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FIXED POINTS OF MULTIVALUED MAPPINGS WITH ELC^K VALUES

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Abstract. We prove some fixed point theorems for the Hausdorff continuous multivalued mappings with equilocally connected values in dimension $n-1$ or $n-2$ on n-dimensional discs and closed manifolds.

1. Introduction

The aim of this paper is to find some conditions which guarantee that the mapping $f: X \to 2^X$ with compact nonempty values has a fixed point $x \in X$. The space X will be regarded as a disc or a closed oriented topological manifold. This is well known that

- the upper semicontinuous (u.s.c.) mappings with acyclic values satisfy the Lefschetz (and in particular Brouwer's) Fixed Point Theorem (see [7]),
- there is a fixed point free mapping of the disc D^2 with values homeomorphic to S^1, which is continuous with respect to the Hausdorff metric ϱ_s (see [17]).

Górniewicz conjectured that the lack of the acyclicity of the values can be compensated in the fixed point theory by the stronger continuity of the mapping,

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e.g. with respect to the Borsuk metric of continuity ϱ_c or Borsuk metric of homotopy ϱ_h, ($\varrho_h \geq \varrho_c \geq \varrho_s$) (see [1]). This idea leads us to get some fixed point theorems for ϱ_s-continuous mappings with equilocally connected values (in the homotopy sense) in dimension $(\dim(X) - i)$ for $i = 1$. We apply the Górniewicz method of spheric mappings, to pass from the case $i = 1$ to $i = 2$ (see [8]).

2. Results

Our first result solves a problem formulated in [8] and called the Górniewicz conjecture in [14].

Theorem 2.1. *There exists a fixed point free ϱ_c-continuous mapping of D^4 with compact connected values.*

The proof is based on the Jezierski example of a fixed point free ϱ_s-continuous mapping of D^2 with values being finite sets (see [12]).

The next task consists in replacing ϱ_c by ϱ_h.

Recall that the family $\{X_\lambda : \lambda \in \Lambda\}$ is eLC^k (equilocally connected in dimension k) if and only if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for all λ, $x \in X_\lambda$, $r = 0, \ldots, k$, every map $\omega : S^r \to K(x, \delta(\varepsilon)) \cap X_\lambda$ has a continuous extension $\overline{\omega} : D^{r+1} \to K(x, \varepsilon) \cap X_\lambda$.

Theorem 2.2. *For every mapping $f : D^n \to 2^{D^n}$ the following conditions*

(a) *f is ϱ_h-continuous,*
(b) *f is ϱ_s-continuous and $\{f(x) : x \in D^n\}$ is eLC^{n-1},*

are equivalent. Under each of these conditions, f has a continuous single-valued selector and a fixed point.

Theorem 2.2(b) is close to the Michael Theorem in [13, Theorem 1.2] (note that we do not assume $f(x)$ to be C^{n-1}, but D^n is the very special space and ϱ_s-continuity is stronger than l.s.c.). The proof of the next result is based on the concept of the spheric mapping (see [8]). We recall the notation used in [8].

Let Y be a compact subset of $\mathbb{R}^n$. Then $B(Y)$ denotes the sum of all bounded components of $\mathbb{R}^n \setminus Y$; $D(Y)$ – the unbounded component of $\mathbb{R}^n \setminus Y$; $\widetilde{Y} = Y \cup B(Y)$. The bounded components of $\mathbb{R}^n \setminus Y$ are these with compact closure, which is important when we forget the metric in $\mathbb{R}^n$. To shorten notation, we use $Bf(x)$, $Df(x)$, $\widetilde{f}(x)$ instead of $B(f(x))$, $D(f(x))$, $\widetilde{f(x)}$.

Figure 1 shows a 1-dimensional continuum Y in $\mathbb{R}^2$ shaped as two hearts joined by two wedding rings and a 2-dimensional continuum $\widetilde{Y}$ which has a form of the gingerbread "katarzynka" baked in the town Toruń as a souvenir connected with a beautiful ancient legend.

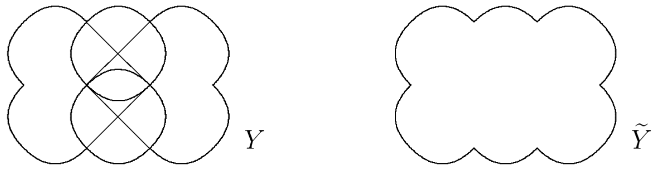

Figure 1

Theorem 2.3. *If*

(a) $f: D^n \to 2^{D^n}$ *is* ϱ_s*-continuous,*

(b) $\{f(x) : x \in D^n\}$ *is* eLC^{n-2}*,*

(c) $\{\widetilde{f}(x) : x \in D^n\}$ *is* eLC^{n-1}*,*

then f *has a fixed point.*

Author does not know, if the assumption (c) in the Theorem 2.3 is necessary for $n > 2$.

Problem 2.4. *Let* $n \geq 2$*. Is it true that if* $\{Y_\lambda : \lambda \in \Lambda\}$ *is* eLC^{n-2} *in* $\mathbb{R}^n$*, then* $\{\widetilde{Y_\lambda} : \lambda \in \Lambda\}$ *is* eLC^{n-1}*?*

(Author does not know the answer to this question even for the one point set Λ, when the letter e in eLC can be omitted).

For $n = 2$ the assumption (c) is superfluous, which can be proved without solving Problem 2.4.

Theorem 2.5. *If* $f: D^2 \to 2^{D^2}$ *is* ϱ_s*-continuous and* $\{f(x) : x \in D^2\}$ *is* eLC^0 *then* f *has a fixed point.*

We will give two different proofs of Theorem 2.5: the first shows that f is approximable by the singlevalued continuous mappings, the latter – that f is spheric with $\widetilde{f}$ permissible in the sense of [4].

Example 2.6. Let us recall, that every ϱ_c-continuous mapping $f: D^2 \to 2^{D^2}$ with compact connected values has a fixed point (see [8, Theorem 4.4]). The map f defined by $f(x) = \{y \in D^2 : \|y\| \geq \|x\|\}$ is not ϱ_c-continuous but satisfies the assumptions of Theorem 2.5.

Let Γ_f denote the graph $\{(x, y) : y \in f(x)\}$ and $p: \Gamma_f \to D^n$ – the projection $p(x, y) = x$. It appears that many conditions on the multivalued mapping f in fixed point theorems are equivalent to some fibre properties of p.

Example 2.7. Let $U = \{x \in D^n : f(x)$ is not a one point set$\}$. Every ϱ_s-continuous mapping $f: D^n \to 2^{D^n}$ such that $p: \Gamma_{f|U} \to U$ is a locally trivial fibration with the fibre S^{n-2} has a fixed point, see [15], [16]. For $n \neq 6$ we can assume equivalently, that f is ϱ_c-continuous and takes values which are one point sets or $(n-2)$ - dimensional spheres embedded in D^n, (see [15] and the references given there). For such mappings $\{f(x) : x \in D^n\}$ is eLC^{n-3}.

Example 2.8. There is a fixed point free ϱ_s-continuous mapping $f\colon D^n \to 2^{D^n}$ such that $\{f(x) : x \in D^n\}$ is eLC^{n-3}. Set

$$f(x) = \{y \in S^{n-1} : \langle y, x\rangle \leq (1 - \|x\|)\|x\|\}.$$

We can now formulate our main results for multivalued mappings on the manifolds. Let $L(\,\cdot\,;\,\cdot\,)$ denote the Lefschetz number.

Theorem 2.9. *Let M be a metrizable compact connected n-dimensional topological manifold without boundary. Suppose that M is K-oriented for a field K. Let*

(a) *$f\colon M \to 2^M$ be ϱ_s-continuous with connected values,*
(b) *$s\colon M \to M$ be continuous with $L(s;K) \neq 0$,*
(c) *$W\colon M \to 2^M$ be u.s.c.*

and such that $W(x)$ is homeomorphic to the disc D^n, $s(x) \in W(x)$ and $f(x) \subset \operatorname{int}_M(W(x))$ for every $x \in M$. Assume that $\{f(x) : x \in M\}$ is eLC^{n-1}, (which forces $p\colon \Gamma_f \to M$ to be a Hurewicz fibration with a fibre $F \simeq f(x)$). If the fibration p is orientable with respect to $H_*(\,\cdot\,;K)$ and

$$H_{n-1}(M \times F; K) = H_{n-1}(M; K),$$

then f has a fixed point.

Definition 2.10. The function s in Theorem 2.9 will be called a *positioning function* for f.

The positioning function is defined to be a selector of the map W only for simplicity of the formulation of Theorem 2.9. As well we can assume s to be a sufficiently close graph – approximation of W. It seems, that the existence of the pair (s, W) is a proper assumption which makes it legitimate to apply the notion of the Lefschetz number to find fixed points of f, nevertheless $L(s;K)$ is not uniquely determined by f. Note, that the inclusion $f(x) \subset \operatorname{int}_M(W(x)) \cong \mathbb{R}^n$ makes $\widetilde{f}(x)$ well defined.

Corollary 2.11. *Let $f\colon M \to 2^M$ be ϱ_s-continuous with $\widetilde{f}$ satisfying all other assumptions on f in Theorem* 2.9. *If $\{f(x) : x \in M\}$ is eLC^{n-2} and the positioning function for f is not homotopic to the identity on M, then f has a fixed point.*

2.1. Proof of Theorem 2.1. We shall define a fixed point free ϱ_c-continuous mapping $f\colon D^4 \to 2^{D^4}$ with compact connected values. Recall that

$$\varrho_c(X, Y) = \max\{d_c(X, Y), d_c(Y, X)\}$$

with $d_c(X,Y) = \inf\{\max\{\|\alpha(x) - x\| : x \in X\}\}$, where the infimum is taken over all continuous functions $\alpha: X \to Y$; $(X, Y \subset D^4)$. The disc D^4 will be identified with $D^2 \times D^2$. This is well known, [12], that there exists a ϱ_s-continuous homotopy $H: S^1 \times I \to 2^{S^1}$ joining $H(z,0) = \{z_0\}$ and $H(z,1) = \{z\}$ such that $H(z,t)$ is a finite subset of S^1 which has at most 3 elements for every $(z,t) \in S^1 \times I$. The multivalued retraction $r: D^2 \to 2^{S^1}$ is the standard one:

$$r(x) = \begin{cases} \{z_0\} & \text{for } \|x\| \le 1/2, \\ H(x/\|x\|, 2\|x\| - 1) & \text{for } \|x\| \in [1/2, 1]. \end{cases}$$

We define $J: D^2 \to 2^{S^1}$ by

$$J(x) = \begin{cases} -r(3x) & \text{for } \|x\| \le 1/3, \\ \{-x/\|x\|\} & \text{for } \|x\| \in [1/3, 1]. \end{cases}$$

Of course, J is ϱ_s-continuous and has finite values. Since $\varrho_s = \varrho_c$ on finite sets, J is ϱ_c-continuous. The mapping J is fixed point free, moreover $x \notin [1/2,1]J(x)$ for every $x \in D^2$. This is easy to check that the join of sets $A \subset S^1 \times \{0\}$ and $B \subset \{0\} \times S^1$ in $D^2 \times D^2$ is well defined by

$$A * B = \{(1-t)a + tb : t \in [0,1],\ a \in A,\ b \in B\}.$$

Let $\phi_1, \phi_2, f: D^2 \times D^2 \to 2^{D^2 \times D^2}$ be given by

$$\phi_1(x,y) = J(x) \times \{0\}, \quad \phi_2(x,y) = \{0\} \times J(y), \quad f(p) = \phi_1(p) * \phi_2(p).$$

We check now that f is ϱ_c-continuous. Take an $\varepsilon > 0$. Since ϕ_i is ϱ_c-continuous, there is a positive δ such that

$$\|p - q\| < \delta \Rightarrow \varrho_c\left(\phi_i(p), \phi_i(q)\right) < \varepsilon,$$

for $i = 1, 2$. Fix $p, q \in D^2 \times D^2$ with $\|p - q\| < \delta$. By the definition of ϱ_c, there is a continuous map $\alpha_i: \phi_i(p) \to \phi_i(q)$ such that $\|\alpha_i(v) - v\| < \varepsilon$, for every $v \in \phi_i(p)$. Let $\alpha_1 * \alpha_2: f(p) \to f(q)$ be the join of maps α_1 and α_2. Take $x = (1-t)u_1 + tu_2 \in f(p)$ with $u_i \in \phi_i(p)$. Thus

$$\begin{aligned} \|\alpha_1 * \alpha_2(x) - x\| &= \|(1-t)\alpha_1(u_1) + t\alpha_2(u_2) - (1-t)u_1 - tu_2\| \\ &\le (1-t)\|\alpha_1(u_1) - u_1\| + t\|\alpha_2(u_2) - u_2\| < \varepsilon. \end{aligned}$$

Hence $d_c(f(p), f(q)) < \varepsilon$. Likewise, $\varrho_c(f(p), f(q)) < \varepsilon$.

The mapping f is fixed point free. Otherwise, there is $(x,y) \in D^2 \times D^2$ such that

$$(x,y) \in f(x,y) = \{((1-t)a, tb) : t \in [0,1],\ a \in J(x),\ b \in J(y)\}.$$

Thus $x = (1-t)a \in [1/2,1]J(x)$, for $t \in [0,1/2]$ and $y = tb \in [1/2,1]J(y)$ for $t \in [1/2,1]$, a contradiction.

The values of f being joins of some finite sets are compact and connected (these are graphs of 4 homotopy types: $\bullet$, $\bigcirc$, $\ominus$, $\oplus$). One can check that $\{f(p) : p \in D^2 \times D^2\}$ is not eLC^0.

2.2. Proof of Theorem 2.2. Let $f: D^n \to 2^{D^n}$ be ϱ_h-continuous. By [1], the ϱ_h-continuity of f is equivalent to the conjunction of two conditions:

(a) f is ϱ_s-continuous,

(b) $\{f(x) : x \in D^n\}$ is equally locally contractible.

According to [1, Proof, p. 200], the projection $p: \Gamma_f \to D^n$ is strongly regular in the sense of [6, Definition, p. 373]. By [6, Theorem 1], p is the Hurewicz fibration. Since D^n is contractible, p has a section. The second coordinate of this section is a continuous selector of f, which proves Theorem 2.2(a). Since $f(x) \subset \mathbb{R}^n$, (b) is equivalent to

(b') $\{f(x) : x \in D^n\}$ is eLC^{n-1},

(see [1, Proof, pp. 187–188]), which shows that conditions (a) and (b) in Theorem 2.2 are equivalent.

2.3. Preparation for proving Theorem 2.3.

Lemma 2.12. *Let X be a compact* ANR *and $x \in \mathbb{R}^n \setminus X$. Then $x \in B(X)$ if and only if there is a singular $(n-1)$-cycle Z_{n-1} in X with rational coefficients, which does not bound in $\mathbb{R}^n \setminus \{x\}$.*

Proof. Choose $r_1, r_2 > 0$ such that

$$\breve{D}_1 \overset{\text{def}}{=} \{y \in \mathbb{R}^n : \|y - x\| < r_1\} \subset \mathbb{R}^n \setminus X$$

and

$$D_2 \overset{\text{def}}{=} \{y \in \mathbb{R}^n : \|y - x\| \le r_2\} \supset X.$$

The part "if" does not require that X is ANR. By assumption, the homomorphism $j_*: H_{n-1}X \to H_{n-1}(\mathbb{R}^n \setminus \{x\})$ (induced by inclusion) is nontrivial. Suppose, contrary to our claim that $x \in D(X)$. Fix in $\mathbb{R}^n$ a point $y \notin D_2$. Since $D(X)$ is a domain in $\mathbb{R}^n$, there are points $z_0 = x, z_1, \ldots, z_q = y$ such that each interval $z_i z_{i+1}$ lies in $D(X)$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{j_{i*}} & \mathbb{R}^n \setminus \{z_i\} \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle \cong T_{i*}} \\ X & \xrightarrow[j_{i+1*}]{} & \mathbb{R}^n \setminus \{z_{i+1}\} \end{array}$$

with $T_i(z) = z + z_{i+1} - z_i$ is homotopy commutative for $i = 0, \ldots, q-1$. Indeed, $H(z,t) = (1-t)z + tT_i(z)$ is a homotopy $H: j_{i+1} \simeq T_i \circ j_i$. It follows that $j_{q*} : H_{n-1}X \to H_{n-1}(\mathbb{R}^n \setminus \{y\})$ is a nontrivial homomorphism. But $X \subset D_2 \subset \mathbb{R}^n \setminus \{y\}$ and $H_{n-1}D_2 = 0$, a contradiction.

We now prove "only if". Let $A = D_2 \setminus \breve{D}_1$, $S^n = \mathbb{R}^n \cup \{\infty\}$, $x \in B(X)$. Consider the following commutative diagram

$$\begin{array}{ccc}
\check{H}^{n-1}A & \xrightarrow{\ \beta^*\ } & \check{H}^{n-1}X \\
\cong\downarrow & & \cong\downarrow \text{Alexander duality} \\
H_1(S^n, S^n \setminus A) & \longrightarrow & H_1(S^n, S^n \setminus X) \\
\cong\downarrow & & \cong\downarrow \text{excision} \\
H_1(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \longrightarrow & H_1(\mathbb{R}^n, \mathbb{R}^n \setminus X) \\
\cong\downarrow & & \cong\downarrow \widetilde{\partial} \\
\widetilde{H}_0(\mathbb{R}^n \setminus A) & \xrightarrow[\ \widetilde{\alpha}_*\ ]{} & \widetilde{H}_0(\mathbb{R}^n \setminus X)
\end{array}$$

with inclusions $\beta: X \to A$ and $\alpha: \mathbb{R}^n \setminus A \to \mathbb{R}^n \setminus X$. We have

$$\alpha: (\mathbb{R}^n \setminus D_2) \cup \breve{D}_1 \to D(X) \cup B(X),$$

$\mathbb{R}^n \setminus D_2 \subset D(X)$, $\breve{D}_1 \subset B_\mu \subset B(X) = \bigcup_\lambda B_\lambda$, where $\{B_\lambda\}$ is a family of all bounded components of $\mathbb{R}^n \setminus X$. Thus $\alpha_*: H_0(\mathbb{R}^n \setminus A) \to H_0(\mathbb{R}^n \setminus X)$ is the homomorphism

$$(s, t) \in Q \oplus Q \to Q \oplus \bigoplus_\lambda Q \ni (s, i_\mu(t)),$$

where $i_\mu: Q \to \bigoplus_\lambda Q$ denotes the μ-th canonical inclusion. Choose $y \in \mathbb{R}^n \setminus D_2$. Since $\widetilde{H}_0(\mathbb{R}^n \setminus A) = \operatorname{coker}(H_0\{y\} \to H_0(\mathbb{R}^n \setminus A))$ and the same is true for X in place of A, $\widetilde{\alpha}_* = i_\mu \neq 0$. Consequently $\beta^* \neq 0$. Since A and X are compact ANRs and the (co)homology coefficients are in Q, it follows that $\beta_*: H_{n-1}X \to H_{n-1}A$ is nontrivial. Clearly, the same holds for the composition $j_*: H_{n-1}X \to H_{n-1}A \to H_{n-1}(\mathbb{R}^n \setminus \{x\})$, which proves the lemma. □

Lemma 2.13. *Let X, X_1, $X_2, \ldots$ be compact subsets of $\mathbb{R}^n$ such that $\{X_k : k = 1, 2, \ldots\}$ is eLC^{n-2} and* $\lim_{k\to\infty} \varrho_s(X_k, X) = 0$. *Then*

$$\forall x \in B(X)\ \exists k_0\ \forall k > k_0 \quad x \in B(X_k).$$

Proof. Let η denote a positive number. Fix $x \in B(X)$ and a compact polyhedron P such that

$$X \subset P \subset O_\eta(X) \stackrel{\text{def}}{=} \{p \in \mathbb{R}^n : \operatorname{dist}(p, X) < \eta\}.$$

Clearly, $\varrho_s(X, P) < \eta$. Since $D(P) \subset D(X)$, $x \in \widetilde{P}$. Assuming that $\eta < \operatorname{dist}(x, X)$ gives $x \in B(P)$. By Lemma 2.12, there is a singular $(n-1)$-cycle

$Z_{n-1} = \sum_\sigma c_\sigma \sigma$ in P with $c_\sigma \in Q$, which does not bound in $\mathbb{R}^n \setminus \{x\}$. There is no loss of generality in assuming that

$$\forall \sigma \ (c_\sigma \neq 0 \Rightarrow \operatorname{diam}(\sigma(\Delta_{n-1})) < \eta),$$

(we apply to Z_{n-1} the multiple barycentric subdivision, if necessary).

Choose $k_0 \in N$ with $\varrho_s(X_k, X) < \eta$ for every $k > k_0$ and take such a k. Hence $\varrho_s(X_k, P) < 2\eta$. The main point of this proof is the construction of the $(n-1)$-cycle $\phi(Z_{n-1})$ in X_k, which is homologous to Z_{n-1} in $\mathbb{R}^n \setminus \{x\}$. Finding such $\phi(Z_{n-1})$ will complete the proof, by Lemma 2.12.

Let $\delta: \mathbb{R}_+ \to \mathbb{R}_+$ be a function, which appears in the definition of the eLC^{n-2} property for the family $\{X_k : k \in N\}$. Set $\mu(\varepsilon) = \delta(\varepsilon/4)$. Let $\mu^{(r)}$ denote the r-th iteration of the function μ. Fix the positive numbers

$$\varepsilon < \operatorname{dist}(x, X) \quad \text{and} \quad \eta < \min\left\{\frac{1}{5}\mu^{(n-1)}(\varepsilon), \frac{1}{8}\varepsilon\right\}.$$

Let Ξ be the set of all simplices of the cycle Z_{n-1} and all their faces. Thus $\Xi = \bigcup_{i=0}^{n-1} \Xi_i$, $\Xi_{n-1} = \{\sigma : c_\sigma \neq 0\}$, $\Xi_{i-1} = \{\tau \circ F_p^i : \tau \in \Xi_i,\ 0 \le p \le i\}$ for $1 \le i \le n-1$; ($F_p^i : \Delta_{i-1} \to \Delta_i$ is the p -th face mapping).

Our strategy is to make a copy $\phi(\tau)$ in X_k of every simplex $\tau \in \Xi$. Take $\tau \in \Xi_0$. By an obvious convention, $\tau \in P$. We choose $\phi(\tau)$ to be any point of X_k such that $\|\phi(\tau) - \tau\| < 2\eta$.

Take $\tau \in \Xi_1$. We have

$$\begin{aligned}
\|\phi(\tau \circ F_1^1) - \phi(\tau \circ F_0^1)\| &\le \|\phi(\tau \circ F_1^1) - \tau \circ F_1^1\| \\
&\quad + \|\tau \circ F_1^1 - \tau \circ F_0^1\| + \|\tau \circ F_0^1 - \phi(\tau \circ F_0^1)\| \\
&< 2\eta + \operatorname{diam}(\tau(\Delta_1)) + 2\eta < 5\eta < \mu^{(n-1)}(\varepsilon) \\
&= \delta(\mu^{(n-2)}(\varepsilon)/4).
\end{aligned}$$

We choose $\phi(\tau): \Delta_1 \to X_k$ to be any path joining $\phi(\tau \circ F_1^1)$ and $\phi(\tau \circ F_0^1)$ in $K\left(\phi(\tau \circ F_0^1),\ \mu^{(n-2)}(\varepsilon)/4\right) \cap X_k$.

Suppose that ϕ is defined on Ξ_{i-1} for an $i \le n-1$ in this way, that $\phi(\tau)(\Delta_{i-1})$ lies in an open ball of the radius $\mu^{(n-i)}(\varepsilon)/4$ in X_k, for every $\tau \in \Xi_{i-1}$.

Take $\tau \in \Xi_i$. We have $\operatorname{diam}\left(\phi(\tau \circ F_p^i)(\Delta_{i-1})\right) < \mu^{(n-i)}(\varepsilon)/2$ for $p = 0, \dots, i$. Define $\omega: \partial\Delta_i \to X_k$ by $\omega(F_p^i(x)) = \phi(\tau \circ F_p^i)(x)$. Clearly,

$$\operatorname{diam}(\omega(\partial\Delta_i)) < \mu^{(n-i)}(\varepsilon) = \delta(\mu^{(n-i-1)}(\varepsilon)/4.$$

Take any point $q \in \omega(\partial\Delta_i)$. We choose $\phi(\tau)$ to be a continuous extension $\widetilde{\omega}: \Delta_i \to X_k$ of ω such that $\widetilde{\omega}(\Delta_i) \subset K(q, \mu^{(n-i-1)}(\varepsilon)/4)$. In particular,

$$\phi(\tau) \circ F_p^i = \phi(\tau \circ F_p^i) \quad \text{for } \tau \in \Xi_i.$$

This condition on Ξ_{i-1} makes ω well defined. Since $F^i_p \circ F^{i-1}_q = F^i_q \circ F^{i-1}_{p-1}$ for $q < p$ (see [11]),

$$\begin{aligned}\omega(F^i_p \circ F^{i-1}_q(y)) &= \phi(\tau \circ F^i_p)(F^{i-1}_q(y)) = \phi(\tau \circ F^i_p \circ F^{i-1}_q)(y)\\ &= \phi(\tau \circ F^i_q \circ F^{i-1}_{p-1})(y) = \phi(\tau \circ F^i_q)(F^{i-1}_{p-1}(y))\\ &= \omega(F^i_q \circ F^{i-1}_{p-1}(y)).\end{aligned}$$

The induction completes the construction of ϕ on Ξ. In particular,

$$\operatorname{diam}(\phi(\sigma)(\Delta_{n-1})) < \mu^{(0)}(\varepsilon)/2 = \varepsilon/2.$$

Now, we define the $(n-1)$-chain $\phi(Z_{n-1})$ in X_k to be $\sum_\sigma c_\sigma \phi(\sigma)$. Since

$$\partial Z_{n-1} = \sum_\sigma \sum_{p=0}^{n-1} (-1)^p c_\sigma \cdot \sigma \circ F^{n-1}_p = 0,$$

we see that

$$\partial \phi(Z_{n-1}) = \sum_\sigma \sum_{p=0}^{n-1} (-1)^p c_\sigma \cdot \phi(\sigma) \circ F^{n-1}_p = \sum_\sigma \sum_{p=0}^{n-1} (-1)^p c_\sigma \cdot \phi(\sigma \circ F^{n-1}_p) = 0.$$

What is left is to show that the cycle $\phi(Z_{n-1})$ is homologous to Z_{n-1} in $\mathbb{R}^n \setminus \{x\}$.

We follow the notation of [11]: $E_0, \dots, E_q$ – the vertices of Δ_q; $\delta_q = \mathrm{id}_{\Delta_q}$; $S_q(Y)$ – the group of the singular q – chains in Y (with rational coefficients); $P_q \colon S_q(Y) \to S_{q+1}(Y \times I)$ – the homomorphism defined by

$$\begin{aligned}P_q(\sigma) &= S_{q+1}(\sigma \times \mathrm{id}) \circ P_q(\delta_q), \quad \text{for } \sigma\colon \Delta_q \to Y,\\ P_q(\delta_q) &= \sum_{i=0}^{q} (-1)^i \cdot ((E_0,0)\dots(E_i,0)(E_i,1)\dots(E_q,1)).\end{aligned}$$

Let $\lambda_t \colon Y \to Y \times I$ be given by $\lambda_t(y) = (y,t)$. By [11],

$$\partial \circ P_q + P_{q-1} \circ \partial = S_q(\lambda_1) - S_q(\lambda_0).$$

Now, we define $G_q(\sigma,\tau) \colon \Delta_q \times I \to \mathbb{R}^n \setminus \{x\}$ by

$$G_q(\sigma,\tau)(E,t) = (1-t)\sigma(E) + t\tau(E),$$

for all q-simplices σ, τ in $\mathbb{R}^n \setminus \{x\}$ such that the above expression takes values apart from $\{x\}$. Note that

$$\begin{aligned}&\operatorname{dist}((1-t)\sigma(E) + t\phi(\sigma)(E), X)\\ &\qquad \le t\|\phi(\sigma)(E) - \phi(\sigma)(E_0)\| + t\|\phi(\sigma)(E_0) - \sigma(E_0)\|\\ &\qquad\quad + t\|\sigma(E_0) - \sigma(E)\| + \operatorname{dist}(\sigma(E), X)\\ &\qquad \le \varepsilon/2 + 2\eta + \eta + \varrho_s(P,X) < \varepsilon/2 + 4\eta < \varepsilon < \operatorname{dist}(x,X),\end{aligned}$$

for every $\sigma \in \Xi_q$ and $E \in \Delta_q$.

It follows that $G_q(\sigma,\phi(\sigma))$ is well defined for every $\sigma \in \Xi_q$. Clearly, $\sigma = G_q(\sigma,\phi(\sigma)) \circ \lambda_0$ and $\phi(\sigma) = G_q(\sigma,\phi(\sigma)) \circ \lambda_1$. Moreover,

$$G_q(\sigma,\tau) \circ (F \times id) = G_{q-1}(\sigma \circ F, \tau \circ F),$$

for any $F\colon \Delta_{q-1} \to \Delta_q$. Thus

$$\begin{aligned}\phi(\sigma) - \sigma &= S_q(G_q(\sigma,\phi(\sigma))) \circ (S_q(\lambda_1) - S_q(\lambda_0))(\delta_q)\\ &= S_q(G_q(\sigma,\phi(\sigma))) \circ (\partial P_q + P_{q-1}\partial)(\delta_q)\\ &= \partial S_{q+1}(G_q(\sigma,\phi(\sigma)))P_q(\delta_q) + S_q(G_q(\sigma,\phi(\sigma)))P_{q-1}\partial(\delta_q).\end{aligned}$$

The second summand is equal to

$$\begin{aligned}&\sum_{j=0}^{q}(-1)^j S_q(G_q(\sigma,\phi(\sigma))) \circ P_{q-1}(F_j^q)\\ &\qquad= \sum_{j=0}^{q}(-1)^j S_q(G_q(\sigma,\phi(\sigma))) \circ S_q(F_j^q \times \mathrm{id}) \circ P_{q-1}(\delta_{q-1})\\ &\qquad= \sum_{j=0}^{q}(-1)^j S_q\left(G_{q-1}(\sigma \circ F_j^q, \phi(\sigma) \circ F_j^q)\right) \circ P_{q-1}(\delta_{q-1}).\end{aligned}$$

Take $q = n-1$. The cycle $\phi(Z_{n-1}) - Z_{n-1}$ is homologous in $\mathbb{R}^n \setminus \{x\}$ to

$$\sum_\sigma c_\sigma \cdot \sum_{j=0}^{q}(-1)^j S_q\left(G_{q-1}(\sigma \circ F_j^q, \phi(\sigma \circ F_j^q))\right) \circ P_{q-1}(\delta_{q-1}),$$

which equals to zero, because

$$\sum_\sigma c_\sigma \cdot \sum_{j=0}^{q}(-1)^j \sigma \circ F_j^q = \partial Z_{n-1} = 0. \qquad \square$$

2.4. Proof of Theorem 2.3. This proof is based on the notion of the spheric mapping. There are various definitions of spheric mappings, [8], [9], [2], [3], which lead to the similar proofs of the corresponding fixed point theorems. We will prove that $f\colon D^n \to 2^{D^n}$ satisfying assumptions of our theorem is spheric in the following sense:

(1) f is u.s.c. and compact-valued,
(2) the graph Γ_{Bf} is open in $D^n \times \mathbb{R}^n$,
(3) $\widetilde{f}$ has a fixed point.

The only point which needs our attention is (2). Indeed, if f is ϱ_s-continuous then f is u.s.c. and l.s.c.; if f is u.s.c. then $\widetilde{f}$ is u.s.c. ([8]), if f is l.s.c. and (2) then $\widetilde{f}$ is l.s.c.; if $\widetilde{f}$ is u.s.c. and l.s.c. then $\widetilde{f}$ is ϱ_s-continuous. Theorem 2.2(b) now yields (3).

Suppose, (2) is false. Then

$$\exists (x,y) \in \Gamma_{Bf}\ \exists \{(x_k,y_k)\} \lim_{k\to\infty} (x_k,y_k) = (x,y) \quad \text{and} \quad \forall k\ (x_k,y_k) \notin \Gamma_{Bf}.$$

Thus $y \in Bf(x)$, $y_k \in f(x_k) \cup Df(x_k)$. Since $\lim_{k\to\infty} \varrho_s(f(x_k), f(x)) = 0$, Lemma 2.13 shows that $y \in Bf(x_k)$ for $k > k_0$. By connectedness of the interval yy_k, there is $c_k \in yy_k$ such that $c_k \in f(x_k)$ for $k > k_0$. This gives $y \in f(x)$, a contradiction.

2.5. Two proofs of Theorem 2.5.

Proof I. Let us identify D^2 with I^2 and consider a ϱ_s - continuous mapping $f: I^2 \to I^2$ with eLC^0 values. Fix $\eta > 0$. Choose $\varepsilon > 0$ small enough that for all $x \in I^2$ and $y, y' \in f(x)$ with $\|y - y'\| < 4\varepsilon$ there is a path $\sigma: I \to I^2$ from y to y' in $f(x)$ satisfying $\operatorname{diam}(\sigma(I)) < \eta$. Take $\delta > 0$ such that $\varrho_s(f(x), f(x')) < \varepsilon$ for all x, x' with $\|x - x'\| < \delta$. We assume that $\delta < \varepsilon < \eta$.

Let us divide I^2 into squares, each with the edge of the same length less than δ. Our purpose is to find a single-valued continuous map $s: I^2 \to I^2$ which approximates f.

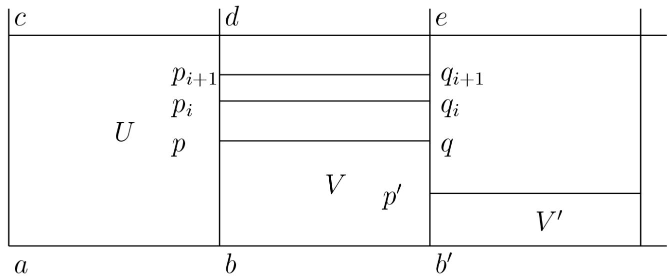

Figure 2

We follow the notation of the Figure 2. Fix $A \in f(a)$. Then choose $B \in f(b)$, $C \in f(c)$ and $D, P \in f(d)$ such that $\|B - A\|$, $\|C - A\|$, $\|D - C\|$, $\|P - B\|$ are all less than ε. It follows that $\|D - P\| < 4\varepsilon$.

Set $p = (b + d)/2$ and $s(a) = A$, $s(b) = B$, $s(c) = C$, $s(d) = D$, $s(p) = P$. Find a path $\sigma: I \to I^2$ from P to D in $f(d)$ with $\operatorname{diam}(\sigma(I)) < \eta$. Choose $r_0 = 0, r_1, \dots, r_k = 1$ in I such that $\operatorname{diam}(\sigma([r_i, r_{i+1}])) < \varepsilon$ for $i < k$. Set $p_i = p + i/k \cdot (d - p)$ and $s((1-t)p_i + tp_{i+1}) = \sigma((1-t)r_i + tr_{i+1})$ for $t \in I$. Define $P_i = s(p_i)$ and note, that $\|P_{i+1} - P_i\| < \varepsilon$. Extend s to be linear on the intarvals ab, ac, cd, bp; e.g. $s((1-t)b + tp) = (1-t)B + tP$.

Let U be the square $abdc$. Clearly, $\operatorname{diam}(s(\partial U)) < 3\varepsilon + \eta$. Since the convex sets are AR's, there is an extension $s: U \to \operatorname{conv}(s(\partial U))$. Obviously, $\operatorname{diam}(s(U)) < 3\varepsilon + \eta$. Let $q = (b' + e)/2$ and V be the rectangle $bb'qp$ in I^2.

Choose $Q \in f(e)$ such that $\|Q - P\| < \varepsilon$ and repeat the construction of s on V after that on U. We stress that $s(q) = Q \in f(e)$, moreover s maps the interval $p'q$ into $f(e)$, where $p' = (b' + q)/2$.

Since $P_i \in f(d)$, there is $Q_i \in f(e)$ with $\|Q_i - P_i\| < \varepsilon$ for $i = 0, \ldots, k$ and $Q_0 = Q$. Set $E = Q_k$, $q_i = q + i/k \cdot (e - q)$, $s(q_i) = Q_i$. Thus $\|Q_{i+1} - Q_i\| < 3\varepsilon$. Find a path $\alpha_i \colon I \to I^2$ from Q_i to Q_{i+1} in $f(e)$ with $\operatorname{diam}(\alpha_i(I)) < \eta$.

Let V_i be the rectangle $p_i q_i q_{i+1} p_{i+1}$. We extend s to be linear on intervals $p_i q_i$ and by $s((1 - t)q_i + tq_{i+1}) = \alpha_i(t)$ on $q_i q_{i+1}$. Thus $\operatorname{diam}(s(\partial V_i)) < 2\varepsilon + \eta$. Clearly, there is an extension $s \colon V_i \to \operatorname{conv}(s(\partial V_i))$ with $\operatorname{diam}(s(V_i)) < 2\varepsilon + \eta$ for $i = 0, \ldots, k - 1$.

The map s is now defined on U and on the square $bb'ed = V \cup \bigcup_{i=0}^{k-1} V_i$. In the same manner we extend s, square by square, on the first row of our subdivision of I^2. It is worth pointing out that passing to the third square, we forget points q_i and define $p'_j = p' + j/k' \cdot (e - p')$ with k' such that $\operatorname{diam}(s(p'_j p'_{j+1})) < \varepsilon$ for $j = 0, \ldots, k' - 1$. The definition of s on the other rows is straightforward.

It remains to prove that $s \colon I^2 \to I^2$ approximates f. For every $x \in I^2$ there are $R = r^0 r^1 r^2 r^3$ and $T = t^0 t^1 t^2 t^3$ such that:

- R is a rectangle, T is a square and $x \in R \subset T$,
- $\operatorname{diam}(T) < \sqrt{2} \cdot \delta$ and $\operatorname{diam}(s(R)) < 3\varepsilon + \eta$,
- $s(r^2) \in f(t^2)$.

Write $O_\varepsilon J = \{v \in I^2 : \operatorname{dist}(v, J) < \varepsilon\}$ for $J \subset I^2$. Thus $s(x) \in s(R) \subset O_{3\varepsilon+\eta}\{s(r^2)\} \subset O_{4\eta} f(t^2) \subset O_{4\eta} f(O_{2\delta}\{x\}) \subset O_{6\eta} f(x)$. □

Proof II. Another way of proving Theorem 2.5 is analysis similar to that in the proof of Theorem 2.3. The only difference is the argument which shows that $\widetilde{f}$ has a fixed point. We will see that the values of $\widetilde{f}$ have a fixed finite number of acyclic components. Therefore $\widetilde{f}$ is in a class of mappings which is equipped with the fixed point index, [4].

Let $\operatorname{nc}(X)$ denote the number of the components of the space X. Since $f(x)$ is compact and LC^0, $\operatorname{nc}(f(x)) < \infty$. By the Alexander duality, $\check{H}^i(\widetilde{f}(x)) = \widetilde{H}_{1-i}(Df(x)) = 0$ for $i \geq 1$.

It suffices to show that $\operatorname{nc}(\widetilde{f}(x))$ is finite and does not depend on x. Since every component of $\widetilde{f}(x)$ contains a point of the set $f(x)$, we have $\operatorname{nc}(\widetilde{f}(x)) \leq \operatorname{nc}(f(x)) < \infty$. Because $\{f(x) : x \in D^2\}$ is eLC^0, there is an $\varepsilon > 0$ such that the distance of any two components of $f(x)$ is not less than ε, for every $x \in D^2$. The same is true for the components of $\widetilde{f}(x)$. Indeed, if C, C' are two components of $\widetilde{f}(x)$, then

$$\partial C \subset f(x), \quad \partial C' \subset f(x), \quad \operatorname{dist}(C, C') = \operatorname{dist}(\partial C, \partial C') \geq \varepsilon.$$

Since $\widetilde{f}$ is ϱ_s-continuous, there is $\delta > 0$ such that $\mathrm{nc}(\widetilde{f}(x')) \geq \mathrm{nc}(\widetilde{f}(x))$ whenever $\|x - x'\| < \delta$. Thus $\mathrm{nc}(\widetilde{f}(x')) = \mathrm{nc}(\widetilde{f}(x))$ for every $x' \in O_\delta\{x\}$. The connectedness of D^2 finishes the proof. □

2.6. Proof of Theorem 2.9. Let us consider the composition

$$M \xrightarrow{D} M^2 \xrightarrow{1\times s} M^2 \xrightarrow{j} (M^2, M^2 \setminus \Delta)$$

with $M^2 = M \times M$, $D(x) = (x,x)$, j – an inclusion and Δ – the diagonal in M^2.

Let $U \in H^n\big(M^2, M^2 \setminus \Delta\big)$ denote a K-orientation class of the manifold M and $\lambda(s) = D^* \circ (1 \times s)^* \circ j^*(U)$ – the Lefschetz class of the positioning function s for f. The following diagram

$$\begin{array}{ccc} \Gamma_f & \xrightarrow{i} & M^2 \\ {\scriptstyle p}\downarrow & \simeq & \uparrow{\scriptstyle 1\times s} \\ M & \xrightarrow[D]{} & M^2 \end{array}$$

is homotopy commutative. This is because mappings $i(x,y) = (x,y)$ (with $y \in f(x)$) and $(1 \times s) \circ D \circ p(x,y) = (x, s(x))$ are two continuous selectors of so called J-mapping $\Psi(x,y) = \{x\} \times W(x)$, see [10], between compact ANRs. This is due to the fact that if $p\colon E \to B$ is a Hurewicz fibration with fibre F and any two of E, B, F are ANRs, then the third is also, [5], [6].

To obtain a contradiction, suppose that f is fixed point free. From this the following diagram

$$\begin{array}{ccccc} \Gamma_f & \xrightarrow{i} & M^2 & \xrightarrow{j} & (M^2, M^2 \setminus \Delta) \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle k} & & \\ M^2 \setminus \Delta & \xrightarrow[\mathrm{id}]{} & M^2 \setminus \Delta & & \end{array}$$

commutes, (h, k-inclusions).

Since $L(s;K) \neq 0$, we see that $\lambda(s) \neq 0$. By our diagrams,

$$p^*(\lambda(s)) = p^* D^* (1 \times s)^* j^*(U) = i^* j^*(U) = h^* k^* j^*(U) = 0.$$

(The last equality follows from the long exact sequence of the pair $(M^2, M^2 \setminus \Delta)$.) Hence $p^*\colon H^n(M) \to H^n(\Gamma_f)$ is not a monomorphism. Equivalently,

$$p_*\colon H_n(\Gamma_f) \to H_n(M) \quad \text{is not an epimorphism.}$$

On the other hand, p_* can be described in terms of the Leray–Serre spectral sequence as the composition

$$H_n(\Gamma_f) \xrightarrow{\text{onto}} E^\infty_{n,0} \xrightarrow{\mu} E^2_{n,0} \cong H_n(M),$$

(see [18]). The monomorphism μ is the composition of inclusions

$$E^{r+1}_{n,0} = \ker(E^r_{n,0} \to E^r_{n-r,r-1}) \subset E^r_{n,0},$$

for $r = n, \dots, 2$. Clearly, $E^{n+1}_{n,0} = E^{\infty}_{n,0}$. By assumption,

$$0 = H_{n-r}(M;K) \otimes_K H_{r-1}(F;K) = E^2_{n-r,r-1},$$

which suffices to conclude that $E^r_{n-r,r-1} = 0$ and μ is an isomorphism. Thus p_* is an epimorphism, a contradiction.

2.7. Proof of the Corollary 2.11. Suppose, contrary to our claim, that f is fixed point free. By Theorem 2.9, there is an $x_0 \in \widetilde{f}(x_0)$. Thus $x_0 \in Bf(x_0)$. Since $\{f(x) : x \in M\}$ is eLC^{n-2}, the graph Γ_{Bf} is an open subset of $M \times M$. If $x \in \widetilde{f}(x)$ for every $x \in M$, then both mappings id_M and s are continuous selectors of the J-mapping W, [10]. Hence $id_M \simeq s$, which contradicts our assumption. Otherwise, both $\{x \in M : x \notin \widetilde{f}(x)\}$ and $\{x \in M : x \in Bf(x)\}$ are nonempty open subsets of M, contrary to the connectedness of M.

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