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MATTER AND ELECTROMAGNETIC FIELDS: REMARKS ON THE DUALISTIC AND UNITARIAN STANDPOINTS

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Abstract. The study of the relation of matter and the electromagnetic field is a classical, intriguing problem both from physical and mathematical point of view. This relation can be interpreted from two different standpoints which, following [5], are called *unitarian standpoint* and *dualistic standpoint*.

In this paper we briefly describe two models which are related to the unitarian and the dualistic standpoint respectively. For each model it is possible to prove the existence of solitary waves which can be interpreted as matter particles.

1. Electromagnetic fields and matter

First we recall some basic facts on Maxwell equations. The Maxwell equations for an electromagnetic field $\mathbf{E} = \mathbf{E}(t,x)$, $\mathbf{H} = \mathbf{H}(t,x)$ ($t \in \mathbb{R}$, $x \in \mathbb{R}^3$ are the time and space variables, respectively) are

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}, \tag{1.1}$$

$$\nabla \cdot \mathbf{E} = \rho, \tag{1.2}$$

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{1.3}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{1.4}$$

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$\rho = \rho(t,x)$ and $\mathbf{J} = \mathbf{J}(t,x)$ are, respectively, a scalar and a vector valued function which represent the charge and the current density of an external source.

In the empty space

$$\rho = 0, \quad \mathbf{J} = 0.$$

The first three equations (1.1)–(1.3) are respectively the Ampère, Gauss and Faraday laws.

Observe that from (1.1) we get

$$\frac{\partial \nabla \cdot \mathbf{E}}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

then, using (1.2), we get that ρ and $\mathbf{J}$ are related by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Now let $\mathbf{A}$, φ be the gauge potentials related to $\mathbf{E}$ and $\mathbf{H}$ by

$$\mathbf{H} = \nabla \times \mathbf{A} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi. \tag{1.5}$$

The first two Maxwell equations (1.1) and (1.2) can be written as follows

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi\right) + \nabla \times (\nabla \times \mathbf{A}) = \mathbf{J}, \tag{1.6}$$

$$\nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi\right) = -\rho. \tag{1.7}$$

And the other two equations (1.3) and (1.4) are obviously satisfied.

Let $\chi = \chi(t,x)$ be a scalar function, then it is easily verified that the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ and equations (1.6), (1.7) do not change under the gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla \chi, \quad \varphi \to \varphi - \frac{\partial \chi}{\partial t}.$$

The equations (1.6), (1.7) have a variational structure, namely they are the Euler equations of the functional

$$S_m(\varphi, \mathbf{A}) = \int L_m \, dx \, dt \tag{1.8}$$

where L_m is the Lagrangian

$$\begin{aligned} L_m &= \frac{1}{2}\left(\left|\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi\right|^2 - |\nabla \times \mathbf{A}|^2\right) + (\mathbf{J}|\mathbf{A}) - \rho \cdot \varphi \\ &= \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{H}|^2) + (\mathbf{J}|\mathbf{A}) - \rho \cdot \varphi. \end{aligned} \tag{1.9}$$

The energy of the electromagnetic field is given by (see [8])

$$\begin{aligned}(1.10)\qquad \mathcal{E} &= \int \left(\frac{\partial L_m}{\partial(\partial \mathbf{A}/\partial t)} \cdot \frac{\partial \mathbf{A}}{\partial t} - L_m \right) dx \\ &= \int \left(\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi \Big| \frac{\partial \mathbf{A}}{\partial t} \right) - \frac{1}{2} \left| \frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi \right|^2 + \frac{1}{2} |\nabla \times \mathbf{A}|^2 \right) dx \\ &\quad + \int (-(\mathbf{J}|\mathbf{A}) + \rho\varphi)\, dx.\end{aligned}$$

The problem of energy divergence. Now consider the electrostatic case, i.e. assume $\mathbf{A} = 0$, $\mathbf{J} = 0$, $\varphi = \varphi(x)$. Then

$$(1.11)\qquad \mathcal{E} = \int \left(-\frac{1}{2} |\nabla\varphi|^2 + \rho\varphi \right) dx.$$

We can give a simpler expression to $\mathcal{E}$ exploiting the fact that φ solves the equation (see (1.7))

$$-\Delta\varphi = \rho.$$

Infact, multiplying both sides of the above equation by φ and integrating, we have

$$(1.12)\qquad \int |\nabla\varphi|^2\, dx = \int \rho\varphi\, dx.$$

Inserting (1.12) in (1.11) we get

$$\mathcal{E} = \frac{1}{2} \int |\nabla\varphi|^2\, dx = \frac{1}{2} \int |\mathbf{E}|^2\, dx$$

which is the usual expression for the electrostatic energy.

Now suppose that we want to model matter particles as dimensionless points. In this model, the density ρ of a particle located at 0 is the Dirac measure. Then $\varphi = 1/|x|$ and the energy $\mathcal{E}$ diverges. As a consequence, the inertial mass of the particle diverges. The difficulties presented by this problem touch one of the most fundamental aspects of physics, the nature of an elementary particle. Although partial solutions, workable within limited areas can be given, the basic problems remain unsolved ([9, Section 17.1, p. 579], see also [10], [12]).

The divergence of the energy could be avoided if particles are supposed to have a space extension, namely, if matter is modelled as a field. Particles are usually stable; then they need to be described by solutions of field equations whose energy travels as a localized packet and which preserve this localization property under perturbations. These kind of solutions are usually called solitary waves (or solitons).

In order to build a field equation which presents the existence of solitary waves, there are two possible choices:

- (Dualistic standpoint) The matter is described as a field ψ which is the source of the electromagnetic field $(\mathbf{E}, \mathbf{H})$ and it is itself influenced by

$(\mathbf{E}, \mathbf{H})$. However it is not part of the electromagnetic field. In Section 2 we consider the case in which ψ is a complex field related to the nonlinear Klein–Gordon equation. In this case we get an Abelian gauge theory (cf. e.g. Section 1.4 in [14]).

- (Unitarian standpoint) There is only one physical entity, the electromagnetic field: the matter and the electromagnetic field have the same nature and the particles are solitary waves of the field. In Section 3 we present a unitarian field theory based on a nonlinear perturbation of the usual Maxwell equations in the spirit of the ideas of Born and Infeld [5].

The models we introduce in the next two sections are based only on two very general principles:

(IP) the Invariance with respect to the Poincaré group,

(NL) the presence of NonLinear terms which make the existence of solitary waves possible.

Here, the principles of Quantum Mechanics are not involved. Our analysis is purely mathematical and it aims to understand the consequences of (IP) and (NL). Thus, the function ψ does not need to be interpreted as the Ψ-function of quantum mechanics, even if the relation with quantum mechanics comes as an interesting and intriguing problem.

2. A model for the dualistic standpoint

In this section, following [2], we analyse a model for the dualistic standpoint. The linear second order equation for a scalar field which satisfies (IP) (see [3]) is the Klein–Gordon equation

$$\Box\psi + \Omega^2\psi = 0, \quad \Box = \frac{\partial^2}{\partial t^2} - \Delta$$

where $\psi = \psi(t, x)$ is a complex function.

The simplest nonlinear term which can be added to the Klein–Gordon equation is a homogeneous term $|\psi|^{p-2}\psi$. Then we get:

$$\Box\psi + \Omega^2\psi - |\psi|^{p-2}\psi = 0. \tag{2.1}$$

It is well known that, when $p \in (2, 6)$ and $|\omega| < |\Omega|$, the above equation admits standing wave solutions of frequency ω (see [11], [4]), namely solutions of the type

$$\psi(t, x) = u(x)e^{-i\omega t}, \quad \text{where } u, \omega \text{ real.}$$

The Lagrangian for (2.1) is

$$\mathcal{L}_0 = \frac{1}{2}\left[\left|\frac{\partial\psi}{\partial t}\right|^2 - |\nabla\psi|^2 - \Omega^2|\psi|^2\right] + \frac{1}{p}|\psi|^p. \tag{2.2}$$

Now we want to couple the complex field ψ with an electromagnetic field represented by the gauge potentials $(\mathbf{A},\varphi)$. This interaction is described by replacing the derivatives $\partial/\partial t$, $\partial/\partial x_j$ $(j=1,2,3)$ with respect to the time and space variables by the so called (Weyl) covariant derivatives

$$D_t=\frac{\partial}{\partial t}+iq\varphi,\quad D_j=\frac{\partial}{\partial x_j}-iqA_j$$

where A_j $(j=1,2,3)$ are the component of $\mathbf{A}$, i is the imaginary unit and q is a coupling constant. Also we will use the notation

$$\mathbf{D}=\nabla-iq\mathbf{A}.$$

Then the Lagrangian (2.2) becomes

$$\begin{aligned}\mathcal{L}_1&=\frac{1}{2}(|D_t\psi|^2-|\mathbf{D}\psi|^2-\Omega^2|\psi|^2)+\frac{1}{p}|\psi|^p\\&=\frac{1}{2}\left(\left|\frac{\partial\psi}{\partial t}+iq\varphi\psi\right|^2-|\nabla\psi-iq\mathbf{A}\psi|^2-\Omega^2|\psi|^2\right)+\frac{1}{p}|\psi|^p.\end{aligned}\tag{2.3}$$

Now we set

$$\psi(t,x)=u(t,x)\,e^{iS(t,x)},\quad \text{for } u,S\in\mathbb{R}.$$

Then (2.3) becomes

$$\mathcal{L}_1=\frac{1}{2}[u_t^2-|\nabla u|^2]+\frac{1}{2}[|\nabla S-q\mathbf{A}|^2-(S_t+q\varphi)^2+\Omega^2]u^2+\frac{1}{p}|u|^p\tag{2.4}$$

and the new action functional is

$$\mathcal{A}_1=\mathcal{A}_1(u,S,A,\varphi)=\iint\mathcal{L}_1\,dx\,dt.$$

Now by (2.4) it is clear that $\mathcal{L}_1$ is invariant under the combined gauge transformation

$$\mathbf{A}\to\mathbf{A}+\nabla\chi,\quad \varphi\to\varphi-\frac{\partial\chi}{\partial t},\quad S\to S+q\chi\tag{2.5}$$

where $\chi=\chi(x,t)$ is any smooth real map on the space-time.

Now observe that the use of the Weyl covariant derivatives (i.e. the interaction with an electromagnetic field) permits to get a “better” invariance of the new lagrangian $\mathcal{L}_1$. In fact in the original lagrangian $\mathcal{L}_0$ (2.2) we are allowed to change the phase only by a constant θ $(\psi\to e^{i\theta}\psi)$. The new lagrangian $\mathcal{L}_1$ presents a stronger invariance property since we are allowed to change the phase S (see (2.5)) by a function $q\chi(t,x)$ $(\psi\to e^{iq\chi(t,x)}\psi)$ of the point (t,x) in the space-time.

If the electromagnetic field is given, then we know (up to a gauge) the potentials $\mathbf{A}$, φ and the only unknowns of our problem are u and S (i.e. ψ). The equations for u, S

$$\delta\mathcal{A}_{1,u,S}=0$$

can be obtained by taking the variations of $\mathcal{A}_1$ with respect to u, S. These equations are

$$\Box u + [|\nabla S - q\mathbf{A}|^2 - (S_t + q\varphi)^2 + \Omega^2]u - |u|^{p-2}u = 0, \tag{2.6}$$

$$\frac{\partial}{\partial t}[(S_t + q\varphi)u^2] - \nabla \cdot [(\nabla S - q\mathbf{A})u^2] = 0. \tag{2.7}$$

The first equation (2.6) describes the dynamics of the field

$$\psi(t,x) = u(t,x)e^{iS(t,x)}.$$

The second equation is a continuity equation for the charge and current densities ρ and $\mathbf{J}$

$$\rho = q(S_t + q\varphi)u^2 = q \operatorname{Im}(\overline{\psi} D_t \psi), \tag{2.8}$$

$$\mathbf{J} = -q(\nabla S - q\mathbf{A})u^2 = -q \operatorname{Im}(\overline{\psi} \mathbf{D} \psi). \tag{2.9}$$

Observe that ρ and $\mathbf{J}$ depend not only on the field ψ but also on the gauge potentials $\mathbf{A}$, φ.

Now assume that $\mathbf{A}$, φ are not given but they are also unknowns of the problem. We obtain other two equations by coupling (2.6), (2.7) with the Maxwell equations.

To do this we cosider the lagrangian of the electromagnetic field $\mathbf{E}$, $\mathbf{H}$

$$\mathcal{L}_2 = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{H}|^2) = \frac{1}{2}|\mathbf{A}_t + \nabla\varphi|^2 - \frac{1}{2}|\nabla \times \mathbf{A}|^2.$$

Then the total action is

$$\mathcal{A} = \iint \mathcal{L}_1 + \mathcal{L}_2.$$

The variations of $\mathcal{A}$ with respect to φ and $\mathbf{A}$ give

$$\nabla \cdot (\mathbf{A}_t + \nabla\varphi) = q(S_t + q\varphi)u^2, \tag{2.10}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}(\mathbf{A}_t + \nabla\varphi) = q(\nabla S - q\mathbf{A})u^2. \tag{2.11}$$

These two equations are just the Maxwell equations in presence of a source ρ, $\mathbf{J}$ given by (2.8) and (2.9).

Concluding we get a system of 4 equations (2.6), (2.7), (2.10), (2.11) whose unknown are u, S, $\mathbf{A}$, φ.

2.1. Existence of standing waves. Now we consider the electrostatic case, namely we look for solutions of (2.6)–(2.11) of the following type

$$u = u(x), \quad S = \omega t, \quad \mathbf{A} = 0, \quad \varphi = \varphi(x).$$

These solutions consist of a standing wave $\psi(t,x) = u(x)e^{-i\omega t}$ which interacts with an electrostatic field described by $\varphi(x)$. With this ansatz, (2.7), (2.11) are satisfied and (2.6), (2.10) become (in the sequel we take for simplicity $q = 1$)

$$-\Delta u + [\Omega^2 - (\omega+\varphi)^2]u - |u|^{p-2}u = 0, \tag{2.12}$$

$$\Delta\varphi = (\omega+\varphi)u^2. \tag{2.13}$$

The following theorem can be proved [2]:

THEOREM 2.1. *Assume that* $4 < p < 6$ *and* $|\omega| < |\Omega|$. *Then there exist infinitely many solutions* (u,φ) *of* (2.12)–(2.13) *such that*

$$u \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx < \infty.$$

In [7] existence results for (2.12)–(2.13) have been obtained for $2 < p < 6$. In [6] the critical case $p = 6$ has been examined.

In the following we shall indicate the main steps of the proof of Theorem 2.1

Step 1. Let D denote the completion of C_0^∞ with respect to the norm

$$\|\varphi\| = \sqrt{\int |\nabla\varphi|^2\,dx}.$$

We look for critical points $(u,\,\varphi) \in H^1(\mathbb{R}^3) \times D$ of

$$F(u,\varphi) = \frac{1}{2}\int [|\nabla u|^2 - |\nabla\varphi|^2]\,dx + \frac{1}{2}\int [\Omega^2 - (\omega+\varphi)^2]u^2 - \frac{1}{p}\int |u|^p\,dx.$$

Observe that F is strongly indefinite, i.e. it is unbounded both from below and from above on infinite dimensional subspaces. Moreover, it is not even to avoid these difficulties we perform a reduction method.

Consider the partial derivatives $F'_\varphi(u,\varphi)$ and $F'_u(u,\varphi)$ of F in (u,φ), defined by

$$\begin{aligned} \langle F'_\varphi(u,\varphi),\zeta\rangle &= \left\langle F'(u,\varphi), \begin{bmatrix} 0 \\ \zeta \end{bmatrix} \right\rangle, \quad \zeta \in D, \\ \langle F'_u(u,\varphi),\xi\rangle &= \left\langle F'(u,\varphi), \begin{bmatrix} \xi \\ 0 \end{bmatrix} \right\rangle, \quad \xi \in H^1(\mathbb{R}^3). \end{aligned}$$

Now we fix $u \in H^1(\mathbb{R}^3)$ and take the partial derivative F'_φ of $F(u,\varphi)$ with respect to φ. It can be shown that there exists only one $\varphi \in D$ such that

$$F'_\varphi(u,\varphi) = 0. \tag{2.14}$$

More explicitly such φ solves the equation

$$\Delta\varphi = (\omega+\varphi)u^2. \tag{2.15}$$

Consider the map $\Phi\colon H^1(\mathbb{R}^3) \to D$ such that $\Phi(u) = \varphi$ solves (2.15). Now consider the functional

$$J(u) = F(u, \Phi(u)), \quad u \in H^1(\mathbb{R}^3).$$

Let u be a critical point of J. We shall show that $(u, \Phi(u))$ is a critical point of F. In fact for all $\zeta \in H^1(\mathbb{R}^3)$, using (2.14), we get

$$\begin{aligned} 0 = \langle J'(u), \zeta\rangle &= \langle F'_u(u, \Phi(u)), \zeta\rangle + \langle F'_\varphi(u, \Phi(u)), \Phi'(u)\zeta\rangle \\ &= \langle F'_u(u, \Phi(u)), \zeta\rangle \end{aligned} \tag{2.16}$$

then, using (2.16) and (2.14), for all $\begin{bmatrix}\zeta\\ \xi\end{bmatrix} \in H^1(\mathbb{R}^3) \times D$, we have

$$\begin{aligned} \left\langle F'(u, \Phi(u)), \begin{bmatrix}\zeta\\ \xi\end{bmatrix}\right\rangle &= \left\langle F'(u, \Phi(u)), \begin{bmatrix}\zeta\\ 0\end{bmatrix}\right\rangle + \left\langle F'(u, \Phi(u)), \begin{bmatrix}0\\ \xi\end{bmatrix}\right\rangle \\ &= \langle F'_u(u, \Phi(u)), \zeta\rangle + \langle F'_\varphi(u, \Phi(u)), \xi\rangle = 0. \end{aligned}$$

So $(u, \Phi(u))$ is a critical point of F. Clearly the viceversa holds, i.e. if (u, φ) is a critical point of F, then φ solves (2.15).

Step 2. So we are reduced to find critical points of $J(u) = F(u, \Phi(u))$. Easy calculations show that

$$J(u) = \frac{1}{2}\int (|\nabla u|^2 + |\nabla\Phi(u)|^2 + u^2\Phi(u)^2 + (\Omega^2 - \omega^2)u^2)\, dx - \frac{1}{p}\int |u|^p\, dx.$$

Observe that the fuctional J is even.

J is invariant with respect to the space translations, namely under the group action $u(x) \to u(x+a)$ ($a \in \mathbb{R}^3$).This causes a lack of compactness. To overcome this difficulty we restrict ourselves to radial functions $u = u(r)$, $r = |x|$. More precisely we shall consider the functional J on the subspace

$$H^1_r = \{u \in H^1(\mathbb{R}^3) : u = u(r),\ r = |x|\}.$$

H^1_r is a natural constraint for J, namely any critical point $u \in H^1_r$ of $J|_{H^1_r}$ is also a critical point of J. Then we are reduced to look for critical points of $J|_{H^1_r}$.

We recall (see [11], [4]) that, for $6 > p > 2$, H^1_r is compactly embedded into $L^p(\mathbb{R}^3)$. As a consequence, it can be shown that $J\,|_{H^1_r}$ satisfies the Palais–Smale compactness condition.

Finally the conclusion follows by using a well known equivariant version of the mountain pass theorem for even functionals.

2.2. Travelling solitary waves. Set

$$\mathbf{v} = (v, 0, 0), \quad \gamma = \frac{1}{\sqrt{1 - v^2}}.$$

By the Lorentz invariance of

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$$

given any solution $(\psi(t,x), \mathbf{A}(t,x), \varphi(t,x))$ of (2.6)–(2.7), (2.10)–(2.11) (with $\psi(t,x) = u(t,x)e^{iS(t,x)}$), we can get a family of solutions $(\psi_{\mathbf{v}}(t,x), \mathbf{A}_{\mathbf{v}}(t,x), \varphi_{\mathbf{v}}(t,x))$ just making a Lorentz transformation:

$$\psi_{\mathbf{v}}(t,x) = \psi(t',x'), \quad \varphi_{\mathbf{v}}(t,x) = \varphi'(t',x'), \quad \mathbf{A}_{\mathbf{v}}(t,x) = \mathbf{A}'(t',x') \tag{2.17}$$

where

$$t' = \gamma(t - vx_1), \qquad x' = \begin{bmatrix} \gamma(x_1 - vt) \\ x_2 \\ x_3 \end{bmatrix}$$

and

$$\varphi' = \gamma(\varphi + vA_1), \qquad \mathbf{A}' = (\gamma(A_1 + vt), A_2, A_3).$$

In particular, given the standing wave solution (whose existence is guaranteed by Theorem 2.1)

$$(\psi(t,x), \varphi(t,x), \mathbf{A}(t,x)) = (u(x)e^{-i\omega t}, \varphi(x), \mathbf{0})$$

we obtain a *travelling solitary wave*

$$\psi_{\mathbf{v}}(t,x_1,x_2,x_3) = u(x') \exp[-i\gamma\omega(t - vx_1)], \tag{2.18}$$

$$\varphi_{\mathbf{v}}(t,x) = \frac{\varphi(x)}{\sqrt{1-v^2}}, \tag{2.19}$$

$$\mathbf{A}_{\mathbf{v}}(t,x) = \begin{bmatrix} v\varphi(x')/\sqrt{1-v^2} \\ 0 \\ 0 \end{bmatrix}. \tag{2.20}$$

In particular, equation (2.18) can be written as follows

$$\psi_{\mathbf{v}}(t,x_1,x_2,x_3) = u\left(\frac{x_1 - vt}{\sqrt{1-v^2}}, x_2, x_3\right) e^{i(\mathrm{k}_{\mathbf{v}} \cdot \mathrm{x} - \omega_{\mathbf{v}} t)}, \tag{2.21}$$

with

$$\omega_{\mathbf{v}} = \gamma\omega, \quad \mathrm{k}_{\mathbf{v}} = \gamma\omega\mathbf{v}, \quad \mathrm{x} = (x_1, x_2, x_3). \tag{2.22}$$

This solution represents a solitary wave which travels with velocity $\mathbf{v} = (v,0,0)$ in the x_1 direction.

It is well known that the expression $\begin{bmatrix} \gamma \\ \gamma\mathbf{v} \end{bmatrix}$ is a 4-vector in the Minkowsky space (called 4-velocity); then also

$$\begin{bmatrix} \omega_{\mathbf{v}} \\ \mathrm{k}_{\mathbf{v}} \end{bmatrix} = \omega \begin{bmatrix} \gamma \\ \gamma\mathbf{v} \end{bmatrix} \tag{2.23}$$

is a 4-vector. On the other hand, also the energy-momentum $(E_{\mathbf{v}}, \mathbf{P}_{\mathbf{v}})$ of the solution $(\psi_{\mathbf{v}}, \varphi_{\mathbf{v}}, \mathbf{A}_{\mathbf{v}})$ is a four vector. This vector, for $\mathbf{v} = 0$, has the form $\begin{bmatrix} E_0 \\ 0 \end{bmatrix}$, thus in the moving frame it takes the form

$$\begin{bmatrix} E_{\mathbf{v}} \\ \mathbf{P}_{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \gamma E_0 \\ \gamma \mathbf{v} E_0 \end{bmatrix}. \tag{2.24}$$

Comparing (2.23) and (2.24), we get

$$E_{\mathbf{v}} = \frac{E_0}{\omega}\omega_{\mathbf{v}}, \quad \mathbf{P}_{\mathbf{v}} = \frac{E_0}{\omega}\mathrm{k}_{\mathbf{v}}.$$

If we set $\hbar = E_0/\omega$ we get the expressions

$$E_{\mathbf{v}} = \hbar\omega_{\mathbf{v}}, \tag{2.25}$$

$$\mathbf{P}_{\mathbf{v}} = \hbar\mathrm{k}_{\mathbf{v}}, \tag{2.26}$$

which formally are nothing else but the De Broglie relations. It is interesting to note that (2.25) and (2.26) are consequences of (IP) and (NL) and, in this context, they do not depend on the axioms of Quantum Mechanics.

3. A model for the unitarian standpoint

In [5] Born and Infeld introduce a new formulation of the Maxwell equations; they replace the usual Lagrangian density of the electromagnetic fields $\mathbf{E}$, $\mathbf{H}$

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{H}^2) = \frac{1}{2}\left(\left|\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right|^2 - |\nabla \times \mathbf{A}|^2\right) \tag{3.1}$$

with a modified Lagrangian

$$\mathcal{L} = 1 - \sqrt{1 - (\mathbf{E}^2 - \mathbf{H}^2)} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{H}^2) + o(\mathbf{E}^2 - \mathbf{H}^2). \tag{3.2}$$

Clearly the above Lagrangian defines a nonlinear theory of electromagnetism and the Maxwell theory is recovered for $\mathbf{E}, \mathbf{H} \to 0$. In this framework pointwise particles have finite energy (see [5] and Section 12.1 in [14]). However it can be shown [13] that, in the electrostatic case, the only finite energy solution of the Euler–Lagrange equation relative to (3.2) is the trivial one. This means that there is no self-induced electrostatic field. Then Born–Infeld theory is not unitarian.

Here we report some results contained in [1] where a unitarian field theory, based on a semilinear perturbation of (3.1), has been introduced.

We modify the usual Maxwell action in the empty space

$$S_m(\varphi\mathbf{A}) = \frac{1}{2}\iint \left[\left|\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right|^2 - |\nabla \times \mathbf{A}|^2\right] dx\, dt$$

in the following way:

$$\mathcal{S} = \frac{1}{2}\iint \left[\left|\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right|^2 - |\nabla \times \mathbf{A}|^2 + W(|\mathbf{A}|^2 - \varphi^2)\right] dx\, dt \tag{3.3}$$

where $W\colon \mathbb{R} \to \mathbb{R}$.

The argument of W is $|\mathbf{A}|^2 - |\varphi|^2$ in order to make this expression invariant for the Poincaré group and the equations consistent with special relativity.

Making the variation of $\mathcal{S}$ with respect to $\delta\mathbf{A}$, $\delta\varphi$ respectively, we get the equations

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right) + \nabla\times(\nabla\times\mathbf{A}) = W'(|\mathbf{A}|^2 - \varphi^2)\mathbf{A}, \tag{3.4}$$

$$-\nabla\cdot\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right) = W'(|\mathbf{A}|^2 - \varphi^2)\varphi. \tag{3.5}$$

If we set

$$\mathbf{H} = \nabla\times\mathbf{A}, \tag{3.6}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi, \tag{3.7}$$

$$\rho = W'(|\mathbf{A}|^2 - \varphi^2)\varphi, \tag{3.8}$$

$$\mathbf{J} = W'(|\mathbf{A}|^2 - \varphi^2)\mathbf{A}, \tag{3.9}$$

we get the equations:

$$\nabla\times\mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}(\mathbf{A},\varphi), \tag{3.10}$$

$$\nabla\cdot\mathbf{E} = \rho(\mathbf{A},\varphi), \tag{3.11}$$

$$\nabla\times\mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0, \tag{3.12}$$

$$\nabla\cdot\mathbf{H} = 0, \tag{3.13}$$

which, formally, are the Maxwell equations in the presence of matter if we interpret $\rho(\mathbf{A},\varphi)$ as the charge density and $\mathbf{J}(\mathbf{A},\varphi)$ as the current density. Notice that ρ and $\mathbf{J}$ are functions of the gauge potentials, so that we are in the presence of a unitarian theory. Hereafter the system (3.4)–(3.5) (or (3.10)–(3.13)) will be called SME.

We now make the following assumption on W:

(W1) there exists two positive constants $\varepsilon_1, \varepsilon_2 \ll 1$ such that

$$|W'(s)| \le \varepsilon_1 |s| \quad \text{for } |s| \le 1, \tag{3.14}$$

$$|W'(s)| \ge 1 \qquad \text{for } |s| \ge 1 + \varepsilon_2. \tag{3.15}$$

We set

$$\Omega_t(\mathbf{A},\varphi) = \{x \in \mathbb{R}^3 : ||\mathbf{A}(t,x)|^2 - \varphi(t,x)^2| \ge 1\}.$$

Ω_t represents the portion of space filled with matter at time t. Assumption (3.14) implies that ρ and $\mathbf{J}$ become negligible outside Ω_t and the above equations can be interpreted as the Maxwell equations in the empty space. Assumption (3.15) implies that ρ and $\mathbf{J}$ become strong inside Ω_t, at least in the region where $||\mathbf{A}(t,x)|^2 - \varphi(t,x)^2| \ge 1 + \varepsilon_2$.

3.1. Invariants of motion. In this section, we will assume that SME have sufficiently smooth solutions and we will analyze some of their properties. Also we will assume that these solutions are sufficiently small at infinity in such a way that we can perform integrations by part with null "boundary" terms. The main invariants of the motion of SME, namely the energy and the momentum can be calculated by using Noether's theorem. A direct calculation shows that they have the following expressions:

- (Energy)

$$\mathcal{E}(\mathbf{A},\varphi)=\frac{1}{2}\int\left(\left|\frac{\partial\mathbf{A}}{\partial t}\right|^2-|\nabla\varphi|^2+|\nabla\times\mathbf{A}|^2-W(|\mathbf{A}|^2-\varphi^2)\right)dx. \tag{3.16}$$

- (Momentum)

$$\mathbf{P}(\mathbf{A},\varphi)=\int\sum_{i=1}^{3}\left(\frac{\partial A_i}{\partial t}+\frac{\partial\varphi}{\partial x_i}\right)\nabla A_i\,dx. \tag{3.17}$$

Another invariant is the

- (Charge)

$$C(\mathbf{A},\varphi)=\int\rho(\mathbf{A},\varphi)\,dx=\int W'(|\mathbf{A}|^2-\varphi^2)\varphi\,dx. \tag{3.18}$$

In fact, if we take the divergence in (3.10) and the derivative with respect to t in (3.11), we easily get the continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{J}=0.$$

We can express the energy by a more meaningful expression which will be useful later:

Proposition 3.1. *The energy of the solutions of* SME *is*

$$\begin{aligned}\mathcal{E}(\mathbf{A},\varphi)&=\int\left(\frac{1}{2}|\mathbf{E}|^2+\frac{1}{2}|\mathbf{H}|^2-W'(\sigma)\varphi^2-\frac{1}{2}W(\sigma)\right)dx\\&=\frac{1}{2}\int(|\mathbf{E}|^2+|\mathbf{H}|^2)\,dx-\int\left(\rho\varphi+\frac{1}{2}W(\sigma)\right)dx\end{aligned}$$

where $\sigma=|\mathbf{A}|^2-\varphi^2$.

Proof. If we multiply equation (3.11) by φ and integrate in x we get

$$\int\left(\frac{\partial\mathbf{A}}{\partial t}\cdot\nabla\varphi+|\nabla\varphi|^2\right)dx-\int W'(\sigma)\varphi^2\,dx=0.$$

We add this expression to $\mathcal{E}(\mathbf{A},\varphi)$. Then

$$\begin{aligned}\mathcal{E}(\mathbf{A},\varphi) &= \int\bigg(\frac{1}{2}\left|\frac{\partial\mathbf{A}}{\partial t}\right|^2 + \frac{1}{2}|\nabla\varphi|^2 + \nabla\varphi\cdot\frac{\partial\mathbf{A}}{\partial t}\\ &\qquad + \frac{1}{2}|\nabla\times\mathbf{A}|^2 - W'(\sigma)\varphi^2 - \frac{1}{2}W(\sigma)\bigg)\,dx\\ &= \int\bigg(\frac{1}{2}\left|\frac{\partial\mathbf{A}}{\partial t} + \nabla\varphi\right|^2 + \frac{1}{2}|\nabla\times\mathbf{A}|^2 - W'(\sigma)\varphi^2 - \frac{1}{2}W(\sigma)\bigg)\,dx\\ &= \int\bigg(\frac{1}{2}|\mathbf{E}|^2 + \frac{1}{2}|\mathbf{H}|^2 - W'(\sigma)\varphi^2 - \frac{1}{2}W(\sigma)\bigg)\,dx.\end{aligned}$$

The second expression for the energy is obtained just using (3.8). □

The term

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\mathbf{E}|^2 + |\mathbf{H}|^2)\,dx$$

represents the energy of the electromagnetic field, while

$$-\int_{\mathbb{R}^3}\bigg(\frac{1}{2}W(\sigma) + W'(\sigma)\varphi^2\bigg)\,dx = -\int_{\mathbb{R}^3}\bigg(\rho\varphi + \frac{1}{2}W(\sigma)\bigg)\,dx \tag{3.19}$$

represents the energy of the matter (short range field such as the nuclear fields). It can be interpreted as bond energy and it is "concentrated" essentially in Ω_t.

3.2. Existence of static solutions. In this section we are interested in the static solutions of (3.4)–(3.5), namely in the solutions $\mathbf{A}$, φ, depending only on the space variable x, of the following equations:

$$\nabla\times(\nabla\times\mathbf{A}) = W'(|\mathbf{A}|^2 - \varphi^2)\mathbf{A}, \tag{3.20}$$

$$-\Delta\varphi = W'(|\mathbf{A}|^2 - \varphi^2). \tag{3.21}$$

The static solutions are critical points of the energy functional:

$$\mathcal{E}(\mathbf{A},\varphi) = \frac{1}{2}\int(|\nabla\times\mathbf{A}|^2 - |\nabla\varphi|^2 - W(|\mathbf{A}|^2 - \varphi^2))\,dx. \tag{3.22}$$

Proposition 3.2. *If $(\mathbf{A},\varphi)$ is a finite energy, static solution of* SME*, then*

$$\mathcal{E}(\mathbf{A},\varphi) = \frac{1}{3}\int(|\nabla\times\mathbf{A}|^2 - |\nabla\varphi|^2)\,dx = \int W(|\mathbf{A}|^2 - \varphi^2)\,dx.$$

Proof. Let $\lambda > 0$ and set

$$\varphi_\lambda(x) = \varphi(\lambda^{-1}x), \quad \mathbf{A}_\lambda(x) = \mathbf{A}(\lambda^{-1}x);$$

then, setting $y = \lambda^{-1}x$, we have

$$\begin{aligned}
&\mathcal{E}(\mathbf{A}_\lambda, \varphi_\lambda) \\
&\quad= \frac{1}{2}\int (|\nabla_x \times \mathbf{A}_\lambda(x)|^2 - |\nabla_x \varphi_\lambda(x)|^2)\, dx - \frac{1}{2}\int W(|\mathbf{A}_\lambda(x)|^2 - \varphi_\lambda(x)^2)\, dx \\
&\quad= \frac{\lambda}{2}\int (|\nabla_y \times \mathbf{A}(y)|^2 - |\nabla_y \varphi(y)|^2)\, dy - \frac{\lambda^3}{2}\int W(|\mathbf{A}(y)|^2 - \varphi(y)^2)\, dy.
\end{aligned}$$

Since u is a critical point of $\mathcal{E}(\mathbf{A}_\lambda, \varphi_\lambda)$,

$$\frac{d}{d\lambda}\mathcal{E}(\mathbf{A}_\lambda, \varphi_\lambda)\bigg|_{\lambda=1} = 0. \tag{3.23}$$

Let us compute this expression explicitly:

$$\frac{d}{d\lambda}\mathcal{E}(\mathbf{A}_\lambda, \varphi_\lambda) = \frac{1}{2}\int |\nabla_y \times \mathbf{A}(y)|^2 - |\nabla_y \varphi(y)|^2\, dy - \frac{3}{2}\lambda^2 \int W(|\mathbf{A}(y)|^2 - \varphi(y)^2)\, dy.$$

For $\lambda = 1$, using (3.23), we get

$$\frac{1}{3}\int (|\nabla \times \mathbf{A}|^2 - |\nabla \varphi|^2)\, dx = \int W(|\mathbf{A}|^2 - \varphi^2)\, dx. \tag{3.24}$$

Then

$$\begin{aligned}
\mathcal{E}(\mathbf{A}, \varphi) &= \frac{1}{2}\int (|\nabla \times \mathbf{A}|^2 - |\nabla \varphi|^2)\, dx - \frac{1}{2}\int W(|\mathbf{A}|^2 - \varphi^2)\, dx \\
&= \frac{1}{3}\int (|\nabla \times \mathbf{A}|^2 - |\nabla \varphi|^2)\, dx.
\end{aligned}$$

And, by (3.24), we have also

$$\mathcal{E}(\mathbf{A}, \varphi) = \int W(|\mathbf{A}|^2 - \varphi^2)\, dx. \qquad \square$$

In order to get the simplest static solutions, we make the following ansatz:

- $\varphi \neq 0$, $\mathbf{A} = 0$,
- $\varphi = 0$, $\mathbf{A} \neq 0$.

With these ansatz, we obtain the following equations:

- Electrostatic equation:

$$-\Delta\varphi = W'(-\varphi^2)\varphi. \tag{3.25}$$

- Magnetostatic equation:

$$\nabla \times (\nabla \times \mathbf{A}) = W'(|\mathbf{A}|^2)\mathbf{A}. \tag{3.26}$$

They correspond to the critical points respectively of the functionals

$$\begin{aligned} \mathcal{E}(\varphi) &= -\frac{1}{2}\int (|\nabla\varphi|^2 + W(-\varphi^2))\,dx, \\ \mathcal{E}(\mathbf{A}) &= \frac{1}{2}\int (|\nabla\times\mathbf{A}|^2 - W(|\mathbf{A}|^2))\,dx. \end{aligned} \tag{3.27}$$

In order to get solutions we need the following technical assumptions:

(W2) there exist positive constants $c_2, c_3 > 0$ such that

$$\begin{aligned} |W'(s)| &\le c_2|s|^{p/2-1}, \quad p < 6 \text{ for } |s| \ge 1, \\ |W'(s)| &\le c_3|s|^{q/2-1}, \quad q > 6 \text{ for } |s| \le 1. \end{aligned}$$

We have the following result:

Theorem 3.3. *Assume that W satisfies* (W2). *Then* (3.25) *possesses a finite energy, nontrivial solution if and only if there exists s_0 such that*

$$W(s_0) < 0. \tag{3.28}$$

Proof. Since W satisfies (W2) and (3.28), the if part follows from Theorem 4 in [4]. The only if part follows from the Pohozaev identity (see Proposition 1 in [4]). □

Unfortunately, by Proposition 3.2, the energy (rest mass) of the solutions of (3.25)

$$\mathcal{E}(\varphi) = -\frac{1}{3}\int |\nabla\varphi|^2\,dx = \int W(-\varphi^2)\,dx$$

is negative; they are not physically acceptable for our program.

Thus, if we want to avoid negative energy solutions, we are forced to assume

(W^+) $W(s) \ge 0$.

More exactly we have the following

Proposition. *Assume that W satisfies* (W^+) *and let* $(\mathbf{A}, \varphi)$ *be a finite energy, non trivial solution of the system* (3.20)–(3.21) *then:*

(a) $A(x) \ne 0$,
(b) *the total energy* $\mathcal{E}(\mathbf{A}, \varphi)$, *in* (3.22) *is positive,*
(c) *the bond energy* (3.19) *is negative.*

Proof. $A(x) \ne 0$ is a consequence of Theorem 3.3. In fact, arguing by contradiction, assume that $\mathbf{A} = 0$. Then φ solves (3.25). So, since W satisfies (W^+), by Theorem 3.3 we get that also $\varphi = 0$. This contradicts the assumption that $(\mathbf{A}, \varphi)$ is nontrivial.

Since $W(s) \geq 0$, by Proposition 3.2 we deduce that the total energy $\mathcal{E}(\mathbf{A}, \varphi)$ is positive. Consider now the bond energy

$$-\int\left(\rho(\sigma)\varphi + \frac{1}{2}W(\sigma)\right)dx, \quad \sigma = |\mathbf{A}|^2 - \varphi^2, \quad \rho(\sigma) = W'(\sigma)\varphi.$$

Now, $\int W(\sigma)\,dx$ is positive by (W^+); moreover by (3.21) we easily derive

$$\int \rho(\sigma)\varphi\,dx = \int W'(\sigma)\varphi^2\,dx = \int |\nabla\varphi|^2\,dx \geq 0. \qquad \square$$

Remark 3.5. The property (a) of the above proposition implies that any static solution carries a magnetic moment, even when $\varphi = 0$. Thus any "particle" is sensitive to external magnetic field even if it has no charge. This can be interpreted as the classical analogous of the spin. In the following we shall prove the existence of static solutions when $W \geq 0$.

In order to get solutions we need to make some other technical assumptions:

(W3) There are constants M_1 and M_2 such that

$$\begin{aligned} W(s) &\geq M_1|s|^{p/2}, \quad 2 < p < 6 \text{ for } |s| \geq 1, \\ W(s) &\geq M_2|s|^{q/2}, \quad q > 6 \text{ for } |s| \leq 1. \end{aligned}$$

(W4) $W \in C^2$ and $W''(s) > 0$ for $s \neq 0$.

Clearly (W3) imply (W^+). Moreover, given $\varepsilon_1, \varepsilon_2 > 0$, it is possible to choose suitable constants in (W3) and (W2) such that (W1) holds.

We have the following result

Theorem. *If* (W2)–(W4) *hold, then equation* (3.26) *has a nontrivial, finite energy solution. This solutions has radial symmetry, namely*

$$\mathbf{A}(x) = g^{-1}\mathbf{A}(gx) \quad \textit{for all } g \in O(3)$$

where $O(3)$ *is the orthogonal group in* $\mathbb{R}^3$.

The proof of above theorem is quite involved and it can be found in ([1]). Here we give an heuristic argument of the proof. Any possible critical point of (3.27) has infinite Morse index; namely, the second variation

$$\mathcal{E}''(\mathbf{A})[v]^2 = \int (|\nabla \times v|^2 - W'(|\mathbf{A}|^2)|v|^2 - 2W''(|\mathbf{A}|^2)(\mathbf{A}\cdot v)^2)\,dx$$

is negative definite on the infinite dimensional subspace

$$\{v = \nabla\varphi : \varphi \in C_0^\infty\},$$

so the usual tools of critical point theory cannot be used. On the other hand, it is not possible to work in the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) since the nonlinear term of the functional is not gauge invariant. To avoid this difficulty, we split any vector field $\mathbf{A}\colon \mathbb{R}^3 \to \mathbb{R}^3$ as follows

$$\mathbf{A} = u + v = u + \nabla w \tag{3.29}$$

where $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a divergence free vector field ($\nabla \cdot u = 0$) and $v\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a potential vector field $v = \nabla w$ (w scalar field). Since W is strictly convex, for every u with $\nabla \cdot u = 0$, we can find w_0 which minimizes the functional

$$w \mapsto \int W(|u + \nabla w|^2)\,dx$$

and set $w_0 = \Phi(u)$. Replacing (3.29) in (3.27) with $w = \Phi(u)$, we get a new functional

$$J(u) := \mathcal{E}(u, \Phi(u)) = \frac{1}{2}\int (|\nabla u|^2 - W(|u + \nabla\Phi(u)|^2))\,dx$$

which depends only on u. This functional has the Mountain Pass geometry. Then, we expect the existence of a nontrivial critical point u_0. Now, if J and the map $u \to \Phi(u)$ were sufficiently smooth in suitable function spaces, the field

$$\mathbf{A} = u_0 + \nabla[\Phi(u_0)]$$

would solve equation (3.25). However the lack of smoothness does not allow to carry out a rigorous simple proof directly

Remark 3.7. It is still an open question to prove the existence of solutions of (3.20)–(3.21) with $\varphi \neq 0$ under the assumption $W \geq 0$.

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