Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 24, 2004, 107–145

C^m-SMOOTHNESS OF INVARIANT FIBER BUNDLES

Christian Pötzsche — Stefan Siegmund

Abstract. The method of invariant manifolds, now called the Hadamard–Perron Theorem, was originally developed by Lyapunov, Hadamard and Perron for time-independent maps and differential equations at a hyperbolic fixed point. It was then extended from hyperbolic to non-hyperbolic systems, from time-independent and finite-dimensional to time-dependent and infinite-dimensional equations. The generalization of an invariant manifold for a discrete dynamical system (mapping) to a time-variant difference equation is called an invariant fiber bundle. While in the hyperbolic case the smoothness of the invariant fiber bundles is easily obtained with the contraction principle, in the non-hyperbolic situation the smoothness depends on a spectral gap condition, is subtle to prove and proofs were given under various assumptions by basically three different approaches, so far: (1) A lemma of Henry, (2) the fiber-contraction theorem, or (3) fixed point theorems for scales of embedded Banach spaces.

In this paper we present a new self-contained and basic proof of the smoothness of invariant fiber bundles which relies only on Banach's fixed point theorem. Our result extends previous versions of the Hadamard–Perron Theorem and generalizes it to the time-dependent, not necessarily hyperbolic, infinite-dimensional, non-invertible and parameter-dependent case. Moreover, we show by an example that our gap-condition is sharp.

2000 *Mathematics Subject Classification.* 39A11, 37B55.

Key words and phrases. Hadamard–Perron Theorem, difference equations, nonautonomous, invariant fiber bundles, smoothness.

Research supported by the "Graduiertenkolleg: Nichtlineare Probleme in Analysis, Geometrie und Physik" (GRK 283) financed by the DFG and the State of Bavaria (Christian Pötzsche).

©2004 Juliusz Schauder Center for Nonlinear Studies

1. Introduction

One of the basic tasks of the theory of dynamical systems is to study the qualitative, asymptotic and long-term behavior of solutions or orbits. A main tool turned out to be invariant manifold theory providing a *dynamical skeleton* of orbits converging with a certain exponential rate to a given rest point or reference orbit. In this paper we consider time-dependent, not necessarily hyperbolic, infinite-dimensional, non-invertible and parameter-dependent difference equations. Invariant fiber bundles are the generalization of invariant manifolds to this situation. It is crucial to allow our difference equations to depend on a parameter, since this allows to construct invariant foliations as in [4] and also to apply our result to discretization theory of time-invariant difference equations. From the point of view of applications it is indispensable to treat difference equations which are non-invertible. The fact that we will consider invariant fiber bundles which contain the zero solution is no restriction, in fact, every invariant fiber bundle through an arbitrary reference solution $k \mapsto z(k)$ of a given difference equation is an invariant fiber bundle of the time-variant difference equation which we get from the (time-depending) transformation $x \mapsto x - z(k)$; this allows e.g. the treatment of the invariant manifolds of an almost periodic orbit of a map. But also discretization problems of semiflows are in the scope of applications, since we allow the state space to be infinite-dimensional. The technical difficulties of the proof of our main result (Theorem 3.5) are due to the fact that we allow our reference solution to be non-hyperbolic. This flexibility turns out to be crucial in continuous time applications (see e.g. [13]) when it is necessary not only to split into stable and unstable manifold but to have a finer decomposition at hand which provides a more detailed picture of the dynamics. We expect the same to be true for discrete time applications and provide a theorem which is flexible and strong enough to apply to various situations without the need to be reproven for every explicit problem.

The existence of invariant fiber bundles in our general situation has been proven by Aulbach and the authors in [2], where also the C^1-smoothness of the invariant fiber bundles was showed. Although stable and unstable fiber bundles are in the same smoothness class as the system, an arbitrary fiber bundle is only C^1 in general. However, sometimes a system restricted to one of its invariant fiber bundles carries relevant information and therefore it is important to know the maximal smoothness class of an invariant fiber bundle. It is known that a gap condition on the spectrum of the linearization along the reference orbit has to be satisfied in order to get higher order smoothness of the invariant fiber bundles. But it is also well-known from the theory of ordinary differential equations that the differentiability of invariant manifolds is technically hard to prove. For a modern approach using sophisticated fixed point theorems see [21], [20], [16] or [9]. Another approach to the smoothness of invariant manifolds is essentially based on a lemma by Henry (cf. [6, Lemma 2.1]) or methods of a more differential topological nature (cf. [10], [17] or [18]), namely the C^m-section theorem for fiber contracting maps. In [5] and [19] the problem of higher order smoothness is tackled directly. Other contemporary theorems on the smoothness of invariant fiber bundles of difference equations are contained in the articles [9], [7] and in the monograph [11]. The first two papers deal only with autonomous systems and apply a fixed point result on scales of Banach spaces and the fiber contraction theorem, respectively. In [11, Theorem 6.2.8, pp. 242–243] the so-called Hadamard–Perron Theorem is proved via a graph transformation technique for a time-dependent family of C^m-diffeomorphisms on a finite-dimensional space. Using a different method of proof, our main results Theorem 3.5 and Theorem 4.1 generalize this version of the Hadamard–Perron Theorem to not necessarily hyperbolic, non-invertible, infinite-dimensional and parameter-dependent difference equations. We would like to point out that the hyperbolic theory is already elegantly and didactically well presented in the survey [22] and the exposition [8].

Our contribution consists in treating also the technical non-hyperbolic case. We tried hard to give a clear and accessible "ad hoc" proof of the maximal smoothness class of pseudo-hyperbolic invariant fiber bundles. Moreover, we give an example that shows that our gap conditions are sharp. The smoothness proof is basically derived from [19] and needs no technical tools beyond the contraction mapping principle, the Neumann series and Lebesgue's theorem. C^m-smoothness of invariant fiber bundles is proved by induction over m. The induction over the smoothness class m is the key for understanding the structure of the problem. Our focus it not to hide the core of the proof by omitting the technical induction argument as it is usually done in the literature. To our understanding this is one of the reasons why the Hadamard–Perron Theorem has been reproven by so many authors for similar situations over the years. The induction argument of the proof is crucial because it is needed to rigorously compute the higher order derivatives of compositions of maps, the so-called *derivative tree.* It turned out to be advantageous to use two different representations of the derivative tree, namely a *totally unfolded derivative tree* to show that a fixed point operator is well-defined and to compute explicit global bounds for the higher order derivatives of the fiber bundles and besides a *partially unfolded derivative tree* to elaborate the induction argument in a recursive way.

The structure of this paper is as follows: In Section 2 we present the notation and basic results.

Section 3 is devoted to the C^1-smoothness of invariant fiber bundles. We will also state our main assumptions here and prove some preparatory lemmas which will also be needed later. The C^1-smoothness follows without any gap condition from the main result of this section which is Theorem 3.5. Our proof may seem long and intricate and in fact it would be if we would like to show only the C^1-smoothness, but in its structure it already contains the main idea of the induction argument for the C^m-case and we will profit then from being rather detailed in the C^1-case.

Section 4 contains our main result (Theorem 4.1), stating that for every *spectral gap* (α,β) the pseudo-stable fiber bundle (which corresponds to the *spectrum* in $(-\infty,\alpha)$) is of class C^{m_s} if $\alpha^{m_s} < \beta$ and the pseudo-unstable fiber bundle (which corresponds to the *spectrum* in (β,∞)) is of class C^{m_r} if $\alpha < \beta^{m_r}$. Example 4.2 shows that these gap conditions are sharp.

2. Preliminaries

$\mathbb{N}$ denotes the positive integers and a *discrete interval* I is defined to be the intersection of a real interval with the integers $\mathbb{Z} = \{0, \pm 1, \dots\}$. For an integer $\kappa \in \mathbb{Z}$ we define $\mathbb{Z}_\kappa^+ := [\kappa, \infty) \cap \mathbb{Z}$, $\mathbb{Z}_\kappa^- := (-\infty, \kappa] \cap \mathbb{Z}$.

The Banach spaces $\mathcal{X}, \mathcal{Y}$ are all real or complex throughout this paper and their norm is denoted by $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ or simply by $\|\cdot\|$. If $\mathcal{X}$ and $\mathcal{Y}$ are isometrically isomorphic we write $\mathcal{X} \cong \mathcal{Y}$. $\mathcal{L}_n(\mathcal{X};\mathcal{Y})$ is the Banach space of n-linear continuous operators from $\mathcal{X}^n$ to $\mathcal{Y}$ for $n \in \mathbb{N}$,

$$\mathcal{L}_0(\mathcal{X};\mathcal{Y}) := \mathcal{Y}, \quad \mathcal{L}(\mathcal{X};\mathcal{Y}) := \mathcal{L}_1(\mathcal{X};\mathcal{Y}), \quad \mathcal{L}(\mathcal{X}) := \mathcal{L}_1(\mathcal{X};\mathcal{X}),$$

$I_{\mathcal{X}}$ the identity map on $\mathcal{X}$ and $\mathcal{GL}(\mathcal{X})$ the multiplicative group of bijective mappings in $\mathcal{L}(\mathcal{X})$. On the product space $\mathcal{X} \times \mathcal{Y}$ we always use the maximum norm

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_{\mathcal{X}\times\mathcal{Y}} := \max\{\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}}\}. \tag{2.1}$$

We write DF for the Fréchet derivative of a mapping F and if $F: (x,y) \mapsto F(x,y)$ depends differentiable on more than one variable, then the partial derivatives are denoted by $\partial F/\partial x$ and $\partial F/\partial y$, respectively. Now we quote the two versions of the higher order chain rule for Fréchet derivatives on which our smoothness proof is based. Thereto let $\mathcal{Z}$ be a further Banach space over $\mathbb{R}$ or $\mathbb{C}$. With given $j, l \in \mathbb{N}$ we write

$$\begin{aligned} P_j^<(l) := \{(N_1,\dots,N_j) \subset \{1,\dots,l\}^j \mid N_i \neq \emptyset,\ N_1 \cup \dots \cup N_j = \{1,\dots,l\}, \\ N_i \cap N_k = \emptyset \text{ for } i \neq k,\ \max N_i < \max N_{i+1}\} \end{aligned}$$

for the set of *ordered partitions* of $\{1,\dots,l\}$ with length j, we write $\#N$ for the cardinality of a finite set $N \subset \mathbb{N}$. In case $N = \{n_1,\dots,n_k\} \subseteq \{1,\dots,l\}$ for $k \in \mathbb{N}$, $k \leq l$, we abbreviate $D^k g(x) x_N := D^k g(x) x_{n_1} \dots x_{n_k}$ for vectors $x, x_1, \dots, x_l \in \mathcal{X}$, where $g: \mathcal{X} \to \mathcal{Y}$ is l-times continuously differentiable.

Theorem 2.1 (Chain rule). *Given $m \in \mathbb{N}$ and two mappings $f: \mathcal{Y} \to \mathcal{Z}$, $g: \mathcal{X} \to \mathcal{Y}$ which are m-times continuously differentiable. Then also the composition $f \circ g: \mathcal{X} \to \mathcal{Z}$ is m-times continuously differentiable and for $l \in \{1, \dots, m\}$, $x \in \mathcal{X}$ the derivatives possess the representations as a so-called partially unfolded derivative tree*

$$D^l(f \circ g)(x) = \sum_{j=0}^{l-1} \binom{l-1}{j} D^j[Df(g(x))] \cdot D^{l-j} g(x) \tag{2.2}$$

and as a so-called totally unfolded derivative tree

$$\begin{aligned} D^l(f \circ g)(x)x_1 \dots x_l \\ = \sum_{j=1}^{l} \sum_{(N_1, \dots, N_j) \in P_j^<(l)} D^j f(g(x)) D^{\# N_1} g(x) x_{N_1} \dots D^{\# N_j} g(x) x_{N_j} \end{aligned} \tag{2.3}$$

for any $x_1, \dots, x_l \in \mathcal{X}$.

Proof. A proof of (2.2) follows by an easy induction argument (cf. [19, B.3 Satz, p. 266]), while (2.3) is shown in [15, Theorem 2]. ☐

We use the notation

$$x' = f(k, x, p) \tag{2.4}$$

to denote the parameter-dependent difference equation $x(k+1) = f(k, x(k), p)$, with the right-hand side $f: I \times \mathcal{X} \times \mathcal{P} \to \mathcal{X}$, where I is a discrete interval and $\mathcal{P}$ is a topological space. Let $\lambda(k; \kappa, \xi, p)$ denote the *general solution* of equation (2.4), i.e. $\lambda(\,\cdot\,; \kappa, \xi, p)$ solves (2.4) and satisfies the initial condition $\lambda(\kappa; \kappa, \xi, p) = \xi$ for $\kappa \in I$, $\xi \in \mathcal{X}$, $p \in \mathcal{P}$. In forward time λ can be defined recursively as

$$\lambda(k; \kappa, \xi, p) := \begin{cases} \xi & \text{for } k = \kappa, \\ f(k-1, \lambda(k-1; \kappa, \xi, p), p) & \text{for } k > \kappa. \end{cases}$$

Given an operator sequence $A: I \to \mathcal{L}(\mathcal{X})$ we define the *evolution operator* $\Phi(k, \kappa) \in \mathcal{L}(\mathcal{X})$ of the linear equation $x' = A(k)x$ as the mapping given by

$$\Phi(k, \kappa) := \begin{cases} I_{\mathcal{X}} & \text{for } k = \kappa, \\ A(k-1) \dots A(\kappa) & \text{for } k > \kappa \end{cases}$$

and if $A(k)$ is invertible (in $\mathcal{L}(\mathcal{X})$) for $k \leq \kappa$ then

$$\Phi(k, \kappa) := A(k)^{-1} \dots A(\kappa - 1)^{-1} \quad \text{for } k < \kappa.$$

Now we introduce a notion describing exponential growth of sequences or solutions of difference equations. For a $\gamma > 0$, a Banach space $\mathcal{X}$, a discrete interval I, $\kappa \in I$ and $\lambda: I \to \mathcal{X}$ we say that

(a) λ is γ^+-*quasibounded* if I is unbounded above and if

$$\|\lambda\|^+_{\kappa,\gamma} := \sup_{k \in \mathbb{Z}^+_\kappa} \|\lambda(k)\|\gamma^{\kappa-k} < \infty,$$

(b) λ is γ^--*quasibounded* if I is unbounded below and if

$$\|\lambda\|^-_{\kappa,\gamma} := \sup_{k \in \mathbb{Z}^-_\kappa} \|\lambda(k)\|\gamma^{\kappa-k} < \infty,$$

(c) λ is $\gamma^\pm$-*quasibounded* if $I = \mathbb{Z}$ and if $\sup_{k\in\mathbb{Z}} \|\lambda(k)\|\gamma^{\kappa-k} < \infty$.

$\ell^+_{\kappa,\gamma}(\mathcal{X})$ and $\ell^-_{\kappa,\gamma}(\mathcal{X})$ denote the sets of all γ^+- and γ^--quasibounded functions $\lambda: I \to \mathcal{X}$, they are Banach spaces with the norms $\|\cdot\|^+_{\kappa,\gamma}$ and $\|\cdot\|^-_{\kappa,\gamma}$, and satisfy the following properties (cp. also Lemma 3.3 in [2]).

Lemma 2.2. *For real constants γ, δ with $0 < \gamma \le \delta$, $m \in \mathbb{N}$, $\kappa \in \mathbb{Z}$ and Banach spaces $\mathcal{X}, \mathcal{Y}$ the following statements are valid:*

(a) *The Banach spaces $\ell^+_{\kappa,\gamma}(\mathcal{X}) \times \ell^+_{\kappa,\gamma}(\mathcal{Y})$ and $\ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ are isometrically isomorphic,*

(b) *$\ell^+_{\kappa,\gamma}(\mathcal{X}) \subseteq \ell^+_{\kappa,\delta}(\mathcal{X})$ and $\|\lambda\|^+_{\kappa,\delta} \le \|\lambda\|^+_{\kappa,\gamma}$ for $\lambda \in \ell^+_{\kappa,\gamma}(\mathcal{X})$,*

(c) *with the abbreviation $\ell^0_{\kappa,\gamma} := \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$, $\ell^m_{\kappa,\gamma} := \ell^+_{\kappa,\gamma}(\mathcal{L}_m(\mathcal{X}; \mathcal{X} \times \mathcal{Y}))$, the Banach spaces $\ell^m_{\kappa,\gamma}$ and $\mathcal{L}(\mathcal{X}; \ell^{m-1}_{\kappa,\gamma})$ are isometrically isomorphic.*

3. C^1-smoothness of invariant fiber bundles

We begin this section by stating our frequently used main assumptions.

Hypothesis 3.1. *Let us consider the system of parameter-dependent difference equations*

$$\begin{cases} x' = A(k)x + F(k, x, y, p), \\ y' = B(k)y + G(k, x, y, p), \end{cases} \tag{3.1}$$

where $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces, $\mathcal{P}$ is a topological space satisfying the first axiom of countability, the discrete interval I is unbounded to the right, $A: I \to \mathcal{L}(\mathcal{X})$, $B: I \to \mathcal{GL}(\mathcal{Y})$ and the mappings $F: I \times \mathcal{X} \times \mathcal{Y} \times \mathcal{P} \to \mathcal{X}$, $G: I \times \mathcal{X} \times \mathcal{Y} \times \mathcal{P} \to \mathcal{Y}$ are m-times, $m \in \mathbb{N}$, continuously differentiable with respect to (x, y). Moreover we assume:

(a) *Hypothesis on linear part: The evolution operators $\Phi(k, l)$ and $\Psi(k, l)$ of the linear systems $x' = A(k)x$ and $y' = B(k)y$, respectively, satisfy for all $k, l \in I$ the estimates*

$$\begin{aligned} \|\Phi(k,l)\|_{\mathcal{L}(\mathcal{X})} &\le K_1 \alpha^{k-l} \quad \text{for } k \ge l, \\ \|\Psi(k,l)\|_{\mathcal{L}(\mathcal{Y})} &\le K_2 \beta^{k-l} \quad \text{for } l \ge k, \end{aligned} \tag{3.2}$$

with real constants $K_1, K_2 \geq 1$ and α, β with $0 < \alpha < \beta$.

(b) (Hypothesis on perturbation) *We have*

$$F(k,0,0,p) \equiv 0, \qquad G(k,0,0,p) \equiv 0 \quad \text{on } I \times \mathcal{P}, \tag{3.3}$$

and the partial derivatives of F and G are globally bounded, i.e. for $n \in \{1,\ldots,m\}$ we assume

$$\begin{aligned} |F|_n &:= \sup_{(k,x,y,p)\in I\times\mathcal{X}\times\mathcal{Y}\times\mathcal{P}} \left\| \frac{\partial^n F}{\partial(x,y)^n}(k,x,y,p) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{X})} < \infty, \\ |G|_n &:= \sup_{(k,x,y,p)\in I\times\mathcal{X}\times\mathcal{Y}\times\mathcal{P}} \left\| \frac{\partial^n G}{\partial(x,y)^n}(k,x,y,p) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{Y})} < \infty, \end{aligned} \tag{3.4}$$

and additionally for some real $\sigma_{\max} > 0$ we require

$$\max\{|F|_1, |G|_1\} < \frac{\sigma_{\max}}{\max\{K_1, K_2\}}. \tag{3.5}$$

Furthermore, we choose a fixed real number

$$\sigma \in (\max\{K_1,K_2\}\max\{|F|_1,|G|_1\}, \sigma_{\max}).$$

Remark 3.2. In [2] difference equations of the type (3.1) are considered without an explicit parameter-dependence. Anyhow, every result from [2] remains applicable since all the above estimates in Hypothesis 3.1 are uniform in $p \in \mathcal{P}$.

Lemma 3.3. *We assume Hypothesis* 3.1 *for $m = 1$, $\sigma_{\max} = (\beta-\alpha)/2$ and choose $\kappa \in I$. Moreover, let $(\mu,\nu), (\overline{\mu},\overline{\nu})\colon \mathbb{Z}_\kappa^+ \to \mathcal{X}\times\mathcal{Y}$ be solutions of* (3.1) *such that their difference $(\mu,\nu)-(\overline{\mu},\overline{\nu})$ is γ^+-quasibounded for any $\gamma \in (\alpha+\sigma, \beta-\sigma)$. Then, for all $k \in \mathbb{Z}_\kappa^+$, the estimate*

$$\left\| \begin{pmatrix} \mu \\ \nu \end{pmatrix}(k) - \begin{pmatrix} \overline{\mu} \\ \overline{\nu} \end{pmatrix}(k) \right\|_{\mathcal{X}\times\mathcal{Y}} \leq K_1 \frac{\gamma-\alpha}{\gamma-\alpha-K_1|F|_1} \gamma^{k-\kappa} \|\mu(\kappa)-\overline{\mu}(\kappa)\|_{\mathcal{X}}, \tag{3.6}$$

holds.

Proof. Choose an arbitrary $p \in \mathcal{P}$ and $\kappa \in I$. First of all the difference $\mu - \overline{\mu} \in \ell^+_{\kappa,\gamma}(\mathcal{X})$ is a solution of the inhomogeneous difference equation

$$x' = A(k)x + F(k,(\mu,\nu)(k),p) - F(k,(\overline{\mu},\overline{\nu})(k),p), \tag{3.7}$$

where the inhomogeneity is γ^+-quasibounded

$$\|F(\,\cdot\,,(\mu,\nu)(\,\cdot\,),p) - F(\,\cdot\,,(\overline{\mu},\overline{\nu})(\,\cdot\,),p)\|^+_{\kappa,\gamma} \overset{(3.4)}{\leq} |F|_1 \left\| \begin{pmatrix} \mu \\ \nu \end{pmatrix} - \begin{pmatrix} \overline{\mu} \\ \overline{\nu} \end{pmatrix} \right\|^+_{\kappa,\gamma}$$

by Hypothesis 3.1(b). Applying [1, Lemma 3.3] to the equation (3.7) yields

$$\|\mu-\overline{\mu}\|^+_{\kappa,\gamma} \leq K_1\|\mu(\kappa)-\overline{\mu}(\kappa)\| + \frac{K_1|F|_1}{\gamma-\alpha} \left\| \begin{pmatrix} \mu \\ \nu \end{pmatrix} - \begin{pmatrix} \overline{\mu} \\ \overline{\nu} \end{pmatrix} \right\|^+_{\kappa,\gamma}; \tag{3.8}$$

note that our definition of $\|\cdot\|^+_{\kappa,\gamma}$ is slightly different from [1, Definition 3.1(a)]. Because of $K_1|F|_1/(\gamma-\alpha) < 1$ (cf. (3.5)), without loss of generality we can assume $\nu \neq \overline{\nu}$ from now on. Analogously the difference $\nu - \overline{\nu} \in \ell^+_{\kappa,\gamma}(\mathcal{Y})$ is a solution of the linear equation

$$y' = B(k)y + G(k,(\mu,\nu)(k),p) - G(k,(\overline{\mu},\overline{\nu})(k),p),$$

where the inhomogeneity is also γ^+-quasibounded

$$\|G(\,\cdot\,,(\mu,\nu)(\,\cdot\,),p) - G(\,\cdot\,,(\overline{\mu},\overline{\nu})(\,\cdot\,),p)\|^+_{\kappa,\gamma} \overset{(3.4)}{\leq} |G|_1 \left\| \begin{pmatrix}\mu\\ \nu\end{pmatrix} - \begin{pmatrix}\overline{\mu}\\ \overline{\nu}\end{pmatrix} \right\|^+_{\kappa,\gamma}$$

by Hypothesis 3.1(b). Now using the result [1, Lemma 3.4(a)] yields

$$\|\nu - \overline{\nu}\|^+_{\kappa,\gamma} \leq \frac{K_2|G|_1}{\beta-\gamma} \left\| \begin{pmatrix}\mu\\ \nu\end{pmatrix} - \begin{pmatrix}\overline{\mu}\\ \overline{\nu}\end{pmatrix} \right\|^+_{\kappa,\gamma},$$

and since we have $K_2|G|_1/(\beta-\gamma) < 1$ (cf. assumption (3.5)) as well as $\nu \neq \overline{\nu}$ we get the inequality

$$\|\nu - \overline{\nu}\|^+_{\kappa,\gamma} < \max\{\|\mu-\overline{\mu}\|^+_{\kappa,\gamma}, \|\nu-\overline{\nu}\|^+_{\kappa,\gamma}\}$$

by (2.1). Consequently we obtain $\|\mu-\overline{\mu}\|^+_{\kappa,\gamma} = \|(\mu,\nu)-(\overline{\mu},\overline{\nu})\|^+_{\kappa,\gamma}$, which leads to

$$\left\| \begin{pmatrix}\mu\\ \nu\end{pmatrix} - \begin{pmatrix}\overline{\mu}\\ \overline{\nu}\end{pmatrix} \right\|^+_{\kappa,\gamma} \overset{(3.8)}{\leq} K_1\|\mu(\kappa)-\overline{\mu}(\kappa)\| + \frac{K_1|F|_1}{\gamma-\alpha} \left\| \begin{pmatrix}\mu\\ \nu\end{pmatrix} - \begin{pmatrix}\overline{\mu}\\ \overline{\nu}\end{pmatrix} \right\|^+_{\kappa,\gamma}.$$

This, in turn, immediately implies the estimate (3.6). □

Now we collect some crucial results from the earlier paper [2]. In particular we can characterize the quasibounded solutions of (3.1) quite easily as fixed points of an appropriate operator.

LEMMA 3.4 the operator $\mathcal{T}_\kappa$). *We assume Hypothesis* 3.1 *for* $m=1$, $\sigma_{\max} = (\beta-\alpha)/2$ *and choose* $\kappa \in I$. *Then for arbitrary* $\gamma \in [\alpha+\sigma, \beta-\sigma]$ *and* $\xi \in \mathcal{X}$, $p \in \mathcal{P}$, *the mapping* $\mathcal{T}_\kappa \colon \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y}) \times \mathcal{X} \times \mathcal{P} \to \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$,

$$(\mathcal{T}_\kappa(\mu,\nu;\xi,p))(k) := \begin{pmatrix} \Phi(k,\kappa)\xi + \displaystyle\sum_{n=\kappa}^{k-1} \Phi(k,n+1)F(n,(\mu,\nu)(n),p) \\ -\displaystyle\sum_{n=k}^{\infty} \Psi(k,n+1)G(n,(\mu,\nu)(n),p) \end{pmatrix}, \tag{3.9}$$

for $k \in \mathbb{Z}^+_\kappa$, *has the following properties:*

(a) $\mathcal{T}_\kappa(\,\cdot\,;\xi,p)$ *is a uniform contraction in* $\xi \in \mathcal{X}$, $p \in \mathcal{P}$ *with Lipschitz constant*

$$L := \frac{\max\{K_1,K_2\}}{\sigma} \max\{|F|_1,|G|_1\} < 1, \tag{3.10}$$

(b) *the unique fixed point* $(\mu_\kappa, \nu_\kappa)(\xi, p) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ *of* $\mathcal{T}_\kappa(\,\cdot\,; \xi, p)$ *does not depend on* $\gamma \in [\alpha + \sigma, \beta - \sigma]$ *and is globally Lipschitzian:*

$$\left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi, p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\bar{\xi}, p) \right\|^+_{\kappa,\gamma} \leq \frac{K_1}{1 - L} \|\xi - \bar{\xi}\|_{\mathcal{X}} \quad \text{for } \xi, \bar{\xi} \in \mathcal{X},\ p \in \mathcal{P}, \tag{3.11}$$

(c) *a function* $(\mu, \nu) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ *is a solution of the difference equation* (3.1) *with* $\mu(\kappa) = \xi$*, if and only if it is a solution of the fixed point equation*

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \mathcal{T}_\kappa(\mu, \nu; \xi, p). \tag{3.12}$$

Proof. See [2], in particular the proof of Theorem 4.11 in the quoted paper for (a), (b), and Lemma 4.10 for (c). □

Having all preparatory results at hand we may now head for our main theorem in the C^1-case. As mentioned in the introduction, invariant fiber bundles are generalizations of invariant manifolds to non-autonomous equations. In order to be more precise, for fixed parameters $p \in \mathcal{P}$, we call a subset $S(p)$ of the extended state space $I \times \mathcal{X} \times \mathcal{Y}$ an *invariant fiber bundle* of (3.1), if it is *positively invariant*, i.e. for any tuple $(\kappa, \xi, \eta) \in S(p)$ one has $(k, \lambda(k; \kappa, \xi, \eta, p)) \in S(p)$ for all $k \geq \kappa$, $k \in I$, where λ denotes the general solution of (3.1).

Theorem 3.5 (C^1-smoothness of invariant fiber bundles). *We assume Hypothesis* 3.1 *for* $m = 1$, $\sigma_{\max} = (\beta - \alpha)/2$ *and let* λ *denote the general solution of* (3.1). *Then the following statements are valid:*

(a) *There exists a uniquely determined mapping* $s \colon I \times \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$ *whose graph* $S(p) := \{(\kappa, \xi, s(\kappa, \xi, p)) : \kappa \in I,\ \xi \in \mathcal{X}\}$ *can be characterized dynamically for any parameter* $p \in \mathcal{P}$ *and any constant* $\gamma \in [\alpha + \sigma, \beta - \sigma]$ *as* $S(p) = \{(\kappa, \xi, \eta) \in I \times \mathcal{X} \times \mathcal{Y} : \lambda(\,\cdot\,; \kappa, \xi, \eta, p) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})\}$. *Furthermore we have*

(a_1) $s(\kappa, 0, p) \equiv 0$ *on* $I \times \mathcal{P}$,

(a_2) *the graph* $S(p)$, $p \in \mathcal{P}$, *is an invariant fiber bundle of* (3.1). *Additionally* s *is a solution of the invariance equation*

$$s(\kappa + 1, A(\kappa)\xi + F(\kappa, \xi, s(\kappa, \xi, p), p), p) = B(\kappa)s(\kappa, \xi, p) + G(\kappa, \xi, s(\kappa, \xi, p), p)$$

for $(\kappa, \xi, p) \in I \times \mathcal{X} \times \mathcal{P}$,

(a_3) $s \colon I \times \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$ *is continuous and continuously differentiable in the second argument with globally bounded derivative*

$$\left\| \frac{\partial s}{\partial \xi}(\kappa, \xi, p) \right\|_{\mathcal{L}(\mathcal{X};\mathcal{Y})} \leq \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1, K_2\} \max\{|F|_1, |G|_1\}}$$

for $(\kappa, \xi, p) \in I \times \mathcal{X} \times \mathcal{P}$.

The graph $S(p)$, $p \in \mathcal{P}$, is called the pseudo-stable fiber bundle of (3.1).

(b) *In case $I = \mathbb{Z}$ there exists a uniquely determined mapping $r: I \times \mathcal{Y} \times \mathcal{P} \to \mathcal{X}$ whose graph $R(p) := \{(\kappa, r(\kappa,\eta,p),\eta) : \kappa \in I,\ \eta \in \mathcal{Y}\}$ can be characterized dynamically for any parameter $p \in \mathcal{P}$ and any constant $\gamma \in [\alpha+\sigma, \beta-\sigma]$ as*

$$R(p) = \{(\kappa,\xi,\eta) \in I \times \mathcal{X} \times \mathcal{Y} : \lambda(\,\cdot\,;\kappa,\xi,\eta,p) \in \ell^{-}_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})\}. \tag{3.13}$$

Furthermore we have

(b_1) *$r(\kappa,0,p) \equiv 0$ on $I \times \mathcal{P}$,*

(b_2) *the graph $R(p)$, $p \in \mathcal{P}$, is an invariant fiber bundle of* (3.1). *Additionally r is a solution of the invariance equation*

$$r(\kappa+1, B(\kappa)\eta + G(\kappa, r(\kappa,\eta,p),\eta,p),p) = A(\kappa)r(\kappa,\eta,p) + F(\kappa, r(\kappa,\eta,p),\eta,p)$$

for $(\kappa,\eta,p) \in I \times \mathcal{Y} \times \mathcal{P}$,

(b_3) *$r: I \times \mathcal{Y} \times \mathcal{P} \to \mathcal{X}$ is continuous and continuously differentiable in the second argument with globally bounded derivative*

$$\left\| \frac{\partial r}{\partial \eta}(\kappa,\eta,p) \right\|_{\mathcal{L}(\mathcal{Y};\mathcal{X})} \le \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1,K_2\}\max\{|F|_1,|G|_1\}}$$

for $(\kappa,\eta,p) \in I \times \mathcal{Y} \times \mathcal{P}$.

The graph $R(p)$, for $p \in \mathcal{P}$, is called the pseudo-unstable fiber bundle of (3.1).

(c) *In case $I = \mathbb{Z}$ only the zero solution of equation* (3.1) *is contained both in $S(p)$ and $R(p)$, i.e. $S(p) \cap R(p) = \mathbb{Z} \times \{0\} \times \{0\}$ for $p \in \mathcal{P}$ and hence the zero solution is the only $\gamma^{\pm}$-quasibounded solution of* (3.1) *for $\gamma \in [\alpha+\sigma, \beta-\sigma]$.*

Remark 3.6. Since we did not assume invertibility of the difference equation (3.1) one has to interpret the dynamical characterization (3.13) of the pseudo-unstable fiber bundle $R(p)$, $p \in \mathcal{P}$, as follows. A point $(\kappa,\xi,\eta) \in I \times \mathcal{X} \times \mathcal{Y}$ is contained in $R(p)$ if and only if there exists a γ^{-}-quasibounded solution $\lambda(\,\cdot\,;\kappa,\xi,\eta,p): I \to \mathcal{X}\times\mathcal{Y}$ of (3.1) satisfying the initial condition $x(\kappa) = \xi$, $y(\kappa) = \eta$. In this case the solution $\lambda(\,\cdot\,;\kappa,\xi,\eta,p)$ is uniquely determined.

Proof of Theorem 3.5. (a) Our main intention in the current proof is to show the continuity and the partial differentiability assertion (a_3) for the mapping $s: I \times \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$. Any other statement of Theorem 3.5(a) follows from [2, Proof of Theorem 4.11]. Nevertheless we reconsider the main ingredients in our argumentation.

Using [2, Proof of Theorem 4.11] we know that for any triple $(\kappa,\xi,p) \in I \times \mathcal{X} \times \mathcal{P}$ there exists exactly one $s(\kappa,\xi,p) \in \mathcal{Y}$ such that $\lambda(\,\cdot\,;\kappa,\xi,s(\kappa,\xi,p),p) \in \ell^{+}_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ for every $\gamma \in [\alpha+\sigma,\beta-\sigma]$. Then the function $s(\,\cdot\,,p): I \times \mathcal{X} \to \mathcal{Y}$,

$p \in \mathcal{P}$, defines the invariant fiber bundle $S(p)$, if we set $s(\kappa, \xi, p) := (\nu_\kappa(\xi, p))(\kappa)$, where $(\mu_\kappa, \nu_\kappa)(\xi, p) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ denotes the unique fixed point of the operator $\mathcal{T}_\kappa(\,\cdot\,; \xi, p)\colon \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \to \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ introduced in Lemma 3.4 for any $\xi \in \mathcal{X}$, $p \in \mathcal{P}$ and $\gamma \in [\alpha + \sigma, \beta - \sigma]$. Here and in the following one should be aware of the estimate

$$\max\left\{\frac{K_1|F|_1}{\gamma - \alpha}, \frac{K_2|G|_1}{\beta - \gamma}\right\} \overset{(3.10)}{\leq} L < 1. \tag{3.14}$$

The further proof of part (a$_3$) will be divided into several steps. For notational convenience we introduce the abbreviations $\mu_\kappa(k; \xi, p) := (\mu_\kappa(\xi, p))(k)$ and $\nu_\kappa(k; \xi, p) := (\nu_\kappa(\xi, p))(k)$.

Step 1 For every $\gamma \in (\alpha+\sigma, \beta-\sigma]$ the mappings $(\mu_\kappa, \nu_\kappa)\colon \mathbb{Z}^+_\kappa \times \mathcal{X} \times \mathcal{P} \to \mathcal{X} \times \mathcal{Y}$ and $s\colon I \times \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$ are continuous.

By Hypothesis 3.1 the parameter space $\mathcal{P}$ satisfies the first axiom of countability. Consequently [14, Theorem 1.1(b), p. 190] implies that in order to prove the continuity of $(\mu_\kappa, \nu_\kappa)(\kappa; \xi_0, \cdot\,)\colon \mathcal{P} \to \mathcal{X} \times \mathcal{Y}$, it suffices to show for arbitrary but fixed $\kappa \in I$, $\xi_0 \in \mathcal{X}$ and $p_0 \in \mathcal{P}$ the following limit relation:

$$\lim_{p \to p_0} \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\kappa; \xi_0, p) = \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\kappa; \xi_0, p_0). \tag{3.15}$$

For any parameter $p \in \mathcal{P}$ we obtain, by using the equations (3.9) and (3.12)

$$\begin{aligned}
&\left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (k; \xi_0, p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (k; \xi_0, p_0) \right\| \\
&\overset{(3.2)}{\leq} \max\Bigg\{ K_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \\
&\qquad \cdot \|F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p), p) - F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_0)\|, \\
&\qquad K_2 \sum_{n=k}^{\infty} \beta^{k-n-1} \|G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p), p) - G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_0)\| \Bigg\}
\end{aligned}$$

for $k \in \mathbb{Z}^+_\kappa$.

Subtraction and addition of the expressions $\|F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p)\|$ and $\|G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p)\|$, respectively, leads to

$$\left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (k; \xi_0, p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (k; \xi_0, p_0) \right\| \leq \max\{a + b, c + d\} \quad \text{for } k \in \mathbb{Z}^+_\kappa,$$

where (cf. (3.4))

$$\begin{aligned}
a &:= K_1 \sum_{k=\kappa}^{k-1} \alpha^{k-n-1} \|F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p) - F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_0)\|, \\
b &:= K_1 |F|_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n; \xi_0, p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n; \xi_0, p_0) \right\|,
\end{aligned}$$

$$c := K_2 \sum_{n=k}^{\infty} \beta^{k-n-1} \|G(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p) - G(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p_0)\|,$$

$$d := K_2 |G|_1 \sum_{n=k}^{\infty} \beta^{k-n-1} \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n;\xi_0,p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n;\xi_0,p_0) \right\|.$$

Now and in the further progress of this proof, we often use the relation

$$\max\{a+b,c+d\} \le a+c+\max\{b,d\}, \tag{3.16}$$

which is valid for arbitrary reals $a,b,c,d \ge 0$, and obtain the estimate

$$\begin{aligned} \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (k;\xi_0,p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (k;\xi_0,p_0) \right\| \gamma^{\kappa-k} &\le a\gamma^{\kappa-k} + c\gamma^{\kappa-k} \\ &+ \max\left\{ \frac{K_1|F|_1}{\gamma-\alpha}, \frac{K_2|G|_1}{\beta-\gamma} \right\} \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0,p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma}^{+} \end{aligned}$$

for $k \in \mathbb{Z}_\kappa^+$. Hence, by passing over to the least upper bound for $k \in \mathbb{Z}_\kappa^+$, we get (cf. (3.10))

$$\left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0,p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma}^{+} \overset{(3.14)}{\le} \frac{\max\{K_1,K_2\}\gamma^\kappa}{1-L} \sup_{k\in\mathbb{Z}_\kappa^+} U(k,p)$$

with the mapping

$$\begin{aligned} U(k,p) := \frac{\alpha^{k-1}}{\gamma^k} \sum_{n=\kappa}^{k-1} \alpha^{-n} &\|F(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p) \\ &- F(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p_0)\| \\ &+ \frac{\beta^{k-1}}{\gamma^k} \sum_{n=k}^{\infty} \beta^{-n} \|G(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p) \\ &- G(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p_0)\|. \end{aligned} \tag{3.17}$$

Therefore it suffices to prove

$$\lim_{p\to p_0} \sup_{k\in\mathbb{Z}_\kappa^+} U(k,p) = 0 \tag{3.18}$$

to show the limit relation (3.15). We proceed indirectly. Assume (3.18) does not hold. Then there exists an $\varepsilon > 0$ and a sequence $(p_i)_{i\in\mathbb{N}}$ in $\mathcal{P}$ with $\lim_{i\to\infty} p_i = p_0$ and $\sup_{k\in\mathbb{Z}_\kappa^+} U(k,p_i) > \varepsilon$ for $i \in \mathbb{N}$. This implies the existence of a sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ such that

$$U(k_i,p_i) > \varepsilon \quad \text{for } i \in \mathbb{N}. \tag{3.19}$$

From now on we consider $\alpha+\sigma < \gamma$, choose a fixed growth rate $\delta \in (\alpha+\sigma,\gamma)$ and remark that the inequality $\delta/\gamma < 1$ will play an important role below. Because of

Hypothesis 3.1(b) and the inclusion $(\mu_\kappa, \nu_\kappa)(\xi_0, p) \in \ell^+_{\kappa,\delta}(\mathcal{X}\times\mathcal{Y})$ we get (cf. (3.4))

$$\|F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p)\| \overset{(3.3)}{\leq} |F|_1 \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0, p_0) \right\|^+_{\kappa,\delta} \delta^{n-\kappa} \quad \text{for } n \in \mathbb{Z}^+_\kappa,$$
$$\|G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p)\| \overset{(3.3)}{\leq} |G|_1 \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0, p_0) \right\|^+_{\kappa,\delta} \delta^{n-\kappa} \quad \text{for } n \in \mathbb{Z}^+_\kappa,$$

and the triangle inequality leads to

$$\begin{aligned} U(k,p) &\overset{(3.17)}{\leq} 2|F|_1 \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0, p_0) \right\|^+_{\kappa,\delta} \frac{\alpha^{k-1}}{\gamma^k} \sum_{n=\kappa}^{k-1} \left(\frac{\delta}{\alpha}\right)^n \\ &\quad + 2|G|_1 \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0, p_0) \right\|^+_{\kappa,\delta} \frac{\beta^{k-1}}{\gamma^k} \sum_{n=k}^{\infty} \left(\frac{\delta}{\beta}\right)^n \\ &\leq \frac{2\max\{|F|_1, |G|_1\}}{\delta^\kappa} \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0, p_0) \right\|^+_{\kappa,\delta} \left(\frac{1}{\delta-\alpha} + \frac{1}{\beta-\delta}\right) \left(\frac{\delta}{\gamma}\right)^k \end{aligned}$$

for $k \in \mathbb{Z}^+_\kappa$. Because of $\delta/\gamma < 1$, passing over to the limit $k \to \infty$ yields $\lim_{k\to\infty} U(k,p) = 0$ uniformly in $p \in \mathcal{P}$, and taking into account (3.19) the sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^+_\kappa$ has to be bounded above, i.e. there exists an integer $K > \kappa$ with $k_i \leq K$ for all $i \in \mathbb{N}$. Hence we can deduce

$$\begin{aligned} &U(k_i,p_i) \\ &\overset{(3.17)}{\leq} \frac{\alpha^{\kappa-1}}{\gamma^\kappa} \sum_{n=\kappa}^{K-1} \alpha^{-n} \|F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_i) - F(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_0)\| \\ &\quad + \frac{\beta^{K-1}}{\gamma^K} \sum_{n=\kappa}^{\infty} \beta^{-n} \|G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_i) - G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_0)\| \end{aligned}$$

for $i \in \mathbb{N}$, where the first finite sum tends to zero for $i \to \infty$ by the continuity of F. Continuity of G implies

$$\lim_{i\to\infty} G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_i) = G(n, (\mu_\kappa, \nu_\kappa)(n; \xi_0, p_0), p_0)$$

and with the Theorem of Lebesgue[1] we get the convergence of the infinite sum to zero for $i \to \infty$. Thus we derived the relation $\lim_{i\to\infty} U(k_i, p_i) = 0$, which obviously contradicts (3.19). Up to now we have shown the continuity of $(\mu_\kappa, \nu_\kappa)(\xi_0, \cdot): \mathcal{P} \to \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ and by properties of the evaluation map (see [2, Lemma 3.4]) that (3.15) holds. On the other hand, Lemma 3.4(b) gives us the

[1]To apply this result from integration theory, one has to write the infinite sum as an integral over piecewise-constant functions and use the Lipschitz estimate of G, which is implied by (3.4), to get an integrable majorant.

Lipschitz estimate

$$\begin{aligned}\left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\kappa;\xi,p_0) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\kappa;\xi_0,p_0) \right\| \\ \leq \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi,p_0) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma}^{+} \overset{(3.11)}{\leq} \frac{K_1}{1-L} \|\xi-\xi_0\|\end{aligned}$$

for any $\xi \in \mathcal{X}$ and e.g. [3, Lemma B.4] together with the discrete topology on $\mathbb{Z}_\kappa^+$, implies the continuity of the fixed point mapping $(\mu_\kappa,\nu_\kappa)\colon \mathbb{Z}_\kappa^+ \times \mathcal{X} \times \mathcal{P} \to \mathcal{X} \times \mathcal{Y}$. With a view to the definition of $s\colon I \times \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$, its continuity readily follows.

Step 2. Let $\gamma \in [\alpha+\sigma,\beta-\sigma]$, $\xi \in \mathcal{X}$ and $p \in \mathcal{P}$ be arbitrary. By formal differentiation of the fixed point equation (cf. (3.9), (3.12))

$$\begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (k;\xi,p) = \begin{pmatrix} \Phi(k,\kappa)\xi + \sum_{n=\kappa}^{k-1} \Phi(k,n+1) F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \\ -\sum_{n=k}^{\infty} \Psi(k,n+1) G(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \end{pmatrix} \tag{3.20}$$

for $k \in \mathbb{Z}_\kappa^+$, with respect to $\xi \in \mathcal{X}$, we obtain another fixed point equation

$$\begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi,p) = \mathcal{T}_\kappa^1((\mu_\kappa^1,\nu_\kappa^1)(\xi,p);\xi,p) \tag{3.21}$$

for the formal partial derivative $(\mu_\kappa^1,\nu_\kappa^1)$ of $(\mu_\kappa,\nu_\kappa)\colon \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^+(\mathcal{X} \times \mathcal{Y})$ with respect to ξ, where the right-hand side of (3.20) is given by

$$\begin{aligned}&(\mathcal{T}_\kappa^1(\mu^1,\nu^1;\xi,p))(k) := \\ &\begin{pmatrix} \Phi(k,\kappa) + \sum_{n=\kappa}^{k-1} \Phi(k,n+1) \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \begin{pmatrix} \mu^1 \\ \nu^1 \end{pmatrix}(n) \\ -\sum_{n=k}^{\infty} \Psi(k,n+1) \frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \begin{pmatrix} \mu^1 \\ \nu^1 \end{pmatrix}(n) \end{pmatrix}\end{aligned} \tag{3.22}$$

for $k \in \mathbb{Z}_\kappa^+$. Here (μ^1,ν^1) is a mapping from $\mathbb{Z}_\kappa^+$ to $\mathcal{L}(\mathcal{X};\mathcal{X} \times \mathcal{Y})$ and in the following we investigate this operator $\mathcal{T}_\kappa^1$.

Step 3. For every $\gamma \in [\alpha+\sigma,\beta-\sigma]$ the operator $\mathcal{T}_\kappa^1\colon \ell_{\kappa,\gamma}^1 \times \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^1$ is well-defined and, for $(\mu^1,\nu^1) \in \ell_{\kappa,\gamma}^1$, $\xi \in \mathcal{X}$, $p \in \mathcal{P}$, satisfies the estimate

$$\|\mathcal{T}_\kappa^1(\mu^1,\nu^1;\xi,p)\|_{\kappa,\gamma}^+ \leq K_1 + L \left\| \begin{pmatrix} \mu^1 \\ \nu^1 \end{pmatrix} \right\|_{\kappa,\gamma}^+. \tag{3.23}$$

Thereto choose arbitrary sequences $(\mu^1, \nu^1) \in \ell^1_{\kappa,\gamma}$ and $\xi \in \mathcal{X}$, $p \in \mathcal{P}$. Now using (3.2), (3.4) it is

$$
\begin{aligned}
(3.24)\quad & \|\mathcal{T}^1_\kappa(\mu^1, \nu^1; \xi, p)(k)\|_{\mathcal{L}(\mathcal{X};\mathcal{X}\times\mathcal{Y})}\gamma^{\kappa-k} \\
&\overset{(3.22)}{\leq} \max\left\{K_1\left(\frac{\gamma}{\alpha}\right)^{\kappa-k} + K_1|F|_1\gamma^{\kappa-k}\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\left\|\begin{pmatrix}\mu^1\\ \nu^1\end{pmatrix}(n)\right\|,\right. \\
&\qquad \left. K_2|G|_1\gamma^{\kappa-k}\sum_{n=k}^{\infty}\beta^{k-n-1}\left\|\begin{pmatrix}\mu^1\\ \nu^1\end{pmatrix}(n)\right\|\right\} \\
&\overset{(3.16)}{\leq} K_1 + \max\left\{\frac{K_1|F|_1}{\alpha}\sum_{n=\kappa}^{k-1}\left(\frac{\alpha}{\gamma}\right)^{k-n}, \frac{K_2|G|_1}{\beta}\sum_{n=k}^{\infty}\left(\frac{\beta}{\gamma}\right)^{k-n}\right\}\left\|\begin{pmatrix}\mu^1\\ \nu^1\end{pmatrix}\right\|^+_{\kappa,\gamma} \\
&\leq K_1 + \max\left\{\frac{K_1|F|_1}{\gamma-\alpha}, \frac{K_2|G|_1}{\beta-\gamma}\right\}\left\|\begin{pmatrix}\mu^1\\ \nu^1\end{pmatrix}\right\|^+_{\kappa,\gamma} \overset{(3.14)}{\leq} K_1 + L\left\|\begin{pmatrix}\mu^1\\ \nu^1\end{pmatrix}\right\|^+_{\kappa,\gamma}
\end{aligned}
$$

for $k \in \mathbb{Z}^+_\kappa$, and passing over to the least upper bound over $k \in \mathbb{Z}^+_\kappa$ implies our claim $\mathcal{T}^1_\kappa(\mu^1, \nu^1; \xi, p) \in \ell^1_{\kappa,\gamma}$, as well as the estimate (3.23).

Step 4. For every $\gamma \in [\alpha+\sigma, \beta-\sigma]$ the operator

$$\mathcal{T}^1_\kappa(\,\cdot\,; \xi, p)\colon \ell^1_{\kappa,\gamma} \to \ell^1_{\kappa,\gamma}$$

is a uniform contraction in $\xi \in \mathcal{X}$, $p \in \mathcal{P}$, moreover, the fixed point $(\mu^1_\kappa, \nu^1_\kappa)(\xi, p)$ in $\ell^1_{\kappa,\gamma}$ does not depend on $\gamma \in [\alpha+\sigma, \beta-\sigma]$ and satisfies

$$\left\|\begin{pmatrix}\mu^1_\kappa\\ \nu^1_\kappa\end{pmatrix}(\xi, p)\right\|^+_{\kappa,\gamma} \leq \frac{K_1}{1-L} \quad \textit{for } \xi \in \mathcal{X},\ p \in \mathcal{P}.$$

Let $\xi \in \mathcal{X}$ and $p \in \mathcal{P}$ be arbitrary. Completely analogous to the estimate (3.24) we get

$$\|\mathcal{T}^1_\kappa(\mu^1, \nu^1; \xi, p) - \mathcal{T}^1_\kappa(\overline{\mu}^1, \overline{\nu}^1; \xi, p)\|^+_{\kappa,\gamma} \overset{(3.14)}{\leq} L\|\begin{pmatrix}\mu^1\\ \nu^1\end{pmatrix} - \begin{pmatrix}\overline{\mu}^1\\ \overline{\nu}_1\end{pmatrix}\|^+_{\kappa,\gamma}$$

for $(\mu^1, \nu^1), (\overline{\mu}^1, \overline{\nu}^1) \in \ell^1_{\kappa,\gamma}$. Taking (3.10) into account, consequently Banach's fixed point theorem guarantees the unique existence of a fixed point $(\mu^1_\kappa, \nu^1_\kappa)(\xi, p) \in \ell^1_{\kappa,\gamma}$ of $\mathcal{T}^1_\kappa(\,\cdot\,; \xi, p)\colon \ell^1_{\kappa,\gamma} \to \ell^1_{\kappa,\gamma}$. This fixed point is independent of the growth constant $\gamma \in [\alpha+\sigma, \beta-\sigma]$ because with Lemma 2.2(b) and (c) we have the inclusion $\ell^1_{\kappa,\alpha+\sigma} \subseteq \ell^1_{\kappa,\gamma}$ and every mapping $\mathcal{T}^1_\kappa(\,\cdot\,; \xi, p)\colon \ell^1_{\kappa,\gamma} \to \ell^1_{\kappa,\gamma}$ has the same fixed point as the restriction $\mathcal{T}^1_\kappa(\,\cdot\,; \xi, p)\big|_{\ell^1_{\kappa,\alpha+\sigma}}$. Finally the fixed point identity (3.20) and (3.23) lead to the estimate (3.25).

Step 5. For every $\gamma \in (\alpha+\sigma, \beta-\sigma]$ and $p \in \mathcal{P}$ the mapping $(\mu_\kappa, \nu_\kappa)(\,\cdot\,, p)\colon \mathcal{X} \to \ell^+_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ is differentiable with derivative

$$\frac{\partial}{\partial\xi}\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix} = \begin{pmatrix}\mu^1_\kappa\\ \nu^1_\kappa\end{pmatrix}\colon \mathcal{X}\times\mathcal{P} \to \ell^1_{\kappa,\gamma}. \tag{3.26}$$

Let $\xi \in \mathcal{X}$ and $p \in \mathcal{P}$ be arbitrary. In relation (3.26), as well as in the subsequent considerations we are using the isomorphism between the spaces $\ell^1_{\kappa,\gamma}$ and $\mathcal{L}(\mathcal{X}; \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}))$ from Lemma 2.2(c) and identify them. To show the claim above, we define the following four quotients

$$\begin{aligned} \Delta\mu(n,h) &:= \frac{\mu_\kappa(n; \xi + h, p) - \mu_\kappa(n; \xi, p) - \mu^1_\kappa(n; \xi, p)h}{\|h\|}, \\ \Delta\nu(n,h) &:= \frac{\nu_\kappa(n; \xi + h, p) - \nu_\kappa(n; \xi, p) - \nu^1_\kappa(n; \xi, p)h}{\|h\|} \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &\Delta F(n, x, y, h_1, h_2) \\ &\quad := \frac{F(n, x + h_1, y + h_2, p) - F(n, x, y, p) - \frac{\partial F}{\partial(x,y)}(n, x, y, p) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}{\|(h_1, h_2)\|}, \end{aligned}$$

$$\begin{aligned} &\Delta G(n, x, y, h_1, h_2) \\ &\quad := \frac{G(n, x + h_1, y + h_2, p) - G(n, x, y, p) - \frac{\partial G}{\partial(x,y)}(n, x, y, p) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}}{\|(h_1, h_2)\|} \end{aligned}$$

for integers $n \in I$ and $x \in \mathcal{X}$, $h, h_1 \in \mathcal{X} \setminus \{0\}$, $y \in \mathcal{Y}$, $h_2 \in \mathcal{Y} \setminus \{0\}$. Thereby obviously the inclusion $(\Delta\mu, \Delta\nu)(\,\cdot\,, h) \in \ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ holds. To prove the differentiability we have to show the limit relation $\lim_{h \to 0}(\Delta\mu, \Delta\nu)(\,\cdot\,, h) = 0$ in $\ell^+_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$. For this consider $\alpha + \sigma < \gamma$, a growth rate $\delta \in (\alpha + \sigma, \gamma)$ and from Lemma 3.3 we obtain

$$\frac{1}{\|h\|} \left\| \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n; \xi + h, p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n; \xi, p) \right\| \overset{(3.6)}{\leq} K_1 \frac{\delta - \alpha}{\delta - \alpha - K_1 |F|_1} \delta^{n-\kappa} \tag{3.28}$$

for $n \in \mathbb{Z}^+_\kappa$. Moreover, using the fixed point equations (3.20) for μ_κ and (3.20) for μ^1_κ it results (cf. (3.9), (3.22))

$$\begin{aligned} \|\Delta\mu(k,h)\| = \frac{1}{\|h\|} \bigg\| \sum_{n=\kappa}^{k-1} \Phi(k, n+1) \bigg[&F(n, (\mu_\kappa, \nu_\kappa)(n; \xi + h, p), p) \\ &- F(n, (\mu_\kappa, \nu_\kappa)(n; \xi, p), p) \\ &- \frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi, p), p) \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix} (n; \xi, p) h \bigg] \bigg\| \end{aligned}$$

for $k \in \mathbb{Z}^+_\kappa$, where subtraction and addition of the expression

$$\frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi, p), p) \left[\begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n; \xi + h, p) - \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix} (n; \xi, p) \right]$$

in the above brackets implies the estimate

$$
\begin{aligned}
\|\Delta\mu(k,h)\| \leq{}& \frac{1}{\|h\|}\Bigg\|\sum_{n=\kappa}^{k-1}\Phi(k,n+1)\Bigg\{F(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h,p),p)\\
&- F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\\
&- \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\\
&\cdot\left[\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h,p)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi,p)\right]\Bigg\}\Bigg\|\\
&+\frac{1}{\|h\|}\Bigg\|\sum_{n=\kappa}^{k-1}\Phi(k,n+1)\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\\
&\cdot\left[\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h,p)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi,p)-\begin{pmatrix}\mu_\kappa^1\\ \nu_\kappa^1\end{pmatrix}(n;\xi,p)h\right]\Bigg\|\\
\stackrel{(3.4)}{\leq}{}& \sum_{n=\kappa}^{k-1}\|\Phi(k,n+1)\|\\
&\cdot\|\Delta F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\|\\
&\cdot\frac{1}{\|h\|}\left\|\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h,p)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi,p)\right\|\\
&+|F|_1\sum_{n=\kappa}^{k-1}\|\Phi(k,n+1)\|\left\|\begin{pmatrix}\Delta\mu\\ \Delta\nu\end{pmatrix}(n,h)\right\|\\
\stackrel{(3.2)}{\leq}{}& K_1\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\|\Delta F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)\\
&-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\|\\
&\cdot\frac{1}{\|h\|}\left\|\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h,p)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi,p)\right\|\\
&+K_1|F|_1\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\left\|\begin{pmatrix}\Delta\mu\\ \Delta\nu\end{pmatrix}(n,h)\right\|
\end{aligned}
$$

for $k\in\mathbb{Z}_\kappa^+$ and together with (3.28) we get

$$
\begin{aligned}
\|\Delta\mu(k,h)\|\leq{}& K_1|F|_1\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\left\|\begin{pmatrix}\Delta\mu\\ \Delta\nu\end{pmatrix}(n,h)\right\|+\frac{K_1^2(\delta-\alpha)}{\delta-\alpha-K_1|F|_1}\alpha^{k-1}\delta^{-\kappa}\\
&\cdot\sum_{n=\kappa}^{k-1}\left(\frac{\delta}{\alpha}\right)^n\|\Delta F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\|
\end{aligned}
$$

for $k \in \mathbb{Z}_\kappa^+$. Now we analogously derive a similar estimate for the norm of the second component $\|\Delta\nu(k,h)\|$ and obtain

$$\|\Delta\nu(k,h)\| \leq K_2|G|_1 \sum_{n=k}^{\infty} \beta^{k-n-1} \left\| \begin{pmatrix} \Delta\mu \\ \Delta\nu \end{pmatrix} (n,h) \right\| + \frac{K_1K_2(\delta-\alpha)}{\delta-\alpha-K_1|F|_1} \beta^{k-1}\delta^{-\kappa}$$
$$\cdot \sum_{n=k}^{\infty} \left(\frac{\delta}{\beta}\right)^n \|\Delta G(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\|$$

for $k \in \mathbb{Z}_\kappa^+$. Consequently for the norm $\|(\Delta\mu,\Delta\nu)(k,h)\|$ one gets the inequality

$$\left\| \begin{pmatrix} \Delta\mu \\ \Delta\nu \end{pmatrix} (k,h) \right\| \overset{(2.1)}{=} \max\{\|\Delta\mu(k,h)\|, \|\Delta\nu(k,h)\|\} \leq \max\{a+b, c+d\}$$

for $k \in \mathbb{Z}_\kappa^+$ with

$$\begin{aligned} a &:= \frac{K_1^2(\delta-\alpha)}{\delta-\alpha-K_1|F|_1} \alpha^{k-1}\delta^{-\kappa} \sum_{n=\kappa}^{k-1} \left(\frac{\delta}{\alpha}\right)^n \\ &\quad \cdot \|\Delta F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\|, \\ b &:= K_1|F|_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \left\| \begin{pmatrix} \Delta\mu \\ \Delta\nu \end{pmatrix} (n,h) \right\|, \\ c &:= \frac{K_1K_2(\delta-\alpha)}{\delta-\alpha-K_1|F|_1} \beta^{k-1}\delta^{-\kappa} \sum_{n=k}^{\infty} \left(\frac{\delta}{\beta}\right)^n \\ &\quad \cdot \|\Delta G(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\|, \\ d &:= K_2|G|_1 \sum_{n=k}^{\infty} \beta^{k-n-1} \left\| \begin{pmatrix} \Delta\mu \\ \Delta\nu \end{pmatrix} (n,h) \right\|. \end{aligned}$$

We are using the relation (3.16) again, and obtain the estimate

$$\left\| \begin{pmatrix} \Delta\mu \\ \Delta\nu \end{pmatrix} (k,h) \right\| \gamma^{\kappa-k} \overset{(3.14)}{\leq} a\gamma^{\kappa-k} + c\gamma^{\kappa-k} + L \left\| \begin{pmatrix} \Delta\mu \\ \Delta\nu \end{pmatrix} (h) \right\|_{\kappa,\gamma}^{+} \quad \text{for } k \in \mathbb{Z}_\kappa^+.$$

By passing over to the least upper bound for $k \in \mathbb{Z}_\kappa^+$ we get (cf. (3.10))

$$\left\| \begin{pmatrix} \Delta\mu \\ \Delta\nu \end{pmatrix} (h) \right\|_{\kappa,\gamma}^{+} \leq \frac{K_1\max\{K_1,K_2\}}{1-L} \frac{\delta-\alpha}{\delta-\alpha-K_1|F|_1} \left(\frac{\gamma}{\delta}\right)^\kappa \sup_{k\in\mathbb{Z}_\kappa^+} V(k,h)$$

with

$$\begin{aligned} V(k,h) &:= \frac{\alpha^{k-1}}{\gamma^k} \sum_{n=\kappa}^{k-1} \left(\frac{\delta}{\alpha}\right)^n \\ &\quad \cdot \|\Delta F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\| \\ &\quad + \frac{\beta^{k-1}}{\gamma^k} \sum_{n=k}^{\infty} \left(\frac{\delta}{\beta}\right)^n \\ &\quad \cdot \|\Delta G(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\| \end{aligned} \tag{3.29}$$

for $k \in \mathbb{Z}_\kappa^+$. Thus to prove the above claim in the present Step 5, we only have to show the limit relation

$$\lim_{h \to 0} \sup_{k \in \mathbb{Z}_\kappa^+} V(k,h) = 0, \tag{3.30}$$

which will be done indirectly. Suppose (3.30) is not true. Then there exists an $\varepsilon > 0$ and a sequence $(h_i)_{i \in \mathbb{N}}$ in $\mathcal{X}$ with $\lim_{i \to \infty} h_i = 0$ such that $\sup_{k \in \mathbb{Z}_\kappa^+} V(k,h_i) > \varepsilon$ for $i \in \mathbb{N}$. This implies the existence of a further sequence $(k_i)_{i \in \mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ with

$$V(k_i,h_i) > \varepsilon \quad \text{for } i \in \mathbb{N}. \tag{3.31}$$

Using the estimates $\|\Delta F(n,x,y,h_1,h_2)\| \leq 2|F|_1$ and $\|\Delta G(n,x,y,h_1,h_2)\| \leq 2|G|_1$, which result from (3.4) in connection with [12, Corollary 4.3], it follows

$$\begin{aligned} V(k,h) &\overset{(3.29)}{\leq} \frac{2|F|_1}{\alpha}\left(\frac{\alpha}{\gamma}\right)^k \sum_{n=\kappa}^{k-1}\left(\frac{\delta}{\alpha}\right)^n + \frac{2|G|_1}{\beta}\left(\frac{\beta}{\gamma}\right)^k \sum_{n=k}^{\infty}\left(\frac{\delta}{\beta}\right)^n \\ &\leq \left(\frac{2|F|_1}{\delta-\alpha} + \frac{2|G|_1}{\beta-\delta}\right)\left(\frac{\delta}{\gamma}\right)^k \end{aligned}$$

for $k \in \mathbb{Z}_\kappa^+$ and the right-hand side of this estimate converges to 0 for $k \to \infty$, i.e. we have $\lim_{k \to \infty} V(k,h) = 0$ uniformly in $h \in \mathcal{X}$. Because of (3.31) the sequence $(k_i)_{i \in \mathbb{N}}$ has to be bounded in $\mathbb{Z}_\kappa^+$, i.e. there exists an integer $K > \kappa$ with $k_i \leq K$ for any $i \in \mathbb{N}$. Now we obtain

$$\begin{aligned} V(k_i,h_i) &\overset{(3.29)}{\leq} \frac{\alpha^{\kappa-1}}{\gamma^\kappa} \sum_{n=\kappa}^{K-1}\left(\frac{\delta}{\alpha}\right)^n \\ &\quad \cdot \|\Delta F(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h_i,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\| \\ &\quad + \frac{\beta^{K-1}}{\gamma^K} \sum_{n=\kappa}^{\infty}\left(\frac{\delta}{\beta}\right)^n \\ &\quad \cdot \|\Delta G(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h_i,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p))\| \end{aligned} \tag{3.32}$$

for $i \in \mathbb{N}$ and because of Step 1 we have

$$\lim_{i \to \infty} \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix}(n;\xi+h_i,p) = \begin{pmatrix} \mu_\kappa \\ \nu_\kappa \end{pmatrix}(n;\xi,p) \quad \text{for } n \in \mathbb{Z}_\kappa^+,\ \xi \in \mathcal{X},\ p \in \mathcal{P},$$

as well as using the partial differentiability of F and G

$$\lim_{(h_1,h_2) \to (0,0)} \left\| \begin{pmatrix} \Delta F \\ \Delta G \end{pmatrix}(n,x,y,h_1,h_2) \right\| = 0,$$

which leads to the limit relation

$$\lim_{i \to \infty} \left\| \begin{pmatrix} \Delta F \\ \Delta G \end{pmatrix}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),(\mu_\kappa,\nu_\kappa)(n;\xi+h_i,p)-(\mu_\kappa,\nu_\kappa)(n;\xi,p)) \right\| = 0$$

for $n \in \mathbb{Z}_\kappa^+$. Therefore the finite sum in (3.32) tends to 0 for $i \to \infty$. Using Lebesgue's theorem, also the infinite sum in (3.32) converges to 0 for $i \to \infty$ and we finally have $\lim_{i\to\infty} V(k_i, h_i) = 0$, which contradicts (3.31). Hence the claim in Step 5 is true, where (3.26) follows by the uniqueness of Fréchet derivatives.

Step 6. For every $\gamma \in (\alpha+\sigma, \beta-\sigma]$ the mapping $\partial(\mu_\kappa,\nu_\kappa)/\partial\xi \colon \mathcal{X}\times\mathcal{P} \to \ell^1_{\kappa,\gamma}$ is continuous.

With a view to (3.26) it is sufficient to show the continuity of the mapping $(\mu^1_\kappa, \nu^1_\kappa)\colon \mathcal{X}\times\mathcal{P} \to \ell^1_{\kappa,\gamma}$. To do this, we fix any $\xi_0 \in \mathcal{X}$, $p_0 \in \mathcal{P}$ and choose $\xi \in \mathcal{X}$, $p \in \mathcal{P}$ arbitrarily. Using the fixed point equation (3.20) for $(\mu^1_\kappa, \nu^1_\kappa)$ we obtain the estimate (cf. (3.22))

$$
\begin{aligned}
&\left\| \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(k;\xi,p) - \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(k;\xi_0,p_0) \right\| \\
&\quad\overset{(3.2)}{\le} \max\left\{ K_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \left\| \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi,p) \right.\right. \\
&\qquad \left. - \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p_0)\begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi_0,p_0) \right\|, \\
&\qquad K_2 \sum_{n=k}^{\infty} \beta^{k-n-1} \left\| \frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi,p) \right. \\
&\qquad \left.\left. - \frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p_0)\begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi_0,p_0) \right\| \right\}
\end{aligned}
$$

for $k \in \mathbb{Z}_\kappa^+$, where subtraction and addition of the expressions

$$
\begin{aligned}
&\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi_0,p_0), \\
&\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi_0,p_0),
\end{aligned}
$$

respectively, in the corresponding norms and the use of (3.4) leads to

$$
\left\| \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(k;\xi,p) - \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(k;\xi_0,p_0) \right\| \le \max\{a+b, c+d\}
$$

for $k \in \mathbb{Z}_\kappa^+$, with the abbreviations

$$
\begin{aligned}
a &:= K_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \|\widehat{F}(n,\xi,p)\| \left\| \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi_0,p_0) \right\|, \\
b &:= K_1 |F|_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \left\| \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi,p) - \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi_0,p_0) \right\|, \\
c &:= K_2 \sum_{n=k}^{\infty} \beta^{k-n-1} \|\widehat{G}(n,\xi,p)\| \left\| \begin{pmatrix} \mu^1_\kappa \\ \nu^1_\kappa \end{pmatrix}(n;\xi_0,p_0) \right\|,
\end{aligned}
$$

$$d := K_2|G|_1 \sum_{n=k}^{\infty} \beta^{k-n-1} \left\| \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (n;\xi,p) - \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (n;\xi_0,p_0) \right\|$$

and

$$\begin{aligned} \widehat{F}(n,\xi,p) &:= \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \\ &\quad - \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p_0), \\ \widehat{G}(n,\xi,p) &:= \frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \\ &\quad - \frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0),p_0). \end{aligned} \tag{3.33}$$

With the aid of the relation (3.16) one obtains

$$\begin{aligned} &\left\| \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (k;\xi,p) - \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (k;\xi_0,p_0) \right\| \gamma^{\kappa-k} \\ &\qquad \overset{(3.14)}{\leq} a\gamma^{\kappa-k} + c\gamma^{\kappa-k} + L \left\| \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi,p) - \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma}^+ \end{aligned} \tag{3.34}$$

for $k \in \mathbb{Z}_\kappa^+$. We define $\gamma_1 := \alpha + \sigma$ to get $(\mu_\kappa^1, \nu_\kappa^1)(\xi_0,p_0) \in \ell_{\kappa,\gamma_1}^1$. In the sums a and c we can estimate the mapping $(\mu_\kappa^1, \nu_\kappa^1)(\xi_0,p_0)$ using its γ_1^+-norm, which yields

$$\begin{aligned} a &\leq K_1 \gamma_1^{-\kappa} \alpha^{k-1} \left\| \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma_1}^+ \sum_{n=\kappa}^{k-1} \left(\frac{\gamma_1}{\alpha}\right)^n \|\widehat{F}(n,\xi,p)\| \quad \text{for } k \in \mathbb{Z}_\kappa^+, \\ c &\leq K_2 \gamma_1^{-\kappa} \beta^{k-1} \left\| \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma_1}^+ \sum_{n=k}^{\infty} \left(\frac{\gamma_1}{\beta}\right)^n \|\widehat{G}(n,\xi,p)\| \quad \text{for } k \in \mathbb{Z}_\kappa^+. \end{aligned}$$

Now we substitute these expressions into (3.34) and pass over to the supremum over $k \in \mathbb{Z}_\kappa^+$ to derive

$$\begin{aligned} &\left\| \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi,p) - \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma}^+ \\ &\qquad \overset{(3.14)}{\leq} \frac{\max\{K_1,K_2\}}{1-L} \left\| \begin{pmatrix} \mu_\kappa^1 \\ \nu_\kappa^1 \end{pmatrix} (\xi_0,p_0) \right\|_{\kappa,\gamma_1}^+ \left(\frac{\gamma}{\gamma_1}\right)^\kappa \sup_{k \in \mathbb{Z}_\kappa^+} W(k,\xi,p) \end{aligned}$$

with

$$\begin{aligned} W(k,\xi,p) &:= \frac{1}{\alpha}\left(\frac{\alpha}{\gamma}\right)^k \sum_{n=\kappa}^{k-1} \left(\frac{\gamma_1}{\alpha}\right)^n \|\widehat{F}(n,\xi,p)\| \\ &\qquad + \frac{1}{\beta}\left(\frac{\beta}{\gamma}\right)^k \sum_{n=k}^{\infty} \left(\frac{\gamma_1}{\beta}\right)^n \|\widehat{G}(n,\xi,p)\|. \end{aligned} \tag{3.35}$$

Therefore it is sufficient to prove the following limit relation

$$\lim_{(\xi,p)\to(\xi_0,p_0)} \sup_{k\in\mathbb{Z}_\kappa^+} W(k,\xi,p) = 0 \tag{3.36}$$

to show the claim in the present Step 6. We proceed indirectly and assume the equation (3.36) does not hold. Then there exists an $\varepsilon > 0$ and a sequence $((\xi_i,p_i))_{i\in\mathbb{N}}$ in $\mathcal{X}\times\mathcal{P}$ with $\lim_{i\to\infty}(\xi_i,p_i) = (\xi_0,p_0)$ and

$$\sup_{k\in\mathbb{Z}_\kappa^+} W(k,\xi_i,p_i) > \varepsilon \quad \text{for } i\in\mathbb{N}, \tag{3.37}$$

which moreover leads to the existence of a sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ such that

$$W(k_i,\xi_i,p_i) > \varepsilon \quad \text{for } i\in\mathbb{N}. \tag{3.38}$$

Apart from this, we get (cf. (3.4), (3.33))

$$\begin{aligned} W(k,\xi,p) &\overset{(3.35)}{\le} \frac{2|F|_1}{\alpha}\left(\frac{\alpha}{\gamma}\right)^k \sum_{n=\kappa}^{k-1}\left(\frac{\gamma_1}{\alpha}\right)^n + \frac{2|G|_1}{\beta}\left(\frac{\beta}{\gamma}\right)^k \sum_{n=k}^{\infty}\left(\frac{\gamma_1}{\beta}\right)^n \\ &\le \left(\frac{2|F|_1}{\gamma_1-\alpha} + \frac{2|G|_1}{\beta-\gamma_1}\right)\left(\frac{\gamma_1}{\gamma}\right)^k \end{aligned}$$

for $k\in\mathbb{Z}_\kappa^+$, and since $\gamma_1/\gamma < 1$, the right-hand side of this estimate converges to 0 for $k\to\infty$ which yields $\lim_{k\to\infty} W(k,\xi,p) = 0$ uniformly in $(\xi,p)\in\mathcal{X}\times\mathcal{P}$. Because of (3.38) the sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ has to be bounded above, i.e. there exists an integer $K > \kappa$ with $k_i \le K$ for all $i\in\mathbb{N}$ and this is used to obtain

$$\begin{aligned} W(k_i,\xi_i,p_i) \le{}& \frac{1}{\alpha}\left(\frac{\alpha}{\gamma}\right)^\kappa \sum_{n=\kappa}^{K-1}\left(\frac{\gamma_1}{\alpha}\right)^n \|\widehat{F}(n,\xi,p)\| \\ &+ \frac{1}{\beta}\left(\frac{\beta}{\gamma}\right)^K \sum_{n=\kappa}^{\infty}\left(\frac{\gamma_1}{\beta}\right)^n \|\widehat{G}(n,\xi,p)\|. \end{aligned} \tag{3.39}$$

The continuity of $(\mu_\kappa,\nu_\kappa)(n,\,\cdot\,)$ from Step 1 yields $\lim_{i\to\infty}(\mu_\kappa,\nu_\kappa)(n;\xi_i,p_i) = (\mu_\kappa,\nu_\kappa)(n;\xi_0,p_0)$ for $n\in\mathbb{Z}_\kappa^+$ and therefore the finite sum in (3.39) tends to 0 for $i\to\infty$ by (3.33) and the continuity of $\partial F/\partial(x,y)$. By the continuity of $\partial G/\partial(x,y)$ the infinite sum in (3.39) does the same and we can apply Lebesgue's Theorem, which finally implies $\lim_{i\to\infty} W(k_i,\xi_i,p_i) = 0$. Of course this contradicts (3.38) and consequently we have shown the above claim in Step 6.

Step 7. We have the identity $s(\kappa,\xi,p) = \nu_\kappa(\xi,p)(\kappa)$ for $\kappa\in I$, $\xi\in\mathcal{X}$, $p\in\mathcal{P}$ and by well-known properties of the evaluation map (see [2, Lemma 3.4]) it follows that the mapping $s\colon I\times\mathcal{X}\times\mathcal{P}\to\mathcal{Y}$ is continuously differentiable with respect to its second variable.

(b) Since part (b) of the theorem can be proved along the same lines as part (a) we present only a rough sketch of the proof. Analogously to Lemma 3.4, for initial values $\eta\in\mathcal{Y}$ and parameters $p\in\mathcal{P}$, the γ^--quasibounded solutions of the system (3.1) may be characterized as the fixed points of a mapping $\overline{\mathcal{T}}_\kappa\colon \ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y}) \times \mathcal{Y} \times \mathcal{P} \to \ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$,

$$(\overline{\mathcal{T}}_\kappa(\mu,\nu;\eta,p))(k) := \begin{pmatrix} \displaystyle\sum_{n=-\infty}^{k-1} \Phi(k,n+1)F(n,(\mu,\nu)(n),p) \\ \displaystyle\Psi(k,\kappa)\eta - \sum_{n=k}^{\kappa-1} \Psi(k,n+1)G(n,(\mu,\nu)(n),p) \end{pmatrix} \tag{3.40}$$

for $k \in \mathbb{Z}^-_\kappa$. Here the variation of constant formula in backward time and [1, Lemma 3.2(a)] are needed. Now $\overline{\mathcal{T}}_\kappa$ can be treated just as $\mathcal{T}_\kappa$ in (a). In order to prove the counterpart of Lemma 3.3 the two results [1, Lemmas 3.3, 3.4(a)] have to be replaced by [1, Lemmas 3.2(a), 3.5]. It follows from the assumption (3.5) that also $\overline{\mathcal{T}}_\kappa$ is a contraction on the space $\ell^-_{\kappa,\gamma}(\mathcal{X} \times \mathcal{Y})$ and if $(\mu_\kappa,\nu_\kappa)(\eta,p) \in \ell^-_{\kappa,\gamma}(\mathcal{X}\times\mathcal{Y})$ denotes its unique fixed point, we define the function $r\colon I\times\mathcal{Y}\times\mathcal{P} \to \mathcal{X}$ by $r(\kappa,\eta,p) := \big(\mu_\kappa(\eta,p)\big)(\kappa)$. The claimed properties of r can be proved along the lines of part (a).

(c) The proof of part (c) can be done just as in [1, Theorem 4.1(c)] and hence the proof of Theorem 3.5 is complete. □

4. Higher order smoothness of invariant fiber bundles

In [2] a higher order smoothness result for the fiber bundles S or R in a nearly hyperbolic situation is proved, i.e. if the growth rates α,β and the real $\sigma_{\max}$ from Hypothesis 3.1 satisfy $\alpha + \sigma_{\max} \le 1$ or $1 \le \beta - \sigma_{\max}$, respectively. Now we weaken this assumption and replace it by the so-called gap-condition. However, in contrast to [2], we cannot use the uniform contraction principle here.

THEOREM 4.1 (C^m-smoothness of invariant fiber bundles). *We assume Hypothesis* 3.1. *Then the statements of Theorem* 3.5 *hold and moreover, the mappings s and r satisfy the following statements:*

(a) *Under the gap-condition*

$$\alpha^{m_s} < \beta \tag{4.1}$$

for some integer $m_s \in \{1,\dots,m\}$ and if

$$\sigma_{\max} = \min\left\{\frac{\beta-\alpha}{2}, \alpha\left(\sqrt[m_s]{\frac{\alpha+\beta}{\alpha+\alpha^{m_s}}}\right) - 1\right\},$$

the mapping $s\colon I \times \mathcal{X} \times \mathcal{P} \to \mathcal{Y}$ is m_s-times continuously differentiable in the second argument with globally bounded derivatives

$$\left\|\frac{\partial^n s}{\partial \xi^n}(\kappa,\xi,p)\right\|_{\mathcal{L}_n(\mathcal{X};\mathcal{Y})} \le C_n \quad \text{for } n \in \{1,\dots,m_s\},\ (\kappa,\xi,p) \in I \times \mathcal{X} \times \mathcal{P},$$

where in particular $C_1 := \sigma K_1/(\sigma - \max\{K_1|F|_1, K_2|G|_1\})$,

(b) *in case $I = \mathbb{Z}$, under the gap-condition $\alpha < \beta^{m_r}$ for some integer $m_r \in \{1, \dots, m\}$ and if* $\sigma_{\max} = \min\{(\beta-\alpha)/2, \beta(1-\sqrt[m_r]{(\alpha+\beta)/(\beta+\beta^{m_r})})\}$, *the mapping $r: I \times \mathcal{Y} \times \mathcal{P} \to \mathcal{X}$ is m_r-times continuously differentiable in the second argument with globally bounded derivatives*

$$\left\| \frac{\partial^n r}{\partial \eta^n}(\kappa, \eta, p) \right\|_{\mathcal{L}_n(\mathcal{Y};\mathcal{X})} \le C_n \quad \text{for } n \in \{1, \dots, m_r\},\ (\kappa, \eta, p) \in I \times \mathcal{Y} \times \mathcal{P},$$

where in particular $C_1 := \sigma K_2/(\sigma - \max\{K_1|F|_1, K_2|G|_1\})$,

(c) *the global bounds $C_2, \dots, C_m \ge 0$ can be determined recursively using the formula*

$$C_n := \max\left\{ K_1 \sum_{j=2}^{n} |F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(n)} \prod_{i=1}^{j} C_{\#N_i}, \right. \tag{4.2}$$

$$\left. K_2 \sum_{j=2}^{n} |G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(n)} \prod_{i=1}^{j} C_{\#N_i} \right\} \Big/ (\sigma - \max\{K_1, K_2\} \max\{|F|_1, |G|_1\})$$

for $n \in \{2, \dots, m\}$.

The following example shows that the gap-condition (4.2) is sharp, i.e. the invariant fiber bundle S from Theorem 3.5(a) is not C^m in general, even if the non-linearities F and G are C^∞-functions.

Example 4.2. The two-dimensional autonomous difference equation

$$\begin{cases} x' = ex, \\ y' = e^m y + e^m x^m \Theta_\rho(x^2 + y^2), \end{cases} \tag{4.3}$$

satisfies Hypothesis 3.1 with $\alpha = e$, $\beta = e^m$ and $K_1 = K_2 = 1$, where $\Theta_\rho: [0,\infty) \to [0,1]$ is a C^∞-cut-off-function with $\Theta_\rho(t) = 1$ for $t \in [0,\rho]$ and $\Theta_\rho(t) = 0$ for $t \in [2\rho,\infty)$. Here we choose the real constant $\rho > 0$ small enough such that condition (3.5) is satisfied with

$$\sigma_{\max} = \min\left\{ \frac{\beta-\alpha}{2}, \alpha\left(\sqrt[m-1]{\frac{\alpha+\beta}{\alpha+\alpha^{m-1}}} - 1 \right) \right\}.$$

Now for every $c \in \mathbb{R}$ the sets

$$S_c := \{(\xi,\eta) \in B_\rho(0,0) \setminus \{(0,0)\} : \eta = (\xi^m/2)\ln \xi^2 + c\xi^m\} \cup \{(0,0)\}$$

contain the origin and are positively invariant with respect to (4.3), i.e. $\mathbb{Z} \times S_c$ is an invariant fiber bundle. Additionally, each point $(\xi,\eta) \in B_\rho(0,0)$, $\xi \ne 0$, is contained in exactly one of the sets S_c, namely for $c = \eta/\xi^m - \ln \xi^2/2$. Hence the restriction of the pseudo-stable fiber bundle S from Theorem 4.1 to $\mathbb{Z} \times B_\rho(0,0)$ has the form $\mathbb{Z} \times S_c$ for some $c \in \mathbb{R}$. On the other hand, each S_c is a graph of a C^{m-1}-function $s_c(\xi) = \eta$, but s_c fails to be m-times continuously differentiable. Note that in the present example the gap-condition $\alpha < \beta^{m_s}$ is only fulfilled for $m_s \in \{1, \dots, m-1\}$. A similar example demonstrating this smoothness deficiency for the pseudo-unstable fiber bundle R can be found in [2, Example 4.13].

Remark 4.3. Hypothesis 3.1(b) on the non-linearities can be relaxed in the way that the partial derivatives of F and G of order 2 to up to m may be allowed to grow exponentially in k. More precisely, if for each integer $n \in \{2, \dots, m\}$ we assume that for $k \in I$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $p \in \mathcal{P}$ the estimates

$$\left\| \frac{\partial^n F}{\partial(x,y)^n}(k,x,y,p) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{X})} < M\gamma_n^{|k|},$$

$$\left\| \frac{\partial^n G}{\partial(x,y)^n}(k,x,y,p) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{Y})} < M\gamma_n^{|k|}$$

hold with positive constants $M, \gamma_2, \dots, \gamma_n$, then Theorem 4.1 is true provided a stronger gap-condition holds which becomes more and more restrictive as the growth rates $\gamma_2, \dots, \gamma_n$ become larger. This can be seen along the lines of the following proof of Theorem 4.1. One has to balance the growth rates of the evolution operators $\Phi(k,l)$ and $\Psi(k,l)$ with the growth rates $\gamma_2, \dots, \gamma_n$ of the non-linearities.

Proof of Theorem 4.1. (a) Since the proof is quite involved we subdivide it into six steps and use the conventions and notation from the proof of Theorem 3.5. We choose $\kappa \in I$.

Step 1. Let $\gamma \in [\alpha+\sigma, \beta-\sigma]$ and $\xi \in \mathcal{X}$, $p \in \mathcal{P}$ be arbitrary. By formal differentiation of the fixed point equation (3.20) with respect to $\xi \in \mathcal{X}$ using the higher order chain rule from Theorem 2.1, we obtain another fixed point equation

$$\begin{pmatrix} \mu_\kappa^l \\ \nu_\kappa^l \end{pmatrix}(\xi,p) = \mathcal{T}_\kappa^l((\mu_\kappa^l,\nu_\kappa^l)(\xi,p);\xi,p) \tag{4.4}$$

for the formal partial derivative $(\mu_\kappa^l, \nu_\kappa^l)$ of

$$(\mu_\kappa,\nu_\kappa)\colon \mathcal{X}\times\mathcal{P} \to \ell_{\kappa,\gamma}^+(\mathcal{X}\times\mathcal{Y})$$

of order $l \in \{2, \dots, m_s\}$, where the right-hand side of (4.4) is given by

$$\begin{aligned} &(\mathcal{T}_\kappa^l(\mu^l,\nu^l;\xi,p))(k) \\ &:= \begin{pmatrix} \displaystyle\sum_{n=\kappa}^{k-1} \Phi(k,n+1)\left[\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\begin{pmatrix}\mu^l\\ \nu^l\end{pmatrix}(n) + R_1^l(n,\xi,p)\right] \\ \displaystyle -\sum_{n=k}^{\infty} \Psi(k,n+1)\left[\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\begin{pmatrix}\mu^l\\ \nu^l\end{pmatrix}(n) + R_2^l(n,\xi,p)\right] \end{pmatrix} \end{aligned} \tag{4.5}$$

for $k \in \mathbb{Z}_\kappa^+$. Here (μ^l, ν^l) is a mapping from $\mathbb{Z}_\kappa^+$ to $\mathcal{L}_l(\mathcal{X}; \mathcal{X} \times \mathcal{Y})$. The remainder $R^l = (R_1^l, R_2^l)$ has the following two representations as a partially unfolded derivative tree

$$(4.6)\quad R^l(n,\xi,p) \overset{(2.2)}{=} \sum_{j=1}^{l-1} \binom{l-1}{j} \frac{\partial^j}{\partial \xi^j} \left[\frac{\partial(F,G)}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \right] \begin{pmatrix} \mu_\kappa^{l-j} \\ \nu_\kappa^{l-j} \end{pmatrix} (n;\xi,p),$$

which is appropriate for the induction in the subsequent Step 4, and as a totally unfolded derivative tree

$$(4.7)\quad R^l(n,\xi,p) \overset{(2.3)}{=} \sum_{j=2}^{l} \sum_{(N_1,\dots,N_j)\in P_j^<(l)} \frac{\partial^j(F,G)}{\partial(x,y)^j}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p) \cdot \begin{pmatrix} \mu_\kappa^{\#N_1} \\ \nu_\kappa^{\#N_1} \end{pmatrix}(n;\xi,p)\dots\begin{pmatrix} \mu_\kappa^{\#N_j} \\ \nu_\kappa^{\#N_j} \end{pmatrix}(n;\xi,p),$$

which enables us to obtain explicit global bounds for the higher order derivatives in Step 2. For our forthcoming considerations it is crucial that R^l does not depend on $(\mu_\kappa^l, \nu_\kappa^l)$. In the following steps we will solve the fixed point equation (4.4) for the operator $\mathcal{T}_\kappa^l$. As a preparation we define for every $l \in \{1, \dots, m_s\}$ the abbreviations

$$\gamma_l := \max\{\alpha+\sigma, (\alpha+\sigma)^l\} = \begin{cases} \alpha+\sigma & \text{if } \alpha+\sigma<1, \\ (\alpha+\sigma)^l & \text{if } \alpha+\sigma \geq 1. \end{cases}$$

Because of the gap-condition (4.2) and with our choice of $\sigma_{\max}$, it is easy to see that one has the inclusion $\gamma_1, \dots, \gamma_{m_s} \in [\alpha+\sigma, \beta-\sigma)$, which in case $\alpha+\sigma<1$ follows from $\sigma < (\beta-\alpha)/2$ and otherwise essentially results from $(\alpha+\sigma)^{m_s} < \beta - \sigma$, which in turn is implied by

$$\begin{aligned}(\alpha+\sigma)^{m_s} + \alpha + \sigma &= \alpha^{m_s}\left(1+\frac{\sigma}{\alpha}\right)^{m_s} + \alpha\left(1+\frac{\sigma}{\alpha}\right) \\ &\leq \alpha^{m_s}\left(1+\frac{\sigma}{\alpha}\right)^{m_s} + \alpha\left(1+\frac{\sigma}{\alpha}\right)^{m_s} < \alpha+\beta\end{aligned}$$

if $\sigma < \alpha(\sqrt[m_s]{(\alpha+\beta)/(\alpha+\alpha^{m_s})} - 1)$.

Now we formulate for $\overline{m} \in \{1, \dots, m_s\}$ the induction hypotheses

$A(\overline{m})$: For any $l \in \{1, \dots, \overline{m}\}$ and $\gamma \in [\gamma_l, \beta-\sigma)$ the operator $\mathcal{T}_\kappa^l \colon \ell_{\kappa,\gamma}^l \times \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^l$ satisfies:

(a) It is well-defined,

(b) $\mathcal{T}_\tau^l(\,\cdot\,;\xi,p)$ is a uniform contraction in $\xi \in \mathcal{X}$, $p \in \mathcal{P}$,

(c) the unique fixed point $(\mu_\kappa^l, \nu_\kappa^l)(\,\cdot\,;\xi,p) = (\mu_\kappa^l, \nu_\kappa^l)(\xi,p)$ of $\mathcal{T}_\kappa^l(\,\cdot\,;\xi,p)$ is globally bounded in the γ_l^+-norm

$$\left\| \begin{pmatrix} \mu_\kappa^l \\ \nu_\kappa^l \end{pmatrix} (n;\xi,p) \right\| \le C_l \gamma_l^{n-\kappa} \quad \text{for } n \in \mathbb{Z}_\kappa^+,\ \xi \in \mathcal{X},\ p \in \mathcal{P}$$

with the constants $C_l \ge 0$ given in (4.2),

(d) if $\gamma_l < \gamma$ then $(\mu_\kappa^{l-1}, \nu_\kappa^{l-1})\colon \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^l$ is continuously partially differentiable with respect to $\xi \in \mathcal{X}$ with derivative

$$\frac{\partial}{\partial \xi} \begin{pmatrix} \mu_\kappa^{l-1} \\ \nu_\kappa^{l-1} \end{pmatrix} = \begin{pmatrix} \mu_\kappa^l \\ \nu_\kappa^l \end{pmatrix} \colon \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^l.$$

For $\overline{m} = 1$ the proof of Theorem 3.5 implies the induction hypothesis $A(1)$ with $C_1 = K_1/(1-L)$ (cf. (3.25)).

Now we assume $A(\overline{m}-1)$ for an $\overline{m} \in \{2, \dots, m_s\}$ and we will prove $A(\overline{m})$ in the following five steps.

Step 2. For every $\gamma \in [\gamma_{\overline{m}}, \beta - \sigma)$ the operator $\mathcal{T}_\kappa^{\overline{m}}\colon \ell_{\kappa,\gamma}^{\overline{m}} \times \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^{\overline{m}}$ is well-defined and satisfies the estimate

$$\begin{aligned}
(4.8)\quad \|\mathcal{T}_\kappa^{\overline{m}}(\mu^{\overline{m}}, \nu^{\overline{m}};\xi,p)\|_{\kappa,\gamma}^+ &\le L \left\| \begin{pmatrix} \mu^{\overline{m}} \\ \nu^{\overline{m}} \end{pmatrix} \right\|_{\kappa,\gamma}^+ \\
&+ \max\Bigg\{ \frac{K_1}{\sigma} \sum_{j=2}^{\overline{m}} |F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})} \prod_{i=1}^j C_{\# N_i}, \\
&\qquad \frac{K_2}{\sigma} \sum_{j=2}^{\overline{m}} |G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})} \prod_{i=1}^j C_{\# N_i} \Bigg\}
\end{aligned}$$

for $(\mu^{\overline{m}}, \nu^{\overline{m}}) \in \ell_{\kappa,\gamma}^{\overline{m}}$, $\xi \in \mathcal{X}$, $p \in \mathcal{P}$, i.e. $A(\overline{m})$(a) *holds.*

Let $l \in \{2, \dots, \overline{m}\}$, $\xi \in \mathcal{X}$, $p \in \mathcal{P}$ be arbitrary and choose $\gamma \in [\gamma_l, \beta - \sigma)$. Using the estimate $\gamma_{\# N_1} \cdot s\gamma_{\# N_j} \le \gamma_l$ for any ordered partition $(N_1, \dots, N_j) \in P_j^<(l)$ of length $j \in \{2, \dots, l\}$, from (3.2), (3.4) and $A(\overline{m}-1)$(c) we obtain the inequality

$$\begin{aligned}
(4.9)\quad & \|R^l(k,\xi,p)\| \\
&\overset{(4.7)}{\le} \max\Bigg\{ K_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \sum_{j=2}^l |F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)} \prod_{i=1}^j C_{\# N_i} \gamma_{\# N_i}^{n-\kappa}, \\
&\qquad K_2 \sum_{n=k}^{\infty} \beta^{k-n-1} \sum_{j=2}^l |G|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)} \prod_{i=1}^j C_{\# N_i} \gamma_{\# N_i}^{n-\kappa} \Bigg\} \\
&\le \max\Bigg\{ K_1 \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \gamma_l^{n-\kappa} \sum_{j=2}^l |F|_j \sum_{(N_1,\dots,N_j)\in P_j^<(l)} \prod_{i=1}^j C_{\# N_i},
\end{aligned}$$

$$K_2\sum_{n=k}^{\infty}\beta^{k-n-1}\gamma_l^{n-\kappa}\sum_{j=2}^{l}|G|_j\sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j}C_{\#N_i}\Big\}$$

$$\le\max\Big\{\frac{K_1}{\gamma_l-\alpha}\sum_{j=2}^{l}|F|_j\sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j}C_{\#N_i},$$

$$\frac{K_2}{\beta-\gamma_l}\sum_{j=2}^{l}|G|_j\sum_{(N_1,\dots,N_j)\in P_j^<(l)}\prod_{i=1}^{j}C_{\#N_i}\Big\}\gamma^{k-\kappa}$$

for $k\in\mathbb{Z}_\kappa^+$. Now let $\gamma\in[\gamma_{\overline{m}},\beta-\sigma)$ be arbitrary but fixed, and $(\mu^{\overline{m}},\nu^{\overline{m}})\in\ell_{\kappa,\gamma}^{\overline{m}}$. With the aid of the above estimate (4.9) we obtain

$$\|\mathcal{T}_\kappa^{\overline{m}}(\mu^{\overline{m}},\nu^{\overline{m}};\xi,p)(k)\|\gamma^{\kappa-k} \tag{4.10}$$

$$\overset{(4.5)}{\le}\max\Big\{K_1|F|_1\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\gamma^{n-\kappa}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}$$

$$+\frac{K_1}{\gamma_{\overline{m}}-\alpha}\sum_{j=2}^{\overline{m}}|F|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})}\prod_{i=1}^{j}C_{\#N_i}\gamma^{k-\kappa},$$

$$K_2|G|_1\sum_{n=k}^{\infty}\beta^{k-n-1}\gamma^{n-\kappa}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}$$

$$+\frac{K_2}{\beta-\gamma_{\overline{m}}}\sum_{j=2}^{\overline{m}}|G|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})}\prod_{i=1}^{j}C_{\#N_i}\gamma^{k-\kappa}\Big\}\gamma^{\kappa-k}$$

$$\le\max\Big\{\frac{K_1|F|_1}{\gamma-\alpha}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}$$

$$+\frac{K_1}{\gamma_{\overline{m}}-\alpha}\sum_{j=2}^{\overline{m}}|F|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})}\prod_{i=1}^{j}C_{\#N_i},$$

$$\frac{K_2|G|_1}{\beta-\gamma}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}$$

$$+\frac{K_2}{\beta-\gamma_{\overline{m}}}\sum_{j=2}^{\overline{m}}|G|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})}\prod_{i=1}^{j}C_{\#N_i}\Big\}$$

$$\overset{(3.14)}{\le}L\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}$$

$$+\max\Big\{\frac{K_1}{\gamma_{\overline{m}}-\alpha}\sum_{j=2}^{\overline{m}}|F|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})}\prod_{i=1}^{j}C_{\#N_i},$$

$$\frac{K_2}{\beta-\gamma_{\overline{m}}}\sum_{j=2}^{\overline{m}}|G|_j\sum_{(N_1,\dots,N_j)\in P_j^<(\overline{m})}\prod_{i=1}^{j}C_{\#N_i}\Big\}$$

for $k\in\mathbb{Z}_\kappa^+$, and passing over to the least upper bound over $k\in\mathbb{Z}_\kappa^+$ implies our claim $\mathcal{T}_\kappa^{\overline{m}}(\mu^{\overline{m}},\nu^{\overline{m}};\xi,p)\in\ell_{\kappa,\gamma}^{\overline{m}}$. In particular the estimate (4.8) is a consequence of (4.10) and the choice of $\gamma_{\overline{m}}\in[\alpha+\sigma,\beta-\sigma)$.

Step 3. For every $\gamma\in[\gamma_{\overline{m}},\beta-\sigma)$ the operator $\mathcal{T}_\kappa^{\overline{m}}(\,\cdot\,;\xi,p)\colon\ell_{\kappa,\gamma}^{\overline{m}}\to\ell_{\kappa,\gamma}^{\overline{m}}$ is a uniform contraction in $\xi\in\mathcal{X}$, $p\in\mathcal{P}$. Moreover, the fixed point $(\mu_\kappa^{\overline{m}},\nu_\kappa^{\overline{m}})(\xi,p)\in\ell_{\kappa,\gamma}^{\overline{m}}$ does not depend on $\gamma\in[\gamma_{\overline{m}},\beta-\sigma)$ and satisfies

$$\left\|\begin{pmatrix}\mu_\kappa^{\overline{m}}\\ \nu_\kappa^{\overline{m}}\end{pmatrix}(\xi,p)\right\|_{\kappa,\gamma}^{+}\le C_{\overline{m}}\quad\textit{for } \xi\in\mathcal{X},\ p\in\mathcal{P},\tag{4.11}$$

i.e. $A(\overline{m})$(b) *and* (c) *holds.*

Choose $\gamma\in[\gamma_{\overline{m}},\beta-\sigma)$ arbitrarily but fixed, and let $(\mu^{\overline{m}},\nu^{\overline{m}}),(\overline{\mu}^{\overline{m}},\overline{\nu}^{\overline{m}})\in\ell_{\kappa,\gamma}^{\overline{m}}$, $\xi\in\mathcal{X}$, $p\in\mathcal{P}$. Keeping in mind that the remainder $R^{\overline{m}}$ does not depend on $(\mu^{\overline{m}},\nu^{\overline{m}})$ or $(\overline{\mu}^{\overline{m}},\overline{\nu}^{\overline{m}})$, respectively, from (3.2) and (3.4) we obtain the Lipschitz estimate

$$\begin{aligned}
&\|\mathcal{T}_\kappa^{\overline{m}}(\mu^{\overline{m}},\nu^{\overline{m}};\xi,p)(k)-\mathcal{T}_\kappa^{\overline{m}}(\overline{\mu}^{\overline{m}},\overline{\nu}^{\overline{m}};\xi,p)(k)\|\gamma^{\kappa-k}\\
&\quad\overset{(4.5)}{\le}\max\Big\{K_1|F|_1\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}(n)-\begin{pmatrix}\overline{\mu}^{\overline{m}}\\ \overline{\nu}^{\overline{m}}\end{pmatrix}(n)\Big\|,\\
&\qquad K_2|G|_1\sum_{n=k}^{\infty}\beta^{k-n-1}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}(n)-\begin{pmatrix}\overline{\mu}^{\overline{m}}\\ \overline{\nu}^{\overline{m}}\end{pmatrix}(n)\Big\|\Big\}\gamma^{\kappa-k}\\
&\quad\le\max\Big\{K_1|F|_1\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\gamma^{n-\kappa},K_2|G|_1\sum_{n=k}^{\infty}\beta^{k-n-1}\gamma^{n-\kappa}\Big\}\\
&\qquad\cdot\gamma^{\kappa-k}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}-\begin{pmatrix}\overline{\mu}^{\overline{m}}\\ \overline{\nu}^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}\\
&\quad\le\max\Big\{\frac{K_1|F|_1}{\gamma-\alpha},\frac{K_2|G|_1}{\beta-\gamma}\Big\}\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}-\begin{pmatrix}\overline{\mu}^{\overline{m}}\\ \overline{\nu}^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}\\
&\quad\overset{(3.14)}{\le}L\Big\|\begin{pmatrix}\mu^{\overline{m}}\\ \nu^{\overline{m}}\end{pmatrix}-\begin{pmatrix}\overline{\mu}^{\overline{m}}\\ \overline{\nu}^{\overline{m}}\end{pmatrix}\Big\|_{\kappa,\gamma}^{+}
\end{aligned}$$

for $k\in\mathbb{Z}_\kappa^+$, and passing over to the least upper bound over $k\in\mathbb{Z}_\kappa^+$ together with (3.10) implies our claim. Therefore Banach's fixed point theorem guarantees the unique existence of a fixed point $(\mu_\kappa^{\overline{m}},\nu_\kappa^{\overline{m}})(\xi,p)\in\ell_{\kappa,\gamma}^{\overline{m}}$ of the mapping $\mathcal{T}_\kappa^{\overline{m}}(\,\cdot\,;\xi,p)\colon\ell_{\kappa,\gamma}^{\overline{m}}\to\ell_{\kappa,\gamma}^{\overline{m}}$. It can be seen along the same lines as in Step 4 in the proof of Theorem 3.5 that $(\mu_\kappa^{\overline{m}},\nu_\kappa^{\overline{m}})(\xi,p)$ does not depend on $\gamma\in[\gamma_{\overline{m}},\beta-\sigma)$. The fixed point identity (4.4) for $(\mu_\kappa^{\overline{m}},\nu_\kappa^{\overline{m}})(\xi,p)$ together with (4.8) and (3.10) finally implies (4.11).

Step 4. For every $\gamma \in (\gamma_{\overline{m}}, \beta - \sigma)$ and $p \in \mathcal{P}$ the mapping

$$(\mu_\kappa^{\overline{m}-1}, \nu_\kappa^{\overline{m}-1})(\,\cdot\,, p)\colon \mathcal{X} \to \ell_{\kappa,\gamma}^{\overline{m}}$$

is differentiable with derivative

$$\frac{\partial}{\partial \xi}\begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} = \begin{pmatrix} \mu_\kappa^{\overline{m}} \\ \nu_\kappa^{\overline{m}} \end{pmatrix} : \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^{\overline{m}}. \tag{4.12}$$

Let $\gamma \in (\gamma_{\overline{m}}, \beta-\sigma)$ and $p \in \mathcal{P}$ be fixed. First we show that $(\mu_\kappa^{\overline{m}-1}, \nu_\kappa^{\overline{m}-1})(\,\cdot\,, p)$ is differentiable and then we prove that the derivative is given by

$$(\mu_\kappa^{\overline{m}}, \nu_\kappa^{\overline{m}})(\,\cdot\,, p)\colon \mathcal{X} \to \mathcal{L}(\mathcal{X}; \ell_{\kappa,\gamma}^{\overline{m}-1}) \cong \ell_{\kappa,\gamma}^{\overline{m}}$$

(cf. Lemma 2.2(c)). Thereto choose $\xi \in \mathcal{X}$ arbitrarily, but fixed. From now on for the rest of the proof of the present Step 4 we suppress the p-dependence of the mappings under consideration; nevertheless $p \in \mathcal{P}$ is arbitrary. Using the fixed point equation (4.4) for $(\mu_\kappa^{\overline{m}-1}, \nu_\kappa^{\overline{m}-1})$ we get for $h \in \mathcal{X}$ the identity

$$\begin{aligned}
&\begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix}(k; \xi + h) - \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix}(k; \xi) \\
&\overset{(4.5)}{=} \begin{pmatrix} \displaystyle\sum_{n=\kappa}^{k-1} \Phi(k, n+1)\Big[\frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi + h)) \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix}(n; \xi + h) + R_1^{\overline{m}-1}(n, \xi + h)\Big] \\
\displaystyle -\sum_{n=k}^{\infty} \Psi(k, n+1)\Big[\frac{\partial G}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi + h)) \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix}(n; \xi + h) + R_2^{\overline{m}-1}(n, \xi + h)\Big] \end{pmatrix} \\
&\quad - \begin{pmatrix} \displaystyle\sum_{n=\kappa}^{k-1} \Phi(k, n+1)\Big[\frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi)) \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix}(n; \xi) + R_1^{\overline{m}-1}(n, \xi)\Big] \\
\displaystyle -\sum_{n=k}^{\infty} \Psi(k, n+1)\Big[\frac{\partial G}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi)) \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix}(n; \xi) + R_2^{\overline{m}-1}(n, \xi)\Big] \end{pmatrix}
\end{aligned}$$

for $k \in \mathbb{Z}_\kappa^+$.

This leads to

$$\begin{aligned}
(4.13)\qquad &\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(k;\xi+h)-\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(k;\xi)\\
&-\begin{pmatrix}\displaystyle\sum_{n=\kappa}^{k-1}\Phi(k,n+1)\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h))\\
\qquad\cdot\left[\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi)\right]\\
\displaystyle -\sum_{n=k}^{\infty}\Psi(k,n+1)\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h))\\
\qquad\cdot\left[\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi)\right]\end{pmatrix}\\
&=\begin{pmatrix}\displaystyle\sum_{n=\kappa}^{k-1}\Phi(k,n+1)\left[\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h))\right.\\
\qquad\left.-\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi))\right]\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)\\
\displaystyle -\sum_{n=k}^{\infty}\Psi(k,n+1)\left[\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h))\right.\\
\qquad\left.-\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi))\right]\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)\end{pmatrix}\\
&+\begin{pmatrix}\displaystyle\sum_{n=\kappa}^{k-1}\Phi(k,n+1)[R_1^{\overline{m}-1}(n,\xi+h)-R_1^{\overline{m}-1}(n,\xi)]\\
\displaystyle -\sum_{n=k}^{\infty}\Psi(k,n+1)[R_2^{\overline{m}-1}(n,\xi+h)-R_2^{\overline{m}-1}(n,\xi)]\end{pmatrix}
\end{aligned}$$

for $k\in\mathbb{Z}_\kappa^+$. With sequences $(\mu^{\overline{m}-1},\nu^{\overline{m}-1})\in\ell_{\kappa,\gamma}^{\overline{m}-1}$ and $h\in\mathcal{X}$ we define the operators $\mathcal{K}\in\mathcal{L}(\ell_{\kappa,\gamma}^{\overline{m}-1})$, $\mathcal{E}\in\mathcal{L}(\mathcal{X};\ell_{\kappa,\gamma}^{\overline{m}-1})$, $\mathcal{J}\colon\mathcal{X}\to\ell_{\kappa,\gamma}^{\overline{m}-1}$ as follows

$$\begin{aligned}
&\left(\mathcal{K}\begin{pmatrix}\mu^{\overline{m}-1}\\ \nu^{\overline{m}-1}\end{pmatrix}\right)(k)\\
&:=\begin{pmatrix}\displaystyle\sum_{n=\kappa}^{k-1}\Phi(k,n+1)\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi))\begin{pmatrix}\mu^{\overline{m}-1}\\ \nu^{\overline{m}-1}\end{pmatrix}(n)\\
\displaystyle -\sum_{n=k}^{\infty}\Psi(k,n+1)\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi))\begin{pmatrix}\mu^{\overline{m}-1}\\ \nu^{\overline{m}-1}\end{pmatrix}(n)\end{pmatrix},
\end{aligned}$$

$$(\mathcal{E}h)(k) := \begin{pmatrix} \sum_{n=\kappa}^{k-1} \Phi(k,n+1) R_1^{\overline{m}}(n,\xi)h \\ -\sum_{n=k}^{\infty} \Psi(k,n+1) R_2^{\overline{m}}(n,\xi)h \end{pmatrix}$$

and

(4.14) $(\mathcal{J}(h))(k)$

$$:= \begin{pmatrix} \begin{aligned} &\sum_{n=\kappa}^{k-1} \Phi(k,n+1) \bigg\{ \bigg[\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h)) \\ &\qquad - \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi)) \bigg] \\ &\cdot \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (n;\xi+h) + R_1^{\overline{m}-1}(n,\xi+h) - R_1^{\overline{m}-1}(n,\xi) - R_1^{\overline{m}}(n,\xi)h \bigg\} \\[2ex] &-\sum_{n=k}^{\infty} \Psi(k,n+1) \bigg\{ \bigg[\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h)) \\ &\qquad - \frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi)) \bigg] \\ &\cdot \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (n;\xi+h) + R_2^{\overline{m}-1}(n,\xi+h) - R_2^{\overline{m}-1}(n,\xi) - R_2^{\overline{m}}(n,\xi)h \bigg\} \end{aligned} \end{pmatrix}$$

for $k \in \mathbb{Z}_\kappa^+$. In the subsequent lines we will show that $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{J}$ are well-defined. Using (3.2) and (3.4) it is easy to see that $\mathcal{K}\colon \ell_{\kappa,\gamma}^{\overline{m}-1} \to \ell_{\kappa,\gamma}^{\overline{m}-1}$ is linear and satisfies the estimate

$$\bigg\| \mathcal{K} \begin{pmatrix} \mu^{\overline{m}-1} \\ \nu^{\overline{m}-1} \end{pmatrix} \bigg\|_{\kappa,\gamma}^{+} \le \max\left\{ \frac{K_1|F|_1}{\gamma-\alpha}, \frac{K_2|G|_1}{\beta-\gamma} \right\} \bigg\| \begin{pmatrix} \mu^{\overline{m}-1} \\ \nu^{\overline{m}-1} \end{pmatrix} \bigg\|_{\kappa,\gamma}^{+} \overset{(3.14)}{\le} L \bigg\| \begin{pmatrix} \mu^{\overline{m}-1} \\ \nu^{\overline{m}-1} \end{pmatrix} \bigg\|_{\kappa,\gamma}^{+},$$

which in turn gives us

$$\|\mathcal{K}\|_{\mathcal{L}(\ell_{\kappa,\gamma}^{\overline{m}-1})} \overset{(3.10)}{<} 1. \tag{4.15}$$

Keeping in mind that $\mathcal{E}h = \mathcal{T}_\kappa^{\overline{m}}(0;\xi,p)h$ (cf. (4.5)), our Step 2 yields the inclusion $\mathcal{E}h \in \ell_{\kappa,\gamma}^{\overline{m}-1}$, while $\mathcal{E}$ is obviously linear and continuous, hence $\mathcal{E} \in \mathcal{L}(\mathcal{X};\ell_{\kappa,\gamma}^{\overline{m}-1})$. Arguments similar to those in Step 2, together with (4.9), lead to $\mathcal{J}(h) \in \ell_{\kappa,\gamma}^{\overline{m}-1}$ for any $h \in \mathcal{X}$. Because of (4.13) we obtain

$$\bigg[\begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi+h) - \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi) \bigg] - \mathcal{K} \bigg[\begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi+h) - \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi) \bigg] = \mathcal{E}h + \mathcal{J}(h)$$

for $h \in \mathcal{X}$. Using the Neumann series (cf. [12, Theorem 2.1, p. 74]) and the estimate (4.15), the linear mapping $I_{\ell_{\kappa,\gamma}^{\overline{m}-1}} - \mathcal{K} \in \mathcal{L}(\ell_{\kappa,\gamma}^{\overline{m}-1})$ is invertible and this implies

$$\begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi + h) - \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi) = [I_{\ell_{\kappa,\gamma}^{\overline{m}-1}} - \mathcal{K}]^{-1}[\mathcal{E}h + \mathcal{J}(h)] \quad \text{for } h \in \mathcal{X}.$$

Consequently it remains to show $\lim_{h\to 0} \mathcal{J}(h)/\|h\| = 0$ in $\ell_{\kappa,\gamma}^{\overline{m}-1}$, because then one gets

$$\lim_{h\to 0} \frac{1}{\|h\|} \left\| \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi + h) - \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (\xi) - [I_{\ell_{\kappa,\gamma}^{\overline{m}-1}} - \mathcal{K}]^{-1}\mathcal{E}h \right\|_{\kappa,\gamma}^{+} = 0,$$

i.e. the claim of the present Step 4 follows.

Nevertheless the proof of $\lim_{h\to 0} \|\mathcal{J}(h)\|_{\kappa,\gamma}^{+}/\|h\| = 0$ needs a certain technical effort. Thereto we use the fact that due to the induction hypothesis $A(\overline{m}-1)(d)$ the remainder

$$R^{\overline{m}-1}(n,\xi) \overset{(4.6)}{=} \sum_{j=1}^{\overline{m}-2} \begin{pmatrix} \overline{m}-2 \\ j \end{pmatrix} \frac{\partial^j}{\partial \xi^j} \left[\frac{\partial(F,G)}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n;\xi)) \right] \begin{pmatrix} \mu_\kappa^{\overline{m}-1-j} \\ \nu_\kappa^{\overline{m}-1-j} \end{pmatrix} (n;\xi)$$

is partially differentiable with respect to $\xi \in \mathcal{X}$, where the derivative is given by

$$\begin{aligned} \frac{\partial R^{\overline{m}-1}}{\partial \xi}(n,\xi) &\overset{(4.6)}{=} R^{\overline{m}}(n,\xi) \\ &\quad - \frac{\partial^2(F,G)}{\partial(x,y)^2}(n, (\mu_\kappa, \nu_\kappa)(n;\xi)) \begin{pmatrix} \mu_\kappa^{1} \\ \nu_\kappa^{1} \end{pmatrix} (n;\xi) \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (n;\xi). \end{aligned}$$

Using the abbreviation

$$\begin{aligned} \Delta R^{\overline{m}-1}(n,\xi,h) &:= \frac{1}{\|h\|} \Bigg\{ R^{\overline{m}-1}(n,\xi+h) - R^{\overline{m}-1}(n,\xi) \\ &- \left[R^{\overline{m}}(n,\xi) - \frac{\partial^2(F,G)}{\partial(x,y)^2}(n, (\mu_\kappa, \nu_\kappa)(n;\xi)) \begin{pmatrix} \mu_\kappa^{1} \\ \nu_\kappa^{1} \end{pmatrix} (n;\xi) \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (n;\xi) \right] h \Bigg\} \end{aligned}$$

we obtain the limit relation

$$\lim_{h\to 0} \Delta R^{\overline{m}-1}(n,\xi,h) = 0 \quad \text{for } n \in \mathbb{Z}_\kappa^+.$$

Now we prove estimates for the components $\mathcal{J}_1$ and $\mathcal{J}_2$ of $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2)$ separately. For $k \in \mathbb{Z}_\kappa^+$ we get

$$\begin{aligned} (\mathcal{J}_1(h))(k) \overset{(4.14)}{=} \sum_{n=\kappa}^{k-1} \Phi(k,n+1) \Bigg\{ & \left[\frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n;\xi+h)) \right. \\ & \left. - \frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n;\xi)) \right] \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (n;\xi+h) \\ & - \frac{\partial^2 F}{\partial(x,y)^2}(n, (\mu_\kappa, \nu_\kappa)(n;\xi)) \begin{pmatrix} \mu_\kappa^{1} \\ \nu_\kappa^{1} \end{pmatrix} (n;\xi) \\ & \cdot \begin{pmatrix} \mu_\kappa^{\overline{m}-1} \\ \nu_\kappa^{\overline{m}-1} \end{pmatrix} (n;\xi) h + \Delta R_1^{\overline{m}-1}(n,\xi,h)\|h\| \Bigg\}, \end{aligned}$$

where subtraction and addition of the expression

$$\frac{\partial^2 F}{\partial(x,y)^2}(n,(\mu_\kappa,\nu_\kappa)(n;\xi)) \cdot \left[\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi)-\begin{pmatrix}\mu_\kappa^1\\ \nu_\kappa^1\end{pmatrix}(n;\xi)h\right]\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)$$

leads to

$$\begin{aligned}
&(\mathcal{J}_1(h))(k) \\
&= \sum_{n=\kappa}^{k-1}\Phi(k,n+1)\bigg\{\bigg[\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi+h)) \\
&\quad - \frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi)) - \frac{\partial^2 F}{\partial(x,y)^2}(n,(\mu_\kappa,\nu_\kappa)(n;\xi)) \\
&\quad \cdot\left(\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi)\right)\bigg]\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h) \\
&\quad + \frac{\partial^2 F}{\partial(x,y)^2}(n,(\mu_\kappa,\nu_\kappa)(n;\xi)) \\
&\quad \cdot\left[\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi)-\begin{pmatrix}\mu_\kappa^1\\ \nu_\kappa^1\end{pmatrix}(n;\xi)h\right]\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h) \\
&\quad + \frac{\partial^2 F}{\partial(x,y)^2}(n,(\mu_\kappa,\nu_\kappa)(n;\xi))\begin{pmatrix}\mu_\kappa^1\\ \nu_\kappa^1\end{pmatrix}(n;\xi) \\
&\quad \cdot\left[\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi)\right]h+\Delta R_1^{\overline{m}-1}(n,\xi,h)\|h\|\bigg\}
\end{aligned}$$

for $k\in\mathbb{Z}_\kappa^+$. Using the quotient

$$\begin{aligned}
\Delta\frac{\partial F}{\partial(x,y)}(n,x,y,h_1,h_2) := \frac{1}{\|(h_1,h_2)\|}\bigg(&\frac{\partial F}{\partial(x,y)}(n,x+h_1,y+h_2) \\
&-\frac{\partial F}{\partial(x,y)}(n,x,y)-\frac{\partial^2 F}{\partial(x,y)^2}(n,x,y)\begin{pmatrix}h_1\\ h_2\end{pmatrix}\bigg)
\end{aligned}$$

for $n\in I$ and $x\in\mathcal{X}$, $y\in\mathcal{Y}$, $h_1\in\mathcal{X}\setminus\{0\}$ and $h_2\in\mathcal{Y}\setminus\{0\}$, we obtain the estimate

$$\begin{aligned}
&\|(\mathcal{J}_1(h))(k)\| \le \sum_{n=\kappa}^{k-1}\|\Phi(k,n+1)\| \\
&\quad\cdot\bigg[\bigg\|\Delta\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi),(\mu_\kappa,\nu_\kappa)(n;\xi+h)-(\mu_\kappa,\nu_\kappa)(n;\xi))\bigg\| \\
&\quad\cdot\left\|\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi)\right\|\left\|\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)\right\| \\
&\quad+\left\|\frac{\partial^2 F}{\partial(x,y)^2}(n,(\mu_\kappa,\nu_\kappa)(n;\xi))\right\| \\
&\quad\cdot\left\|\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi)-\begin{pmatrix}\mu_\kappa^1\\ \nu_\kappa^1\end{pmatrix}(n;\xi)h\right\|\left\|\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)\right\|
\end{aligned}$$

$$+\left\|\frac{\partial^2 F}{\partial(x,y)^2}(n,(\mu_\kappa,\nu_\kappa)(n;\xi))\right\|\left\|\begin{pmatrix}\mu_\kappa^1\\ \nu_\kappa^1\end{pmatrix}(n;\xi)\right\|$$
$$\cdot\left\|\left[\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi)\right]h\right\|+\|\Delta R_1^{\overline{m}-1}\|(n;\xi,h)\|h\|\Bigg]$$

for $k\in\mathbb{Z}_\kappa^+$. With Hypothesis 3.1(b) (cf. (3.2), (3.4)), the abbreviations (3.27) and the induction hypothesis $A(\overline{m}-1)$(c) we therefore get

$$\begin{aligned}\|(\mathcal{J}_1(h))(k)\|\le K_1\sum_{n=\kappa}^{k-1}\alpha^{k-n-1}\Bigg[&\left\|\Delta\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi),(\mu_\kappa,\nu_\kappa)(n;\xi+h)\right.\\ &\left.-(\mu_\kappa,\nu_\kappa)(n;\xi))\right\|\frac{1}{\|h\|}\\ &\cdot\left\|\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa\\ \nu_\kappa\end{pmatrix}(n;\xi)\right\|C_{\overline{m}-1}\gamma_{\overline{m}-1}^{n-\kappa}\\ &+|F|_2\left\|\begin{pmatrix}\Delta\mu_\kappa\\ \Delta\nu_\kappa\end{pmatrix}(n,h)\right\|C_{\overline{m}-1}\gamma_{\overline{m}-1}^{n-\kappa}\\ &+|F|_2C_1\gamma_1^{n-\kappa}\left\|\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi)\right\|\\ &+\|\Delta R_1^{\overline{m}-1}(n,\xi,h)\|\Bigg]\|h\|\end{aligned}$$

for $k\in\mathbb{Z}_\kappa^+$. Rewriting this estimate and using Lemma 3.3 we obtain

$$\begin{aligned}\frac{\|\mathcal{J}_1(h)\|_{\kappa,\gamma}^+}{\|h\|}\overset{(3.6)}{\le}&\frac{K_1^2C_{\overline{m}-1}}{\alpha\gamma_{\overline{m}-1}^\kappa}\frac{\gamma-\alpha}{\gamma-\alpha-K_1|F|_1}\sup_{k\in\mathbb{Z}_\kappa^+}V_1(k,h)\\ &+\frac{K_1|F|_2C_{\overline{m}-1}}{\alpha}\left(\frac{\gamma}{\gamma_{\overline{m}-1}}\right)^\kappa\sup_{k\in\mathbb{Z}_\kappa^+}V_2(k,h)\\ &+\frac{K_1|F|_2C_1}{\alpha}\left(\frac{\gamma}{\gamma_1}\right)^\kappa\sup_{k\in\mathbb{Z}_\kappa^+}V_3(k,h)+\frac{K_1}{\alpha}\gamma^\kappa\sup_{k\in\mathbb{Z}_\kappa^+}V_4(k,h)\end{aligned}$$

with

$$\begin{aligned}V_1(k,h):=&\left(\frac{\alpha}{\gamma}\right)^k\sum_{n=\kappa}^{k-1}\left(\frac{\gamma\gamma_{\overline{m}-1}}{\alpha}\right)^n\\ &\cdot\left\|\Delta\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi),(\mu_\kappa,\nu_\kappa)(n;\xi+h)-(\mu_\kappa,\nu_\kappa)(n;\xi))\right\|,\\ V_2(k,h):=&\left(\frac{\alpha}{\gamma}\right)^k\sum_{n=\kappa}^{k-1}\left(\frac{\gamma_{\overline{m}-1}}{\alpha}\right)^n\left\|\begin{pmatrix}\Delta\mu\\ \Delta\nu\end{pmatrix}(n,h)\right\|,\\ V_3(k,h):=&\left(\frac{\alpha}{\gamma}\right)^k\sum_{n=\kappa}^{k-1}\left(\frac{\gamma_1}{\alpha}\right)^n\left\|\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi+h)-\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi)\right\|,\\ V_4(k,h):=&\left(\frac{\alpha}{\gamma}\right)^k\sum_{n=\kappa}^{k-1}\alpha^{-n}\|\Delta R_1^{\overline{m}-1}(n,\xi,h)\|.\end{aligned}$$

Similarly to Step 4 in the proof of Theorem 3.5 we get

$$\lim_{h\to 0}\sup_{k\in\mathbb{Z}_\kappa^+} V_i(k,h) = 0 \quad \text{for } i \in \{1,\dots,4\},$$

proving that

$$\lim_{h\to 0}\frac{\|\mathcal{J}_1(h)\|_{\kappa,\gamma}^+}{\|h\|} = 0.$$

Completely analogous one shows

$$\lim_{h\to 0}\frac{\|\mathcal{J}_2(h)\|_{\kappa,\gamma}^+}{\|h\|} = 0$$

and therefore we have verified the differentiability of the mapping

$$(\mu_\kappa^{\overline{m}-1},\nu_\kappa^{\overline{m}-1})(\,\cdot\,,p)\colon \mathcal{X}\to \ell_{\kappa,\gamma}^{\overline{m}-1}$$

for any $p\in\mathcal{P}$. Finally we derive for any parameter $p\in\mathcal{P}$ that the derivative

$$\frac{\partial}{\partial\xi}\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(\,\cdot\,,p)\colon \mathcal{X}\to\mathcal{L}(\mathcal{X};\ell_{\kappa,\gamma}^{\overline{m}-1})\cong \ell_{\kappa,\gamma}^{\overline{m}}$$

is the fixed point mapping

$$(\mu_\kappa^{\overline{m}},\nu_\kappa^{\overline{m}})(\,\cdot\,,p)\colon \mathcal{X}\to \ell_{\kappa,\gamma}^{\overline{m}}$$

of $\mathcal{T}_\kappa^{\overline{m}}(\,\cdot\,;\,\cdot\,,p)$. From the fixed point equation (4.4) for $(\mu_\kappa^{\overline{m}-1},\nu_\kappa^{\overline{m}-1})$ we obtain by partial differentiation with respect to $\xi\in\mathcal{X}$ the identity

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(k;\xi,p)\\
&\overset{(4.5)}{=}\begin{pmatrix}\displaystyle\sum_{n=\kappa}^{k-1}\Phi(k,n+1)\frac{\partial F}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\frac{\partial}{\partial\xi}\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi,p)\\ \displaystyle -\sum_{n=k}^{\infty}\Phi(k,n+1)\frac{\partial G}{\partial(x,y)}(n,(\mu_\kappa,\nu_\kappa)(n;\xi,p),p)\frac{\partial}{\partial\xi}\begin{pmatrix}\mu_\kappa^{\overline{m}-1}\\ \nu_\kappa^{\overline{m}-1}\end{pmatrix}(n;\xi,p)\end{pmatrix}\\
&\quad+\begin{pmatrix}\displaystyle\sum_{n=\kappa}^{k-1}\Phi(k,n+1)R_1^{\overline{m}}(n,\xi,p)\\ \displaystyle -\sum_{n=k}^{\infty}\Psi(k,n+1)R_2^{\overline{m}}(n,\xi,p)\end{pmatrix}
\end{aligned}$$

for $k\in\mathbb{Z}_\kappa^+$. Hence the derivative $\partial(\mu_\kappa^{\overline{m}-1},\nu_\kappa^{\overline{m}-1})(\xi,p)/\partial\xi\in\mathcal{L}(\mathcal{X};\ell_{\kappa,\gamma}^{\overline{m}-1})\cong\ell_{\kappa,\gamma}^{\overline{m}}$ (cf. Lemma 2.2(c)) is a fixed point of $\mathcal{T}_\kappa^{\overline{m}}(\,\cdot\,;\xi,p)$, which in turn is unique by Step 3, and consequently (4.12) holds.

Step 5. For every $\gamma\in(\gamma_{\overline{m}},\beta-\sigma)$ the mapping

$$\frac{\partial^{\overline{m}}(\mu_\kappa,\nu_\kappa)}{\partial\xi^{\overline{m}}}\colon \mathcal{X}\times\mathcal{P}\to\ell_{\kappa,\gamma}^{\overline{m}}$$

is continuous, i.e. $A(\overline{m})(d)$ holds.

Because of (4.12) it suffices to prove the continuity of the mapping

$$(\mu_\kappa^{\overline{m}}, \nu_\kappa^{\overline{m}}) : \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^{\overline{m}}$$

and this is analogous to Step 1 in the proof of Theorem 3.5 by adding and subtracting the expressions

$$\frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi, p), p) \begin{pmatrix} \mu_\kappa^{\overline{m}} \\ \nu_\kappa^{\overline{m}} \end{pmatrix} (n; \xi_0, p_0),$$
$$\frac{\partial G}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \xi, p), p) \begin{pmatrix} \mu_\kappa^{\overline{m}} \\ \nu_\kappa^{\overline{m}} \end{pmatrix} (n; \xi_0, p_0),$$

in the corresponding estimates. Thus we have verified $A(\overline{m})$.

Step 6. In the preceding five steps we have shown that $(\mu_\kappa, \nu_\kappa) : \mathcal{X} \times \mathcal{P} \to \ell_{\kappa,\gamma}^{+}(\mathcal{X} \times \mathcal{P})$ is m_s-times continuously partially differentiable with respect to its first argument. With the identity $s(\kappa, \xi, p) = \nu_\kappa(\xi, p)(\kappa)$ the claim follows from properties of the evaluation map (see [2, Lemma 3.4]) and the global bound for the derivatives can be obtained using the fact

$$\left\| \frac{\partial^n s}{\partial \xi^n}(\kappa, \xi, p) \right\| = \left\| \frac{\partial^n \nu_\kappa}{\partial \xi^n}(\xi, p)(\kappa) \right\| \leq \|\nu_\kappa^n(\xi, p)\|_{\kappa,\gamma}^{+} \overset{(4.11)}{\leq} C_n \tag{4.16}$$

for $\xi \in \mathcal{X}$, $p \in \mathcal{P}$, and $n \in \{1, \dots, m_s\}$. Hereby the expression for C_1 is a consequence of (3.25).

(b) The smoothness proof of the mapping $r : I \times \mathcal{Y} \times \mathcal{P} \to \mathcal{X}$ is dual to the above considerations for s. A formal differentiation of the identity (3.40) with respect to $\eta \in \mathcal{Y}$ gives us a fixed point equation

$$(\mu_\kappa^l, \nu_\kappa^l)(\eta, p) = \overline{\mathcal{T}}_\kappa^l((\mu_\kappa^l, \nu_\kappa^l)(\eta, p); \eta, p)$$

with the right-hand side

$$(4.17) \quad (\overline{\mathcal{T}}_\kappa^l(\mu^l, \nu^l; \eta, p))(k)$$
$$:= \begin{pmatrix} \displaystyle\sum_{n=-\infty}^{k-1} \Phi(k, n+1) \left[\frac{\partial F}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \eta, p), p) \begin{pmatrix} \mu^l \\ \nu^l \end{pmatrix}(n) + \overline{R}_1^l(n, \eta, p) \right] \\ \displaystyle -\sum_{n=k}^{\kappa-1} \Psi(k, n+1) \left[\frac{\partial G}{\partial(x,y)}(n, (\mu_\kappa, \nu_\kappa)(n; \eta, p), p) \begin{pmatrix} \mu^l \\ \nu^l \end{pmatrix}(n) + \overline{R}_2^l(n, \eta, p) \right] \end{pmatrix}$$

for $k \in \mathbb{Z}_\kappa^-$ and parameters $p \in \mathcal{P}$, where the remainder $\overline{R}^l = (\overline{R}_1^l, \overline{R}_2^l)$ allows representations analogous to (4.6) and (4.7). We omit the further details.

(c) The recursion for the global bounds $C_n \geq 0$ of $\|\partial^n s(\kappa, \xi, p)/\partial \xi^n\|$ for $n \in \{2, \dots, m\}$, in (4.2) is an obvious consequence of the estimate (4.10) from Step 2 of part (a) in the present proof. A dual argument shows that the solution of the fixed point equation for (4.17) is globally bounded by C_n as well, and an estimate analogous to (4.16) gives us the global bounds for the partial derivatives of r. Hence we have shown the assertion (c) and the proof of Theorem 4.1 is finished. □

References

[1] B. Aulbach, *The fundamental existence theorem on invariant fiber bundles*, J. Differential Equations Appl. **3** (1998), 501–537.

[2] B. Aulbach, C. Pötzsche and S. Siegmund, *A smoothness theorem for invariant fiber bundles*, J. Dynam. Differential Equations **14** (2002), 519–547.

[3] B. Aulbach and T. Wanner, *Integral manifolds for Carathéodory type differential equations in Banach spaces*, Six Lectures on Dynamical Systems (B. Aulbach and F. Colonius, eds.), World Scientific, Singapore–New Jersey–London–Hong Kong, 1996, pp. 45–119.

[4] ______, *Invariant foliations and decoupling of nonautonomous difference equations*, J. Differential Equations Appl. (to appear).

[5] N. Castañeda and R. Rosa, *Optimal estimates for the uncoupling of differential equations*, J. Dynam. Differential Equations **8** (1996), 103–139.

[6] S.-N. Chow and K. Lu, *C^k centre unstable manifolds*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 303–320.

[7] M. S. ElBialy, *On pseudo-stable and pseudo-unstable manifolds for maps*, J. Math. Anal. Appl. **232** (1999), 229–258.

[8] N. Fenichel, *Hyperbolicity and exponential dichotomy for dynamical systems*, Dynamics Reported New Series (. U. Kirchgraber and H.-O. Walther, eds.), vol. 5, Springer, Berlin, 1996, pp. 1–25.

[9] S. Hilger, *Smoothness of invariant manifolds*, J. Funct. Anal. **106** (1992), 95–129.

[10] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics, vol. 583, Springer–Verlag, Berlin–Heidelberg–New York, 1977.

[11] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995.

[12] S. Lang, *Real and Functional Analysis*, Graduate Texts in Mathematics, vol. 142, Springer–Verlag, Berlin–Heidelberg–New York, 1993.

[13] S. Maier-Paape and T. Wanner, *Spinodal decomposition for the Cahn–Hilliard equation in higher dimensions: nonlinear dynamics*, Arch. Rational Mech. Anal. **151** (2000), 187–219.

[14] J. R. Munkres, *Topology – A First Course*, Prentice Hall, Eaglewood Cliffs, New Jersey, 1975.

[15] K. P. Rybakowski, *Formulas for higher-order finite expansions of composite maps*, The Mathematical Heritage of C. F. Gauß (G. M. Rassias,, ed.), World Scientific, Singapore–New Jersey–London–Hong Kong, 1991, pp. 652–669.

[16] ______, *An abstract approach to smoothness of invariant manifolds*, Appl. Anal. **49** (1993), 119–150.

[17] M. Shub, *Global Stability of Dynamical Systems*, Springer–Verlag, Berlin–Heidelberg–New York, 1987.

[18] S. Siegmund, *On the differentiability of integral manifolds*, Diploma thesis (1996), Universität Augsburg. (German)

[19] ______, *Spectral theory, smooth foliations and normal forms for differential equations of Carathéodory-type*, Ph. D. Dissertation (1999), Universität Augsburg. (German)

[20] A. Vanderbauwhede, *Center manifolds, normal forms and elementary bifurcations*, Dynamics Reported (U. Kirchgraber and H.-O. Walther, eds.), vol. 2, B. G. Teubner, Stuttgart, 1989, pp. 89–169.

[21] A. Vanderbauwhede and S.A. Van Gils, *Center manifolds and contractions on a scale of Banach spaces*, J. Funct. Anal. **72** (1987), 209–224.

[22] J.-C. Yoccoz, *Introduction to hyperbolic dynamics*, Real and Complex Dynamical Systems (R. Branner and P. Hjorth, eds.), Kluwer Academic Publishers, Dordrecht–Boston, 1995, pp. 265–291.

Manuscript received December 17, 2003

Christian Pötzsche
Institut für Mathematik
Universität Augsburg
86135 Augsburg, GERMANY

E-mail address: poetzsche@math.uni-augsburg.de

Stefan Siegmund
FB Mathematik
Johann Wolfgang Goethe-Universität
60325 Frankfurt am Main, GERMANY

E-mail address: siegmund@math.uni-frankfurt.de