup} |u|^q + \sum_{j\in J}\nu_j \delta_{x_j}$$

in the space of measures for some positive constants $\mu_j$ and $\nu_j$ with $x_j \in \partial\Omega$. Let $x_j$ be fixed. Testing (6.4) by family of $C^1$-functions concentrating at $x_j$ we get

$$\mu_j = Q(x_j)\nu_j.$$

We always have the inequality $S_1 \nu_j^{2/q} \leq \mu_j$. If $\nu_j > 0$ for some $j \in J$, then

$$\frac{S_1^{N-1}}{Q(x_j)^{N-1}} \leq \nu_j.$$

On the other hand we have

$$I_\lambda(u_n) - \frac{1}{2}\langle I'_\lambda(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{q}\right)\int_{\partial\Omega} Q|u_n|^q\, dS_x.$$

Letting $n \to \infty$ we obtain

$$\begin{aligned} c &= \frac{1}{2(N-1)}\int_{\partial\Omega} Q|u|^q\, dS_x + \frac{1}{2(N-1)}\sum_{j\in J} Q(x_j)\nu_j \\ &\geq \frac{S_1^{N-1}Q(x_j)}{2(N-1)Q(x_j)^{N-1}} \geq \frac{S_1^{N-1}}{2(N-1)Q_M^{N-2}} \end{aligned}$$

and we have arrived at a contradiction. Hence $\nu_j = 0$ for every $j \in J$. This yields $u_n \to u$ in $L^q(\partial\Omega)$. By the Sobolev embedding theorems we also have that $u_n \to u$ in $L^2(\Omega)$. Combining these two facts with (6.4), we see that $\{u_n\}$ is relatively compact in $H^1(\Omega)$. □

We now establish the existence result using the min-max principle based on a topological linking [27]. Let $E^- = \text{span}\{e_1, \dots, e_l\}$, where $e_1, \dots, e_l$ are eigenfunctions corresponding to eigenvalues $\lambda_1, \dots, \lambda_{k-1}$. We have the orthogonal decomposition $H^1(\Omega) = E^- \oplus E^+$. Let $w \in E^+ - \{0\}$ and define a set

$$M = \{u \in H^1(\Omega) : u = v + sw,\ v \in E^-,\ s \geq 0,\ \|u\| \leq R\}.$$

Lemma 6.2. *There exist constants $\alpha > 0$, $\rho > 0$ and $R > \rho$ (depending on $w$) such that*

$$I_\lambda(u) \geq \alpha \quad \text{for all } u \in E^+ \cap \partial B(0, \rho)$$

*and*

$$I_\lambda(u) \leq 0 \quad \text{for all } u \in \partial M.$$

The proof is standard and is omitted.

We now define

$$Z_\varepsilon = E^- \oplus \mathbb{R}W_{\varepsilon,y} = E^- \oplus \mathbb{R}W^+_{\varepsilon,y},$$

where $W^+_{\varepsilon,y}$ denotes the projection of $W_{\varepsilon,y}$ onto $E^+$. From now on we use $W^+_{\varepsilon,y}$ in the definition of $M$.

Theorem 6.3. *Suppose that the parameter $\lambda$ satisfies* (6.2) *and that $Q$ achieves its maximum at $y \in \partial\Omega$ with $H(y) > 0$ and moreover,*

$$|Q(y) - Q(x)| = o(|x - y|)$$

*for $x$ near $y$. If $\lambda_{k-1} < \lambda < \lambda_k$, then problem* (6.1) *has a solution for $N \geq 3$ and if $\lambda = \lambda_{k-1}$ a solution exits for $N \geq 5$.*

Proof. First we observe that

$$\max_{0 \leq t < \infty} I_\lambda(tu) = \frac{(\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx)^{N-1}}{2(N-1)(\int_{\partial\Omega} Q|u|^q\,dS_x)^{N-2}}$$

for $u \in H^1(\Omega)$ with $u \neq 0$ on $\partial\Omega$. Therefore if

$$m_\varepsilon = \sup_{\substack{u \in Z_\varepsilon \\ \int_{\partial\Omega} Q|u|^q\,dS_x = 1}} \int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx < \frac{S_1}{Q_M^{(N-2)/(N-1)}}, \tag{6.7}$$

then

$$\sup_{u \in M} I_\lambda(u) < \frac{S_1^{N-1}}{2(N-1)Q_M^{N-2}}.$$

Hence it is sufficient to show that (6.7) holds. In what follows, we assume for simplicity that $y = 0$ and let $W_\varepsilon = W_{\varepsilon,0}$. Since

$$\int_\Omega (|\nabla W_\varepsilon^-|^2 - \lambda (W_\varepsilon^-)^2)\,dx \leq 0,$$

we see that

$$\int_\Omega |\nabla W_\varepsilon^-|^2\,dx \le \lambda \int_\Omega (W_\varepsilon^-)^2\,dx \le \lambda \int_\Omega W_\varepsilon^2\,dx \to 0$$

as $\varepsilon \to 0$. Therefore

$$\int_{\partial\Omega} (W_\varepsilon^-)^q\,dS_x \le C\bigg(\int_\Omega (|\nabla W_\varepsilon^-|^2 + (W_\varepsilon^-)^2)\,dx\bigg)^{q/2} \to 0$$

as $\varepsilon \to 0$. Suppose that $\int_{\partial\Omega} Q|u|^q\,dS_x = 1$. We write $u = u^- + sW_\varepsilon = (u^- + sW_\varepsilon^-) + sW_\varepsilon^+$. It follows from the above argument that $\|u^-\|_{q,\partial\Omega} \le C_3$ and $0 < s \le C_3$ for some constant $C_3 > 0$. We now deduce from the convexity of $\int_{\partial\Omega} Q|u|^q\,dS_x$ that

$$\begin{aligned} 1 = \int_{\partial\Omega} Q|u|^q\,dS_x &\ge \|sW_\varepsilon\|^q_{\partial\Omega,Q,q} + q\int_{\partial\Omega} Qu^-(sW_\varepsilon)^{q-1}\,dS_x \\ &\ge \|sW_\varepsilon\|^q_{\partial\Omega,Q,q} - C_4\|W_\varepsilon\|^{q-1}_{q-1,\partial\Omega}\|u^-\|_{q,\partial\Omega}. \end{aligned}$$

Since $\|W_\varepsilon\|^{q-1}_{q-1,\partial\Omega} = O(\varepsilon^{(N-2)/2})$, we deduce from the above inequality that

$$\|sW_\varepsilon\|^q_{\partial\Omega,Q,q} \le 1 + C_4\varepsilon^{(N-2)/2} \tag{6.8}$$

for some constant $C_4 > 0$. Since all norms on $E^-$ are equivalent we get the following estimate

$$\begin{aligned} &\int_\Omega (\nabla W_\varepsilon \nabla u^- - \lambda W_\varepsilon u^-)\,dx \\ &\qquad \le (\|\nabla W_\varepsilon\|_1 + \lambda\|W_\varepsilon\|_1)\|u^-\|_2 = O(\varepsilon^{(N-2)/2})\|u^-\|_2. \end{aligned} \tag{6.9}$$

We now estimate the surface integral. It follows from the assumption $Q$ that

$$\int_{\partial\Omega} Q(x)W_\varepsilon(x)^q\,dS_x = Q_M\int_{\partial\Omega} W_\varepsilon^q\,dS_x + o(\varepsilon). \tag{6.10}$$

Using (6.9) we can write

$$\begin{aligned} \int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx \le{}& (\lambda_{k-1} - \lambda)\int_\Omega (u^-)^2\,dx + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &+ s^2\int_\Omega (|\nabla W_\varepsilon|^2 - \lambda W_\varepsilon^2)\,dx \\ ={}& -(\lambda - \lambda_{k-1})\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &+ \frac{\int_\Omega (|\nabla W_\varepsilon|^2 - \lambda W_\varepsilon^2)\,dx}{(\int_{\partial\Omega} Q(x)W_\varepsilon^q\,dS_x)^{2/q}}\bigg(s^q\int_{\partial\Omega} Q(x)W_\varepsilon^q\,dS_x\bigg)^{2/q}. \end{aligned}$$

Since $\int_\Omega W_\varepsilon^2\,dx = O(\varepsilon^2)$, we deduce from (1.4), (6.8) and (6.10) that $m_\varepsilon < S_1/Q_M^{(N-2)/(N-1)}$ for $\varepsilon > 0$ sufficiently small and this completes the proof. □

## References


[1] R. Adams, *Sobolev spaces*, Academic Press, New York–San Francisco–London, 1975.

[2] Adimurthi and G. Mancini, *The Neumann problem for elliptic equations with critical nonlinearity*, A tribute in honour of G. Prodi, Scuola Norm. Sup. Pisa (1991), 9–25.

[3] ______, *Effect of geometry and topology of the boundary in critical Neumann problem*, J. Reine Angew. Math. **456** (1994), 1–18.

[4] Adimurthi, G. Mancini and S. L. Yadava, *The role of the mean curvature in a semilinear Neumann problem involving critical exponent*, Comm. in Partial Differential Equations **20** (1995), 591–631.

[5] Adimurthi, F. Pacella and S. L. Yadava, *Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity*, J. Funct. Anal. **113** (1993), 318–350.

[6] ______, *Characterization of concentration points and $L^\infty$-estimates for solutions of a semilinear Neumann problem involving the critical Sobolev exponent*, Differential Integral Equations **8** (1995), 31–68.

[7] Adimurthi and S. L. Yadava, *Critical Sobolev exponent problem in $\mathbb{R}^N$ $(N \geq 4)$ with Neumann boundary condition*, Proc. Indian Acad. Sci. **100** (1990), 275–284.

[8] J. Chabrowski, *On the nonlinear Neumann problem with indefinite weight and Sobolev critical nonlinearity*, Bull. Polish Acad. Sci. Math. **50** (2002), 323–333.

[9] ______, *Mean curvature and least energy solutions for the critical Neumann problem with weight*, Boll. Un. Mat. Ital. B (8) **5** (2002), 715–733.

[10] J. Chabrowski and M. Willem, *Least energy solutions of a critical Neumann problem with weight*, Calc. Var. **15** (2002), 121–131.

[11] P. Cherrier, *Problémes de Neumann nonlinèaires sur les variétés Riemanniennes*, J. Funct. Anal. **57** (1984), 154–207.

[12] J. F. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. **37** (1988), 687–698.

[13] M. Grossi and F. Pacella, *Positive solutions of nonlinear elliptic equations with critical Sobolev exponent and mixed boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **116** (1990), 23–43.

[14] C. Gui and N. Ghoussoub, *Multi-peak solutions for semilinear Neumann problem involving the critical Sobolev exponent*, Math. Z. **229** (1998), 443–474.

[15] Y. Y. Li and M. Zhu, *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), 383–417.

[16] P. L. Lions, *The concentration-compactness principle in the calculus of variations, The limit case*, Rev. Math. Iberoamericana **1** (1985), 145–201, 45–120.

[17] P. L. Lions, F. Pacella and M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. **37** (1988), 301–324.

[18] W. M. Ni, X. B. Pan and L. Takagi, *Singular behavior of least energy solutions of a semilinear Neumann problem involving critical Sobolev exponent*, Duke Math. J. **67** (1992), 1–20.

[19] W. M. Ni and L. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), 819–851.

[20] X. J. Wang, *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents*, J. Differential Equations **93** (1991), 283–310.

[21] Z. Q. Wang, *On the shape of solutions for a nonlinear Neumann problem in symmetric domains*, Lecture in Appl. Math. **29** (1993), 433–442.

[22] ______, *Remarks on a nonlinear Neumann problem with critical exponent*, Houston J. Math. **20** (1994), 671–694.

[23] ———, *High-energy and multipeaked solutions for a nonlinear Neumann problem with critical exponents*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 1013–1029.

[24] ———, *The effect of the domain geometry on number of positive solutions of Neumann problems with critical exponents*, Differential Integral Equations **8** (1995), 1533–1554.

[25] ———, *Construction of multi-peaked solutions for a nonlinear Neumann problem with critical exponent in symmetric domains*, Nonlinear Anal. **27** (1996), 1281–1306.

[26] ———, *Existence and nonexistence of G-least energy solutions for a nonlinear Neumann problem with critical exponent in symmetric domains*, Calc. Var. Partial Differential Equations **8** (1999), 109–122.

[27] M. WILLEM, *Min-Max Theorems*, Birkhäser, Boston, 1996.

[28] M. ZHU, *Sobolev inequalities with interior norms*, Calc. Var. Partial Differential Equations **8** (1999), 27–43.

*Manuscript received October 21, 2003*

JAN CHABROWSKI
Department of Mathematics
University of Queensland
St. Lucia 4072, Qld, AUSTRALIA

*E-mail address*: jhc@maths.uq.edu.au

JIANFU YANG
Institute of Physics and Mathematics
Chinese Academy of Sciences
PO Box 71010
Wuhan 430071, P.R. of CHINA