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# ORTHOGONAL TRAJECTORIES ON STATIONARY SPACETIMES UNDER INTRINSIC ASSUMPTIONS

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Abstract. Using global variational methods and coordinate free assumptions, we obtain existence and multiplicity results on stationary Lorentzian manifolds for solutions to the Lorentz force equation joining two spacelike submanifolds. Some examples and applications are provided.

## 1. Introduction

Let $(\mathcal{M}, g)$ be a Lorentzian manifold, that is $\mathcal{M}$ is a (connected), finite dimensional, smooth manifold with $\dim \mathcal{M} \geq 2$ and $g$ is a smooth, symmetric, two covariant tensor field such that, for any $z \in \mathcal{M}$, the bilinear form $g(z)[\,\cdot\,,\,\cdot\,]$ induced on $T_z\mathcal{M}$ is non-degenerate and of index $\nu(g) = 1$. In the remainder of the article, for simplicity of notation, $g$ will be also denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$. The points of $\mathcal{M}$ are called *events*.

Before introducing our problem we recall some basic notions of Lorentzian geometry (we refer to [5], [13], [16], [18], [21] for the background material used in the sequel). If $z \in \mathcal{M}$, a tangent vector $\zeta \in T_z\mathcal{M}$ is called timelike (respectively

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lightlike; spacelike) if $\langle \zeta, \zeta \rangle < 0$ (respectively $\langle \zeta, \zeta \rangle = 0$, $\zeta \neq 0$; $\langle \zeta, \zeta \rangle > 0$ or $\zeta = 0$). A submanifold $P$ of $\mathcal{M}$ is said spacelike (respectively, timelike; lightlike) if $g$ is positive definite on $P$ (respectively, $g$ is non-degenerate of index 1 on $P$; $g$ is degenerate on $P$).

Let $A$ be a smooth vector field on $\mathcal{M}$. We shall study the existence and the multiplicity of trajectories under the action of $A$ i.e. solutions of

$$\nabla_{\dot z} \dot z = ((A'(z))^* - A'(z))[\dot z] \tag{1.1}$$

where $\nabla$ denotes the covariant derivative relative to the Levi–Civita connection of the metric tensor $g$, $A'$ is the differential of the vector field $A$ and $(A'(z))^*$ denotes, for any $z \in \mathcal{M}$, the adjoint operator of $A'(z)$ on $T_z\mathcal{M}$ with respect to $g$. When $A \equiv 0$ solutions of (1.1) are geodesics. For this case, the problem of geodesic connectedness has been recently widely studied (see for example the survey [20]); in this context, lightlike geodesics joining two submanifolds has special physical interest [7].

In the general case for $A$, solutions of (1.1) joining two fixed events $p, q \in \mathcal{M}$ have been studied in [1]–[3], [9] where it is assumed that $\mathcal{M}$ is a *standard* static or stationary manifold, that is $\mathcal{M}$ admits a global space-time splitting as $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ where, for any $t \in \mathbb{R}$, $\mathcal{M}_0 \times \{t\}$ is a spacelike submanifold of $\mathcal{M}$ with metric independent of $t$ and, for any $x \in \mathcal{M}_0$, $\{x\} \times \mathbb{R}$ is a timelike submanifold of $\mathcal{M}$. Then, under assumptions of completeness for $\mathcal{M}_0$ and on the growth of the metric coefficients with respect to the given splitting, existence and multiplicity results can be obtained.

Recently (see [4]), some of the quoted results have been generalized to the case of orthogonal solutions of (1.1) joining two given submanifolds of $\mathcal{M}$ according to the following definition.

Definition 1.1. Let $\Sigma_1, \Sigma_2$ be two submanifolds of $\mathcal{M}$. A curve $z\colon [0,1] \to \mathcal{M}$ is called *orthogonal trajectory (under the action of A) joining* $\Sigma_1$ *to* $\Sigma_2$ if

(a) $z$ satisfies (1.1),
(b) $z(0) \in \Sigma_1, z(1) \in \Sigma_2$, $\dot z(0) \in T_{z(0)}\Sigma_1^{\perp}, \dot z(1) \in T_{z(1)}\Sigma_2^{\perp}$.

Aim of this paper is to study orthogonal trajectories joining two submanifolds using global variational methods and under intrinsic (i.e. coordinate free) assumptions. This approach has been introduced in [12] where the authors study the geodesic connectedness of $\mathcal{M}$. Here we extend the techniques and the results in [12] in two directions: (a) we consider a non-trivial field $A$; (b) we study orthogonal trajectories according to Definition 1.1. Moreover, some discussions, complementary to the ones in [12], are introduced in the appendixes.

We recall that a Lorentzian manifold $(\mathcal{M}, g)$ is stationary if it is endowed with a smooth timelike Killing vector field $Y$. A vector field $Y$ is Killing if

$L_Y g = 0$, where $L_Y g$ denotes the Lie derivative of $g$ with respect to $Y$. It is well known that $Y$ is Killing if the stages of its local flows are isometries i.e. if the metric tensor $g$ is invariant by the flow of $Y$.

Consider a $\mathcal{C}^1$-vector field $A$ on $\mathcal{M}$ which is stationary with respect to $g$ that is

$$L_Y A = [Y, A] = \nabla_Y A - \nabla_A Y = 0 \tag{1.2}$$

and two spacelike submanifolds $P_1$ and $P_2$ of $\mathcal{M}$.

We shall consider orthogonal trajectories under the action of $A$ joining $P_1$ and $P_2$. By standard arguments (see e.g. [14]) it can be proved that if $A$ is orthogonal to $P_1$ and $P_2$, that is,

$$\langle A(z), \zeta\rangle = 0 \quad \text{for all } z \in P_i, \text{for all } \zeta \in T_z P_i,\ i = 1, 2 \tag{1.3}$$

then, our problem has a variational structure i.e. orthogonal trajectories joining $P_1$ and $P_2$ are the critical points of the following functional, introduced in [6]

$$F(z) = \frac{1}{2}\int_0^1 \langle \dot z, \dot z\rangle\, ds + \int_0^1 \langle A(z), \dot z\rangle\, ds \tag{1.4}$$

on the manifold of the $\mathcal{C}^1$-curves $z\colon [0,1] \to \mathcal{M}$ such that $z(0) \in P_1$ and $z(1) \in P_2$.

We observe that equation (1.1) has a prime integral: in fact if $z\colon [0,1] \to \mathcal{M}$ is a solution of (1.1), then $E_z \in \mathbb{R}$ exists such that

$$\langle \dot z, \dot z\rangle = E_z \quad \text{on } [0,1] \tag{1.5}$$

(see e.g. [6]). Moreover, when $\mathcal{M}$ is stationary and (1.2) holds, a further conservation law can be derived: indeed if $z\colon [0,1] \to \mathcal{M}$ is a solution of (1.1), $C_z \in \mathbb{R}$ exists such that

$$\langle \dot z + A(z), Y(z)\rangle = C_z \quad \text{on } [0,1] \tag{1.6}$$

(see Section 2).

The functional $F$ defined at (1.4) is strongly indefinite, nevertheless using (1.6) a suitable variational principle allows us to apply variational techniques (see Section 3).

Remark 1.2. Let us set

$$\begin{aligned} C_{P_1,P_2} = \{z \in \mathcal{C}^1([0,1], \mathcal{M}) \mid z(0) \in P_1,\ z(1) \in P_2,& \\ \langle \dot z + A(z), Y(z)\rangle = C_z\}.& \end{aligned} \tag{1.7}$$

We observe that $C_{P_1,P_2}$ may be empty. This possibility is discussed in Appendix B, where some conditions are also given in order to ensure that $C_{P_1,P_2}$ is not empty. Notice that a sufficient condition is to assume that $Y$ is complete i.e. all its flow lines are defined on the whole real axis.

In our problem, the following condition replaces the completeness condition for Riemannian manifolds.

Definition 1.3. Let $F$ be as in (1.4) and $c \in \mathbb{R}$ be such that $c > \inf_{C_{P_1,P_2}} F$. The set $C_{P_1,P_2}$ is said to be *c-precompact* if every sequence $(z_n) \subset C_{P_1,P_2}$ with $F(z_n) \leq c$ has a uniformly convergent subsequence in $\mathcal{M}$. We say that $F$ is *pseudocoercive* on $C_{P_1,P_2}$ if $C_{P_1,P_2}$ is $c$-precompact for any $c > \inf_{C_{P_1,P_2}} F$.

Definition 1.3 allows us to state the following existence result (see Section 5).

Theorem 1.4. *Let $\mathcal{M}$ be a stationary Lorentzian manifold endowed with a timelike, Killing vector field $Y$. Let $P_1$ and $P_2$ be two closed, spacelike submanifolds of $\mathcal{M}$ such that either $P_1$ or $P_2$ is compact. Let $A$ be a smooth vector field on $\mathcal{M}$ satisfying* (1.2), (1.3). *If $C_{P_1,P_2}$ is not empty and, for some $c > \inf_{C_{P_1,P_2}} F$, $C_{P_1,P_2}$ is c-precompact, then at least an orthogonal trajectory joining $P_1$ to $P_2$ in $\mathcal{M}$ exists.*

Under some topological assumptions on $\mathcal{M}$ we can also prove a multiplicity result (see Section 5).

Theorem 1.5. *Let $\mathcal{M}$, $P_1$, $P_2$, $Y$, $A$ satisfy the assumptions of Theorem* 1.4. *Moreover, let $Y$ be complete, $\mathcal{M}$ non-contractible in itself, $P_1$ and $P_2$ contractible in $\mathcal{M}$ and $F$ pseudocoercive on $C_{P_1,P_2}$. Then a sequence $(z_n)$ of orthogonal trajectories joining $P_1$ to $P_2$ in $\mathcal{M}$ exists such that* $\lim_{n\to\infty} F(z_n) = \infty$.

Remark 1.6. We point out that our problem has a physical interpretation. Indeed, the Lorentz world-force law which determines the motion of relativistic particles $\gamma$ submitted to an electromagnetic field is the Euler–Lagrange equation related to the action functional

$$\mathcal{S}(z) = -m_0 c \frac{1}{2} \int_{s_0}^{s_1} \sqrt{-\langle \dot z, \dot z\rangle}\, ds + q \int_{s_0}^{s_1} \langle A\ (z), \dot z\rangle\, ds$$

where $m_0$ is the rest mass of the particle, $q$ is its charge, $c$ is the speed of light (see [15]). In [6] it is proved that for timelike trajectories the search of critical points of $\mathcal{S}$ is equivalent to that of the critical points of $F$. In particular, when $E_\gamma < 0$ (see (1.5)), this constant of the motion turns to be, up to a dimensional factor, the inertial mass (necessarily equal to the gravitational mass), which is determined by the initial conditions, [6]. On the other hand, one of the spacelike submanifolds $P_i$ may represent an astronomical object under an electromagnetic field such as a neutron star. Of course, it is also interesting to consider the timelike submanifold generated by the world-lines of the particles in $P_i$, as we will discuss in what follows.

The previous remark makes clear that from a physical point of view it is interesting to prove existence and multiplicity results for timelike trajectories.

To this aim, we assume that $Y$ is complete and denote by $\psi\colon \mathcal{M}\times\mathbb{R}\to\mathcal{M}$ its flow. Then, for any $t\in\mathbb{R}$ we can consider the submanifold of $\mathcal{M}$ given by

$$P_t = \psi(P_2,t). \tag{1.8}$$

As $Y$ is a timelike, Killing vector field, $P_t$ is an immersed submanifold of $\mathcal{M}$. Thus it is natural to wonder if $P_1$ and $P_t$ can be joined by a timelike orthogonal trajectory.

Defining by $C_{P_1,P_t}$ the set of curves analogous to $C_{P_1,P_2}$ (see (1.7)), the following results hold (see Section 6).

Theorem 1.7. *Let $\mathcal{M}$, $P_1$, $P_2$, $Y$, $A$ satisfy the assumptions of Theorem* 1.4, *assume that $Y$ is complete and let $P_t$ be as in* (1.8). *If $t_0>0$ exists such that, for any $t\in\mathbb{R}$ with $|t|\geq t_0$, $C_{P_1,P_t}$ is $c_0$-precompact for some $c_0>\inf_{C_{P_1,P_t}}F$, then at least a timelike orthogonal trajectory joining $P_1$ to each $P_t$, $|t|\geq t_0$ in $\mathcal{M}$ exists.*

Theorem 1.8. *For any $t\in\mathbb{R}$, let $N(t)$ denote the number of timelike orthogonal trajectories joining $P_1$ and $P_t$. If all the assumptions of Theorem* 1.7 *are satisfied, $\mathcal{M}$ is not contractible in itself, $P_1$ and $P_2$ are contractible in $\mathcal{M}$, then it is*

$$\lim_{|t|\to\infty} N(t)=\infty.$$

Note that, in the previous two theorems, it is not necessary to assume for $C_{P_1,P_t}$ to be non-empty, because of the completeness of $Y$. At any case, if $P_1$ and $P_t$ must be connectable by a timelike geodesic then they must be connectable by a causal curve, and this weak assumption also implies $C_{P_1,P_t}\neq\emptyset$ (see Appendix A).

Remark 1.9. In Appendix A we will test the accuracy of our results by applying it to stationary standard manifolds. Recall that a product manifold $(\mathcal{M}=\mathcal{M}_0\times\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle)$ is a standard stationary Lorentzian manifold when the metric can written as

$$\langle\zeta,\zeta'\rangle=\langle\xi,\xi'\rangle_0+\langle\delta(x),\xi\rangle_0\tau'+\langle\delta(x),\xi'\rangle_0\tau-\beta(x)\tau\tau' \tag{1.9}$$

for any $z=(x,t)\in\mathcal{M}$, $\zeta=(\xi,\tau)$, $\zeta'=(\xi',\tau')\in T_z\mathcal{M}=T_x\mathcal{M}_0\times\mathbb{R}$, where $\langle\,\cdot\,,\,\cdot\,\rangle_0$, $\delta$ and $\beta$ are, respectively, a Riemannian metric on $\mathcal{M}_0$, a smooth vector field and a smooth, positive scalar field on $\mathcal{M}_0$.

We shall state some conditions on $\beta$, $\delta$ and $A$ implying that $C_{P_1,P_2}$ is pseudocoercive, generalizing the ones in [12, Appendix A]. As a consequence, some of the results in [2], [4], [9] can be obtained as particular cases of Theorems 1.4, 1.5, 1.7 and 1.8.

## 2. The functional framework

A Riemannian metric can be defined on $\mathcal{M}$ by setting, for any $z \in \mathcal{M}$ and $\zeta_1, \zeta_2 \in T_z\mathcal{M}$,

$$g_R(z)[\zeta_1, \zeta_2] = \langle \zeta_1, \zeta_2 \rangle_R = \langle \zeta_1, \zeta_2 \rangle - 2\frac{\langle \zeta_1, Y(z)\rangle \langle \zeta_2, Y(z)\rangle}{\langle Y(z), Y(z)\rangle}. \tag{2.1}$$

Using (2.1) it is possible to assume that $\mathcal{M}$ is a submanifold of $\mathbb{R}^k$ for $k$ sufficiently large (see [17]), thus we can define

$$H^1([0,1], \mathcal{M}) = \{z \in H^1([0,1], \mathbb{R}^k) \mid z([0,1]) \subset \mathcal{M}\}$$

where $H^1([0,1], \mathbb{R}^k)$ is the usual Sobolev space. We consider

$$\Omega_{P_1,P_2} = \{z \in H^1([0,1], \mathcal{M}) \mid z(0) \in P_1,\ z(1) \in P_2\}.$$

It is well known that $\Omega_{P_1,P_2}$ is an infinite dimensional Hilbert manifold whose tangent space at $z \in \Omega_{P_1,P_2}$ can be identified with

$$T_z\Omega_{P_1,P_2} = \{\zeta \in H^1([0,1], T\mathcal{M}) \mid \zeta(0) \in T_{z(0)}P_1,\ \zeta(1) \in T_{z(1)}P_2\}.$$

Observe that $T_z\Omega_{P_1,P_2}$ is a Hilbert manifold with respect to the norm

$$\|\zeta\|_* = \left(\int_0^1 \langle \nabla^R_{\dot z}\zeta, \nabla^R_{\dot z}\zeta\rangle_R \, ds\right)^{1/2} \tag{2.2}$$

where $\nabla^R$ is the covariant derivative with respect to $g_R$. The functional $F$ defined at (1.4) is well defined on $\Omega_{P_1,P_2}$ (in fact it is $|\langle \dot z, \dot z\rangle| \leq \langle \dot z, \dot z\rangle_R$) and it is smooth. By using standard arguments (see [6], [14]) the following proposition can be proved.

Proposition 2.1. *Let $z \in \Omega_{P_1,P_2}$ and assume that* (1.3) *holds. Then $z$ is a critical point of $F$ if and only if it is an orthogonal trajectory joining $P_1$ to $P_2$.*

As in the problem of geodesic connectedness, $F$ is a strongly indefinite functional and this fact makes difficult the search of its critical points. Nevertheless, a variational principle based on the conservation law (1.6) allows one to overcome this difficulty.

We recall the following characterization of Killing vector fields: a $\mathcal{C}^1$ vector field $Y$ on $\mathcal{M}$ is Killing if and only if for any couple of $\mathcal{C}^1$ vector fields $W_1$, $W_2$ on $\mathcal{M}$ there results

$$\langle \nabla_{W_1} Y, W_2\rangle = -\langle \nabla_{W_2} Y, W_1\rangle. \tag{2.3}$$

In particular, if $z\colon [0,1] \to \mathcal{M}$ is a smooth curve

$$\langle \dot z, \nabla_{\dot z} Y(z)\rangle = 0 \quad \text{on } [0,1]. \tag{2.4}$$

This implies that, if $z$ is a solution of (1.1), $Y$ is Killing and (1.2) holds, by (2.3) and (2.4)

$$\frac{d}{ds}\langle \dot z + A(z), Y(z)\rangle = \langle \nabla_{\dot z}\dot z + \nabla_{\dot z}A(z), Y(z)\rangle + \langle \dot z + A(z), \nabla_{\dot z}Y(z)\rangle$$
$$= \langle \dot z, \nabla_{Y(z)}A(z)\rangle - \langle \nabla_{\dot z}A(z), Y(z)\rangle + \langle \nabla_{\dot z}A(z), Y(z)\rangle - \langle \dot z, \nabla_{A(z)}Y(z)\rangle = 0$$

then $C_z \in \mathbb{R}$ exists such that (1.6) holds. Thus each solution of (1.1) belongs to the set

$$\mathcal{N}_{P_1,P_2} = \{z \in \Omega_{P_1,P_2} \mid \langle \dot z + A(z), Y(z)\rangle \text{ is constant a.e. in } [0,1]\}.$$

We observe that the curves in $\mathcal{N}_{P_1,P_2}$ are less regular than the ones in $C_{P_1,P_2}$ (see (1.7)). Using standard arguments in Sobolev spaces one can prove that $C_{P_1,P_2}$ is a dense subset of $\mathcal{N}_{P_1,P_2}$. Thus in Definition 1.3 and Theorems 1.4, 1.5, 1.7, 1.8 we can replace $C_{P_1,P_2}$ by $\mathcal{N}_{P_1,P_2}$. The reason for introducing $\mathcal{N}_{P_1,P_2}$ is that it is the suitable space to obtain the Palais–Smale condition for the action functional (see Section 5).

We end this section by proving that $\mathcal{N}_{P_1,P_2}$ is the subset of $\Omega_{P_1,P_2}$ such that the derivative $F'(z)$ vanishes in the directions of the distribution on $\Omega_{P_1,P_2}$ consisting of vector fields parallel to $Y$. More precisely, consider

$$\mathcal{W} = \{(z,\zeta) \in T\Omega_{P_1,P_2} \mid \zeta(s) \parallel Y(z(s)) \text{ for all } s \in [0,1]\}.$$

If $\Pi(z,\zeta) = z$ is the projection of $\mathcal{W}$ onto $\Omega_{P_1,P_2}$, we define

$$\mathcal{W}_z = \Pi^{-1}(z) = \{\zeta \in T_z\Omega_{P_1,P_2} \mid \zeta(s) \parallel Y(z(s)) \text{ for all } s \in [0,1]\}.$$

We remark that, for any $\zeta \in \mathcal{W}_z$ a function $\mu \in H^1([0,1],\mathbb{R})$ exists such that

$$\zeta(s) = \mu(s)Y(z(s)) \quad \text{for all } s \in [0,1].$$

Moreover, $\mu$ satisfies $\mu(0) = 0 = \mu(1)$ so that $\mu \in H^1_0([0,1],\mathbb{R})$. Indeed, if $z \in \Omega_{P_1,P_2}$ we can consider the local flow $\psi$ of $Y$ around $z_0 = z(0) \in P_1$. As $Y$ is timelike and $P_1$ is spacelike, $d\psi(z_0,0)$, the differential of $\psi$ at $z_0$, is injective so that an open neighbourhood $U \subset P_1$ of $z_0$ and $\varepsilon > 0$ exist such that

$$\psi: U \times \,]-\varepsilon,\varepsilon[\, \to V$$

(where $V = \psi(U \times \,]-\varepsilon,\varepsilon[)$) is a diffeomorphism. Let $T: V \to \,]-\varepsilon,\varepsilon[$ be the projection of $\psi^{-1}$ on $]-\varepsilon,\varepsilon[$, that is for any $q \in V$, $q = \psi(z,t)$, $T(q) = t$. As $T^{-1}(0) = U$, we get, for any $z \in U$,

$$T_zU = \{\zeta \in T_zV \mid \langle \nabla T(z), \zeta\rangle = 0\}.$$

In particular $\zeta(0) = \mu(0)Y(z_0) \in T_{z_0}U$ and

$$\langle \nabla T(z), Y(z)\rangle = 1 \quad \text{for all } z \in V$$

thus $\mu(0) = 0$. We can apply this argument again, with obvious changes, to obtain $\mu(1) = 0$.

Proposition 2.2. *Assume that* $Y$ *is a timelike Killing vector field and* (1.2) *holds. Then*

$$\mathcal{N}_{P_1,P_2} = \{z \in \Omega_{P_1,P_2} \mid F'(z)[\zeta] = 0 \text{ for all } \zeta \in \mathcal{W}_z\}.$$

Proof. Let $(z,\zeta) \in \mathcal{W}$ with $\zeta(s) = \mu(s)Y(z(s))$ for some $\mu \in H_0^1([0,1],\mathbb{R})$. As $Y$ is Killing, by (1.2), (2.3), (2.4), we can compute

$$\begin{aligned} F'(z)[\zeta] &= \int_0^1 [\langle \dot z, \nabla_{\dot z}\zeta\rangle + \langle \nabla_\zeta A(z), \dot z\rangle + \langle A(z), \nabla_{\dot z}\zeta\rangle]\, ds \\ &= \int_0^1 [\mu'\langle \dot z, Y(z)\rangle + \mu\langle \nabla_{Y(z)} A(z), \dot z\rangle \\ &\quad + \mu'\langle A(z), Y(z)\rangle + \mu\langle A(z), \nabla_{\dot z} Y(z)\rangle]\, ds \\ &= \int_0^1 \mu'\langle \dot z + A(z), Y(z)\rangle\, ds. \end{aligned}$$

The last integral is null if and only if $\langle \dot z + A(z), Y(z)\rangle$ is constant a.e. □

## 3. A variational principle

At first we prove a regularity result for $\mathcal{N}_{P_1,P_2}$.

Proposition 3.1. *The set* $\mathcal{N}_{P_1,P_2}$ *is a* $\mathcal{C}^2$ *submanifold of* $\Omega_{P_1,P_2}$.

Proof. Reasoning as in [12, Proposition 3.1], we define the map

$$G\colon \Omega_{P_1,P_2} \to L^2([0,1],\mathbb{R})$$

such that, for any $z \in \Omega_{P_1,P_2}$,

$$G(z) = \langle \dot z + A(z), Y(z)\rangle$$

so it results $\mathcal{N}_{P_1,P_2} = G^{-1}(C)$ where $C$ is the regular submanifold of the functions in $L^2([0,1],\mathbb{R})$ constant a.e. The map $G$ is $\mathcal{C}^2$ and its derivative is given by

$$G'(z)[\zeta] = \langle \nabla_{\dot z}\zeta + \nabla_\zeta A(z), Y(z)\rangle + \langle \dot z + A(z), \nabla_\zeta Y(z)\rangle$$

where $z \in \Omega_{P_1,P_2}$, $\zeta \in T_z\Omega_{P_1,P_2}$. It suffices to prove that for any $z \in \mathcal{N}_{P_1,P_2}$ and $h \in L^2([0,1],\mathbb{R})$ the equation

$$G'(z)[\zeta] = h + c \tag{3.1}$$

has a solution $\zeta \in T_z\Omega_{P_1,P_2}$ for some constant $c \in \mathbb{R}$. We show that (3.1) has a solution

$$\zeta(s) = \mu(s)Y(z(s)) \tag{3.2}$$

for some $\mu \in H_0^1([0,1],\mathbb{R})$. Indeed, as $Y$ is Killing, substituting (3.2) in (3.1), by (2.3) we obtain

$$
\begin{aligned}
(3.3)\quad \mu'\langle Y(z),Y(z)\rangle &+ \mu(\langle \nabla_{\dot z}Y(z) + \nabla_{Y(z)}A(z), Y(z)\rangle \\
&+ \langle \dot z + A(z), \nabla_{Y(z)}Y(z)\rangle) \\
&= \mu'\langle Y(z),Y(z)\rangle + \mu(\langle \nabla_{Y(z)}A(z) - \nabla_{A(z)}Y(z), Y(z)\rangle) \\
&= \mu'\langle Y(z),Y(z)\rangle = h + c
\end{aligned}
$$

As $\langle Y(z),Y(z)\rangle$ is negative on $[0,1]$, equation (3.3) is solved by

$$\mu(s) = \int_0^s \frac{h(r)+c}{\langle Y(z(r)),Y(z(r))\rangle}\,dr.$$

Clearly $\mu(0)=0$ and choosing

$$c = -\bigg(\int_0^1 \frac{h}{\langle Y(z),Y(z)\rangle}\,ds\bigg)\bigg(\int_0^1 \frac{ds}{\langle Y(z),Y(z)\rangle}\bigg)^{-1}$$

we also have $\mu(1)=0$. $\square$

By the previous proposition, (using the Implicit Function Theorem) for any $z \in \mathcal{N}_{P_1,P_2}$, $T_z\mathcal{N}_{P_1,P_2}$ can be identified with the set of all $\zeta$ such that $G'(z)[\zeta] \in T_{G(z)}C$. As $T_{G(z)}C$ can be identified with the set of the constant functions on $[0,1]$, we get the following corollary.

Corollary 3.2. *For any $z \in \mathcal{N}_{P_1,P_2}$ the tangent space $T_z\mathcal{N}_{P_1,P_2}$ is*

$$T_z\mathcal{N}_{P_1,P_2} = \{\zeta \in T_z\Omega_{P_1,P_2} \mid \langle \nabla_{\dot z}\zeta + \nabla_\zeta A(z), Y(z)\rangle + \langle \dot z + A(z), \nabla_\zeta Y(z)\rangle \\ \textit{is constant a.e. on } [0,1]\}.$$

Define a new functional as $J = F_{|\mathcal{N}_{P_1,P_2}}$. By the previous proposition $J$ is smooth. The following variational principle proves that the set of its critical points agrees with the set of the critical points of $F$.

Proposition 3.3. *A curve $z \in \Omega_{P_1,P_2}$ is an orthogonal trajectory joining $P_1$ to $P_2$ if and only if $z$ is a critical point of $J$.*

Proof. By Proposition 2.1 and (1.6), if $z$ is an orthogonal trajectory joining $P_1$ to $P_2$, $z \in \mathcal{N}_{P_1,P_2}$ and it is a critical point of $J$.

Vice-versa, let $z$ be a critical point of $J$. Then $F'(z)$ vanishes on all vectors in $T_z\mathcal{N}_{P_1,P_2}$. By Proposition 2.2, $F'(z)$ vanishes also on any vector field $\zeta(s) = \mu(s)Y(z(s))$ for some $\mu \in H_0^1([0,1],\mathbb{R})$. Then it suffices to show that any vector field $\zeta \in T_z\Omega_{P_1,P_2}$ can be written as

$$\zeta = \mu Y(z) + \widetilde{\zeta}, \quad \mu \in H_0^1([0,1],\mathbb{R}),\ \widetilde{\zeta} \in T_z\mathcal{N}_{P_1,P_2}.$$

To this aim, fixed $\zeta \in T_z\Omega_{P_1,P_2}$, we prove that a function $\mu \in H_0^1([0,1],\mathbb{R})$ exists such that

$$\zeta - \mu Y(z) \in T_z\mathcal{N}_{P_1,P_2}.$$

By Corollary 3.2, (1.2) and (2.3) it is easy to see that $\mu$ has to satisfy the following equation

$$-\mu'\langle Y(z),Y(z)\rangle + \langle \nabla_{\dot z}\zeta + \nabla_\zeta A(z), Y(z)\rangle + \langle \dot z + A(z), \nabla_\zeta Y(z)\rangle = c.$$

Since $\langle Y(z),Y(z)\rangle$ is negative on $[0,1]$, this equation is solved by

$$\mu(s) = \int_0^s \frac{\langle \nabla_{\dot z}\zeta + \nabla_\zeta A(z), Y(z)\rangle + \langle \dot z + A(z), \nabla_\zeta Y(z)\rangle - c}{\langle Y(z),Y(z)\rangle}\,dr$$

and

$$c = \bigg(\int_0^1 \frac{\langle \nabla_{\dot z}\zeta + \nabla_\zeta A(z), Y(z)\rangle + \langle z + A(z), \nabla_\zeta Y(z)\rangle}{\langle Y(z),Y(z)\rangle}\,ds\bigg) \cdot \bigg(\int_0^1 \frac{ds}{\langle Y(z),Y(z)\rangle}\bigg)^{-1},$$

so that $\mu(0) = 0 = \mu(1)$. □

## 4. The properties of $J$

In this section we shall prove that if $\mathcal{N}_{P_1,P_2}$ is $c$-precompact for some $c > \inf_{\mathcal{N}_{P_1,P_2}} J$ then the functional $J$ is bounded from below.

For $z \in \mathcal{N}_{P_1,P_2}$, let $C_z$ be the real number such that (1.6) holds and, as usual, let us set for $c \in \mathbb{R}$

$$J^c = \{z \in \mathcal{N}_{P_1,P_2} \mid J(z) \le c\}.$$

Lemma 4.1. *Let $c > \inf_{\mathcal{N}_{P_1,P_2}} J$ be such that $\mathcal{N}_{P_1,P_2}$ is $c$-precompact. Then $D > 0$ exists such that*

$$|C_z| \le D \quad \textit{for all } z \in J^c. \tag{4.1}$$

Proof. Let $(z_n) \subset J^c$ be a sequence such that

$$\lim_{n\to\infty} |C_{z_n}| = \sup_{z\in J^c} |C_z|.$$

It is sufficient to prove that $(C_{z_n})$ is bounded. By the $c$-precompactness, up to a subsequence, we can assume that $(z_n)$ is uniformly convergent to a curve $z \in \Omega_{P_1,P_2}$ (since $P_1$ and $P_2$ are closed). Thus a compact neighbourhood $\mathcal{U}$ of $z([0,1])$ exists such that $z_n([0,1]) \subset \mathcal{U}$ for $n$ sufficiently large. As every stationary Lorentzian manifold has a local structure of standard type (see [12, Appendix C] and Remark 1.9) we can choose a finite number of local charts of $\mathcal{M}$

$$(U_k, x_k^1, \dots, x_k^{N-1}, t_k)_{k=1,\dots,r}$$

where $N = \dim \mathcal{M}$ such that

- $(U_k)_{k=1,\dots,r}$ is a covering of $\mathcal{U}$ and for any $k = 1, \dots, r$

$$U_k = \Sigma_k \times ]{-\varepsilon_k}, \varepsilon_k[$$

where $\Sigma_k$ is a spacelike hypersurface parameterized by $x_k^1, \dots, x_k^{N-1}$ and $\varepsilon_k$ is a positive number,
- for any $k = 1, \dots, r$ we have

$$Y_{|U_k} = \frac{\partial}{\partial t_k}$$

and setting $\mathbf{x}_k = (x_k^1, \dots, x_k^{N-1})$ the Lorentzian metric on $U_k$ is given by

$$g(\mathbf{x}_k, t_k)[(\xi,\tau),(\xi,\tau)] = \langle \xi, \xi \rangle_0 + 2\langle \delta_k(\mathbf{x}_k), \xi \rangle_0 \tau - \beta(\mathbf{x}_k)\tau^2 \tag{4.2}$$

where $\beta = -\langle Y, Y \rangle$ and $\langle \cdot, \cdot \rangle_0$ denotes the Riemannian metric induced by $g$ on $\Sigma_k$,
- $\max_{k=1,\dots,r}(\sup_{\Sigma_k} \langle \delta_k(\mathbf{x}_k), \delta_k(\mathbf{x}_k) \rangle_0) = D_0 < \infty$,
- a finite sequence $0 = a_0 < a_1 < \dots < a_r = 1$ exists such that for $n$ sufficiently large

$$z_n([a_{k-1}, a_k]) \subset U_k \quad \text{for all } k = 1, \dots, r,$$

- by (1.2), for $k = 1, \dots, r$

$$A(\mathbf{x}_k, t_k) = (A_k^1(\mathbf{x}_k), A_k^2(\mathbf{x}_k)) \quad \text{for all } z_k = (\mathbf{x}_k, t_k) \in U_k.$$

Moreover, we set

$$\Delta_k = \sup_{p_1, p_2 \in U_k} |t_k(p_1) - t_k(p_2)|, \quad \Delta = \max_{k=1,\dots,r} \Delta_k. \tag{4.3}$$

For $n$ large enough, we can write, for any $k = 1, \dots, r$

$$z_n(s) = (\mathbf{x}_{k,n}(s), t_{k,n}(s)) \in U_k \quad s \in [a_{k-1}, a_k]. \tag{4.4}$$

We set for any $(\mathbf{x}_k, v_k) \in T\Sigma_k$

$$E_k(\mathbf{x}_k, v_k) = \frac{1}{2}\langle v_k, v_k \rangle_0 + \langle A_k^1(\mathbf{x}_k), v_k \rangle_0 + \langle \delta_k(\mathbf{x}_k), v_k \rangle_0 A_k^2(\mathbf{x}_k), \tag{4.5}$$

$$G_k(\mathbf{x}_k, v_k) = \langle \delta_k(\mathbf{x}_k), v_k \rangle_0 + \langle \delta_k(\mathbf{x}_k), A_k^1(\mathbf{x}_k) \rangle_0 - \beta(\mathbf{x}_k) A_k^2(\mathbf{x}_k). \tag{4.6}$$

By (1.6) and (4.2)

$$C_{z_n} = G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n}) - \beta(\mathbf{x}_{k,n})\dot{t}_{k,n} \tag{4.7}$$

hence, integrating (4.7) on $[a_{k-1}, a_k]$

$$\begin{aligned} d_{k,n} &= t_{k,n}(a_k) - t_{k,n}(a_{k-1}) \\ &= \int_{a_{k-1}}^{a_k} \dot{t}_{k,n}\, ds = \int_{a_{k-1}}^{a_k} \frac{G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n}) - C_{z_n}}{\beta(\mathbf{x}_{k,n})}\, ds, \end{aligned}$$

so that

$$C_{z_n} = \bigg( \int_{a_{k-1}}^{a_k} \frac{G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})}{\beta(\mathbf{x}_{k,n})}\, ds - d_{k,n} \bigg) \bigg( \int_{a_{k-1}}^{a_k} \frac{1}{\beta(\mathbf{x}_{k,n})}\, ds \bigg)^{-1}. \tag{4.8}$$

Note that by (4.3)

$$|d_{k,n}| \leq \Delta \tag{4.9}$$

and, as $\mathcal{U}$ is compact, $\nu, \mu > 0$ exist such that

$$\nu \leq -\langle Y(z), Y(z) \rangle \leq \mu \quad \text{for all } z \in \mathcal{U} \tag{4.10}$$

then

$$\frac{\nu}{a_k - a_{k-1}} \leq \bigg( \int_{a_{k-1}}^{a_k} \frac{1}{\beta(\mathbf{x}_{k,n})}\, ds \bigg)^{-1} \leq \frac{\mu}{a_k - a_{k-1}}. \tag{4.11}$$

By (4.8)–(4.11)

$$|C_{z_n}| \leq \frac{\mu}{a_k - a_{k-1}} \bigg( \frac{1}{\nu} \int_{a_{k-1}}^{a_k} |G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})|\, ds + \Delta \bigg). \tag{4.12}$$

Thus, it suffices to prove that

$$\int_{a_{k-1}}^{a_k} |G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})|\, ds$$

is bounded (with respect to $n$) for at least one value of $k$. To this aim we compute (using (4.7) and (4.8))

$$\begin{aligned} &\int_{a_{k-1}}^{a_k} \bigg[ \frac{1}{2} \langle \dot{z}_n, \dot{z}_n \rangle + \langle A(z_n), \dot{z}_n \rangle \bigg] ds \\ &= \int_{a_{k-1}}^{a_k} \bigg[ E_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n}) + G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n}) \dot{t}_{k,n} - \frac{1}{2} \beta(\mathbf{x}_{k,n}) (\dot{t}_{k,n})^2 \bigg] ds \\ &= \int_{a_{k-1}}^{a_k} \bigg[ E_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n}) + \frac{1}{2} \frac{G_k^2(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})}{\beta(\mathbf{x}_{k,n})} - \frac{1}{2} \frac{C_{z_n}^2}{\beta(\mathbf{x}_{k,n})} \bigg] ds \\ &= \int_{a_{k-1}}^{a_k} E_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})\, ds + \frac{1}{2} \int_{a_{k-1}}^{a_k} \frac{G_k^2(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})}{\beta(\mathbf{x}_{k,n})}\, ds \\ &\quad + \bigg( \int_{a_{k-1}}^{a_k} \frac{1}{\beta(\mathbf{x}_{k,n})}\, ds \bigg)^{-1} \bigg[ -\frac{1}{2} \bigg( \int_{a_{k-1}}^{a_k} \frac{G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})}{\beta(\mathbf{x}_{k,n})}\, ds \bigg)^2 \\ &\quad + d_{k,n} \int_{a_{k-1}}^{a_k} \frac{G_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})}{\beta(\mathbf{x}_{k,n})}\, ds - \frac{1}{2} d_{k,n}^2 \bigg]. \end{aligned} \tag{4.13}$$

From the Schwartz's inequality, (4.9)–(4.11) we obtain

$$\int_{a_{k-1}}^{a_k}\left[\frac{1}{2}\langle\dot z_n,\dot z_n\rangle+\langle A(z_n),\dot z_n\rangle\right]ds\geq\int_{a_{k-1}}^{a_k}E_k(\mathbf{x}_{k,n},\dot{\mathbf{x}}_{k,n})\,ds$$
$$-\frac{\mu}{\nu}\frac{\Delta}{a_k-a_{k-1}}\int_{a_{k-1}}^{a_k}|G_k(\mathbf{x}_{k,n},\dot{\mathbf{x}}_{k,n})|\,ds-\frac{1}{2}\frac{\mu\Delta^2}{a_k-a_{k-1}}.$$

Summing over $k$ we obtain

$$c\geq J(z_n)\geq\sum_k\bigg(\int_{a_{k-1}}^{a_k}E_k(\mathbf{x}_{k,n},\dot{\mathbf{x}}_{k,n})\,ds-E\int_{a_{k-1}}^{a_k}|G_k(\mathbf{x}_{k,n},\dot{\mathbf{x}}_{k,n})|\,ds\bigg)-F\tag{4.14}$$

for some positive constants $E,F$. As $\mathcal{U}$ is compact

$$E_k(\mathbf{x}_{k,n},\dot{\mathbf{x}}_{k,n})\geq\frac{1}{2}\langle\dot{\mathbf{x}}_{k,n},\dot{\mathbf{x}}_{k,n}\rangle_0-G\sqrt{\langle\dot{\mathbf{x}}_{k,n},\dot{\mathbf{x}}_{k,n}\rangle_0},\tag{4.15}$$

$$|G_k(\mathbf{x}_{k,n},\dot{\mathbf{x}}_{k,n})|\leq H\sqrt{\langle\dot{\mathbf{x}}_{k,n},\dot{\mathbf{x}}_{k,n}\rangle_0}+L,\tag{4.16}$$

for some $G,H,L>0$. Substituting (4.15), (4.16) in (4.14) we get

$$c\geq\sum_k\bigg(\frac{1}{2}\int_{a_{k-1}}^{a_k}(\langle\dot{\mathbf{x}}_{k,n},\dot{\mathbf{x}}_{k,n}\rangle_0-N\sqrt{\langle\dot{\mathbf{x}}_{k,n},\dot{\mathbf{x}}_{k,n}\rangle_0})\,ds\bigg)-P$$

where $N,P>0$, then $R>0$ exists such that

$$\int_{a_{k-1}}^{a_k}\sqrt{\langle\dot{\mathbf{x}}_{k,n},\dot{\mathbf{x}}_{k,n}\rangle_0}\,ds\leq R,\tag{4.17}$$

so, by (4.16) the proof is complete. □

Proposition 4.2. *If $\mathcal{N}_{P_1,P_2}$ is $c$-precompact for some $c>\inf_{\mathcal{N}_{P_1,P_2}}J$, then $J$ is bounded from below in $\mathcal{N}_{P_1,P_2}$.*

Proof. Let $(z_n)$ be a minimizing sequence for $J$. For $n$ sufficiently large, $z_n\in J^c$. By the $c$-precompactness a compact subset $K$ of $\mathcal{M}$ exists such that

$$z_n([0,1])\subset K.$$

We can use local coordinates as in the previous lemma and, by (4.13), (4.1) and (4.10), we have

$$\int_{a_{k-1}}^{a_k}\left[\frac{1}{2}\langle\dot z_n,\dot z_n\rangle+\langle A(z_n),\dot z_n\rangle\right]ds\geq\int_{a_{k-1}}^{a_k}E_k(\mathbf{x}_{k,n},\dot{\mathbf{x}}_{k,n})\,ds-\frac{1}{2}\frac{D^2}{\nu}(a_k-a_{k-1}).\tag{4.18}$$

Moreover, (4.15) and (4.17) hold, then $S > 0$ exists such that

$$\int_{a_{k-1}}^{a_k} E_k(\mathbf{x}_{k,n}, \dot{\mathbf{x}}_{k,n})\,ds \geq -S. \tag{4.19}$$

Finally, summing over $k$ in (4.18) and by (4.19) we get, for some $T > 0$,

$$J(z_n) \geq \sum_k \bigg( -S - \frac{1}{2}\frac{D^2}{\nu}(a_k - a_{k-1})\bigg) = -T. \qquad \square$$

## 5. Proof of Theorems 1.4 and 1.5

We recall that if $(X, h)$ is a Hilbert manifold and $f\colon X \to \mathbb{R}$ is a $\mathcal{C}^1$ functional, $f$ is said to satisfy the Palais–Smale condition at a level $c \in \mathbb{R}$ if every sequence $(x_n) \subset X$ such that

$$\lim_{n\to\infty} f(x_n) = c, \qquad \lim_{n\to\infty} \|f'(x_n)\| = 0 \tag{5.1}$$

has a converging subsequence. The norm $\|\cdot\|$ is the norm induced by $h$ on $T_{x_n}X$.

We also recall that if $z\colon [0,1] \to \mathcal{M}$ is an absolutely continuous curve and $\beta \in L^1([0,1], T\mathcal{M})$ is a vector field along $z$, the covariant integral of $\beta$ is the unique vector field $B$ along $z$ such that

$$\nabla_{\dot z} B = \beta, \quad B(0) = 0. \tag{5.2}$$

Theorem 5.1. *Under the assumptions of Theorem* 1.4*, if $\mathcal{N}_{P_1,P_2}$ is c-pre-compact for some $c > \inf_{\mathcal{N}_{P_1,P_2}} J$, then $J$ satisfies the Palais–Smale condition at any level $c' < c$.*

Proof. Let $(z_n)$ be a sequence in $\mathcal{N}_{P_1,P_2}$ satisfying (5.1) at the level $c' < c$. Reasoning as in Proposition 4.2, we can prove that $(z_n)$ is bounded in $H^1$ then it has a subsequence (again denoted by $(z_n)$) weakly convergent to some $z$ in $H^1$. As $P_1$ and $P_2$ are closed, $z \in \Omega_{P_1,P_2}$. We have to prove that the convergence is strong. Let $(\zeta_n)$ be a bounded sequence in $H^1$ such that, for any $n \in \mathbb{N}$, $\zeta_n \in T_{z_n}\Omega_{P_1,P_2}$. By Proposition 3.3 we can write

$$\zeta_n = \mu_n Y(z_n) + \widetilde{\zeta}_n$$

where $\widetilde{\zeta}_n \in T_{z_n}\mathcal{N}_{P_1,P_2}$ and $\mu_n$ is as in the proof of Proposition 3.3. As $(\zeta_n)$ is bounded and by the definition of $\mu_n$, also $(\widetilde{\zeta}_n)$ is bounded, by (5.1)

$$\lim_{n\to\infty} J'(z_n)[\widetilde{\zeta}_n] = 0.$$

Then, by Proposition 2.2, for any bounded $(\zeta_n) \in T_{z_n}\Omega_{P_1,P_2}$

$$\lim_{n\to\infty} F'(z_n)[\zeta_n] = 0. \tag{5.3}$$

We can express $F'(z_n)$ by using the norm defined at (2.2): there exists a sequence $(\Theta_n)$ such that, for any $n \in \mathbb{N}$, $\Theta_n$ is a vector field along $z_n$ and

$$F'(z_n)[\zeta_n] = \int_0^1 \langle \nabla^R_{\dot{z}_n} \Theta_n, \nabla^R_{\dot{z}_n} \zeta_n \rangle_R \, ds.$$

By (5.3) the sequence of vector fields $A_n = \nabla^R_{\dot{z}_n} \Theta_n$ goes to 0 in $L^2([0,1], T\mathcal{M})$. Using the Christoffel symbols of the metric tensors $g$ and $g_R$ we can write

$$\int_0^1 \langle A_n, \nabla^R_{\dot{z}_n} \zeta_n \rangle_R \, ds = \int_0^1 \langle A_n, \nabla_{\dot{z}_n} \zeta_n + G(z_n)[\dot{z}_n, \zeta_n] \rangle_R \, ds \tag{5.5}$$

where $G(z)[\zeta_1, \zeta_2]$ is a bilinear form in $\zeta_1$, $\zeta_2$ continuous in $z$. Using (2.1), it can be checked that two sequences $B_n$ and $C_n$ going to 0 in $L^2$ exist such that (by (5.4) and (5.5))

$$F'(z_n)[\zeta_n] = \int_0^1 \langle B_n, \nabla_{\dot{z}_n} \zeta_n \rangle \, ds + \int_0^1 \langle C_n, \zeta_n \rangle \, ds. \tag{5.6}$$

On the other hand, we can compute

$$F'(z_n)[\zeta_n] = \int_0^1 \Big[ \langle \dot{z}_n, \nabla_{\dot{z}_n} \zeta_n \rangle + \langle A(z_n), \nabla_{\dot{z}_n} \zeta_n \rangle + \langle (A'(z_n))^*[\dot{z}_n], \zeta_n \rangle \Big] \, ds. \tag{5.7}$$

By (5.6) and (5.7), integrating by parts

$$\begin{aligned} 0 &= \int_0^1 \langle \dot{z}_n + A(z_n) - B_n, \nabla_{\dot{z}_n} \zeta_n \rangle \, ds + \int_0^1 \langle (A'(z_n))^*[\dot{z}_n] - C_n, \zeta_n \rangle \, ds \\ &= \int_0^1 \langle \dot{z}_n + A(z_n) - B_n - S_n, \nabla_{\dot{z}_n} \zeta_n \rangle \, ds + \langle S_n(1), \zeta_n(1) \rangle \end{aligned} \tag{5.8}$$

where $S_n$ is the covariant integral of $(A'(z_n))^*[\dot{z}_n] - C_n$. Setting

$$\omega_n = \dot{z}_n + A(z_n) - B_n - S_n \tag{5.9}$$

by (5.8) we have that $\omega_n$ is $\mathcal{C}^1$ and

$$\nabla_{\dot{z}_n} \omega_n = 0. \tag{5.10}$$

Applying to $(A'(z_n))^*[\dot{z}_n] - C_n$ [12, Lemma 5.1], we get that $S_n$ is bounded in $L^2$ so that, by (5.9) also $\omega_n$ is bounded in $L^2$. This implies that a sequence $(s_n) \subset [0,1]$ exists such that

$$|\omega_n(s_n)| \le c_0 \tag{5.11}$$

for some $c_0 > 0$. Gronwall's Lemma applied to the differential equations (5.10) and (5.11) give the existence of $\gamma_0 > 0$ such that

$$|\omega_n(s)| \le c_0 e^{\gamma_0 \int_0^1 |\dot{z}_n| \, ds} \quad \text{for all } s \in [0,1]. \tag{5.12}$$

It follows that $(\omega_n)$ is bounded in $L^\infty$. Writing equation (5.10) in coordinates it becomes

$$\omega_n' + \Gamma(z_n)[\dot z_n, \omega_n] = 0 \tag{5.13}$$

where $\Gamma$ is a continuous function in $z_n$ (that can be expressed using the Christoffel symbols of $g$) which is linear in the variables $\dot z_n$ and $\omega_n$. From (5.13) we get that $(\omega_n')$ is bounded in $L^2$ and thus $(\omega_n)$ is bounded in $H^1$. It follows that a subsequence of $(\omega_n)$ (still denoted by $(\omega_n)$) is weakly convergent in $H^1$ and, in particular, it is convergent in $L^2$.

Observe now that, as $(\dot z_n)$ is bounded in $L^2$, $z_n(0) \in P_1$, $z_n(1) \in P_2$ and $P_1$ or $P_2$ is compact, $(z_n)$ is uniformly bounded. Moreover, again as in [12, Lemma 5.1], also $S_n$ is bounded in $H^1$ then, up to a subsequence, it is convergent in $L^2$. Finally, by (5.9) we obtain that $(\dot z_n)$ converges in $L^2$, so $(z_n)$ converges in $H^1$ (up to a subsequence) to a curve $z \in \Omega_{P_1,P_2}$. By the $L^2$-convergence, a subsequence of $\langle \dot z_n + A(z_n), Y(z_n)\rangle$ converges pointwise to $\langle \dot z + A(z), Y(z)\rangle$ almost everywhere which implies that $\langle \dot z + A(z), Y(z)\rangle$ is constant a.e. on $[0,1]$ so $z \in \mathcal{N}_{P_1,P_2}$, completing the proof. □

Proposition 5.2. *If $\mathcal{N}_{P_1,P_2}$ is $c$-precompact for some $c > \inf_{\mathcal{N}_{P_1,P_2}} J$, then for any $c' \le c$, $J^{c'}$ is a complete metric subspace of $\mathcal{N}_{P_1,P_2}$.*

Proof. It suffices to consider the $c$-sublevel. As all the curves in $J^c$ are contained in a compact subset of $\mathcal{M}$, we can assume that $\mathcal{M}$ is complete with respect to the metric $g_R$ thus $\Omega_{P_1,P_2}$ is a complete Hilbertian manifold. Let $(z_n)$ be a Cauchy sequence in $J^c$. Then, $(z_n)$ converges to $z \in \Omega_{P_1,P_2}$ and, up to a subsequence, $\langle \dot z_n + A(z_n), Y(z_n)\rangle$ converges pointwise to $\langle \dot z + A(z), Y(z)\rangle$ almost everywhere which implies that $\langle \dot z + A(z), Y(z)\rangle$ is constant a.e. on $[0,1]$ hence $z \in \mathcal{N}_{P_1,P_2}$. By the continuity of $J$, $J(z) \le c$ so $z \in J^c$. □

The Palais–Smale condition and the completeness of the sublevels of $J$ imply the existence of a minimum point for $J$.

Proof of Theorem 1.4. Set $a = \inf_{\mathcal{N}_{P_1,P_2}} J$. By Theorem 5.1, Proposition 5.2 and classical arguments in Critical Point Theory, $a$ is a critical level for $J$ then, by Propositions 2.1 and 3.3, an orthogonal trajectory joining $P_1$ and $P_2$ exists. □

The proof of Theorem 1.5 is based on the Lusternik–Schnirelmann category theory. We recall the following

Definition 5.3. Let $A$ be a subspace of a topological space $X$. The *category* of $A$ in $X$, denoted by $\operatorname{cat}_X A$, is the minimum number of closed and contractible subsets of $X$ covering $A$ (possibly $\infty$). We shall write $\operatorname{cat} X = \operatorname{cat}_X X$.

If we assume that $\mathcal{M}$ is not contractible in itself and $P_1$ and $P_2$ are contractible in $\mathcal{M}$ we can use the following result (see [8], [10]).

THEOREM 5.4. *Let $M$ be a non-contractible in itself $\mathcal{C}^3$ Riemannian manifold. Let $P$ and $Q$ be two submanifolds of $M$ both contractible in $M$. Then a sequence $(K_n)$ exists of compact subsets of $\Omega(P,Q)$ such that*

$$\lim_{n\to\infty} \mathrm{cat}_{\Omega(P,Q)} K_n = \infty.$$

The previous theorem implies that

$$\mathrm{cat}\,\Omega_{P_1,P_2} = \infty. \tag{5.14}$$

Now we prove that if $Y$ is a complete vector field, then there exists a homotopy equivalence between $\Omega_{P_1,P_2}$ and $\mathcal{N}_{P_1,P_2}$. More precisely the following proposition holds.

PROPOSITION 5.5. *Assume that $Y$ is complete. Then a smooth map*

$$\mathcal{F}\colon \Omega_{P_1,P_2} \to \mathcal{N}_{P_1,P_2}$$

*exists such that $\mathcal{F}$ is a strong deformation retract. Moreover,*

$$J(\mathcal{F}(z)) \geq F(z) \quad \textit{for all } z \in \Omega_{P_1,P_2} \tag{5.15}$$

*where the equality holds if and only if $z \in \mathcal{N}_{P_1,P_2}$.*

PROOF. Let $\psi\colon \mathcal{M}\times\mathbb{R} \to \mathcal{M}$ be the flow of $Y$. Let $z\colon[0,1]\to\mathcal{M}$ be a curve such that $z(0)\in P_1$ and $z(1)\in P_2$. We define a curve $w\colon[0,1]\to\mathcal{M}$ by

$$w(s) = \psi(z(s),\phi(s)) \quad \text{for all } s\in[0,1]$$

where $\phi\colon[0,1]\to\mathbb{R}$ will be chosen such that $\langle \dot{w} + A(w), Y(w)\rangle$ is constant on $[0,1]$ and $\phi(0) = 0 = \phi(1)$. This last condition gives that $w(0) = z(0) \in P_1$ and $w(1) = z(1) \in P_2$. Moreover, by (1.2) and as $Y$ is Killing, it is not difficult to prove that, for any $p\in\mathcal{M}$ and $s\in\mathbb{R}$

$$\langle A(\psi(p,s)), Y(\psi(p,s))\rangle = \langle A(p), Y(p)\rangle \tag{5.16}$$

and, by using the property of the flow

$$\psi(\psi(p,s),t) = \psi(p,s+t) \quad \text{for all } p\in\mathcal{M},\ s,t\in\mathbb{R}$$

we also get

$$d_z\psi(z,\phi)[Y(z)] = Y(w) \quad \text{on } [0,1]. \tag{5.17}$$

Thus, computing

$$\dot{w} = d_z\psi(z,\phi)[\dot{z}] + Y(w)\phi' \tag{5.18}$$

by (5.16), (5.17) and as $d_z$ is an isometry, we have

$$\begin{aligned}\langle \dot w + A(w), Y(w)\rangle &= \langle d_z\psi(z,\phi)[\dot z], d_z\psi(z,\phi)[Y(z)]\rangle \\ &\quad + \phi'\langle Y(w), Y(w)\rangle + \langle A(w), Y(w)\rangle \\ &= \langle \dot z, Y(z)\rangle + \phi'\langle Y(z), Y(z)\rangle + \langle A(z), Y(z)\rangle = c.\end{aligned}$$

Then $w \in \mathcal{N}_{P_1,P_2}$ if the function $\phi$ satisfies the following problem

$$\begin{cases} \phi' = \dfrac{c - \langle \dot z + A(z), Y(z)\rangle}{\langle Y(z), Y(z)\rangle}, \\ \phi(0) = 0 = \phi(1). \end{cases} \tag{5.19}$$

We take

$$\phi(s) = \int_0^s \frac{c - \langle \dot z + A(z), Y(z)\rangle}{\langle Y(z), Y(z)\rangle}\, dr \tag{5.20}$$

so that $\phi(0) = 0$ and

$$c = \bigg(\int_0^1 \frac{\langle \dot z + A(z), Y(z)\rangle}{\langle Y(z), Y(z)\rangle}\, ds\bigg)\bigg(\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}\bigg)^{-1}, \tag{5.21}$$

so that $\phi(1) = 0$.

We can define the map $\mathcal{F}\colon \Omega_{P_1,P_2} \to \mathcal{N}_{P_1,P_2}$ by

$$\mathcal{F}(z) = w \quad \text{for all } z \in \Omega_{P_1,P_2}.$$

As [12, Propositions 5.8, 5.9], $\mathcal{F}$ is smooth and it is a strong deformation retract. Notice that if $z \in \mathcal{N}_{P_1,P_2}$, $\phi \equiv 0$ is the unique solution of (5.19), then $\mathcal{F}$ is the identity on $\mathcal{N}_{P_1,P_2}$.

In order to prove (5.15), we observe that, as consequence of (1.2), the flows of $A$ and $Y$ commute that is, for any $p \in \mathcal{M}$, $s \in \mathbb{R}$ and $t \in \mathbb{R}$ with $|t|$ sufficiently small

$$\varphi(\psi(p,s),t) = \psi(\varphi(p,t),s)$$

where $\varphi$ denotes the flow of $A$ (see e.g. [23, Chapter 5, Lemma 13]). Then, differentiating with respect to $t$ it is not difficult to prove that

$$\langle d_z\psi(z,s)[v], A(\psi(z,s))\rangle = \langle v, A(z)\rangle \quad \text{for all } z \in \mathcal{M},\ s \in \mathbb{R},\ v \in T_z\mathcal{M}.$$

Now let $w = \mathcal{F}(z)$. Taking into account (5.16)–(5.19) and as $d_z$ is an isometry, we have on $[0,1]$

$$\begin{aligned}\langle \dot w, \dot w\rangle - \langle \dot z, \dot z\rangle &= (\phi')^2\langle Y(z), Y(z)\rangle + 2\phi'\langle Y(z), \dot z\rangle, \\ \langle A(w), \dot w\rangle - \langle A(z), \dot z\rangle &= \phi'\langle A(z), Y(z)\rangle.\end{aligned}$$

Then, by (5.19), (5.20) and the Hölder inequality

$$J(w) - F(z) = \frac{1}{2}\bigg[\bigg(\int_0^1 \frac{\langle \dot z + A(z), Y(z)\rangle}{\langle Y(z), Y(z)\rangle}\,ds\bigg)^2\bigg(\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}\bigg)^{-1} - \int_0^1 \frac{\langle \dot z + A(z), Y(z)\rangle^2}{\langle Y(z), Y(z)\rangle}\,ds\bigg] \geq 0,$$

so the proof is complete. □

The proof of Theorem 1.5 is an application of the following classical result.

THEOREM 5.6. *Let $X$ be a Hilbert manifold, $f \in C^1(X, \mathbb{R})$ be a functional bounded from below such that, for any $c \geq \inf f$, $f$ satisfies the Palais–Smale condition at the level $c$ and the sublevel $f^c$ is a complete metric subspace of $X$. Then $f$ has at least $\operatorname{cat} X$ critical points. Moreover, if $\operatorname{cat} X = \infty$ there exists a sequence $\{y_n\}$ of critical points of $f$ such that*

$$\lim_{n\to\infty} f(y_n) = \infty.$$

PROOF OF THEOREM 1.5. By Proposition 4.2, Theorem 5.1, Proposition 5.2, $J$ satisfies all the assumptions of Theorem 5.6. □

## 6. Timelike orthogonal trajectories

In this section we shall prove Theorems 1.7, 1.8. To this aim, for all $t \in \mathbb{R}$ we define a map $\mathcal{L}_t$ between the spaces $\mathcal{N}_{P_1,P_2}$ and $\mathcal{N}_{P_1,P_t}$. Let $z$ be a curve in $\mathcal{N}_{P_1,P_2}$ and $C_z$ such that (1.6) holds. We define the curve

$$\mathcal{L}_t(z)(s) = w(s) = \psi(z(s), \phi_t(s)) \quad \text{for all } s \in [0,1] \tag{6.1}$$

where, for any $t \in \mathbb{R}$,

$$\phi_t(s) = t\bigg(\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}\bigg)^{-1}\bigg(\int_0^s \frac{dr}{\langle Y(z), Y(z)\rangle}\bigg). \tag{6.2}$$

As $\phi_t(0) = 0$ and $\phi_t(1) = t$ it is clear that (by (1.8)) $w(0) = z(0) \in P_1$, $w(1) = \psi(z(1), t) \in P_t$ hence $w \in \Omega_{P_1,P_t}$. Moreover, using the properties of the flow $\psi$ already introduced in the previous section and as $Y$ is Killing

$$\begin{aligned} \langle A(w) &+ \dot w, Y(w)\rangle \\ &= \langle A(w), Y(w)\rangle + \langle d_z\psi(z,\phi_t)[\dot z], Y(w)\rangle + \phi_t'\langle Y(w), Y(w)\rangle \\ &= \langle A(z), Y(z)\rangle + \langle \dot z, Y(z)\rangle + \phi_t'\langle Y(z), Y(z)\rangle \\ &= C_z + t\bigg(\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}\bigg)^{-1} = C_w, \end{aligned} \tag{6.3}$$

so that $w \in \mathcal{N}_{P_1,P_t}$.

Proposition 6.1. *Let $t$ be a real number and $\mathcal{L}_t\colon\mathcal{N}_{P_1,P_2} \to \mathcal{N}_{P_1,P_t}$ be as in* (6.1)*. Then, $\mathcal{L}_t$ is a bijection, $\mathcal{L}_t$ is of class $\mathcal{C}^2$ and, for any compact subset $B$ of $\mathcal{N}_{P_1,P_2}$, $c_1, c_2, c_3 > 0$ exist* (*depending only on $B$*) *such that*

$$\sup_{z\in B} J(\mathcal{L}_t(z)) \le c_1 - c_2 t^2 + c_3|t|. \tag{6.4}$$

Proof. Observe that the map $\mathcal{L}_{-t}\colon\mathcal{N}_{P_1,P_t} \to \mathcal{N}_{P_1,P_2}$ defined by $\mathcal{L}_{-t}(z) = w$ where

$$w(s) = \psi(z(s), \phi_{-t}(s)) \quad \text{for all } s \in [0,1]$$

is the inverse of $\mathcal{L}_t$. Clearly, $\mathcal{L}_t$ is of class $\mathcal{C}^2$ (as $\phi_t$ depends smoothly on $z$). Now, let $B$ be a compact subset of $\mathcal{N}_{P_1,P_2}$. As in Lemma 4.1, $D = D(B) > 0$ exist such that

$$|C_z| \le D \quad \text{for all } z \in B$$

and $\mu = \mu(B) > 0$, $\nu = \nu(B) > 0$, $C = C(B) > 0$ exist such that for any $z \in B$

$$-\mu \le \langle Y(z), Y(z)\rangle - \nu, \tag{6.6}$$

$$J(z) \le C. \tag{6.7}$$

Let $z \in B$ and $w = \mathcal{L}_t(z)$. As $Y$ is Killing and (5.16), (5.17), (5.22), (6.2), (6.3) hold, we can compute

$$\begin{aligned} J(w) &= \frac{1}{2}\int_0^1 [\langle \dot z, \dot z\rangle + 2\phi_t'\langle \dot z, Y(z)\rangle + (\phi_t')^2\langle Y(z), Y(z)\rangle]\, ds \\ &\quad + \int_0^1 [\langle \dot z, A(z)\rangle + \phi_t'\langle A(z), Y(z)\rangle]\, ds \\ &= J(z) + \int_0^1 \bigg[\phi_t' C_z + \frac{1}{2}(\phi_t')^2\langle Y(z), Y(z)\rangle\bigg]\, ds \\ &= J(z) + C_z t + \frac{1}{2}t^2\bigg(\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}\bigg)^{-1} \\ &= J(z) + \frac{C_w^2 - C_z^2}{2}\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}. \end{aligned} \tag{6.8}$$

Note that, by (6.3), (6.5), (6.6)

$$|C_w| \ge |t|\bigg(-\int_0^1 \frac{ds}{\langle Y(z), Y(z)\rangle}\bigg)^{-1} - |C_z| \ge \nu|t| - D. \tag{6.9}$$

Then by (6.5)–(6.9)

$$\sup_{z\in B} J(\mathcal{L}_t(z)) \le c + \frac{D^2}{2\nu} - \frac{1}{2\mu}(\nu|t| - D)^2$$

and the proof is complete. $\square$

PROOF OF THEOREM 1.7. Let $z \equiv z_t$ be a minimum point of $J$ on $\mathcal{N}_{P_1,P_t}$ (whose existence is given by Theorem 1.4 applied to $\mathcal{N}_{P_1,P_t}$). We have to prove that, if $|t|$ is sufficiently large

$$\langle \dot z, \dot z\rangle < 0. \tag{6.10}$$

Hereafter, we shall denote by $c_i$, $i = 1, \dots, 19$, suitable positive constants. Let $\gamma \in \mathcal{N}_{P_1,P_2}$ be a fixed curve. By (6.4)

$$J(z) = \min_{z\in\mathcal{N}_{P_1,P_t}} J(z) \le J(\mathcal{L}_t(\gamma)) \le c_1 - c_2t^2 + c_3|t|. \tag{6.11}$$

By (6.11)

$$\frac{1}{2}\langle \dot z, \dot z\rangle = J(z) - \int_0^1 \langle \dot z, A(z)\rangle\, ds \le c_1 - c_2t^2 + c_3|t| - \int_0^1 \langle \dot z, A(z)\rangle\, ds. \tag{6.12}$$

We have to estimate the last term in (6.12). To this aim, we use local coordinates. We can choose a finite number of local charts

$$(U_k, x_k^1, \dots, x_k^{N-1}, t_k)_{k=1,\dots,r}$$

with the same properties listed in the proof of Lemma 4.1, (where $\mathcal{U}$ is a compact neighbourhood of $z([0,1])$). We can write

$$z(s) = (\mathbf{x}_k(s), t_k(s)) \in U_k \quad \text{for all } s \in [a_{k-1}, a_k],\ k = 1, \dots, r.$$

Then

$$\begin{aligned} -\int_0^1 \langle \dot z, A(z)\rangle\, ds &\le \sum_k \int_{a_{k-1}}^{a_k} |\langle \dot z, A(z)\rangle|\, ds \\ &\le \sum_k \int_{a_{k-1}}^{a_k} |\langle \dot{\mathbf{x}}_k, A_k^1(\mathbf{x}_k)\rangle_0 + \langle \delta_k(\mathbf{x}_k), \dot{\mathbf{x}}_k\rangle_0 A_k^2(\mathbf{x}_k)|\, ds \\ &\quad + \sum_k \int_{a_{k-1}}^{a_k} |\langle \delta_k(\mathbf{x}_k), A_k^1(\mathbf{x}_k)\rangle_0 - \beta(\mathbf{x}_k)A_k^2(\mathbf{x}_k)||\dot t_k|\, ds \\ &\le c_4 \sum_k \left( \int_{a_{k-1}}^{a_k} \|\dot{\mathbf{x}}_k\|\, ds + \int_{a_{k-1}}^{a_k} |\dot t_k|\, ds \right) \end{aligned} \tag{6.13}$$

where $\|\cdot\|$ denotes the norm associated to the Riemannian metric $\langle\cdot,\cdot\rangle_0$. Set

$$d_k = t_k(a_k) - t_k(a_{k-1}).$$

By the definition of $G_k$ (given in (4.6))

$$|G_k(\mathbf{x}_k, \dot{\mathbf{x}}_k)| \le c_5\|\dot{\mathbf{x}}_k\| + c_6 \tag{6.14}$$

then, by (4.8) (applied to $z$) and (4.10)

$$|C_z| \le \frac{\mu}{a_k - a_{k-1}} \left( |d_k| + c_7 \left( \int_{a_{k-1}}^{a_k} \|\dot{\mathbf{x}}_k\|\, ds + (a_k - a_{k-1}) \right) \right). \tag{6.15}$$

By (4.10), (6.14), (6.15)

$$|\dot{t}_k| = \left|\frac{G_k(\mathbf{x}_k,\dot{\mathbf{x}}_k) - C_z}{\beta(\mathbf{x}_k)}\right| \\ \le c_8\bigg(\|\dot{\mathbf{x}}_k\| + \frac{1}{a_k - a_{k-1}}|d_k| + \frac{1}{a_k - a_{k-1}}\int_{a_{k-1}}^{a_k}\|\dot{\mathbf{x}}_k\| ds\bigg) + c_9$$

then

$$\int_{a_{k-1}}^{a_k} |\dot{t}_k|\, ds \le c_{10}\bigg(\int_{a_{k-1}}^{a_k}\|\dot{\mathbf{x}}_k\|\, ds + |d_k| + (a_k - a_{k-1})\bigg). \tag{6.16}$$

By the definition of $E_k$ (see (4.5))

$$\int_{a_{k-1}}^{a_k} E_k(\mathbf{x}_k,\dot{\mathbf{x}}_k)\, ds \ge \frac{1}{2}\int_{a_{k-1}}^{a_k}\langle\dot{\mathbf{x}}_k,\dot{\mathbf{x}}_k\rangle_0\, ds - c_{11}\int_{a_{k-1}}^{a_k}\|\dot{\mathbf{x}}_k\|\, ds. \tag{6.17}$$

Thus, as in (4.13) (using also (6.14), (6.17))

$$\begin{aligned} &\int_{a_{k-1}}^{a_k}\bigg[\frac{1}{2}\langle\dot{z},\dot{z}\rangle + \langle A(z),\dot{z}\rangle\bigg]\, ds \\ &\quad\ge \int_{a_{k-1}}^{a_k} E_k(\mathbf{x}_k,\dot{\mathbf{x}}_k)\, ds \\ &\quad\quad - \frac{\mu}{\nu}\frac{|d_k|}{a_k - a_{k-1}}\int_{a_{k-1}}^{a_k}|G_k(\mathbf{x}_k,\dot{\mathbf{x}}_k)|\, ds - \frac{1}{2}\frac{\mu}{\nu}\frac{d_k^2}{a_k - a_{k-1}} \\ &\quad\ge \frac{1}{2}\int_{a_{k-1}}^{a_k}\langle\dot{\mathbf{x}}_k,\dot{\mathbf{x}}_k\rangle_0\, ds - c_{12}\bigg(1 + \frac{|d_k|}{a_k - a_{k-1}}\bigg)\int_{a_{k-1}}^{a_k}\|\dot{\mathbf{x}}_k\|\, ds \\ &\quad\quad - c_{12}\bigg(\frac{d_k^2}{a_k - a_{k-1}} + |d_k|\bigg). \end{aligned} \tag{6.18}$$

By the Schwartz's inequality and (6.18), setting

$$\|\dot{\mathbf{x}}_k\|_2 = \bigg(\int_{a_{k-1}}^{a_k}\langle\dot{\mathbf{x}}_k,\dot{\mathbf{x}}_k\rangle_0\, ds\bigg)^{1/2}$$

it results

$$\begin{aligned} &\int_{a_{k-1}}^{a_k}\bigg[\frac{1}{2}\langle\dot{z},\dot{z}\rangle + \langle A(z),\dot{z}\rangle\bigg]\, ds \\ &\quad\ge \frac{1}{2}\|\dot{\mathbf{x}}_k\|_2^2 - c_{12}\bigg(\bigg(1 + \frac{|d_k|}{a_k - a_{k-1}}\bigg)\|\dot{\mathbf{x}}_k\|_2 + \frac{d_k^2}{a_k - a_{k-1}} + |d_k|\bigg) \\ &\quad\ge \frac{1}{2}\|\dot{\mathbf{x}}_k\|_2^2 - \frac{1}{2}\bigg(\frac{1}{2}\|\dot{\mathbf{x}}_k\|_2^2 + 2c_{12}^2\bigg(1 + \frac{|d_k|}{a_k - a_{k-1}}\bigg)^2\bigg) \\ &\quad\quad - c_{12}\bigg(\frac{d_k^2}{a_k - a_{k-1}} + |d_k|\bigg) \\ &\quad= \frac{1}{4}\|\dot{\mathbf{x}}_k\|_2^2 - c_{13}\bigg(\bigg(1 + \frac{|d_k|}{a_k - a_{k-1}}\bigg)^2 + \frac{d_k^2}{a_k - a_{k-1}} + |d_k|\bigg) \end{aligned} \tag{6.19}$$

$$\geq \frac{1}{4}\|\dot{\mathbf{x}}_k\|_2^2 - c_{14}(1+|d_k|+d_k^2)$$

where the Young's inequality

$$ab \leq \frac{1}{2}\bigg(\varepsilon^2 a^2 + \frac{1}{\varepsilon^2}b^2\bigg) \quad \text{for } a,b>0$$

(with $\varepsilon = 1/\sqrt{2}$) has been applied. By (6.19), for any $k = 1,\dots,r$,

$$\|\dot{\mathbf{x}}_k\|_2^2 \leq \sum_k \|\dot{\mathbf{x}}_k\|_2^2 \leq 4\bigg(J(z) + c_{14}\sum_k (1+|d_k|+d_k^2)\bigg). \tag{6.20}$$

Then, by (6.13), (6.16), (6.20) and the Schwartz's inequality

$$\begin{aligned} -\int_0^1 \langle \dot{z}, A(z)\rangle &\leq c_{15}\bigg(\sum_k \int_{a_{k-1}}^{a_k} \|\dot{\mathbf{x}}_k\| ds + \sum_k |d_k| + 1\bigg) \\ &\leq c_{15}\bigg(\sum_k \|\dot{\mathbf{x}}_k\|_2 + \sum_k |d_k| + 1\bigg) \\ &\leq 2c_{15}\bigg(J(z) + c_{14}\sum_k \Big(1+|d_k|+d_k^2\Big)\bigg)^{1/2} \\ &\quad + c_{15}\bigg(\sum_k |d_k| + 1\bigg). \end{aligned} \tag{6.21}$$

Observe now that, as $z(1) \in P_t = \psi(P_2, t)$ and $Y_{|U_r} = \partial/\partial t_r$ we have

$$z(1) = (x_0, t_0 + t) \quad (x_0, t_0) \in P_2$$

then

$$\sum_k |d_k| \leq c_{16} + |t|, \tag{6.22}$$

$$\sum_k |d_k|^2 \leq c_{17}(t^2 + |t| + 1). \tag{6.23}$$

Finally, by (6.11), (6.12), (6.21)–(6.23)

$$\frac{1}{2}\langle \dot{z}, \dot{z}\rangle \leq c_{18}(1 - t^2 + |t|) + c_{19}(1 + t^2 + |t|)^{1/2}$$

so (6.10) is proved. □

Proof of Theorem 1.8. By Theorem 5.4, for any $n \in \mathbb{N}$ a compact subset $K_n$ of $\Omega_{P_1,P_2}$ exists such that $\mathrm{cat}_{\Omega_{P_1,P_2}} K_n \geq n$.

As the map $\mathcal{F}$ defined in Proposition 5.5 is a strong deformation retract, setting for any $n \in \mathbb{N}$ $\widetilde{K}_n = \mathcal{F}(K_n)$, also $\mathrm{cat}_{\mathcal{N}_{P_1,P_2}} \widetilde{K}_n \geq n$.

Let $c_0$ and $t_0$ be as in the statement of Theorem 1.7 and fix $c < c_0$. By (6.4), for any $n \in \mathbb{N}$, $\bar{t} = \bar{t}(n) \geq 0$ exists such that for any $t$ with $|t| \geq \bar{t}$

$$\mathcal{L}_t(\widetilde{K}_i) \subset J^c \cap \mathcal{N}_{P_1,P_t} \quad \text{for all } i = 1,\dots,n.$$

As $\mathcal{L}_t$ is a homeomorphism, we have

$$\mathrm{cat}_{\mathcal{N}_{P_1,P_t}} \mathcal{L}_t(\widetilde{K}_i) = \mathrm{cat}_{\mathcal{N}_{P_1,P_2}}(\widetilde{K}_i) \quad \text{for all } i = 1, \dots, n.$$

For any $i = 1, \dots, n$, we set

$$A_{t,i} = \{A \subset \mathcal{N}_{P_1,P_t} \mid A \text{ is closed, } \mathrm{cat}_{\mathcal{N}_{P_1,P_t}} A \geq i\}$$

and

$$c_i = \inf_{A \in A_{t,i}} \sup_{z \in A} J(z).$$

From classical arguments in Critical Point Theory, each $c_i$ is a critical value of $J$ on $\mathcal{N}_{P_1,P_t}$ and, if $i \neq j$ exists such that $c_i = c_j$, there are infinitely many critical points of $J$ at the level $c_i$. Then it has been proved the existence of at least $n$ critical points $\{z_1, \dots, z_n\}$ of $J$ on $\mathcal{N}_{P_1,P_t}$ such that, by (6.4)

$$c_i = J(z_i) \leq \sup_{z \in \widetilde{K}_n} J(\mathcal{L}_t(z)) \leq c_1 - c_2 t^2 + c_3 |t|$$

where $c_1$, $c_2$, $c_3$ are positive constants. Moreover,

$$\frac{1}{2}\langle \dot{z}, \dot{z} \rangle = c_i - \int_0^1 \langle \dot{z}_i, A(z_i) \rangle \, ds \tag{6.24}$$

thus, reasoning as in the proof of Theorem 1.7, the integral in (6.24) can be estimated with a term which grows linearly with $t$. Then, if $t$ is sufficiently large each $z_i$ is timelike and the proof is complete. $\square$

## Appendix A. Sufficient conditions for $C_{P_1,P_2} \neq \emptyset$

It is easy to give an example where $C_{P_1,P_2}$ is empty. Filling a small gap in [12, p. 160], consider the following example with $A = 0$. Let $\mathcal{L}^4$ be the 4-dimensional Lorentz–Minkowski space $(+, +, +, -)$ with usual coordinates $(x, y, z, t)$, and let $\Pi$ be the plane $t \equiv 0 \equiv x$. Put $\mathcal{M} = \mathcal{L}^4 \setminus \Pi$, $P_1 = \{(-1, 0, 0, 0)\}$, $P_2 = \{(1, 0, 0, 0)\}$. Assume that a curve $(x(s), y(s), z(s), t(s))$ joins $P_1$ and $P_2$. Necessarily, $t(s)$ cannot be a constant, because $P_1$ and $P_2$ lie in different connected parts of the set $\{(x, y, z, t) \in \mathcal{M} \mid t = 0\}$. Thus, $t(s)$ as well as $\dot{t}(s)$ cannot be constant. But $\dot{t}(s) = -\langle \dot{z}(s), \partial_t \rangle \equiv -\langle \dot{z}(s) + A, Y \rangle$, in contradiction with (1.6).

When $Y$ is complete, it is not necessary to impose that $\mathcal{N}_{P_1,P_2}$ (or $C_{P_1,P_2}$) are not empty. Indeed, we can define a map $\mathcal{F}: \Omega_{P_1,P_2} \to \mathcal{N}_{P_1,P_2}$ as in the proof of Proposition 5.5. Thus the following proposition hold immediately.

Proposition A.1. *Under the assumptions of Theorem* 1.4*, assume that $Y$ is complete. Then $\mathcal{N}_{P_1,P_2}$ (and thus $C_{P_1,P_2}$) are not empty.*

Next we give another sufficient condition. We recall that $P_1$ and $P_2$ are said *causally related* if either $J^+(P_1) \cap P_2$ or $J^+(P_2) \cap P_1$ is non-empty, where $J^+(P_i)$,

$i = 1, 2$ denotes the subset of all the points of $\mathcal{M}$ which can be joined by a causal curve $z$ starting at any point of $P_i$. Recall that, as $Y$ is timelike, then necessarily $\langle Y(z), \dot{z}\rangle \neq 0$ for any causal curve $z$ at any point.

PROPOSITION A.2. *If $P_1$ and $P_2$ are causally related, $\mathcal{N}_{P_1,P_2}$ (and $C_{P_1,P_2}$) are not empty.*

PROOF. Under the above assumptions, a causal curve $z\colon [0,1] \to \mathcal{M}$ exists such that

$$\langle \dot{z}(s), Y(z(s))\rangle < 0 \quad \text{for all } s \in [0,1] \tag{A.1}$$

and, for example, $z(0) \in P_1$, $z(1) \in P_2$. We set $z^*(r) = z(s(r))$ where $s(r)\colon [0,1] \to [0,1]$ is an increasing diffeomorphism to be determined in way that

$$\langle \dot{z}^*(r) + A(z^*(r)), Y(z^*(r))\rangle = C$$

where $C$ is a real constant. As $\dot{z}^*(r) = \dot{s}(r)\dot{z}(s(r))$ and by (A.1), $s$ must satisfy

$$\dot{s}(r) = \frac{C - \langle A(z^*(r)), Y(z^*(r))\rangle}{\langle \dot{z}(s(r)), Y(z(s(r)))\rangle}.$$

Set $\rho(s) = \langle A(z(s)), Y(z(s))\rangle$ and consider $\rho_0 = \rho(s_0)$, its (possible non-unique) minimum in $[0,1]$.

If we assume $C < \rho_0$ then $r(s)$ (the inverse of $s$) must satisfy

$$r(s) = \int_0^s \frac{-\langle \dot{z}, Y(z)\rangle}{\rho - C}\, d\tau.$$

Thus, if we put

$$\Lambda(C) = \int_0^1 \frac{-\langle \dot{z}, Y(z)\rangle}{\rho - C}\, ds,$$

it is enough prove the existence of some $C < \rho_0$ such that $\Lambda(C) = 1$.

This is straightforward from

$$\lim_{C\to-\infty} \Lambda(C) = 0 \qquad \lim_{C\to\rho_0} \Lambda(C) = \infty. \tag{A.2}$$

Indeed, as $c_1, c_2 > 0$ exist such that

$$c_1 \leq -\langle \dot{z}(s), Y(z(s))\rangle \leq c_2 \quad \text{for all } s \in [0,1]$$

we get

$$\Lambda(C) \leq \int_0^1 \frac{c_2}{\rho - C} ds \leq \frac{c_2}{\rho_0 - C}$$

from which the first limit (A.2) follows. For the second one, by using the power expansion of $\rho(s)$ around $s_0$, which yields that

$$\rho(s) - C \leq M|s - s_0|$$

for some positive $M$, we have

$$\lim_{C\to\rho_0}\Lambda(C)\geq\lim_{C\to\rho_0}\int_0^1\frac{c_1}{\rho(s)-C}\,ds=\int_0^1\frac{c_1}{\rho(s)-\rho_0}\,ds=\infty$$

so the proof is complete. □

## Appendix B. The standard stationary case

Next, the case of standard stationary manifolds as in Remark 1.9 will be discussed. Our aim is to find conditions on the coefficients of the metric (1.9) and on the vector field $A$ such that $F$ is pseudocoercive. Our conditions can be stated with a more general language in terms of the spatial growth with respect to a time function as in [12, Appendix A]. Nevertheless, under these conditions the spacetime is always standard stationary. Thus, we prefer to state our result directly on standard spacetimes for the sake of clarity. In fact, recall the following result:

Lemma B.1. *Let $(\mathcal{M},g)$ be a Lorentzian manifold admitting a timelike Killing vector field $Y$ such that*

(a) *the Riemannian metric $g_R$ in* (2.1) *is complete;*
(b) *there are constants $\nu,\mu>0$ such that*

$$0<\nu\leq -g(Y(z),Y(z))\leq\mu\quad \textit{for all } z\in\mathcal{M}.$$

*Then the vector field $Y$ is complete and the Lorentzian metric $g$ is geodesically complete.*

Proof. Condition (a) and the inequality $g_R(Y,Y)=-g(Y,Y)\leq\mu$ imply the completeness of $Y$. For the geodesic completeness of $g$, use (a) and the first inequality in (b) to apply [19, Proposition 2.1]. □

Notice that the hypotheses of [12, Proposition A.3] imply those in Lemma B.1. Thus, $Y$ will be complete. Then, the hypotheses on the time function in [12, Appendix A] imply the existence of a spacelike hypersurface $\Sigma$ which is intersected exactly once by all the flow lines of $Y$. Therefore, moving $\Sigma$ with the flow of $Y$, one sees that the applications in [12, Appendix A] hold only for standard stationary spacetimes.

On the other hand, the geodesic completeness of $g$ in Lemma B.1 poses an interesting question (compare with [12, Introduction and Remark A.5]). The following *static* (i.e. stationary with irrotational $Y$) Lorentzian manifolds are used in the literature as counterexamples to geodesic connectedness: (1) Universal anti-de Sitter spacetime, which is also (geodesically) complete and standard, but *not* globally hyperbolic, (2) the example in [12, Appendix B], which is globally hyperbolic, but neither standard nor complete, and (3) the first example in [22,

p. 925, Counterexamples], which is globally hyperbolic and standard but not complete. Thus, it would be interesting to know if there exists a counterexample being static (or, at least, stationary), globally hyperbolic and complete. We stress that the pseudosphere $S_1^n$ (de Sitter spacetime) is complete, globally hyperbolic and admits a standard splitting (as a Generalized Robertson–Walker spacetime), but it is not stationary. Moreover, it is not difficult to check, by using [11, Section 6, Corollary 5], that a two-dimensional globally hyperbolic standard static spacetime (complete or not)

$$((a,b) \times \mathbb{R}, dx^2 - f^2(x)\, dt^2)$$

is geodesically connected (in fact, global hyperbolicity implies $\int_c^b f = \int_a^c f = \infty$ for $c \in (a,b)$, which is sufficient to ensure Condition A of this reference). For further discussions and related results see also [20], [21].

Thus, let $(\mathcal{M}, \langle \cdot\,,\cdot \rangle)$ be a standard stationary Lorentzian manifold as in Remark 1.9. A complete Killing vector field $Y$ on $\mathcal{M}$ is given by

$$Y(z) = (\mathbf{0}, 1) \equiv \partial_t \quad \text{for all } z \in \mathcal{M}.$$

whose flow $\psi\colon \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ is defined by

$$\psi(z,s) = (x, t+s) \quad \text{for all } z = (x,t) \in \mathcal{M}.$$

We can fix two submanifolds $S_1$, $S_2$ of $\mathcal{M}_0$ and consider the corresponding spacelike submanifolds of $\mathcal{M}$

$$P_1 = S_1 \times \{0\}, \quad P_2 = S_2 \times \{\tau\}$$

where $\tau$ is a fixed real number. In the standard case, equation (1.2) is equivalent to require that the field $A$ does not depend on the time variable, that is

$$A(z) = (A_1(x), A_2(x)) \quad \text{for all } z = (x,t) \in \mathcal{M}.$$

Observe that, as $Y$ is complete, $C_{P_1,P_2}$ is not empty (see Appendix A).

In order to give simple conditions for the pseudocoercitivity of $F$ on $\mathcal{N}_{P_1,P_2}$, let us consider (see Lemma 4.1) for $z = (x,t) \in \mathcal{N}_{P_1,P_2}$

$$F(z) = \int_0^1 \left[ E(x,\dot{x}) + G(x,\dot{x}) - \frac{1}{2}\beta(x)\dot{t}^2 \right] ds$$

where

$$E(x,\dot{x}) = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle_0 + \langle A_1(x), \dot{x} \rangle_0 + \langle \delta(x), \dot{x} \rangle_0 A_2(x), \tag{B.1}$$

$$G(x,\dot{x}) = \langle \delta(x), \dot{x} \rangle_0 + \langle \delta(x), A_1(x) \rangle_0 - \beta(x) A_2(x). \tag{B.2}$$

Proposition B.2. *Let $\mathcal{M}$ be a standard stationary spacetime as in Remark* 1.9. *Assume that $\nu,\mu>0$ exist such that*

$$\nu \le \beta(x) \le \mu \quad \text{for all } x \in \mathcal{M}_0 \tag{B.3}$$

*and that $p,q,r \in [0,1[$, $h \in [0,2[$, $c_i \ge 0$, $i=1,\dots,8$ exist such that, for any $x \in \mathcal{M}_0$,*

$$|A_1(x)| \le c_1[d(x,S_1)]^p + c_2, \tag{B.4}$$

$$|A_2(x)||\delta(x)| \le c_3[d(x,S_1)]^q + c_4, \tag{B.5}$$

$$|\delta(x)| \le c_5[d(x,S_1)]^r + c_6, \tag{B.6}$$

$$|A_2(x)| \le c_7[d(x,S_1)]^h + c_8, \tag{B.7}$$

*where $d$ is the usual distance in $(\mathcal{M}_0,\langle\,\cdot\,,\,\cdot\,\rangle_0)$. Then, $C_{P_1,P_2}$ is pseudocoercive.*

Proof. As in Lemma 4.1, if (B.3) holds

$$F(z) \ge \int_0^1 E(x,\dot x)\,ds - \tau\frac{\mu}{\nu}\int_0^1 |G(x,\dot x)|\,ds - \frac{1}{2}\mu\tau^2.$$

As $x(0) \in S_1$

$$d(x(s),S_1) \le d(x(s),x(0)) \le \int_0^1 |\dot x|dr \le \|\dot x\|_2$$

then, by (B.4)–(B.7) we obtain

$$\begin{aligned} c \ge F(z) \ge{}& \frac{1}{2}\|\dot x\|_2^2 - (c_1\|\dot x\|_2^p + c_2)\|\dot x\|_2 - (c_3\|\dot x\|_2^q + c_4)\|\dot x\|_2 \\ &- \tau\frac{\mu}{\nu}[(c_5\|\dot x\|_2^r + c_6)\|\dot x\|_2 + (c_5\|\dot x\|_2^r + c_6)(c_1\|\dot x\|_2^p + c_2) + \mu(c_7\|\dot x\|_2^h + c_8)] - \frac{1}{2}\mu\tau^2 \end{aligned}$$

then $K_1>0$ exists such that $\|\dot x\|_2 \le K_1$.

Let $C_z$ be such that $\langle \dot z + A(z), Y(z)\rangle = C_z$. As

$$C_z = \bigg(\int_0^1 \frac{G(x,\dot x)}{\beta(x)}\,ds - \tau\bigg)\bigg(\int_0^1 \frac{1}{\beta(x)}\,ds\bigg)^{-1}$$

by (B.2)–(B.7), also $|C_z|$ is bounded. Now, as

$$t(s) = \int_0^s \frac{G(x,\dot x) - C_z}{\beta(x)}\,ds$$

we get $\|t\|_2 \le K_2, \|\dot t\|_2 \le K_3$, where $K_2$, $K_3$ are positive constants. Thus if $(z_n = (x_n,t_n)) \subset \mathcal{N}_{P_1,P_2}$ is such that $F(z_n) \le c$, it results that $(z_n)$ is bounded in $H^1$ and the thesis follows. □

Then, if $S_1$, $S_2$ are closed and at least one of them is compact, from our Theorem 1.4 it follows in particular [4, Theorem 1.3] where standard *static*

($\delta \equiv 0$) Lorentzian manifolds are taken into account. Moreover, if the topological assumptions are verified, from our Theorem 1.5, [4, Theorem 1.4] follows as a particular case.

Finally, we also point out that if in Theorems 1.4, 1.5, 1.7, 1.8, the submanifolds $P_1$ and $P_2$ reduce to a point, then we re-obtain the results in [2] and some of the results in [9] for trajectories under a vectorial potential joining two fixed events.

## References


[1] F. Antonacci, F. Giannoni and P. Magrone, *On the problem of the existence for connecting trajectories under the action of gravitational and electromagnetic fields*, Differential Geom. Appl. **13** (2000), 1–17.

[2] R. Bartolo, *Trajectories connecting two events of a Lorentzian manifold in the presence of a vector field*, J. Differential Equations **153** (1999), 82–95.

[3] ———, *Trajectories under a vectorial potential on stationary manifolds*, Internat. J. Math. Math. Sci. **23** (2003), 1481–1495.

[4] R. Bartolo and A. Germinario, *Trajectories joining two submanifolds under the action of gravitational and electromagnetic fields on static spacetimes*, Math. Phys. Anal. Geom. **5** (2002), 125–143.

[5] J. K. Beem, P. E. Ehrlich and K. L. Easley, *Global Lorentzian Geometry*, Second Edition, Pure App. Math., vol. 202, Marcel Dekker, New York, 1996.

[6] V. Benci and D. Fortunato, *A new variational principle for the fundamental equations of classical physics*, Found. Phys. **28** (1998), 333–352.

[7] A. M. Candela and A. Salvatore, *Light rays joining two submanifolds in space-times*, J. Geom. Phys. **22** (1997), 281–297.

[8] A. Canino, *On p-convex sets and geodesics*, J. Differential Equations **75** (1988), 118–157.

[9] E. Caponio and A. Masiello, *Trajectories for relativistic particles under the action of an electromagnetic force in a stationary space-time*, Nonlin. Anal. **50** (2002), 71–89.

[10] E. Fadell and S. Husseini, *Category of loop spaces of open subsets in Euclidean space*, Nonlinear Anal. **17** (1991), 1153–1161.

[11] J. L. Flores and M. Sánchez, *Geodesic connectedness and conjugate points in GRW spacetimes*, J. Geom. Phys. **36** (2000), 285–314.

[12] F.Giannoni and P. Piccione, *An intrinsic approach to the geodesical connectedness of stationary Lorentzian manifolds*, Comm. Anal. Geom. **7** (1999), 157–197.

[13] S. W. Hawking and G. F. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Univ. Press, London–New York, 1973.

[14] W. Klingenberg, *Riemannian Geometry*, De Gruyter, Berlin, 1982.

[15] L. Landau and L. Lifschitz, *Théorie des champs*, Mir, Moscow, 1970.

[16] A. Masiello, *Variational Methods in Lorentzian Geometry*, Pitman Res. Notes Math. Ser., vol. 309, Longman, London, 1994.

[17] J. Nash, *The embedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20–63.

[18] B. O'Neill, *Semi–Riemannian Geometry with Applications to Relativity*, Academic Press, New York–London, 1983.

[19] A. Romero and M. Sánchez, *On the completeness of certain families of semi-Riemannian manifolds*, Geom. Dedicata **53** (1994), 103–117.

[20] M. Sánchez, *Geodesic connectedness of semi-Riemannian manifolds*, Nonlinear Anal. **47** (2001), 3085–3102.

[21] ———, *Lorentzian manifolds admitting a Killing vector field*, Nonlinear Anal. **30** (1997), 643–654.

[22] ———, *On the geometry of generalized Robertson–Walker spacetimes: geodesics*, Gen. Relativity Gravittion **30** (1998), 915–932.

[23] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 1, Publish or Perish, Inc. Wilmington, Delaware, 1979.

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