Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 25, 2005, 41–68

CONDITIONAL ENERGETIC STABILITY OF GRAVITY SOLITARY WAVES IN THE PRESENCE OF WEAK SURFACE TENSION

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Abstract. For a sequence of values of the total horizontal impulse that converges to 0, there are solitary waves that minimise the energy in a given neighbourhood of the origin in $W^{2,2}(\mathbb{R})$. The problem arises in the framework of the classical Euler equation when a two-dimensional layer of water above an infinite horizontal bottom is considered, at the surface of which solitary waves propagate under the action of gravity and *weak* surface tension. The adjective "weak" refers to the Bond number, which is assumed to be sub-critical ($< 1/3$).

This extends previous results on the conditional energetic stability of solitary waves in the super-critical case, namely those by A. Mielke ([7]) and by the author ([1]). Like in the latter, the method is based on direct minimisation and concentrated compactness, but without relying on "strict sub-additivity", which is still unsettled in the present case. Instead, a complete and careful analysis of minimising sequences is performed that allows us to reach a conclusion, based only on the non-existence of "vanishing" minimising sequences. However, in contrast with [1], we are unable to prove the existence of minimisers for *all* small values of the total horizontal impulse.

In fact more is needed to get stability, namely that every minimising sequence has a subsequence that converges to a global minimiser, after possible shifts in the horizontal direction. This will be obtained as a consequence of the analysis of minimising sequences. Then exactly the same argument as in [1] gives conditional energetic stability and is therefore not repeated.

2000 *Mathematics Subject Classification.* 76B15, 76B25, 76E17, 35J20.

Key words and phrases. Capilary-gravity water waves, solitary waves, stability, variational methods.

Supported by a grant of the Swiss National Science Foundation.

1. Introduction

This work is about the minimising sequences of the functional

$$\mathcal{J}_{\infty,\mu}(w) := \mathcal{K}_\infty(w) + \frac{\mu^2}{\mathcal{L}_\infty(w)}$$

defined for $w \neq 0$ in a small ball

$$U_\infty = \left\{ w \in W^{2,2}(\mathbb{R}) : \|w\|_{W^{2,2}(\mathbb{R})} < r \right\}, \quad r > 0.$$

The parameter $\mu > 0$ is proportional to the total horizontal impulse, $\mathcal{K}_\infty(w)$ is the sum of the capillary and gravity energy, and $\mathcal{L}_\infty(w)$ is proportional to the kinetic energy. The associated Euler equation is

$$\mathcal{K}'_\infty(w) = \frac{\mu^2}{\mathcal{L}_\infty(w)^2}\mathcal{L}'_\infty(w),$$

the solutions of which correspond to solitary capillary-gravity water waves. There are other variational formulations leading to the same equation, for example the critical points of $\mathcal{K}_\infty(w) - \gamma^2\mathcal{L}_\infty(w)$ are solutions to $\mathcal{K}'_\infty(w) = \gamma^2\mathcal{L}'_\infty(w)$, where $\gamma > 0$ is proportional to the propagation speed and is related to μ by $\gamma = \mu\mathcal{L}_\infty(w)^{-1}$ if w is a critical point. The distinctive property of $\mathcal{J}_{\infty,\mu}$ is that the set of its global minimisers is energetically stable as a whole (in some weak sense) provided each minimising sequence tends to it (see [1], [7]).

Let $\{u_n\}$ be a minimising sequence of $\mathcal{J}_{\infty,\mu}$ in $U_\infty \setminus \{0\}$. Its behaviour near the origin is under control because $\mathcal{K}_\infty$ is non negative and $\mathcal{L}_\infty$ is quadratic positive definite, so that $\lim_{w\to 0}\mathcal{J}_{\infty,\mu}(w) = +\infty$. Its behaviour near the boundary ∂U_∞ is less obvious but it can be shown as in [1] that a minimising sequence exists such that

$$\sup_n \|u_n\|_{W^{2,2}(\mathbb{R})} < r. \tag{1.1}$$

The idea is first to deal with periodic waves of large period $P > 0$ and to find minimisers of the corresponding functional $\mathcal{K}_P(w) + \mu^2\mathcal{L}_P(w)^{-1}$ by a regularisation procedure (see [2]). The minimising sequence $\{u_n\} \subset W^{2,2}(\mathbb{R})$ is then built from a sequence $\{w_{P_n}\}$ of periodic solutions with $P_n \to \infty$. As a priori estimates are available for solutions, this gives (1.1).

All minimising sequences satisfying (1.1) are "non-vanishing" in the sense that

$$\liminf_{n\to\infty} \max\{\|u_n\|_{L^\infty(\mathbb{R})}, \|u'_n\|_{L^\infty(\mathbb{R})}\} > 0.$$

This is a particular instance of the general concept of non-vanishing introduced by P. L. Lions in [5], [6] (see also [3]). A direct consequence is the existence of a critical point $w_\infty \neq 0$: let $\{t_n\} \subset \mathbb{R}$ be such that

$$\max\{\|u_n\|_{L^\infty(\mathbb{R})}, \|u'_n\|_{L^\infty(\mathbb{R})}\} = \max\{|u_n(t_n)|, |u'_n(t_n)|\}$$

and observe that $\{u_n(\,\cdot\,+t_n)\}$ is a minimising sequence that has a subsequence converging weakly in $W^{2,2}(\mathbb{R})$ to a limit w_∞ satisfying

$$\max\{|w_\infty(0)|, |w'_\infty(0)|\} > 0.$$

In this paper t denotes a spatial variable and there will be no explicit dependence on time as we are only concerned with stationary solutions.

Up to this point there is no difference between weak and strong surface tension; it is only when dealing with strict sub-additivity that the two cases depart from each other. If

$$c(\mu) := \inf\{\mathcal{J}_{\infty,\mu}(w) : w \in U_\infty \setminus \{0\}\} > 0,$$

then strict sub-additivity means the existence of $\mu_0 > 0$ such that

$$c(\mu_1+\mu_2) < c(\mu_1) + c(\mu_2)$$

for all $\mu_1, \mu_2 > 0$ with $\mu_1 + \mu_2 < \mu_0$. Whereas strict sub-additivity is proved in [1] for strong surface tension, it is still unsettled for weak surface tension (however non-strict sub-additivity is known to hold, see Theorem 4.2). In the theory of compactness by concentration, strict sub-additivity is what forbids a minimising sequence $\{u_n\}$ to split into two parts $\{u_{1,n}\}$ and $\{u_{2,n}\}$ that move apart as $n \to \infty$.[1]

In the weak tension case we need therefore to study minimising sequences that we allow to split into two or more (possibly infinitely many) parts. This leads to the following result (Theorem 4.8): for all small μ, there exists a finite or infinite sequence $\{w_j : 1 \le j < m\} \subset U_\infty \setminus \{0\}$ with $m \in \{2,3,\dots\} \cup \{\infty\}$ such that

$$\sum_{1\le j<m} \|w_j\|^2_{W^{2,2}(\mathbb{R})} < r^2 \quad \text{and} \quad \sum_{1\le j<m} \mathcal{K}_\infty(w_j) + \frac{\mu^2}{\sum_{1\le j<m} \mathcal{L}_\infty(w_j)} = c(\mu).$$

Indeed, given a minimising sequence $\{u_n\} \subset U_\infty \setminus \{0\}$ of $\mathcal{J}_{\infty,\mu}$ that satsifies (1.1), there exists such a sequence $\{w_j : 1 \le j < m\}$ with

$$\lim_{k\to m} \limsup_{q\to\infty} \left\| u_{n_q} - \sum_{1\le j<k} w_j(\,\cdot\,-t_{j,q}) \right\|_{W^{1,2}(\mathbb{R})} = 0$$

and

$$\lim_{q\to\infty} |t_{i,q} - t_{j,q}| = \infty \quad \text{for all } 1 \le i < j < m,$$

for some subsequence $\{u_{n_q}\}$ and some sequences $\{t_{i,q} : q \in \mathbb{N}\} \subset \mathbb{R}$.

Define

$$\widetilde{\mu} = \mu \inf \max_{1\le j<m} \frac{\mathcal{L}_\infty(w_j)}{\sum_{1\le i<m} \mathcal{L}_\infty(w_i)} > 0,$$

[1] E.g. $u_n = u_{1,n} + u_{2,n}$ with $u_{1,n}(t) = w(t+n)$ and $u_{2,n}(t) = w(t-n)$ for some function $w \neq 0$. This kind of splitting is called "dichotomy".

where the infimum is taken over all such m and sequences $\{w_j : 1 \le j < m\}$. Our main result states that every minimising sequence of $\mathcal{J}_{\infty,\widetilde{\mu}}$ that satisfies (1.1) cannot split and therefore conditional energetic stability is established for the value $\widetilde{\mu}$. This leads to a sequence converging to 0 of values of the total horizontal momentum for which conditional energetic stability holds true, that is, Theorem 19 in [1] holds true.[2]

An important preliminary step is to show the inequality $c(\mu) < 2\mu$. Together with the normalisation

$$\inf\left\{\frac{\mathcal{K}''_\infty(0)(w,w)}{2\mathcal{L}_\infty(w)} : w \in W^{2,2}(\mathbb{R}) \setminus \{0\}\right\} = 1,$$

it implies for any minimising sequence $\{u_n\}$ and large enough n that

$$\begin{aligned}\mathcal{M}_\infty(u_n) :=& \mathcal{K}_\infty(u_n) - (1/2)\mathcal{K}''_\infty(0)(u_n,u_n)\\ =& -\frac{1}{2}\mathcal{K}''_\infty(0)(u_n,u_n) - \mu^2\mathcal{L}_\infty(u_n)^{-1} + \mathcal{J}_{\infty,\mu}(u_n)\\ \le& -2\mu\sqrt{(1/2)\mathcal{K}''_\infty(0)(u_n,u_n)/\mathcal{L}_\infty(u_n)} + \mathcal{J}_{\infty,\mu}(u_n)\\ \le& -2\mu + \mathcal{J}_{\infty,\mu}(u_n) < 0,\end{aligned}$$

which is at the core of the non-vanishing of $\{u_n\}$. To prove $c(\mu) < 2\mu$, we shall estimate $\mathcal{J}_{\infty,\mu}(u)$ for u of the type

$$u_w(t) = \alpha\phi(\alpha t)\cos\omega t + \alpha^2\psi(\alpha t)\cos 2\omega t,$$

where $\alpha > 0$ is a small parameter roughly proportional to μ (the exact relationship is to be determined), $\omega > 0$ is a wave number depending on the Bond number that leads to spatial 1:1 resonance ([4]), and $\phi, \psi \in C_0^\infty(\mathbb{R})$ are to be determined so that $c(\mu) < 2\mu$ for small α. For strong surface tension, the test function is of the type

$$u_s(t) = \alpha^2\phi(\alpha t),$$

where now α^3 is roughly proportional to μ. The test function u_w does not give rise to the estimates needed in [1] to show strict sub-additivity.

2. Periodic water waves

We first recall some functional features of the normal derivative operator N that is used in the formulation of the water-wave problem. For more explanations, we refer to [1]. Let L^2_P denote the subspace of $L^2_{\mathrm{loc}}(\mathbb{R})$ made of P-periodic real-valued "functions" and let $W^{s,2}_P := W^{s,2}_{\mathrm{loc}}(\mathbb{R}) \cap L^2_P$ for $s > 0$.

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[2] $\mathcal{L}_\infty$ in [1] is the same as the one used in the present work only up to a positive factor that depends on the Bond number, but this is irrelevant when stating the stability result, for this factor can be integrated into μ^2.

The normal derivative $Nu \in L^2_P$ of $u \in W^{1,2}_P$ is defined by

$$\widehat{Nu}_k = \frac{2\pi k \cosh(2\pi k/P)}{P \sinh(2\pi k/P)} \widehat{u}_k \quad \text{for } k \in \mathbb{Z} \setminus \{0\},\ \widehat{Nu}_0 = \widehat{u}_0, \tag{2.1}$$

and the normal derivative $Nu \in L^2(\mathbb{R})$ of $u \in W^{1,2}(\mathbb{R})$ by

$$\widehat{Nu}(s) = \frac{s \cosh s}{\sinh s} \widehat{u}(s) \quad \text{for } s \in \mathbb{R}^* \text{ and } \widehat{Nu}(0) = \widehat{u}(0). \tag{2.2}$$

The linear operator N is self-adjoint in L^2_P and in $L^2(\mathbb{R})$, positive definite, its spectrum does not contain 0 and it commutes with differentiation in $W^{2,2}_P$ and $W^{2,2}(\mathbb{R})$. Moreover, for all $n \geq 1$, there exists a constant $C_n > 0$ such that

$$|Nu(t)| \leq C_n \{1 + \operatorname{dist}(t, \operatorname{supp}(u))\}^{-n+(1/2)} \left\{ \int_{\operatorname{supp}(u)} (u - u'')^2 \, ds \right\}^{1/2} \tag{2.3}$$

for all $u \in W^{2,2}(\mathbb{R})$ with compact support $\operatorname{supp}(u)$ (C_n is independent of the size of the support).

We can use the same notation for N when it acts on $W^{1,2}_P$ as when it acts on $W^{1,2}(\mathbb{R})$ because it is in fact defined more generally in the space of tempered distributions as the pseudo-differential operator with symbol

$$f(s) := \begin{cases} \dfrac{s \cosh s}{\sinh s} & \text{if } s \neq 0, \\ 1 & \text{at } s = 0. \end{cases} \tag{2.4}$$

It follows that if $u \in W^{2,2}(\mathbb{R})$ has compact support and $v \in W^{1,2}_P$, then

$$\int_{\mathbb{R}} vNu \, dt = \int_{\mathbb{R}} uNv \, dt,$$

where N is defined by (2.2) in the left-hand side and by (2.1) in the right-hand side. Moreover, $v_P \in W^{2,2}_P$ defined by $v_P(t) = \sum_{k\in\mathbb{Z}} u(t + kP)$ satisfies $(Nv_P)(t) = \sum_{k\in\mathbb{Z}} (Nu)(t+kP)$, where the convergence of the series is in $L^\infty_{\text{loc}}(\mathbb{R})$ and in the space of tempered distributions. Also $\lim_{P\to\infty} v_P = u$ uniformly on every bounded interval.

To get periodic water waves of large period $P > 0$, let

$$U_P = \left\{ w \in W^{2,2}_P : \int_{-P/2}^{P/2} (w'' + w)^2 \, dt < R_2^2 \right\}$$

with $R_2 \in (0, 1/2)$, and define the functionals $\mathcal{K}_P, \mathcal{L}_P, \mathcal{M}_P \in C^\infty(U_P, \mathbb{R})$ by

$$\begin{aligned} \mathcal{K}_P(w) &= \int_{-P/2}^{P/2} \left\{ \beta\sqrt{w'^2 + (1 + Nw)^2} - \beta(1 + Nw) + \frac{\lambda}{2} w^2 (1 + Nw) \right\} dt, \\ \mathcal{L}_P(w) &= \frac{1}{2}\Lambda \int_{-P/2}^{P/2} wNw \, dt, \\ \mathcal{M}_P(w) &= \mathcal{K}_P(w) - \frac{1}{2}\mathcal{K}''_P(0)(w, w), \end{aligned} \tag{2.5}$$

where the parameters λ, β are positive and

$$\Lambda = \inf \left\{ \frac{\lambda + \beta s^2}{f(s)} : s \geq 0 \right\} > 0.$$

Since f is continuous and $f(s) = 1 + (1/3)s^2 + O(s^4)$ as $s \to 0$, the infimum is reached at some $s > 0$, denoted by ω, if $\lambda = 1$ and $0 < \beta < 1/3$:

$$\Lambda = \frac{1 + \beta\omega^2}{f(\omega)} \in (0,1), \quad \omega > 0.$$

The factor Λ has been introduced so that the normalisation

$$\inf \left\{ \frac{\mathcal{K}_P''(0)(w,w)}{2\mathcal{L}_P(w)} : w \in W_P^{2,2} \setminus \{0\} \right\} = 1$$

holds if $P \in (2\pi/\omega)\mathbb{Z}$, which we assume from now on (the infimum in the normalisation is then reached at functions in the linear span of $\cos\omega t$ and $\sin\omega t$). In fact $\omega > 0$ is unique. To see this, note that, for all $s > 0$,

$$1 + \beta s^2 - \Lambda f(s) \geq 0$$

with equality at $s = \omega$. Since $f'''(s) < 0$ for all $s > 0$, the third derivative of the map $s \to 1 + \beta s^2 - \Lambda f(s)$ is strictly positive for all $s > 0$, which implies the uniqueness of $\omega > 0$.

The parameters $\lambda, \beta > 0$ can be seen as non-dimensional gravity and surface tension. For $\mu > 0$, a critical point $w \in U_P \setminus \{0\}$ of $\mathcal{K}_P + \mu^2\mathcal{L}_P^{-1}$ satisfy

$$\begin{aligned} 0 = &-\nu^2 Nw + \lambda\{w + wNw + N(w^2/2)\} \\ &- \beta\left\{\frac{w'}{\sqrt{w'^2 + (1+Nw)^2}}\right\}' + \beta N\left\{\frac{1+Nw}{\sqrt{w'^2 + (1+Nw)^2}} - 1\right\}, \end{aligned} \tag{2.6}$$

where $\nu = \sqrt{\Lambda}\mu\mathcal{L}_P(w)^{-1}$ is the non-dimensional propagation speed of the corresponding periodic water wave. The formula for ν is obtained from

$$\frac{\nu^2}{\Lambda} = \frac{\mu^2}{\mathcal{L}_P(w)^2},$$

which is the coefficient in front of $\mathcal{L}_P'(w)$ in the derivative of $\mathcal{K}_P + \mu^2\mathcal{L}_P^{-1}$. The fourth parameter is the depth of the two-dimensional layer of water, which can be chosen to be 1 without loss of generality. Clearly only two parameters among λ, β, ν are mathematically relevant, for we can divide (2.6) by any of these parameters.

In (2.5) and (2.6), we now set $\lambda = 1$ and assume that $0 < \beta < 1/3$ and that $P \in (2\pi/\omega)\mathbb{Z}$.

THEOREM 2.1. *The positive numbers R_2 and $\kappa > 0$ can be chosen independently of μ and $P > 0$, such that, for all $\mu > 0$ small enough and $P > \mathcal{P}_\mu$ in $(2\pi/\omega)\mathbb{Z}$ with $\mathcal{P}_\mu$ large enough (depending on μ), there exists $w_P \in W^{2,2}_P \setminus \{0\}$ satisfying the following properties:*

$$\begin{aligned} &\sup_{\{P>\mathcal{P}_\mu, P\in(2\pi/\omega)\mathbb{Z}\}} \int_{-P/2}^{P/2} (-w_P'' + w_P)^2\, dt \le \kappa^2 \mu < R_2^2, \\ &\sup\{|Nw_P(t)| : P > \mathcal{P}_\mu,\ P \in (2\pi/\omega)\mathbb{Z},\ t \in \mathbb{R}\} < 1/2, \\ &\sup\{|w_P'(t)| : P > \mathcal{P}_\mu,\ P \in (2\pi/\omega)\mathbb{Z},\ t \in \mathbb{R}\} < 1/2, \end{aligned} \tag{2.7}$$

the minimum

$$\min\left\{\mathcal{K}_P(u) + \frac{\mu^2}{\mathcal{L}_P(u)} : u \in W^{2,2}_P,\ 0 < \int_{-P/2}^{P/2} (u - u'')^2\, dt < R_2^2\right\} < 2\mu$$

is attained at $u = w_P$,

$$\sup\{\mathcal{M}_P(w_P) : P > \mathcal{P}_\mu,\ P \in (2\pi/\omega)\mathbb{Z}\} < 0 \tag{2.8}$$

and

$$\begin{aligned} 0 = &- \nu_P^2 N w_P + w_P + w_P N w_P + N(w_P^2/2) \\ &- \beta \left\{ \frac{w_P'}{\sqrt{{w_P'}^2 + (1+Nw_P)^2}} \right\}' + \beta N \left\{ \frac{1+Nw_P}{\sqrt{{w_P'}^2 + (1+Nw_P)^2}} - 1 \right\}, \end{aligned} \tag{2.9}$$

where $\nu_P := 2\mu\{\sqrt{\Lambda}\int_{-P/2}^{P/2} w_p N w_p\, dt\}^{-1} = \mu\sqrt{\Lambda}\mathcal{L}_P(w_P)^{-1} < 2\sqrt{\Lambda}$ and $\beta < 1/3$ has been fixed arbitrarily at the beginning of this section.

The remaining of this section is devoted to its proof, which consists in applying the abstract result of Section 2 in [1] to

$$X_0 = L^2_P, \quad Aw = -w'' + w \quad \text{and} \quad X_n = W^{n,2}_P \quad \text{for } n \ge 0. \tag{2.10}$$

The norms are defined by

$$\|w\|_n^2 = \|w\|_{X_n}^2 = \sum_k \left(1 + \left(\frac{2\pi k}{P}\right)^2\right)^n |\widehat{w}_k|^2;$$

in particular,

$$\|w\|_1^2 = \int_{-P/2}^{P/2} \{w^2 + {w'}^2\}\, dt$$

and

$$\|w\|_2^2 = \int_{-P/2}^{P/2} \{-w'' + w\}^2\, dt = \int_{-P/2}^{P/2} \{{w''}^2 + 2{w'}^2 + w^2\}\, dt.$$

Note that $\max|u| \le \|u\|_1$ if $P \ge 2$, which is assumed from now on. Also $\max|Nu| \le \|u\|_2$ and $\max|u'| \le \|u\|_2$. We choose R_2 so small that

$$\sup|Nw| < 1/2 \quad \text{and} \quad \sup|w'| < 1/2 \tag{2.11}$$

for $w \in U_P$. Note that

$$\mathcal{K}_P(w) = \int_{-P/2}^{P/2} \left\{ \beta\sqrt{w'^2 + (1+Nw)^2} - \beta(1+Nw) + \frac{1}{2}w^2(1+Nw) \right\} dt$$
$$= \int_{-P/2}^{P/2} \left\{ \beta \frac{w'^2}{\sqrt{w'^2 + (1+Nw)^2} + (1+Nw)} + \frac{1}{2}w^2(1+Nw) \right\} dt,$$
$$\mathcal{K}_P(w) \geq \int_{-P/2}^{P/2} \{(2\beta/7)w'^2 + (1/4)w^2\}\, dt \geq \text{const.}\|w\|_1^2,$$
$$\mathcal{K}_P''(0)(w,w) = \int_{-P/2}^{P/2} \{\beta w'^2 + w^2\}\, dt \geq \text{const.}\|w\|_1^2,$$

where the various constants do not depend on μ and P.

To check the assumptions of the abstract theorem of Section 2 in [1], we first modify them by replacing integration over $(-P/2, P/2)$ by integration over $\mathbb{R}$. We are thus looking for $u \in W^{2,2}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |u - u''|^2\, dt < R_2^2 \quad \text{and} \quad \mathcal{K}_\infty(u) + \mu^2 \mathcal{L}_\infty^{-1}(u) < 2\mu, \tag{2.12}$$

where

$$\mathcal{K}_\infty(u) = \beta \int_{\mathbb{R}} \frac{u'^2}{\sqrt{u'^2 + (1+Nu)^2} + (1+Nu)}\, dt + \frac{1}{2}\int_{\mathbb{R}} u^2(1+Nu)\, dt,$$
$$\mathcal{L}_\infty(u) = \frac{1}{2}\Lambda \int_{\mathbb{R}} uNu\, dt.$$

Since, for $|s| < 1$,

$$\sqrt{1+s} = 1 + \frac{1}{2}s - \frac{1}{8}s^2 + \frac{1}{16}s^3 - \frac{5}{128}s^4 + \dots,$$

we get

$$\sqrt{u'^2 + (1+Nu)^2} - (1+Nu) = \frac{1}{2}u'^2 - \frac{1}{2}u'^2 Nu - \frac{1}{8}u'^4 + \frac{1}{2}u'^2(Nu)^2 + \dots$$

We set

$$u(t) = \alpha\phi(\alpha t)\cos\omega t + \alpha^2\psi(\alpha t)\cos 2\omega t, \tag{2.13}$$

where $\phi, \psi \in C_0^\infty(\mathbb{R})$ will be chosen later and $\alpha > 0$ is small. We get

$$u'(t) = -\alpha\omega\phi(\alpha t)\sin\omega t + \alpha^2\phi'(\alpha t)\cos\omega t$$
$$- 2\omega\alpha^2\psi(\alpha t)\sin 2\omega t + \alpha^3\psi'(\alpha t)\cos 2\omega t,$$
$$u''(t) = \alpha^3\phi''(\alpha t)\cos\omega t - 2\alpha^2\phi'(\alpha t)\omega\sin\omega t - \alpha\phi(\alpha t)\omega^2\cos\omega t$$
$$+ \alpha^4\psi''(\alpha t)\cos 2\omega t - 2\alpha^3\psi'(\alpha t)2\omega\sin 2\omega t - \alpha^2\psi(\alpha t)4\omega^2\cos 2\omega t$$

and, for all $n \in \mathbb{N}$ (remember (2.4)),

$$\begin{aligned} Nu(t) = {} & f(\omega)\alpha\phi(\alpha t)\cos\omega t + f(2\omega)\alpha^2\psi(\alpha t)\cos 2\omega t \\ & + (f'(\omega) - f''(\omega)\omega)(\alpha^2\phi'(\alpha t)\sin\omega t) \\ & + (f'(2\omega) - f''(2\omega)2\omega)(\alpha^3\psi'(\alpha t)\sin 2\omega t) \\ & + f''(\omega)(-(1/2)\alpha^3\phi''(\alpha t)\cos\omega t + \alpha^2\phi'(\alpha t)\omega\sin\omega t) \\ & + f''(2\omega)\alpha^3\psi'(\alpha t)2\omega\sin 2\omega t + R(t), \end{aligned}$$

where the remaining term R is a function of t in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\alpha^{-4}R$ is uniformly bounded in $L^\infty(\mathbb{R})$ and $\alpha^{-7/2}R$ is uniformly bounded in $L^2(\mathbb{R})$. Indeed if $\psi \equiv 0$ (for simplicity), we have

$$\{\phi(\alpha\,\cdot)e^{\pm i\omega\,\cdot}\}\widehat{\ }(s) = \alpha^{-1}\widehat{\phi}\left(\frac{s \mp \omega}{\alpha}\right),$$

$$\begin{aligned} \widehat{Nu}\,(s) = {} & \frac{1}{2}\bigg\{f(\omega) + f'(\omega)(s-\omega) \\ & + \frac{1}{2}f''(\omega)(s-\omega)^2 + R_-(s)(s-\omega)^3\bigg\}\widehat{\phi}\left(\frac{s-\omega}{\alpha}\right) \\ & + \frac{1}{2}\bigg\{f(-\omega) + f'(-\omega)(s+\omega) \\ & + \frac{1}{2}f''(-\omega)(s+\omega)^2 + R_+(s)(s+\omega)^3\bigg\}\widehat{\phi}\left(\frac{s+\omega}{\alpha}\right) \\ := {} & \frac{A(\omega)}{2}\left\{\widehat{\phi}\left(\frac{s-\omega}{\alpha}\right) + \widehat{\phi}\left(\frac{s+\omega}{\alpha}\right)\right\} \\ & + \frac{B(\omega)s}{2}\left\{\widehat{\phi}\left(\frac{s-\omega}{\alpha}\right) - \widehat{\phi}\left(\frac{s+\omega}{\alpha}\right)\right\} \\ & + \frac{C(\omega)s^2}{2}\left\{\widehat{\phi}\left(\frac{s-\omega}{\alpha}\right) + \widehat{\phi}\left(\frac{s+\omega}{\alpha}\right)\right\} \\ & + \frac{1}{2}R_-(s)(s-\omega)^3\widehat{\phi}\left(\frac{s-\omega}{\alpha}\right) + \frac{1}{2}R_+(s)(s+\omega)^3\widehat{\phi}\left(\frac{s+\omega}{\alpha}\right) \end{aligned}$$

and

$$\begin{aligned} Nu(t) = {} & A(\omega)u(t) + B(\omega)\{\alpha\phi(\alpha t)\sin wt\}' - C(\omega)u''(t) \\ & + \frac{\alpha^4}{2\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\alpha t s}\{R_-(\alpha s)(s-\omega\alpha^{-1})^3\widehat{\phi}(s-\omega\alpha^{-1})\}\,ds \\ & + \frac{\alpha^4}{2\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\alpha t s}\{R_+(\alpha s)(s+\omega\alpha^{-1})^3\widehat{\phi}(s+\omega\alpha^{-1})\}\,ds, \end{aligned}$$

where $|R_\pm(s)| \le$ const. for all $s \in \mathbb{R}$.

Since the two integrands are absolutely integrable, uniformly in α, the two integrals are bounded functions of t, uniformly in α. The two integrands are also square integrable, uniformly in α, and therefore the two integrals seen as functions of αt are square integrable, uniformly in α. The formula for $Nu(t)$ follows now from

$$\begin{aligned}
&A(\omega) = f(\omega) - \omega f'(\omega) + \frac{1}{2} f''(\omega)\omega^2,\\
&B(\omega) = f'(\omega) - f''(\omega)\omega, \quad C(\omega) = \frac{f''(\omega)}{2},\\
&f(\omega) = A(\omega) + B(\omega)\omega + C(\omega)\omega^2.
\end{aligned}$$

Going back to (2.13), we get the following estimates:

$$\begin{aligned}
\int_{\mathbb{R}} u^2\,dt &= \frac{\alpha}{2}\int_{\mathbb{R}}\phi^2\,dt + \frac{\alpha^3}{2}\int_{\mathbb{R}}\psi^2\,dt + O(\alpha^4),\\
\int_{\mathbb{R}} u'^2\,dt &= \frac{\alpha\omega^2}{2}\int_{\mathbb{R}}\phi^2\,dt + \frac{\alpha^3}{2}\int_{\mathbb{R}}\phi'^2\,dt + 2\alpha^3\omega^2\int_{\mathbb{R}}\psi^2\,dt + O(\alpha^4),\\
\int_{\mathbb{R}} uNu\,dt &= \frac{\alpha}{2} f(\omega)\int_{\mathbb{R}}\phi^2\,dt\\
&\quad + \frac{\alpha^3}{2} f(2\omega)\int_{\mathbb{R}}\psi^2\,dt + f''(\omega)\frac{\alpha^3}{4}\int_{\mathbb{R}}\phi'^2 dt + O(\alpha^4),\\
\int_{\mathbb{R}} u^2 Nu\,dt &= \int \alpha^2\phi^2(\alpha t)\frac{\cos 2\omega t}{2} f(2\omega)\alpha^2\psi(\alpha t)\cos 2\omega t\,dt\\
&\quad + \int_{\mathbb{R}} 2\alpha\phi(\alpha t)\alpha^2\psi(\alpha t)\frac{\cos\omega t}{2} f(\omega)\alpha\phi(\alpha t)\cos\omega t\,dt + O(\alpha^4)\\
&= \frac{\alpha^3}{4}\{2f(\omega) + f(2\omega)\}\int_{\mathbb{R}}\phi^2(t)\psi(t)\,dt + O(\alpha^4),\\
\int_{\mathbb{R}} u'^2 Nu\,dt &= \int \alpha^2\omega^2\phi^2(\alpha t)\frac{-\cos 2\omega t}{2} f(2\omega)\alpha^2\psi(\alpha t)\cos 2\omega t\,dt\\
&\quad + \int_{\mathbb{R}} 2\alpha\omega\phi(\alpha t)2\alpha^2\omega\psi(\alpha t)\frac{\cos\omega t}{2} f(\omega)\alpha\phi(\alpha t)\cos\omega t\,dt + O(\alpha^4)\\
&= \alpha^3\omega^2\{f(\omega) - \frac{1}{4} f(2\omega)\}\int_{\mathbb{R}}\phi^2(t)\psi(t)\,dt + O(\alpha^4),\\
\int_{\mathbb{R}} u'^4\,dt &= \alpha^3\omega^4\int_{\mathbb{R}}\phi(t)^4\sin^4(\omega t/\alpha)\,dt + O(\alpha^4) = \frac{3\alpha^3\omega^4}{8}\int_{\mathbb{R}}\phi^4\,dt + O(\alpha^4),\\
\int_{\mathbb{R}} u'^2(Nu)^2\,dt &= \int_{\mathbb{R}}\alpha^2\omega^2\phi^2(\alpha t)\frac{\sin^2 2\omega t}{4} f^2(\omega)\alpha^2\phi^2(\alpha t)\,dt + O(\alpha^4)\\
&= \frac{\alpha^3\omega^2}{8} f^2(\omega)\int_{\mathbb{R}}\phi^4(t)\,dt + O(\alpha^4).
\end{aligned}$$

Setting $\mu = \mathcal{L}_\infty(u)$, we obtain

$$\begin{aligned}
\mathcal{K}_\infty(u) + \mu^2\mathcal{L}_\infty^{-1}(u) - 2\mu &= \mathcal{K}_\infty(u) - \mathcal{L}_\infty(u)\\
&= \frac{\beta}{4}\left(\alpha\omega^2\int_{\mathbb{R}}\phi^2\,dt + \alpha^3\int_{\mathbb{R}}\phi'^2\,dt + 4\alpha^3\omega^2\int_{\mathbb{R}}\psi^2\,dt\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\alpha^3\beta\omega^2}{2}\left\{f(\omega) - \frac{1}{4}f(2\omega)\right\}\int_{\mathbb{R}}\phi^2\psi\,dt - \frac{3\alpha^3\beta\omega^4}{64}\int_{\mathbb{R}}\phi^4\,dt \\
&+ \frac{\alpha^3\beta\omega^2}{16}f^2(\omega)\int_{\mathbb{R}}\phi^4(t)\,dt + \frac{\alpha}{4}\int_{\mathbb{R}}\phi^2\,dt + \frac{\alpha^3}{4}\int_{\mathbb{R}}\psi^2\,dt \\
&+ \frac{\alpha^3}{8}\{2f(\omega) + f(2\omega)\}\int_{\mathbb{R}}\phi^2\psi\,dt \\
&- \frac{\Lambda}{8}\left\{2\alpha f(\omega)\int_{\mathbb{R}}\phi^2\,dt + 2\alpha^3 f(2\omega)\int_{\mathbb{R}}\psi^2\,dt + f''(\omega)\alpha^3\int_{\mathbb{R}}\phi'^2\,dt\right\} + o(\alpha^3) \\
= {}& \frac{\alpha^3}{4}\{4\beta\omega^2 + 1 - \Lambda f(2\omega)\}\int_{\mathbb{R}}\psi^2\,dt \\
&+ \frac{\alpha^3}{8}\{-\beta\omega^2(4f(\omega) - f(2\omega)) + 2f(\omega) + f(2\omega)\}\int_{\mathbb{R}}\phi^2\psi\,dt \\
&+ \frac{\alpha^3}{64}\{-3\beta\omega^4 + 4\beta\omega^2 f^2(\omega)\}\int_{\mathbb{R}}\phi^4\,dt + \frac{\alpha^3}{8}\{2\beta - \Lambda f''(w)\}\int_{\mathbb{R}}\phi'^2\,dt + o(\alpha^3)
\end{aligned}
$$

because $\beta\omega^2+1-\Lambda f(\omega)=0$ by the definitions of ω and Λ. Since $\beta s^2+1-\Lambda f(s)$ reaches its unique minimum at $s=w$, we also get

$$\beta(2\omega)^2 + 1 - \Lambda f(2\omega) > 0, \quad 2\beta\omega - \Lambda f'(w) = 0 \quad \text{and} \quad 2\beta - \Lambda f''(w) \geq 0.$$

This gives

$$\beta = \frac{f'(\omega)}{2\omega f(\omega) - \omega^2 f'(\omega)} \quad \text{and} \quad \Lambda = \frac{2\omega}{2\omega f(\omega) - \omega^2 f'(\omega)}. \tag{2.14}$$

Note that $2\omega f(\omega)-\omega^2 f'(\omega)>0$ because the derivative of the map $s\to 2sf(s)-s^2f'(s)$ is strictly positive for all $s>0$, as it can be seen from the fact that $f'''(s)<0$ for all $s>0$. Setting

$$
\begin{aligned}
\psi &= x\phi^2, \\
R &= \frac{1}{4}\{4\beta\omega^2 + 1 - \Lambda f(2\omega)\}, \\
S &= \frac{1}{8}\{-\beta\omega^2(4f(\omega) - f(2\omega)) + 2f(\omega) + f(2\omega)\}, \\
T &= \frac{1}{64}\{-3\beta\omega^4 + 4\beta\omega^2 f^2(\omega)\}, \\
U &= \frac{1}{8}\{2\beta - \Lambda f''(\omega)\},
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathcal{K}_\infty(u) + \mu^2\mathcal{L}_\infty^{-1}(u) - 2\mu \\
&\qquad = \alpha^3\{Rx^2 + Sx + T\}\int_{\mathbb{R}}\phi^4 dt + \alpha^3 U\int_{\mathbb{R}}\phi'^2\,dt + o(\alpha^3) < 0
\end{aligned}
\tag{2.15}
$$

if x is such that $Rx^2+Sx+T<0$, ϕ is chosen appropriately (note that $U\geq 0$) and $\alpha>0$ is small enough. Since $R>0$, such a choice of x is possible provided that $S^2 - 4RT > 0$, that is

$$\{-\beta\omega^2(4f(\omega) - f(2\omega)) + 2f(\omega) + f(2\omega)\}^2 \\ - \{4\beta\omega^2 + 1 - \Lambda f(2\omega)\}\{-3\beta\omega^4 + 4\beta\omega^2 f^2(\omega)\} > 0.$$

Thanks to (2.4) and (2.14), this is a rational function of w and e^w, and it has been checked using "maple" that its graph stays above the horizontal axis for all $w > 0$.

For small $\alpha > 0$ and for large enough $P > 0$ in $(2\pi/\omega)\mathbb{Z}$, we now check the assumptions of the abstract theorem with u_P defined by

$$u_P := \sum_{k\in\mathbb{Z}} u(\,\cdot\, + kP).$$

In fact they follow from

$$\begin{aligned} \int_0^P |u_P - u_P''|^2\,dt &= \int_{\mathbb{R}} |u - u''|^2\,dt && \text{for large } P > 0,\\ Nu &= \lim_{P\to\infty} Nu_P && \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}),\\ \sup_{P\ge 2} \|Nu_P\|_{L^\infty(\mathbb{R})} &< 1/2 && \text{if } \alpha > 0 \text{ is small enough},\end{aligned}$$

$$\lim_{P\to\infty} \mathcal{K}_P(u_P) = \mathcal{K}_\infty(u) \quad \text{and} \quad \lim_{P\to\infty} \mathcal{L}_P(u_P) = \mathcal{L}_\infty(u)$$

by Lebesgue's dominated convergence theorem.

3. Solitary water waves

Here we repeat the argument of Section 4 in [1]. Our aim is to find $w \in W^{2,2}(\mathbb{R})$ and $\nu > 0$ such that, almost everywhere,

$$0 = -\nu^2 Nw + \{w + wNw + N(w^2/2)\} \\ - \beta\left\{\frac{w'}{\sqrt{w'^2 + (1+Nw)^2}}\right\}' + \beta N\left\{\frac{1+Nw}{\sqrt{w'^2+(1+Nw)^2}} - 1\right\}. \tag{3.1}$$

Let $\mu > 0$ be fixed and small enough. By (2.7), there exists a sequence $P_n \to \infty$ in $(2\pi/\omega)\mathbb{Z}$ and $w_\infty \in W^{2,2}(\mathbb{R})$ such that $\{\nu_{P_n}\}$ converges to some $\nu_\infty \in (0, 2]$ and, for every bounded interval I, $w_{P_n} \rightharpoonup w_\infty$ weakly in $W^{2,2}(I)$ and $w'_{P_n} \to w'_\infty$ in $L^\infty(I)$ as $n \to \infty$. We can also assume that, for every bounded interval I, $Nw_{P_n} - Nw_{P_m} \to 0$ in $L^\infty(I)$ as $n, m \to \infty$, and therefore that $Nw_{P_n} \to Nw_\infty$ in the space of tempered distributions and in $L^\infty_{\mathrm{loc}}(\mathbb{R})$.

We now multiply (2.9) in which $P = P_n$ by an arbitrary smooth function with compact support and take the limit $n \to \infty$, which shows that $w_\infty \in W^{2,2}(\mathbb{R})$ satisfies equation (3.1). Moreover,

$$\int_{\mathbb{R}} (-w''_\infty + w_\infty)^2\,dt \le O(\mu). \tag{3.2}$$

It remains to discuss how this argument can be modified to yield that $w_\infty \not\equiv 0$. For $P > \mathcal{P}_\mu$ in $(2\pi/\omega)\mathbb{Z}$, we get from

$$\mathcal{M}_P(w_P) = \beta \int_{-P/2}^{P/2} {w_P'}^2 \left(\frac{1}{\sqrt{{w_P'}^2 + (1+Nw_P)^2} + (1+Nw_P)} - \frac{1}{2} \right) dt \\ + \frac{1}{2} \int_{-P/2}^{P/2} w_P^2 N w_P \, dt$$

the estimate

$$\begin{aligned} |\mathcal{M}_P(w_P)| &\le K\beta \int_{-P/2}^{P/2} {w_P'}^2 (|w_P'| + |Nw_P|) \, dt \\ &\quad + \frac{1}{2} \{\max_t |w_P(t)|\} \int_{-P/2}^{P/2} \{w_P^2 + {w_P'}^2\} \, dt \\ &\le \{\max_t |w_P'(t)| + \max_t |w_P(t)|\} \left(2K\beta + \frac{1}{2}\right) \int_{-P/2}^{P/2} \{w_P^2 + {w_P'}^2\} \, dt \end{aligned}$$

for some $K > 0$ independent of P. From (2.8), we deduce that

$$\inf\{\max_t |w_P(t)| + \max_t |w_P'(t)| : P > \mathcal{P}_\mu,\ P \in (2\pi/\omega)\mathbb{Z}\} > 0.$$

We now set $\widehat{w}_P(t) = w_P(t + t_P)$, where t_P is such that

$$\max\{|w_P(t_P)|, |w_P'(t_P)|\} = \max\{\max_t |w_P(t)|, \max_t |w_P'(t)|\}$$

and replace in the previous argument the family $\{w_P\}$ by $\{\widehat{w}_P\}$. The corresponding $w_\infty \in W^{2,2}(\mathbb{R})$ is then not identically 0 because $\max\{|w_\infty(0)|, |w_\infty'(0)|\} > 0$.

4. Analysis of minimising sequences

For $\mu > 0$ let us define the functional

$$\mathcal{J}_{\infty,\mu}(w) = \mathcal{K}_\infty(w) + \frac{\mu^2}{\mathcal{L}_\infty(w)}$$

with $\mathcal{J}_{\infty,\mu}(0) := \infty$, on the set $U_\infty = \{w \in W^{2,2}(\mathbb{R}) : \|w\|_{W^{2,2}(\mathbb{R})} < r\}$ where r is the real number R_2 or any smaller positive number (it can be decreased when needed; R_2 has been introduced in Theorem 2.1).

For $\mu > 0$ small enough, from the sequence of periodic water waves $\{w_{P_n}\}$ we can easily construct as in [1] a sequence $\{u_n\} \in U_\infty \subset W^{2,2}(\mathbb{R})$ that is minimising:

$$\mathcal{K}_\infty(u_n) + \frac{\mu^2}{\mathcal{L}_\infty(u_n)} \to c(\mu) := \inf\{\mathcal{K}_\infty(w) + \mu^2 \mathcal{L}_\infty(w)^{-1} : w \in U_\infty\},$$

that converges weakly in $W^{2,2}(\mathbb{R})$ to $w_\infty \not\equiv 0$, that stays away from the boundary of U_∞:

$$\limsup_{n\to\infty} \|u_n\|_{W^{2,2}(\mathbb{R})} < r/2 \tag{4.1}$$

and such each u_n has compact support. Moreover,

$$\liminf_{n\to\infty} \max\{\max_t |u_n(t)|, \max_t |u_n'(t)|\} > 0$$

because $w_\infty \neq 0$.

When $\beta > 1/3$, we proved in [1] the *strict* sub-additivity of the map $(0,\mu_0) \ni \mu \to c(\mu)$, that is,

$$c(\mu_1 + \mu_2) < c(\mu_1) + c(\mu_2)$$

for all $\mu_1, \mu_2 > 0$ such that $\mu_1 + \mu_2 < \mu_0$. In the present situation in which $0 < \beta < 1/3$, we do not know if *strict* sub-additivity holds, but sub-additivity does hold, as we shall see. But first let us state a useful lemma.

Lemm 4.1. *Consider an increasing sequence* $\{m_n\} \subset \{2,3,\dots\} \cup \{\infty\}$ *and a sequence* (*parametrised by* $n \in \mathbb{N}$) *of sequences* $\{u_{j,n} : 1 \le j < m_n\} \subset U_\infty \setminus \{0\}$ *such that each* $u_{j,n}$ *is compactly supported, the convex hulls (denoted* co*) of the supports of* $u_{i,n}$ *and* $u_{j,n}$ *are disjoint if* $i \neq j$,

$$\lim_{n\to\infty} \inf_{1\le i<j<m_n} \operatorname{dist}(\operatorname{co}\operatorname{supp}(u_{i,n}), \operatorname{co}\operatorname{supp}(u_{j,n})) = \infty$$

and

$$\sum_{1\le j<m_n} \|u_{j,n}\|^2_{W^{2,2}(\mathbb{R})} < r^2 \quad \textit{for all } n \in \mathbb{N}.$$

Under these hypotheses

$$\lim_{n\to\infty} \bigg(\mathcal{L}_\infty\bigg(\sum_{1\le j<m_n} u_{j,n}\bigg) - \sum_{1\le j<m_n} \mathcal{L}_\infty(u_{j,n})\bigg) = 0 \tag{4.2}$$

and

$$\lim_{n\to\infty} \bigg(\mathcal{K}_\infty\bigg(\sum_{1\le j<m_n} u_{j,n}\bigg) - \sum_{1\le j<m_n} \mathcal{K}_\infty(u_{j,n})\bigg) = 0. \tag{4.3}$$

Proof. With the help of (2.3), we get for fixed n and j such that $m_n > j$

$$\begin{aligned}
&\bigg|\sum_{\substack{1\le i<m_n\\ i\neq j}} N u_{i,n}(t)\bigg| \\
&\le C_2\bigg\{1 + \operatorname{dist}\bigg(t, \bigcup_{i\neq j} \operatorname{supp}(u_{i,n})\bigg)\bigg\}^{-2+(1/2)} \bigg\{\sum_{\substack{1\le i<m_n\\ i\neq j}} \|u_{i,n}\|^2_{W^{2,2}(\mathbb{R})}\bigg\}^{1/2} \\
&\le C_2\bigg\{1 + \operatorname{dist}\bigg(t, \bigcup_{i\neq j} \operatorname{supp}(u_{i,n})\bigg)\bigg\}^{-2+(1/2)} r^{1/2}
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} \sup_{t\in \mathrm{supp}(u_{j,n})} \left| N\Big(\sum_{1\le i<m_n} u_{i,n}\Big)(t) - Nu_{j,n}(t)\right| = 0$$

for all $1 \le j < \lim_{n\to\infty} m_n$, uniformly in j. This implies (4.3).

Inequality (2.3) is proved in [1] from

$$Nu(t) = \int_{\mathbb{R}} g(t-s)(u(s) - u''(s))\, ds$$

for all $u \in W^{2,2}(\mathbb{R})$, where g is in $L^2(\mathbb{R})$ and decreases at $\pm\infty$ faster than $|t|^{-k}$ for any $k \ge 1$. Choosing $k = 3$, we obtain for fixed j and large enough n (so that $m_n > j$)

$$\begin{aligned}
&\left|\int_{\mathbb{R}} N\Big(\sum_{i\ne j} u_{i,n}\Big) u_{j,n}\, dt\right| \\
&\quad\le \text{const.} \int_{\mathbb{R}}\int_{\mathbb{R}} |g(s)| \left|\sum_{i\ne j}\{u_{i,n}(t-s) - u''_{i,n}(t-s)\} u_{j,n}(t)\right| ds\, dt \\
&\quad= \text{const.} \int_{|s|\ge \mathrm{dist}(\mathrm{supp}(u_{i,n}),\mathrm{supp}(u_{j,n}))} |g(s)| \\
&\qquad\quad \cdot \left\{\int_{\mathbb{R}} \left|\sum_{i\ne j}\{u_{i,n}(t-s) - u''_{i,n}(t-s)\} u_{j,n}(t)\right| dt\right\} ds \\
&\quad\le \text{const.} \sum_{i\ne j} \mathrm{dist}(\mathrm{supp}(u_{i,n}), \mathrm{supp}(u_{j,n}))^{-2} \|u_{i,n}\|_{W^{2,2}(\mathbb{R})} \|u_{j,n}\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Summing over j,

$$\begin{aligned}
&\left|\sum_j \int_{\mathbb{R}} N\Big(\sum_{i\ne j} u_{i,n}\Big) u_{j,n}\, dt\right| \\
&\le \text{const.} \sum_j \sum_{i\ne j} \mathrm{dist}(\mathrm{supp}(u_{i,n}), \mathrm{supp}(u_{j,n}))^{-2} \|u_{i,n}\|_{W^{2,2}(\mathbb{R})} \|u_{j,n}\|_{W^{2,2}(\mathbb{R})} \\
&:= \text{const.} \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \alpha_{i,j} x_i x_j,
\end{aligned}$$

with

$$\alpha_{i,j} = \begin{cases} \mathrm{dist}(\mathrm{supp}(u_{i,n}), \mathrm{supp}(u_{j,n}))^{-2} & \text{if } 1 \le i, j < m_n \text{ and } i \ne j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x_i = \begin{cases} \|u_{i,n}\|_{W^{2,2}(\mathbb{R})} & \text{if } 1 \le i < m_n, \\ 0 & \text{otherwise.} \end{cases}$$

We perform now a relabelling $\mathbb{Z} \ni p \to i(p)$ (that depends on n) so that if $x_{i(p)} \neq 0$ and $x_{i(p+1)} \neq 0$ then $\operatorname{supp}(u_{i(p),n})$ is to the left of $\operatorname{supp}(u_{i(p+1),n})$, and so that if $x_{i(p)} \neq 0$ and $x_{i(p')} \neq 0$ with $p' > p$ then $x_{i(p+1)} \neq 0$. Hence

$$\begin{aligned}
&\Big|\sum_j \int_{\mathbb{R}} N\Big(\sum_{i\neq j} u_{i,n}\Big) u_{j,n}\, dt\Big| \\
&\qquad \leq \text{const.} \sum_{q\in\mathbb{Z}} \sum_{p\in\mathbb{Z}} \alpha_{i(p),i(p+q)} x_{i(p)} x_{i(p+q)} \\
&\qquad \leq \text{const.} \sum_{q\in\mathbb{Z}} \sup_{p\in\mathbb{Z}} \alpha_{i(p),i(p+q)} \sum_{p\in\mathbb{Z}} x_{i(p)} x_{i(p+q)} \\
&\qquad \leq \text{const.} \sum_{k\in\mathbb{Z}} x_k^2 \sum_{q\in\mathbb{Z}} \sup_{p\in\mathbb{Z}} \alpha_{i(p),i(p+q)} \\
&\qquad \leq \text{const. } r^2 \Big\{ \inf_{1\leq k<l<m_n} \operatorname{dist}(\operatorname{supp}(u_{k,n}), \operatorname{supp}(u_{l,n})) \Big\}^{-2} \sum_{q\neq 0} q^{-2} \\
&\qquad \leq \text{const. } r^2 \Big\{ \inf_{1\leq k<l<m_n} \operatorname{dist}(\operatorname{supp}(u_{k,n}), \operatorname{supp}(u_{l,n})) \Big\}^{-2} \to 0
\end{aligned}$$

as $n \to \infty$. It easily follows that

$$\lim_{n\to\infty} \sum_j \int_{\mathbb{R}} \Big\{ N\Big(\sum_i u_{i,n}\Big) - N(u_{j,n}) \Big\} u_{j,n}\, dt \to 0$$

and that (4.2) holds. □

Theorem 4.2. *There exists $\mu_0 > 0$ such that*

$$c(\mu_1 + \mu_2) \leq c(\mu_1) + c(\mu_2)$$

for all $\mu_1, \mu_2 > 0$ satisfying $\mu_1 + \mu_2 < \mu_0$.

Proof. Let $\{u_{1,n}\}$ and $\{u_{2,n}\}$ be minimising sequences with respect to μ_1 and μ_2, respectively, that satisfy (4.1) and such that each $u_{i,n}$ has compact support. Making shifts in the t-variable, we can further assume that

$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{co}\operatorname{supp}(u_{1,n}), \operatorname{co}\operatorname{supp}(u_{2,n})) = \infty.$$

By Lemma 4.1,

$$\lim_{n\to\infty} (\mathcal{K}_\infty(u_{1,n} + u_{2,n}) - \mathcal{K}_\infty(u_{1,n}) - \mathcal{K}_\infty(u_{2,n})) = 0$$

and

$$\lim_{n\to\infty} (\mathcal{L}_\infty(u_{1,n} + u_{2,n}) - \mathcal{L}_\infty(u_{1,n}) - \mathcal{L}_\infty(u_{2,n})) = 0,$$

which leads to

$$\begin{aligned}
c(\mu_1+\mu_2) &\le \liminf_{n\to\infty} \mathcal{J}_{\infty,\mu_1+\mu_2}(u_{1,n}+u_{2,n})\\
&= \liminf_{n\to\infty} \left\{ \mathcal{K}_\infty(u_{1,n}) + \mathcal{K}_\infty(u_{2,n}) + \frac{(\mu_1+\mu_2)^2}{\mathcal{L}_\infty(u_{1,n})+\mathcal{L}_\infty(u_{2,n})} \right\}\\
&\le \lim_{n\to\infty} \left\{ \mathcal{K}_\infty(u_{1,n}) + \mathcal{K}_\infty(u_{2,n}) + \frac{\mu_1^2}{\mathcal{L}_\infty(u_{1,n})} + \frac{\mu_2^2}{\mathcal{L}_\infty(u_{2,n})} \right\}\\
&= c(\mu_1)+c(\mu_2)
\end{aligned}$$

thanks to the inequality

$$\frac{x^2}{y} + \frac{(1-x)^2}{1-y} \ge 1 \quad \text{if } 0<x,y<1, \tag{4.4}$$

with equality exactly when $x=y$. □

The next theorem deals with what is called *dichotomy* in the standard concentration-compactness principle. Its proof can be found in [1].

THEOREM 4.3. *Let $\{u_n\} \subset U_\infty$ be a sequence that converges weakly in $W^{2,2}(\mathbb{R})$ to some $w_\infty \in U_\infty$ and that satisfies*

$$\limsup_{n\to\infty} \|u_n\|_{W^{2,2}(\mathbb{R})} < r.$$

Replacing $\{u_n\}$ by one of its subsequence if necessary, there exist two sequences $\{u_{1,n}\} \subset U_\infty$ and $\{u_{2,n}\} \subset U_\infty$ such that

(a) *for all $n \in \mathbb{N}$ and $j \in \{1,2\}$, the function $u_{j,n}$ has compact support $\operatorname{supp}(u_{j,n}) \subset \mathbb{R}$,*
(b) $\lim_{n\to\infty} \operatorname{dist}(\operatorname{supp}(u_{1,n}), \operatorname{supp}(u_{2,n})) = \infty$,
(c) $\lim_{n\to\infty} (\mathcal{K}_\infty(u_n) - \mathcal{K}_\infty(u_{1,n}) - \mathcal{K}_\infty(u_{2,n})) = 0$,
(d) $\lim_{n\to\infty} (\mathcal{L}_\infty(u_n) - \mathcal{L}_\infty(u_{1,n}) - \mathcal{L}_\infty(u_{2,n})) = 0$,
(e) $\lim_{n\to\infty} \|u_{1,n}+u_{2,n}-u_n\|_{L^2(\mathbb{R})} = 0$,
(f) $\lim_{n\to\infty} \|u_{1,n}-w_\infty\|_{L^2(\mathbb{R})} = 0$,
(g) $\limsup_{n\to\infty} \|u_{1,n}+u_{2,n}\|_{W^{2,2}(\mathbb{R})} \le \limsup_{n\to\infty} \|u_n\|_{W^{2,2}(\mathbb{R})} < r$,
(h) $\operatorname{supp}(u_{1,n}) \subset \operatorname{supp}(u_n)$ *and* $\operatorname{supp}(u_{2,n}) \subset \operatorname{supp}(u_n)$ *for all $n \in \mathbb{N}$.*

REMARK 4.4. By interpolation, it follows that

$$\lim_{n\to\infty} \|u_{1,n}+u_{2,n}-u_n\|_{W^{1,\infty}(\mathbb{R})} = 0.$$

Hence, for $i=1$ or $i=2$, taking a subsequence if necessary,

$$\liminf_{n\to\infty} \max\{\|u_{i,n}\|_{L^\infty(\mathbb{R})}, \|u_{i,n}'\|_{L^\infty(\mathbb{R})}\} = \liminf_{n\to\infty} \max\{\|u_n\|_{L^\infty(\mathbb{R})}, \|u_n'\|_{L^\infty(\mathbb{R})}\}.$$

Proposition 4.5. *There exists $\kappa > 0$ such that if $\mu \in (0, \mu_0)$ and $\{v_n\} \subset U_\infty$ satisfies $\mathcal{J}_{\infty,\mu}(v_n) \to c(\mu)$, then*

$$\liminf_{n\to\infty} \max\{\|v_n'\|_{L^\infty(\mathbb{R})}, \|v_n\|_{L^\infty(\mathbb{R})}\} \geq \kappa\mu^3.$$

Proof. Inequality (2.15) shows that, for some constant $\widetilde{\kappa} > 0$,

$$c(\mu) - 2\mu \leq -\widetilde{\kappa}\mu^3$$

for all $\mu \in (0, \mu_0)$ with $\mu_0 > 0$ small enough. Hence, defining

$$\mathcal{M}_\infty(v_n) := \mathcal{K}_\infty(v_n) - (1/2)\mathcal{K}_\infty''(0)(v_n, v_n),$$

we get

$$\begin{aligned} -\mathcal{M}_\infty(v_n) &= \frac{1}{2}\mathcal{K}_\infty''(0)(v_n, v_n) + \mu^2\mathcal{L}_\infty(v_n)^{-1} - \mathcal{J}_{\infty,\mu}(v_n) \\ &\geq 2\mu\sqrt{(1/2)\mathcal{K}_\infty''(0)(v_n, v_n)/\mathcal{L}_\infty(v_n)} - \mathcal{J}_{\infty,\mu}(v_n) \geq 2\mu - \mathcal{J}_{\infty,\mu}(v_n) \end{aligned}$$

and

$$\liminf_{n\to\infty} |\mathcal{M}_\infty(v_n)| \geq \liminf_{n\to\infty} \{2\mu - \mathcal{J}_{\infty,\mu}(v_n)\} \geq \widetilde{\kappa}\mu^3.$$

Moreover, as in (3.3),

$$|\mathcal{M}_\infty(v_n)| \leq \{\max_t |v_n'(t)| + \max_t |v(t)|\}\left(2K\beta + \frac{1}{2}\right)\int_{\mathbb{R}} \{v_n^2 + {v_n'}^2\}\, dt$$

and thus

$$\liminf_{n\to\infty} \max\{\|v_n'\|_{L^\infty(\mathbb{R})}, \|v_n\|_{L^\infty(\mathbb{R})}\} \geq \kappa\mu^3. \qquad \square$$

Lemma 4.6. *Let $w \in U_\infty\backslash\{0\}$ be a critical point of $\mathcal{J}_{\infty,\mu}$ such that $\mu \in (0, \mu_0)$ and $\mathcal{J}_{\infty,\mu}(w) < 2\mu$. Then there exists a constant $D > 0$ such that*

$$\|w\|_{W^{2,2}(\mathbb{R})}^2 \leq D\mu.$$

Proof. The critical point w satisfies (3.1) with $\nu^2/\Lambda = \mu^2/\mathcal{L}(w)^2 < 4$ and $\mathcal{K}_\infty(w) < 2\mu$ (because $\mathcal{J}_{\infty,\mu}(w) < 2\mu$). As (3.1) can be written in the form

$$w - \beta w'' = \frac{d}{dt} f(w', Nw) + N(g(w', Nw)) - wNw - N(w^2/2) + \nu^2 Nw,$$

where f, g are smooth functions of order 2 at the origin, we get by multiplying by $w - w''$ and integrating over $\mathbb{R}$ that

$$\begin{aligned} \int_{\mathbb{R}} \{w^2 + (1+\beta)|w'|^2 + \beta|w''|^2\}\, dt &\leq O(r)\int_{\mathbb{R}} \{|w''|^2 + |Nw'||w''|\}\, dt \\ &\quad + \int_{\mathbb{R}} \left|\frac{d}{dt} g(w', Nw)\right| |Nw'|\, dt + O(1)\|w\|_{W^{1,2}(\mathbb{R})}\|w\|_{W^{2,2}(\mathbb{R})} \\ &\leq O(r)\int_{\mathbb{R}} \{|Nw'|^2 + |w''|^2\} dt + O(1)\|w\|_{W^{1,2}(\mathbb{R})}\|w\|_{W^{2,2}(\mathbb{R})} \end{aligned}$$

and therefore, for small enough r,

$$\|w\|^2_{W^{2,2}(\mathbb{R})} \le O(1)\|w\|^2_{W^{1,2}(\mathbb{R})} \le O(1)\mathcal{K}_\infty(w) \le O(1)\mu. \qquad \square$$

Lemma 4.7. *For $\mu_0 > 0$ small enough, let $\{u_n\} \subset U_\infty$ satisfy*

$$\limsup_{n\to\infty} \|u_n\|_{W^{2,2}(\mathbb{R})} < r, \quad \lim_{n\to\infty} \mathcal{J}_{\infty,\mu}(u_n) = c(\mu)$$

for some $\mu \in (0,\mu_0)$ and be such that the following limit exists:

$$L := \lim_{n\to\infty} \mathcal{L}_\infty(u_n) > 0.$$

Then, after shifting in t each u_n and considering a subsequence if needed, the sequence $\{u_n\}$ can be assumed to converge weakly in $W^{2,2}(\mathbb{R})$ to some $w_\infty \in U_\infty$ such that

$$\max\{\|w_\infty\|_{L^\infty(\mathbb{R})}, \|w'_\infty\|_{L^\infty(\mathbb{R})}\} \ge \kappa\mu^3.$$

Let $\{u_{1,n}\}$, $\{u_{2,n}\} \subset U_\infty$ be given by Theorem 4.3, and define

$$\mu_1 = \mu\mathcal{L}_\infty(w_\infty)/L \quad \textit{and} \quad \mu_2 = \mu - \mu_1.$$

Then w_∞ is a global minimiser of $\mathcal{J}_{\infty,\mu_1}$ and

$$c(\mu) = \lim_{n\to\infty}\left\{\mathcal{K}_\infty(w_\infty) + \mathcal{K}_\infty(u_{2,n}) + \frac{\mu^2}{\mathcal{L}_\infty(w_\infty) + \mathcal{L}_\infty(u_{2,n})}\right\}.$$

If $\mu_2 > 0$, the sequence $\{u_{2,n}\}$ is a minimising sequence of $\mathcal{J}_{\infty,\mu_2}$,

$$c(\mu) = c(\mu_1) + c(\mu_2),$$
$$\limsup_{n\to\infty} \|u_{2,n}\|^2_{W^{2,2}(\mathbb{R})} \le \limsup_{n\to\infty} \|u_n\|^2_{W^{2,2}(\mathbb{R})} - \|w_\infty\|^2_{W^{2,2}(\mathbb{R})} < r^2,$$
$$\lim_{n\to\infty} \mathcal{L}_\infty(u_{2,n}) = L - \mathcal{L}_\infty(w_\infty).$$

Proof. For all n, choose t_n such that

$$\max\{|u_n(t_n)|, |u'_n(t_n)|\} = \max\{\max_t |u_n(t)|, \max_t |u'_n(t)|\}.$$

After extracting a subsequence, we can assume that $u_n(\cdot + t_n) \rightharpoonup w_\infty$ weakly in $W^{2,2}(\mathbb{R})$ for some $w_\infty \in W^{2,2}(\mathbb{R})$ such that $\max\{|w_\infty(0)|, |w'_\infty(0)|\} \ge \kappa\mu^3$ by Proposition 4.5. By considering $u_n(\,\cdot\, + t_n)$ instead of u_n, we can assume that $u_n \rightharpoonup w_\infty$. We then apply Theorem 4.3 to $\{u_n\}$, which gives two sequences $\{u_{1,n}\}$ and $\{u_{2,n}\}$ such that

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\mathcal{K}_\infty(w_\infty) + \mathcal{K}_\infty(u_{2,n}) + \frac{\mu^2}{\mathcal{L}_\infty(w_\infty) + \mathcal{L}_\infty(u_{2,n})}\right\} \\
&\quad = \lim_{n\to\infty}\left\{\mathcal{K}_\infty(u_{1,n}) + \mathcal{K}_\infty(u_{2,n}) + \frac{\mu^2}{\mathcal{L}_\infty(u_{1,n}) + \mathcal{L}_\infty(u_{2,n})}\right\} \\
&\quad = \lim_{n\to\infty}\left\{\mathcal{K}_\infty(u_n) + \frac{\mu^2}{\mathcal{L}_\infty(u_n)}\right\} = c(\mu).
\end{aligned}$$

If $\mu_2 = 0$, then $\mu_1 = \mu$, $\mathcal{L}_\infty(u_{2,n}) \to 0$ and $\mathcal{J}_{\infty,\mu}(w_\infty) = c(\mu)$, which ends the proof.

Let $\mu_2 > 0$ and define

$$\mu_{1,n} := \mu \frac{\mathcal{L}_\infty(u_{1,n})}{\mathcal{L}_\infty(u_{1,n}) + \mathcal{L}_\infty(u_{2,n})}, \quad \mu_{2,n} := \mu \frac{\mathcal{L}_\infty(u_{2,n})}{\mathcal{L}_\infty(u_{1,n}) + \mathcal{L}_\infty(u_{2,n})}.$$

Then

$$\begin{aligned}
&\mathcal{J}_{\infty,\mu_1}(w_\infty) + \lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(u_{2,n}) \\
&\quad = \lim_{n\to\infty} \left\{ \mathcal{K}_\infty(u_{1,n}) + \mathcal{K}_\infty(u_{2,n}) + \frac{\mu_{1,n}^2}{\mathcal{L}_\infty(u_{1,n})} + \frac{\mu_{2,n}^2}{\mathcal{L}_\infty(u_{2,n})} \right\} \\
&\quad = \lim_{n\to\infty} \left\{ \mathcal{K}_\infty(u_{1,n}) + \mathcal{K}_\infty(u_{2,n}) + \frac{\mu^2}{\mathcal{L}_\infty(u_{1,n}) + \mathcal{L}_\infty(u_{2,n})} \right\} \\
&\quad = \lim_{n\to\infty} \left\{ \mathcal{K}_\infty(u_n) + \frac{\mu^2}{\mathcal{L}_\infty(u_n)} \right\} = c(\mu),
\end{aligned}$$

which will imply $c(\mu) = c(\mu_1) + c(\mu_2)$ once it is shown that w_∞ is a minimiser of $\mathcal{J}_{\infty,\mu_1}$ and $\{u_{2,n}\}$ is a minimising sequence of $\mathcal{J}_{\infty,\mu_2}$.

Case 1. $\|w_\infty\|_{W^{2,2}(\mathbb{R})} \geq r/2$.

Let us show that w_∞ is a global minimiser of $\mathcal{J}_{\infty,\mu_1}$. Let $\widetilde{w}_\infty$ be in $(1/2)U_\infty$ and observe that

$$\limsup_{n\to\infty} \{ \|\widetilde{w}_\infty\|^2_{W^{2,2}(\mathbb{R})} + \|u_{2,n}\|^2_{W^{2,2}(\mathbb{R})} \} < r^2.$$

For $n \in \mathbb{N}$, let $\widetilde{u}_{1,n}$ have compact support and be such that $\|\widetilde{u}_{1,n} - \widetilde{w}_\infty\|_{W^{2,2}(\mathbb{R})} < 1/n$. By translating in t each $\widetilde{u}_{1,n}$ into some $u^*_{1,n}$ and each $u_{2,n}$ into some $u^*_{2,n}$, we can assume that

$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{co}\operatorname{supp}(u^*_{1,n}), \operatorname{co}\operatorname{supp}(u^*_{2,n})) = \infty.$$

We get by (4.4) and Lemma 4.1

$$\begin{aligned}
&\mathcal{J}_{\infty,\mu_1}(w_\infty) + \lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(u_{2,n}) = c(\mu) \leq \liminf_{n\to\infty} \mathcal{J}_{\infty,\mu}(u^*_{1,n} + u^*_{2,n}) \\
&\quad = \liminf_{n\to\infty} \left\{ \mathcal{K}_\infty(u^*_{1,n}) + \mathcal{K}_\infty(u^*_{2,n}) + \frac{\mu^2}{\mathcal{L}_\infty(u^*_{1,n}) + \mathcal{L}_\infty(u^*_{2,n})} \right\} \\
&\quad \leq \liminf_{n\to\infty} \left\{ \mathcal{K}_\infty(u^*_{1,n}) + \mathcal{K}_\infty(u^*_{2,n}) + \frac{\mu_{1,n}^2}{\mathcal{L}_\infty(u^*_{1,n})} + \frac{\mu_{2,n}^2}{\mathcal{L}_\infty(u^*_{2,n})} \right\} \\
&\quad = \mathcal{J}_{\infty,\mu_1}(\widetilde{w}_\infty) + \lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(u_{2,n})
\end{aligned}$$

and thus $\mathcal{J}_{\infty,\mu_1}(\widetilde{w}_\infty) \geq \mathcal{J}_{\infty,\mu_1}(w_\infty)$. As the infimum of $\mathcal{J}_{\infty,\mu_1}$ over $(1/2)U_\infty$ is the same as the one over U_∞, we deduce that w_∞ is a global minimiser of $\mathcal{J}_{\infty,\mu_1}$.

By the a priori estimate of Lemma 4.6, $\|w_\infty\|^2_{W^{2,2}(\mathbb{R})} \leq D\mu_1$ for some constant $D > 0$ independent of μ. Let $\mu_0 < r^2/(4D)$, so that this first case cannot occur.

Case 2. $\|w_\infty\|_{W^{2,2}(\mathbb{R})} < r/2$.

Consider a sequence of compactly supported functions $\{\widetilde{u}_{2,n} : n \in \mathbb{N}\} \subset U_\infty$ such that $\lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(\widetilde{u}_{2,n}) = c(\mu_2)$ and $\sup_{n\in\mathbb{N}} \|\widetilde{u}_{2,n}\|_{W^{2,2}(\mathbb{R})} < r/2$. For all n, choose $\widetilde{u}_{1,n}$ so that $\|\widetilde{u}_{1,n} - w_\infty\|_{W^{2,2}(\mathbb{R})} < 1/n$ and thus $\|\widetilde{u}_{1,n}\|_{W^{2,2}(\mathbb{R})} < r/2$ for large n (which does not necessarily holds for $\{u_{1,n}\}$). We translate in t each $\widetilde{u}_{2,n}$ into some $u^*_{2,n}$ and each $\widetilde{u}_{1,n}$ into some $u^*_{1,n}$, so that

$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{co}\operatorname{supp}(u^*_{1,n}), \operatorname{co}\operatorname{supp}(u^*_{2,n})) = \infty.$$

Note that $u^*_{1,n} + u^*_{2,n} \in U_\infty$ for large n. We get by (4.4)

$$\begin{aligned}
\mathcal{J}_{\infty,\mu_1}(w_\infty) + \lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(u_{2,n}) &= c(\mu) \le \liminf_{n\to\infty} \mathcal{J}_{\infty,\mu}(u^*_{1,n} + u^*_{2,n}) \\
&= \liminf_{n\to\infty} \left\{ \mathcal{K}_\infty(u^*_{1,n}) + \mathcal{K}_\infty(u^*_{2,n}) + \frac{\mu^2}{\mathcal{L}_\infty(u^*_{1,n}) + \mathcal{L}_\infty(u^*_{2,n})} \right\} \\
&\le \lim_{n\to\infty} \left\{ \mathcal{K}_\infty(u^*_{1,n}) + \mathcal{K}_\infty(u^*_{2,n}) + \frac{\mu^2_{1,n}}{\mathcal{L}_\infty(u^*_{1,n})} + \frac{\mu^2_{2,n}}{\mathcal{L}_\infty(u^*_{2,n})} \right\} \\
&= \mathcal{J}_{\infty,\mu_1}(w_\infty) + \lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(\widetilde{u}_{2,n}) = \mathcal{J}_{\infty,\mu_1}(w_\infty) + c(\mu_2)
\end{aligned}$$

and thus $\lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(u_{2,n}) = c(\mu_2)$ with equalities and limits everywhere. From the case of equality in (4.4), it follows that

$$\frac{\mathcal{L}_\infty(w_\infty)}{\mathcal{L}_\infty(w_\infty) + \lim_{n\to\infty} \mathcal{L}_\infty(\widetilde{u}_{2,n})} = \frac{\mu_1}{\mu} \tag{4.5}$$

if the limit in the denominator exists, which can be assumed.

Conclusion. To show that w_∞ is a global minimiser of $\mathcal{J}_{\infty,\mu_1}$, we argue like in the first case above by considering an arbitrary $\widetilde{w}_\infty \in (1/2)U_\infty$, but with $\{u_{2,n}\}$ replaced by $\{\widetilde{u}_{2,n}\}$ introduced in the second case. This can be done because

$$\limsup_{n\to\infty} \{\|\widetilde{w}_\infty\|^2_{W^{2,2}(\mathbb{R})} + \|\widetilde{u}_{2,n}\|^2_{W^{2,2}(\mathbb{R})}\} < r^2.$$

Namely, for $n \in \mathbb{N}$, let $\widetilde{u}_{1,n}$ have compact support and be such that $\|\widetilde{u}_{1,n} - \widetilde{w}_\infty\|_{W^{2,2}(\mathbb{R})} < 1/n$. By translating in t each $\widetilde{u}_{1,n}$ into some $u^*_{1,n}$ and each $\widetilde{u}_{2,n}$ into some $u^*_{2,n}$, we can assume that

$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{co}\operatorname{supp}(u^*_{1,n}), \operatorname{co}\operatorname{supp}(u^*_{2,n})) = \infty.$$

This time we introduce

$$\mu_{1,n} := \mu \frac{\mathcal{L}_\infty(u_{1,n})}{\mathcal{L}_\infty(u_{1,n}) + \mathcal{L}_\infty(\widetilde{u}_{2,n})} \quad \text{and} \quad \mu_{2,n} := \mu \frac{\mathcal{L}_\infty(\widetilde{u}_{2,n})}{\mathcal{L}_\infty(u_{1,n}) + \mathcal{L}_\infty(\widetilde{u}_{2,n})}$$

that still satisfy $\mu_{1,n} \to \mu_1$ and $\mu_{2,n} \to \mu_2$ because of (4.5). We get as in the first case

$$\mathcal{J}_{\infty,\mu_1}(w_\infty) + \lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(\widetilde{u}_{2,n}) \le \mathcal{J}_{\infty,\mu_1}(\widetilde{w}_\infty) + \lim_{n\to\infty} \mathcal{J}_{\infty,\mu_2}(\widetilde{u}_{2,n})$$

and thus $\mathcal{J}_{\infty,\mu_1}(\widetilde{w}_\infty) \ge \mathcal{J}_{\infty,\mu_1}(w_\infty)$. $\square$

Theorem 4.8. *Let $\{v_n\} \subset U_\infty$ satisfy*

$$\limsup_{n\to\infty} \|v_n\|_{W^{2,2}(\mathbb{R})} < r, \qquad \lim_{n\to\infty} \mathcal{J}_{\infty,\mu}(v_n) = c(\mu)$$

for some $\mu \in (0,\mu_0)$ and be such that the following limit exists:

$$L := \lim_{n\to\infty} \mathcal{L}_\infty(v_n) > 0.$$

Then there exist a finite or infinite sequence $\{w_j : 1 \le j < m\} \subset U_\infty \setminus \{0\}$ with $m \in \{2,3,\dots\} \cup \{\infty\}$, such that $\sum_{1\le j<m} \mathcal{L}_\infty(w_j) = L$ and

$$c(\mu) = \sum_{1\le j<m} \mathcal{K}_\infty(w_j) + \frac{\mu^2}{\sum_{1\le j<m} \mathcal{L}_\infty(w_j)}.$$

Moreover, each w_j is a global minimiser of $\mathcal{J}_{\infty,\mu_j}$ with

$$\mu_j := \mu \frac{\mathcal{L}_\infty(w_j)}{\sum_{1\le i<m} \mathcal{L}_\infty(w_i)},$$

$$c(\mu) = \sum_{1\le j<m} c(\mu_j), \qquad \sum_{1\le j<m} \|w_j\|^2_{W^{2,2}(\mathbb{R})} \le D \sum_{1\le j<m} \mu_j < \frac{r^2}{4}$$

and

$$\max\{\|w_j\|_{L^\infty(\mathbb{R})}, \|w_j'\|_{L^\infty(\mathbb{R})}\} \ge \kappa \bigg(\sum_{j\le i<m} \mu_i\bigg)^3 \quad \textit{for all } 1 \le j < m.$$

Finally there exist a subsequence $\{v_{n_q}\}$ and numbers $t_{j,q}$ for $q \in \mathbb{N}$ and $1 \le j < m$, such that

$$v_{n_q}(\,\cdot\, + t_{j,q}) \rightharpoonup w_j \quad \textit{weakly in } W^{1,2}(\mathbb{R}) \textit{ as } q \to \infty,$$

$$\lim_{k\to m} \limsup_{q\to\infty} \bigg\| v_{n_q} - \sum_{1\le j<k} w_j(\,\cdot\, - t_{j,q}) \bigg\|_{W^{s,2}(\mathbb{R})} = 0 \quad \textit{for all } s \in [0,2)$$

(when $m < \infty$, the first limit means simply that k is replaced by m), and

$$\lim_{q\to\infty} \inf\{|t_{i,q} - t_{j,q}| : 1 \le i < m,\ i \ne j\} = \infty \quad \textit{for all } 1 \le j < m$$

(the limit is not necessarily uniform in j).

Proof. Let us first proof that there exists $m \in \{2,3,\dots\} \cup \{\infty\}$ such that, for any fixed k with $2 \le k < m$, there exist functions w_j ($1 \le j < k$) and a sequence $\{v_{k,n} : n \in \mathbb{N}\}$ satisfying the following properties: each w_j is a global minimiser of $\mathcal{J}_{\infty,\mu_j}$ and $\{v_{k,n}; n \in \mathbb{N}\}$ is a minimising sequence of $\mathcal{J}_{\infty,\nu_k}$, where

$$\mu_j = \mu \mathcal{L}_\infty(w_j)/L \quad \text{for } j = 1,\dots,k-1, \quad \text{and} \quad \nu_k := \mu - \sum_{j=1}^{k-1} \mu_j.$$

We set $\mathcal{J}_{\infty,\nu_k} = \mathcal{K}_\infty$ and $c(\nu_k) = c(0) := 0$ when $\nu_k = 0$. Moreover,

$$
(4.6)\; c(\mu) = \sum_{j=1}^{k-1} c(\mu_j) + c(\nu_k) = \sum_{1\le j<k} \mathcal{K}_\infty(w_j) + \frac{(\mu-\nu_k)^2}{\sum_{1\le j<k}\mathcal{L}_\infty(w_j)} + c(\nu_k),
$$

$$
\max\{\|w_j\|_{L^\infty(\mathbb{R})}, \|w_j'\|_{L^\infty(\mathbb{R})}\} \ge \kappa\bigg(\mu - \sum_{i=1}^{j-1}\mu_i\bigg)^3 \quad \text{for all } 1\le j<k,
$$

$$
\liminf_{n\to\infty} \max\{\|v_{k,n}\|_{L^\infty(\mathbb{R})}, \|v_{k,n}'\|_{L^\infty(\mathbb{R})}\} \ge \kappa\nu_k^3,
$$

$$
\sum_{1\le j<k} \|w_j\|^2_{W^{2,2}(\mathbb{R})} \le D\sum_{1\le j<k}\mu_j < \frac{r^2}{4},
$$

$$
\limsup_{n\to\infty} \|v_{k,n}\|^2_{W^{2,2}(\mathbb{R})} \le \limsup_{n\to\infty}\|v_n\|^2_{W^{2,2}(\mathbb{R})} - \sum_{1\le j<k}\|w_j\|^2_{W^{2,2}(\mathbb{R})} < r^2,
$$

$$
\lim_{n\to\infty}\mathcal{L}_\infty(v_{k,n}) = L - \sum_{1\le j<k}\mathcal{L}_\infty(w_j).
$$

We first apply Lemma 4.7 to the sequence $\{u_n\} := \{v_n\}$, which gives two sequences $\{u_{1,n}\}$ and $\{u_{2,n}\}$, a function $w_\infty \in U_\infty \setminus \{0\}$ and two real numbers $\widetilde{\mu}_1 = \mu\mathcal{L}_\infty(w_\infty)/L$ and $\widetilde{\mu}_2 = \mu - \widetilde{\mu}_1$. Note that $\{u_n\}$ has been possibly replaced by a subsequence and that a shift in t is allowed on each v_n in order to ensure that $w_\infty \ne 0$. We get the above statement for $k = 2$ with $\{v_{2,n}\} := \{u_{2,n}\}$, $w_1 := w_\infty$, $\mu_1 := \widetilde{\mu}_1$ and $\nu_2 := \widetilde{\mu}_2$.

Arguing by induction, assume that, for $k \ge 2$ finite, we have already obtained the functions $w_1, \dots, w_{k-1}$, a sequence $\{v_{k,n}\}$ and real numbers $\mu_1, \dots, \mu_{k-1}, \nu_k$. If $\nu_k = 0$, then $m := k$ and the theorem is fully proved. Let us therefore assume that $\nu_k > 0$ and apply Lemma 4.7 to the sequence $\{u_n\} := \{v_{k,n}\}$, which gives two sequences $\{u_{1,n}\}$ and $\{u_{2,n}\}$, a function $w_\infty \in U_\infty\setminus\{0\}$ and two real numbers $\widetilde{\mu}_1 = \mu\mathcal{L}_\infty(w_\infty)/L$ and $\widetilde{\mu}_2 = \mu - \widetilde{\mu}_1$ such that $c(\nu_k) = c(\widetilde{\mu}_1) + c(\widetilde{\mu}_2)$. Note that $\{v_{k,n}\}$ has been possibly replaced by a subsequence and that a shift in t is allowed on each $v_{k,n}$ in order to ensure that $w_\infty \ne 0$. Then we set $\{v_{k+1,n}\} := \{u_{2,n}\}$, $w_k := w_\infty$, $\mu_k := \widetilde{\mu}_1$ and $\nu_{k+1} := \widetilde{\mu}_2$. We easily get (4.6). By Proposition 4.5,

$$
\liminf_{n\to\infty} \max\{\|v_{k,n}\|_{L^\infty(\mathbb{R})}, \|v_{k,n}'\|_{L^\infty(\mathbb{R})}\} \ge \kappa\nu_k^3
$$

and therefore

$$
\max\{\|w_k\|_{L^\infty(\mathbb{R})}, \|w_k'\|_{L^\infty(\mathbb{R})}\} \ge \kappa\nu_k^3.
$$

The induction does not stop if $\sum_{j=1}^{k-1}\mathcal{L}_\infty(w_j) < L$ for all finite k, in which case we take the limit $k\to\infty$ and set $m = \infty$. From (4.6) we get

$$
\sum_{j=1}^{\infty}\mathcal{K}_\infty(w_j) + \frac{\mu^2}{\sum_{j=1}^{\infty}\mathcal{L}_\infty(w_j)} = \sum_{1\le j<m} c(\mu_j) = c(\mu)
$$

because $c(\nu_k) < 2\nu_k$, $\kappa\nu_k^3 \leq \sqrt{D\mu_k}$ by Lemma 4.6, $\sum_{1\leq j<m}\mu_j < \infty$ and thus

$$\lim_{k\to\infty} c(\nu_k) = \lim_{k\to\infty} \nu_k = \lim_{k\to\infty} \mu_k = 0.$$

This also gives

$$\lim_{k\to\infty}\Big\{L - \sum_{1\leq j<k} \mathcal{L}_\infty(w_j)\Big\} = \lim_{k\to\infty}\frac{L}{\mu}\Big\{\mu - \sum_{1\leq j<k}\mu_j\Big\} = \lim_{k\to\infty}\frac{L\nu_k}{\mu} = 0,$$

$$\mu_j = \frac{\mu\mathcal{L}_\infty(w_j)}{L} = \mu\mathcal{L}_\infty(w_j)\Big\{\sum_{1\leq j<m}\mathcal{L}_\infty(w_j)\Big\}^{-1}$$

and

$$\begin{aligned}\lim_{k\to\infty}\limsup_{n\to\infty}\|v_{k,n}\|^2_{L^2(\mathbb{R})} &\leq 2\lim_{k\to\infty}\lim_{n\to\infty}\mathcal{L}_\infty(v_{k,n})\\ &= 2\lim_{k\to\infty}\Big\{L - \sum_{1\leq j<k}\mathcal{L}_\infty(w_j)\Big\} = 0.\end{aligned}$$

From $\limsup_{n\to\infty}\|v_{k,n}\|_{W^{2,2}(\mathbb{R})} < r$, a standard interpolation gives

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|v_{k,n}\|_{W^{s,2}(\mathbb{R})} = 0 \quad \text{for all } s \in [0,2).$$

When $m = \infty$, the subsequence $\{v_{n_q}\}$ is obtained as a "diagonal" subsequence. □

Theorem 4.9. *Let $\mu \in (0,\mu_0)$ and consider an arbitrary finite or infinite sequence*

$$\{w_j : 1 \leq j < m\} \subset U_\infty \setminus \{0\}, \quad m \in \{2,3,\dots\} \cup \{\infty\},$$

such that $\sum_{1\leq j<m}\|w_j\|^2_{W^{2,2}(\mathbb{R})} < r^2$ and

$$\sum_{1\leq j<m}\mathcal{K}_\infty(w_j) + \frac{\mu^2}{\sum_{1\leq j<m}\mathcal{L}_\infty(w_j)} \leq c(\mu). \tag{4.7}$$

Then there is equality in (4.7), each w_j is a global minimiser of $\mathcal{J}_{\infty,\mu_j}$ with

$$\mu_j = \mu\mathcal{L}_\infty(w_j)\Big(\sum_{1\leq i<m}\mathcal{L}_\infty(w_i)\Big)^{-1}, \quad c(\mu) = \sum_{1\leq j<m} c(\mu_j)$$

and, after relabelling the sequence,

$$\max\{\|w_j\|_{L^\infty(\mathbb{R})}, \|w_j'\|_{L^\infty(\mathbb{R})}\} \geq \kappa\Big(\sum_{j\leq i<m}\mu_i\Big)^3 \tag{4.8}$$

for all $1 \leq j < m$. Define

$$\widetilde{\mu} = \inf \max_{1\leq j<m} \mu_j > 0$$

where the infimum is taken over all such $\{w_j\}$*. Then every minimising sequence* $\{u_n\} \subset U_\infty$ *of* $\mathcal{J}_{\infty,\widetilde{\mu}}$ *that converges weakly in* $W^{2,2}(\mathbb{R})$ *to a non trivial limit and stays away from* ∂U_∞*, converges strongly in* $L^2(\mathbb{R})$ *to this limit.*

Proof. Let m and $\{w_j : 1 \leq j < m\}$ be as in the statement. For $n \in \mathbb{N}$, let $w_{j,n}$ have compact support and be such that $\|w_{j,n} - w_j\|_{W^{2,2}(\mathbb{R})} < 2^{-j}/n$ for $1 \leq j < m$. By translating in t each $w_{j,n}$ into some $w^*_{j,n}$, we can assume that

$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{co}\operatorname{supp}(w^*_{i,n}), \operatorname{co}\operatorname{supp}(w^*_{j,n})) = \infty$$

uniformly in $1 \leq i < j < m$ (this is needed in Lemma 4.1) and

$$\limsup_{n\to\infty} \sum_{1\leq j<m} \|w^*_{j,n}\|^2_{W^{2,2}(\mathbb{R})} < r^2.$$

Then the sequence $\{v_n\}$ defined for large n by $v_n = \sum_{1\leq j<m} w^*_{j,n}$ is a minimising sequence of $\mathcal{J}_{\infty,\mu}$ to which the previous theorem can be applied, giving a sequence $\{\widetilde{w}_j : 1 \leq j < \widetilde{m}\}$, which is nothing else than the original sequence $\{w_j : 1 \leq j < m\}$ after some relabelling and translations in t (and therefore $\widetilde{m} = m$). This proves the first half of the statement.

Consider an increasing sequence $\{m_n\} \subset \{2, 3, \dots\} \cup \{\infty\}$ and a sequence (parametrised by n) of sequences $\{w_{j,n} : 1 \leq j < m_n\}$ such that, in addition to the properties given in the first half of the statement,

$$\widetilde{\mu} = \lim_{n\to\infty} \max_{1\leq j<m_n} \mu_{j,n} \quad \text{with } \mu_{j,n} = \mu \frac{\mathcal{L}_\infty(w_{j,n})}{\sum_{1\leq i<m_n} \mathcal{L}_\infty(w_{i,n})}.$$

Without loss of generality, we assume that the following limit exists:

$$\lim_{n\to\infty} \sum_{1\leq j<m_n} \mathcal{L}_\infty(w_{j,n}).$$

Since, after relabelling in j,

$$\kappa\mu^3 \leq \max\{\|w_{1,n}\|_{L^\infty(\mathbb{R})}, \|w'_{1,n}\|_{L^\infty(\mathbb{R})}\} \leq \sqrt{D\mu_{1,n}}$$

by (4.8) and Lemma 4.6, we get $\widetilde{\mu} \geq (\kappa^2/D)\mu^6$.

Let $\widehat{w}_{j,n}$ have compact support and be such that $\|\widehat{w}_{j,n} - w_{j,n}\|_{W^{2,2}(\mathbb{R})} < 2^{-j}/n$ for $n \in \mathbb{N}$ and $1 \leq j < m_n$. By translating in t each $\widehat{w}_{j,n}$ into some $w^*_{j,n}$, we can assume that

$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{co}\operatorname{supp}(w^*_{i,n}), \operatorname{co}\operatorname{supp}(w^*_{j,n})) = \infty$$

uniformly in $1 \leq i < j < \lim_{n\to\infty} m_n$. Then the sequence $\{v_n\}$ defined for large n by $v_n = \sum_{1\leq j<m_n} w^*_{j,n}$ is a minimising sequence of $\mathcal{J}_{\infty,\mu}$ to which the previous theorem can be applied, giving a sequence $\{w_j; 1 \leq j < m\}$ that is in the considered class and enjoying the additional property

$$\widetilde{\mu} \leq \max_{1\leq j<m} \mu \frac{\mathcal{L}_\infty(w_j)}{\sum_{1\leq i<m} \mathcal{L}_\infty(w_i)} \leq \widetilde{\mu}. \tag{4.9}$$

Indeed, by Lemma 4.1,

$$\sum_{1\le i<m} \mathcal{L}_\infty(w_i) = \lim_{n\to\infty} \mathcal{L}_\infty(v_n)$$
$$= \lim_{n\to\infty} \sum_{1\le i<m_n} \mathcal{L}_\infty(w_{i,n}^*) = \lim_{n\to\infty} \sum_{1\le i<m_n} \mathcal{L}_\infty(w_{i,n}).$$

Moreover, for all $1 \le j < m$ and all $N \in \mathbb{N}$, the last statement of Theorem 4.8 ensures that w_j is in the sequentially weak closure in $W^{1,2}(\mathbb{R})$ of the set

$$\bigcup \{w_{i,n}(\,\cdot\, + t) : n \ge N,\ 1 \le i < m_n,\ t \in \mathbb{R}\}.$$

Hence $\mathcal{L}_\infty(w_j) \le \sup\{\mathcal{L}_\infty(w_{i,n}) : n \ge N,\ 1 \le i < m_n\}$ and, letting $N \to \infty$,

$$\mathcal{L}_\infty(w_j) \le \limsup_{n\to\infty} \max_{1\le i<m_n} \mathcal{L}_\infty(w_{i,n}) = \frac{\widetilde{\mu}}{\mu} \sum_{1\le i<m} \mathcal{L}_\infty(w_i).$$

This proves the second inequality in (4.9) (the first one follows from the definition of $\widetilde{\mu}$).

Let $\{u_n\}$ be a minimising sequence of $\mathcal{J}_{\infty,\widetilde{\mu}}$, and let $\widetilde{m} \in \{2, 3, \dots\} \cup \{\infty\}$ and $\{\widetilde{w}_j : 1 \le j < \widetilde{m}\}$ be given by the previous theorem applied to $\{u_n\}$, so that

$$\sum_{1\le j<\widetilde{m}} \mathcal{L}_\infty(\widetilde{w}_j) < \infty,$$

$$c(\widetilde{\mu}) = \sum_{1\le j<\widetilde{m}} \mathcal{K}_\infty(\widetilde{w}_j) + \frac{\widetilde{\mu}^2}{\sum_{1\le j<m} \mathcal{L}_\infty(\widetilde{w}_j)}$$

and each $\widetilde{w}_j$ is a global minimiser of $\mathcal{J}_{\infty,\widetilde{\mu}_j}$ with

$$\widetilde{\mu}_j = \widetilde{\mu}\mathcal{L}_\infty(\widetilde{w}_j)\Big(\sum_{1\le i<\widetilde{m}} \mathcal{L}_\infty(\widetilde{w}_i)\Big)^{-1}.$$

Suppose for contradiction that $\widetilde{m} > 2$, so that $\max_{1\le j<\widetilde{m}} \widetilde{\mu}_j < \widetilde{\mu}$. Recall that we proved the existence of a sequence $\{w_j : 1 \le j < m\}$ in the considered class such that

$$\max_{1\le j<m} \mu \frac{\mathcal{L}_\infty(w_j)}{\sum_{1\le i<m} \mathcal{L}_\infty(w_i)} = \widetilde{\mu}.$$

We now construct a new sequence $\{W_j : 1 \le j < M\}$ as follows: for each j such that

$$\mu \frac{\mathcal{L}_\infty(w_j)}{\sum_{1\le i<m} \mathcal{L}_\infty(w_i)} = \widetilde{\mu},$$

replace w_j by the full sequence $\{\widetilde{w}_i : 1 \le i < \widetilde{m}\}$; otherwise leave w_j as it is. The new sequence thus obtained by rearrangement satisfies $\sum_j \|W_j\|^2_{W^{2,2}(\mathbb{R})} < r^2$ by Lemma 4.6 and the fact that $\mu_0 < r^2/(4D)$. Moreover, this sequence is in the considered class and

$$\max_{1\le j<M} \mu \frac{\mathcal{L}_\infty(W_j)}{\sum_{1\le i<M} \mathcal{L}_\infty(W_i)} < \widetilde{\mu},$$

which is a contradiction. For simplicity, let us check these two last assertions in the special case that

$$\mu \frac{\mathcal{L}_\infty(w_j)}{\sum_{1\le i<m} \mathcal{L}_\infty(w_i)} = \widetilde{\mu} \quad \text{only when } j=1.$$

We get indeed

$$\begin{aligned} c(\mu) &= \sum_{1\le j<m} \mathcal{K}_\infty(w_j) + \frac{\mu^2}{\mathcal{L}_\infty(w_1)+\sum_{2\le j<m}\mathcal{L}_\infty(w_j)} \\ &= \sum_{1\le j<m} \mathcal{K}_\infty(w_j) + \frac{\widetilde{\mu}^2}{\mathcal{L}_\infty(w_1)} + \frac{(\mu-\widetilde{\mu})^2}{\sum_{2\le j<m}\mathcal{L}_\infty(w_j)} \\ &= \sum_{1\le j<\widetilde{m}} \mathcal{K}_\infty(\widetilde{w}_j) + \sum_{2\le j<m} \mathcal{K}_\infty(w_j) \\ &\quad + \frac{\widetilde{\mu}^2}{\sum_{1\le j<\widetilde{m}}\mathcal{L}_\infty(\widetilde{w}_j)} + \frac{(\mu-\widetilde{\mu})^2}{\sum_{2\le j<m}\mathcal{L}_\infty(w_j)} \\ &\ge \sum_{1\le j<\widetilde{m}} \mathcal{K}_\infty(\widetilde{w}_j) + \sum_{2\le j<m} \mathcal{K}_\infty(w_j) \\ &\quad + \frac{\mu^2}{\sum_{1\le j<\widetilde{m}}\mathcal{L}_\infty(\widetilde{w}_j)+\sum_{2\le j<m}\mathcal{L}_\infty(w_j)} \end{aligned}$$

with equality exactly when

$$\frac{\mathcal{L}_\infty(w_1)}{\sum_{1\le j<m}\mathcal{L}_\infty(w_j)} = \frac{\widetilde{\mu}}{\mu} = \frac{\sum_{1\le j<\widetilde{m}}\mathcal{L}_\infty(\widetilde{w}_j)}{\sum_{1\le j<\widetilde{m}}\mathcal{L}_\infty(\widetilde{w}_j)+\sum_{2\le j<m}\mathcal{L}_\infty(w_j)},$$

that is, exactly when $\mathcal{L}_\infty(w_1) = \sum_{1\le j<\widetilde{m}}\mathcal{L}_\infty(\widetilde{w}_j)$. Hence $\{W_j\}$ is in the considered class and, by the first half of the statement, there is indeed equality. We thus get (recall that $\widetilde{m} > 2$)

$$\frac{\mathcal{L}_\infty(\widetilde{w}_j)}{\sum_{1\le i<\widetilde{m}}\mathcal{L}_\infty(\widetilde{w}_i)+\sum_{2\le i<m}\mathcal{L}_\infty(w_i)} < \frac{\widetilde{\mu}}{\mu} \quad \text{if } 1\le j<\widetilde{m}$$

and

$$\frac{\mathcal{L}_\infty(w_j)}{\sum_{1\le i<\widetilde{m}}\mathcal{L}_\infty(\widetilde{w}_i)+\sum_{2\le i<m}\mathcal{L}_\infty(w_i)} = \frac{\mathcal{L}_\infty(w_j)}{\sum_{1\le i<m}\mathcal{L}_\infty(w_i)} < \frac{\widetilde{\mu}}{\mu}$$

if $2\le j<m$, which contradicts the minimality of $\widetilde{\mu}$. □

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Manuscript received November 6, 2003

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