

SINGULARLY PERTURBED NEUMANN PROBLEMS WITH POTENTIALS

ALESSIO POMPONIO

ABSTRACT. The main purpose of this paper is to study the existence of single-peaked solutions of the Neumann problem

$$\begin{cases} -\varepsilon^2 \operatorname{div}(J(x)\nabla u) + V(x)u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of \mathbb{R}^N , $N \geq 3$, $1 < p < (N + 2)/(N - 2)$ and J and V are positive bounded scalar value potentials. We will show that, for the existence of concentrating solutions, one has to check if at least one between J and V is not constant on $\partial\Omega$. In this case the concentration point is determined by J and V only. In the other case the concentration point is determined by an interplay among the derivatives of J and V calculated on $\partial\Omega$ and the mean curvature H of $\partial\Omega$.

1. Introduction

In this paper we study the following problem:

$$(1.1) \quad \begin{cases} -\varepsilon^2 \operatorname{div}(J(x)\nabla u) + V(x)u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

2000 *Mathematics Subject Classification.* 35B25, 35J20, 35J55.

Key words and phrases. Singularly perturbed Neumann problem, presence of potentials, concentrating solutions.

Supported by MIUR, national project *Variational methods and nonlinear differential equations.*

where Ω is a smooth bounded domain with external normal ν , $N \geq 3$, $1 < p < (N+2)/(N-2)$, $J: \mathbb{R}^N \rightarrow \mathbb{R}$ and $V: \mathbb{R}^N \rightarrow \mathbb{R}$ are C^2 functions.

When $J \equiv 1$ and $V \equiv 1$, then (1.1) becomes

$$(1.2) \quad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Such a problem was intensively studied in several works. For example, W. M. Ni and I. Takagi in [11], [12], show that, for ε sufficiently small, there exists a solution u_ε of (1.2) which concentrates in a point $Q_\varepsilon \in \partial\Omega$ and $H(Q_\varepsilon) \rightarrow \max_{\partial\Omega} H$, here H denotes the mean curvature of $\partial\Omega$. Moreover, in [10], using the Liapunov-Schmidt reduction, Y. Y. Li constructs solutions with single peak and multi-peaks on $\partial\Omega$ located near any stable critical points of H . Since the publications of [11], [12], there have been many works on spike-layer solutions of (1.2), see for example [5]–[9], [14] and references therein.

What happens in presence of potentials J and V ?

In this paper we try to give an answer to this question and we will show that, for the existence of concentrating solutions, one has to check if at least one between J and V is not constant on $\partial\Omega$. In this case the concentration point is determined by J and V only. In the other case the concentration point is determined by an interplay among the derivatives of J and V calculated on $\partial\Omega$ and the mean curvature H .

On J and V we will do the following assumptions:

(J) $J \in C^2(\Omega, \mathbb{R})$, J and D^2J are bounded; moreover,

$$J(x) \geq C > 0 \quad \text{for all } x \in \Omega,$$

(V) $V \in C^2(\Omega, \mathbb{R})$, V and D^2V are bounded; moreover,

$$V(x) \geq C > 0 \quad \text{for all } x \in \Omega.$$

Let us introduce an auxiliary function which will play a crucial role in the study of (1.1). Let $\Gamma: \partial\Omega \rightarrow \mathbb{R}$ be a function so defined:

$$(1.3) \quad \Gamma(Q) = V(Q)^{(p+1)/(p-1)-N/2} J(Q)^{N/2}.$$

Let us observe that by (J) and (V), Γ is well defined.

Our first result is:

THEOREM 1.1. *Let $Q_0 \in \partial\Omega$. Suppose (J) and (V). There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then (1.1) possesses a solution u_ε which concentrates in Q_ε with $Q_\varepsilon \rightarrow Q_0$, as $\varepsilon \rightarrow 0$, provided that one of the two following conditions holds:*

- (a) Q_0 is a non-degenerate critical point of Γ ,
- (b) Q_0 is an isolated local strict minimum or maximum of Γ .

Hence, if J and V are not constant on the boundary $\partial\Omega$, the concentration phenomena depend only by J and V and not by the mean curvature H . Our second result deals with the other case and, more precisely, we will show that, if J and V (and so also Γ) are constant on the boundary, then the concentration phenomena are due by another auxiliary function which depends on the derivatives of J and V on the boundary and by the mean curvature H . Let $\bar{\Sigma}: \partial\Omega \rightarrow \mathbb{R}$ be the function so defined:

$$(1.4) \quad \bar{\Sigma}(Q) \equiv k_1 \int_{\mathbb{R}_{\nu(Q)}^-} J'(Q)[x] |(\nabla \bar{U})(k_2 x)|^2 dx \\ + k_3 \int_{\mathbb{R}_{\nu(Q)}^-} V'(Q)[x] [\bar{U}(k_2 x)]^2 dx - k_4 H(Q),$$

where \bar{U} is the unique solution of

$$\begin{cases} -\Delta \bar{U} + \bar{U} = \bar{U}^p & \text{in } \mathbb{R}^N, \\ \bar{U} > 0 & \text{in } \mathbb{R}^N, \\ \bar{U}(0) = \max_{\mathbb{R}^N} \bar{U}, \end{cases}$$

$\nu(Q)$ is the outer normal in Q at Ω ,

$$\mathbb{R}_{\nu(Q)}^- \equiv \{x \in \mathbb{R}^N : x \cdot \nu(Q) \leq 0\},$$

and, for $i = 1, \dots, 4$, k_i are constants which depend only on J and V and not on Q (see Remark 5.3 for an explicit formula).

Our second result is:

THEOREM 1.2. *Suppose (J) and (V) with J and V constant on the boundary $\partial\Omega$. Let $Q_0 \in \partial\Omega$ be an isolated local strict minimum or maximum of $\bar{\Sigma}$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then (1.1) possesses a solution u_ε which concentrates in Q_ε with $Q_\varepsilon \rightarrow Q_0$, as $\varepsilon \rightarrow 0$.*

EXAMPLE 1.3. Suppose that $J \equiv 1$ and fix any $Q_0 \in \partial\Omega$. For $k \in \mathbb{N}$, let V_k be a bounded smooth function constantly equal to 1 on the $\partial\Omega$ and in the whole Ω , except a little ball tangent at $\partial\Omega$ in Q_0 , with $\nabla V_k(Q_0) = -k\nu(Q_0)$ (see Figure 1).

It is easy to see that, outside a little neighbourhood of Q_0 in $\partial\Omega$, we have

$$\bar{\Sigma}(Q) = -C_1 H(Q), \quad \text{while } \bar{\Sigma}(Q_0) = -C_1 H(Q_0) + kC_2,$$

where

$$C_1 = \frac{1}{2} \bar{B} + \left(\frac{1}{2} - \frac{1}{p+1} \right) \bar{A}, \\ C_2 = -\frac{1}{2} \int_{\{\nu(Q_0) \cdot x \leq 0\}} \nu(Q_0) \cdot x \bar{U}^2 dx.$$

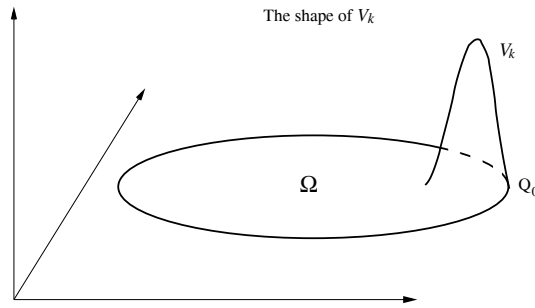


FIGURE 1

Since $C_2 > 0$, we can choose $k \gg 1$ such that Q_0 is the absolute maximum point for $\bar{\Sigma}$ and hence there exists a solution concentrating at Q_0 .

Theorem 1.1 will be proved as a particular case of two multiplicity results in Section 6, where we will prove also Theorem 1.2. The proof of the theorems relies on a finite dimensional reduction, precisely on the perturbation technique developed in [1]–[3]. In Section 2 we give some preliminary lemmas and some estimates which will be useful in Sections 3 and 4, where we perform the Liapunov–Schmidt reduction, and in Section 5, where we make the asymptotic expansion of the finite dimensional functional.

Finally we mention that problem (1.1), but with the Dirichlet boundary conditions, is studied by the author and by S. Secchi in [13], where we show that there are solutions which concentrate in minima of an auxiliary function, which depends only on J and V .

Acknowledgments. The author wishes to thank Professor Antonio Ambrosetti and Professor Andrea Malchiodi for suggesting the problem and for useful discussions.

Notation.

- $\mathbb{R}_+^N \equiv \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$.
- If $\mu \in \mathbb{R}^N$, then $\mathbb{R}_\mu^- \equiv \{x \in \mathbb{R}^N : x \cdot \mu \leq 0\}$, where with $x \cdot \mu$ we denote the scalar product in \mathbb{R}^N between x and μ .
- If $r > 0$ and $x_0 \in \mathbb{R}^N$, $B_r(x_0) \equiv \{x \in \mathbb{R}^N : |x - x_0| < r\}$. We denote with B_r the ball of radius r centered in the origin.
- If $u: \mathbb{R}^N \rightarrow \mathbb{R}$ and $P \in \mathbb{R}^N$, we set $u_P \equiv u(\cdot - P)$.
- If U^Q is the function defined in (2.2), when there is no misunderstanding, we will often write U instead of U^Q . Moreover, if $P = Q/\varepsilon$, then $U_P \equiv U^Q(\cdot - P)$.
- If $Q \in \partial\Omega$, we denote with $\nu(Q)$ the outer normal in Q at Ω and with $H(Q)$ the mean curvature of $\partial\Omega$ in Q .

- If $\varepsilon > 0$, we set $\Omega_\varepsilon \equiv \Omega/\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$.
- We denote with $\|\cdot\|$ and with $(\cdot|\cdot)$, respectively the norm and the scalar product of $H^1(\Omega_\varepsilon)$. While we denote with $\|\cdot\|_+$ and with $(\cdot|\cdot)_+$, respectively the norm and the scalar product of $H^1(\mathbb{R}_+^N)$.
- If $P \in \partial\Omega_\varepsilon$, we set $\partial_{P_i} \equiv \partial/\partial e_i$, where $\{e_1, \dots, e_{N-1}\}$ is an orthonormal basis of $T_P(\partial\Omega_\varepsilon)$. Analogously, if $Q \in \partial\Omega$, we set $\partial_{Q_i} \equiv \partial/\partial \tilde{e}_i$, where $\{\tilde{e}_1, \dots, \tilde{e}_{N-1}\}$ is an orthonormal basis of $T_Q(\partial\Omega)$.

2. Preliminary lemmas and some estimates

First of all we perform the change of variable $x \mapsto \varepsilon x$ and so problem (1.1) becomes

$$(2.1) \quad \begin{cases} -\operatorname{div}(J(\varepsilon x)\nabla u) + V(\varepsilon x)u = u^p & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Of course if u is a solution of (2.1), then $u(\cdot/\varepsilon)$ is a solution of (1.1). Solutions of (2.1) are critical points $u \in H^1(\Omega_\varepsilon)$ of

$$f_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_\varepsilon} V(\varepsilon x) u^2 dx - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1}.$$

The solutions of (2.1) will be found near a U^Q , the unique solution of

$$\begin{cases} -J(Q)\Delta u + V(Q)u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(0) = \max_{\mathbb{R}^N} u \end{cases}$$

for an appropriate choice of $Q \in \partial\Omega$. It is easy to see that

$$(2.2) \quad U^Q(x) = V(Q)^{1/(p-1)} \bar{U}(x\sqrt{V(Q)/J(Q)}),$$

where \bar{U} is the unique solution of

$$\begin{cases} -\Delta \bar{U} + \bar{U} = \bar{U}^p & \text{in } \mathbb{R}^N, \\ \bar{U} > 0 & \text{in } \mathbb{R}^N, \\ \bar{U}(0) = \max_{\mathbb{R}^N} \bar{U}, \end{cases}$$

which is radially symmetric and decays exponentially at infinity with its derivatives.

We remark that U^Q is a solution also of the ‘‘problem to infinity’’:

$$(2.3) \quad \begin{cases} -J(Q)\Delta u + V(Q)u = u^p & \text{in } \mathbb{R}_+^N, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases}$$

The solutions of (2.3) are critical points of the functional defined on $H^1(\mathbb{R}_+^N)$

$$(2.4) \quad F^Q(u) = \frac{1}{2} J(Q) \int_{\mathbb{R}_+^N} |\nabla u|^2 + \frac{1}{2} V(Q) \int_{\mathbb{R}_+^N} u^2 - \frac{1}{p+1} \int_{\mathbb{R}_+^N} |u|^{p+1}.$$

We recall that we will often write U instead of U^Q . If $P = \varepsilon^{-1}Q \in \partial\Omega_\varepsilon$, we set $U_P \equiv U^Q(\cdot - P)$ and $Z^\varepsilon \equiv \{U_P : P \in \partial\Omega_\varepsilon\}$.

LEMMA 2.1. *For all $Q \in \partial\Omega$ and for all ε sufficiently small, if $P = Q/\varepsilon \in \partial\Omega_\varepsilon$, then*

$$(2.5) \quad \|\nabla f_\varepsilon(U_P)\| = O(\varepsilon).$$

PROOF.

$$\begin{aligned} (\nabla f_\varepsilon(U_P)|v) &= \int_{\Omega_\varepsilon} J(\varepsilon x) \nabla U_P \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x) U_P v - \int_{\Omega_\varepsilon} U_P^p v \\ &= \int_{(\Omega-Q)/\varepsilon} J(\varepsilon x + Q) \nabla U \cdot \nabla v_{-P} \\ &\quad + \int_{(\Omega-Q)/\varepsilon} V(\varepsilon x + Q) U v_{-P} - \int_{(\Omega-Q)/\varepsilon} U^p v_{-P} \\ &= \int_{(\Omega-Q)/\varepsilon} J(Q) \nabla U \cdot \nabla v_{-P} \\ &\quad + \int_{(\Omega-Q)/\varepsilon} V(Q) U v_{-P} - \int_{(\Omega-Q)/\varepsilon} U^p v_{-P} \\ &\quad + \int_{(\Omega-Q)/\varepsilon} (J(\varepsilon x + Q) - J(Q)) \nabla U \cdot \nabla v_{-P} \\ &\quad + \int_{(\Omega-Q)/\varepsilon} (V(\varepsilon x + Q) - V(Q)) U v_{-P} \\ &= \int_{(\Omega-Q)/\varepsilon} [-J(Q) \Delta U + V(Q) U - U^p] v_{-P} + J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} v \\ &\quad + \int_{(\Omega-Q)/\varepsilon} (J(\varepsilon x + Q) - J(Q)) \nabla U \cdot \nabla v_{-P} \\ &\quad + \int_{(\Omega-Q)/\varepsilon} (V(\varepsilon x + Q) - V(Q)) U v_{-P}. \end{aligned}$$

Hence, since $U \equiv U^Q$ is solution of (2.3), we get

$$(2.6) \quad \begin{aligned} (\nabla f_\varepsilon(U_P)|v) &= J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} v \\ &\quad + \int_{(\Omega-Q)/\varepsilon} (J(\varepsilon x + Q) - J(Q)) \nabla U \cdot \nabla v_{-P} \\ &\quad + \int_{(\Omega-Q)/\varepsilon} (V(\varepsilon x + Q) - V(Q)) U v_{-P}. \end{aligned}$$

Let us estimate the first of these three terms:

$$\left| J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} v \right| \leq C \|v\|_{L^2(\partial\Omega_\varepsilon)} \left(\int_{\partial\Omega_\varepsilon} \left| \frac{\partial U_P}{\partial \nu} \right|^2 \right)^{1/2}.$$

First of all, we observe that there exist $\varepsilon_0 > 0$ and $C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $v \in H^1(\Omega_\varepsilon)$, we have

$$\|v\|_{L^2(\partial\Omega_\varepsilon)} \leq C\|v\|_{H^1(\Omega_\varepsilon)}.$$

Moreover, after making a translation and rotation, we can assume that Q coincides with the origin \mathcal{O} and that part of $\partial\Omega$ is given by $x_N = \psi(x') = (1/2) \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3)$ for $|x'| < \mu$, where μ is some constant depending only on Ω . Then for $|y'| < \mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N = \Psi(y') = \varepsilon^{-1} \psi(\varepsilon y') = (\varepsilon/2) \sum_{i=1}^{N-1} \lambda_i y_i^2 + O(\varepsilon^2 |y'|^3)$. Then it is easy to see that

$$\frac{\partial U}{\partial \nu}(y', \Psi(y')) = \varepsilon \left[\sum_{i=1}^{N-1} \lambda_i y_i \frac{\partial U}{\partial y_i}(y', 0) - \frac{1}{2} \frac{\partial^2 U}{\partial y_N^2}(y', 0) \sum_{i=1}^{N-1} \lambda_i y_i^2 \right] + O(\varepsilon^2).$$

Let us observe that by the exponential decay of U and of its derivatives, we get:

$$\begin{aligned} \int_{\partial\tilde{\Omega}_\varepsilon} \left| \frac{\partial U}{\partial \nu} \right|^2 &= \varepsilon^2 \int_{\partial\tilde{\Omega}_\varepsilon} \left[\sum_{i=1}^{N-1} \lambda_i y_i \frac{\partial U}{\partial y_i}(y', 0) - \frac{1}{2} \frac{\partial^2 U}{\partial y_N^2}(y', 0) \sum_{i=1}^{N-1} \lambda_i y_i^2 \right]^2 + o(\varepsilon^2) \\ &= O(\varepsilon^2), \end{aligned}$$

where $\partial\tilde{\Omega}_\varepsilon \equiv \partial\Omega_\varepsilon \cap B_{\varepsilon^{-1/2}}$. Therefore

$$(2.7) \quad \left(\int_{\partial\Omega_\varepsilon} \left| \frac{\partial U}{\partial \nu} \right|^2 \right)^{1/2} = \left(\int_{\partial\Omega_\varepsilon \cap B_{\varepsilon^{-1/2}}} \left| \frac{\partial U}{\partial \nu} \right|^2 \right)^{1/2} + o(\varepsilon) = O(\varepsilon).$$

Let us calculate the second term of (2.6). We start observing that, from the assumption D^2J bounded, we infer that

$$|J(\varepsilon x + Q) - J(Q)| \leq \varepsilon |J'(Q)| |x| + c_1 \varepsilon^2 |x|^2,$$

and so, using again the exponential decay of U and of its derivatives,

$$\begin{aligned} (2.8) \quad &\int_{(\Omega-Q)/\varepsilon} (J(\varepsilon x + Q) - J(Q)) \nabla U \cdot \nabla v_{-P} \\ &\leq \|v\| \left(\int_{(\Omega-Q)/\varepsilon} |J(\varepsilon x + Q) - J(Q)|^2 |\nabla U|^2 \right)^{1/2} \\ &\leq c_2 \|v\| \left[\int_{\mathbb{R}_+^N} \varepsilon^2 |J'(Q)|^2 |x|^2 |\nabla U|^4 + \int_{\mathbb{R}_+^N} \varepsilon^4 |x|^4 |\nabla U|^4 \right]^{1/2} = O(\varepsilon) \|v\|. \end{aligned}$$

Analogously, we can say that:

$$(2.9) \quad \int_{(\Omega-Q)/\varepsilon} (V(\varepsilon x + Q) - V(Q)) U v_{-P} = O(\varepsilon) \|v\|.$$

Now the conclusion follows immediately by (2.6)–(2.9). □

We here present some useful estimates that will be used in the sequel.

PROPOSITION 2.2. *Let $P = Q/\varepsilon \in \partial\Omega_\varepsilon$. Then we have:*

$$(2.10) \quad \int_{\Omega_\varepsilon} U_P^{p+1} = \int_{\mathbb{R}_+^N} (U^Q)^{p+1} - \varepsilon \frac{H(Q)}{2} \int_{\mathbb{R}^{N-1}} [U^Q(y', 0)]^{p+1} |y'|^2 dy' + o(\varepsilon),$$

$$(2.11) \quad \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} U_P = -\varepsilon \frac{(N-1)H(Q)}{4} \int_{\mathbb{R}^{N-1}} [U^Q(y', 0)]^2 dy' + o(\varepsilon),$$

$$(2.12) \quad J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + V(Q) \int_{\Omega_\varepsilon} U_P^2 = \int_{\mathbb{R}_+^N} (U^Q)^{p+1} - \varepsilon \frac{H(Q)}{2} \int_{\mathbb{R}^{N-1}} [U^Q(y', 0)]^{p+1} |y'|^2 dy' - \varepsilon J(Q) \frac{(N-1)H(Q)}{4} \int_{\mathbb{R}^{N-1}} [U^Q(y', 0)]^2 dy' + o(\varepsilon),$$

$$(2.13) \quad \int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla U_P|^2 = J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + \varepsilon \int_{\mathbb{R}_\nu^-(Q)} J'(Q)[x] |\nabla U^Q|^2 + o(\varepsilon),$$

$$(2.14) \quad \int_{\Omega_\varepsilon} V(\varepsilon x) U_P^2 = V(Q) \int_{\Omega_\varepsilon} U_P^2 + \varepsilon \int_{\mathbb{R}_\nu^-(Q)} V'(Q)[x] (U^Q)^2 + o(\varepsilon).$$

Moreover, we have

$$(2.15) \quad \int_{\Omega_\varepsilon} U_P^p \partial_{P_i} U_P = \varepsilon \frac{1}{p+1} \overline{C} \partial_{Q_i} \Gamma(Q) + o(\varepsilon),$$

$$(2.16) \quad \partial_{P_i} \left[J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + V(Q) \int_{\Omega_\varepsilon} U_P^2 \right] = \varepsilon \overline{C} \partial_{Q_i} \Gamma(Q) + o(\varepsilon).$$

where $\overline{C} = \int_{\mathbb{R}_+^N} \overline{U}^{p+1}$ and Γ is defined in (1.3).

PROOF. The first two formulas can be proved repeating the arguments of Lemma 1.2 of [10]. Equation (2.12) follows easily by (2.10) and (2.11) observing that

$$J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + V(Q) \int_{\Omega_\varepsilon} U_P^2 = \int_{\Omega_\varepsilon} U_P^{p+1} + J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} U_P.$$

Let us prove (2.13). Arguing as in the proof of (2.8), we infer:

$$\begin{aligned} \int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla U_P|^2 &= \int_{(\Omega-Q)_\varepsilon} J(\varepsilon x + Q) |\nabla U^Q|^2 \\ &= J(Q) \int_{(\Omega-Q)_\varepsilon} |\nabla U^Q|^2 + \varepsilon \int_{(\Omega-Q)_\varepsilon} J'(Q)[x] |\nabla U^Q|^2 + o(\varepsilon) \\ &= J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + \varepsilon \int_{\mathbb{R}_\nu^-(Q)} J'(Q)[x] |\nabla U^Q|^2 + o(\varepsilon). \end{aligned}$$

We can prove equation (2.14) repeating the arguments of (2.13). Since

$$\int_{\Omega_\varepsilon} U_P^p \partial_{P_i} U_P = \frac{1}{p+1} \partial_{P_i} \int_{\Omega_\varepsilon} U_P^{p+1},$$

equations (2.15) and (2.16) follow easily because, as observed by [10], the error terms $O(\varepsilon)$ in (2.10) and (2.12) become of order $o(\varepsilon)$ after applying ∂_{P_i} to them. \square

3. Invertibility of $D^2 f_\varepsilon$ on $(T_{U_P} Z^\varepsilon)^\perp$

In this section we will show that $D^2 f_\varepsilon$ is invertible on $(T_{U_P} Z^\varepsilon)^\perp$, where $T_{U_P} Z^\varepsilon$ denotes the tangent space to Z^ε at U_P .

Let $L_{\varepsilon, Q}: (T_{U_P} Z^\varepsilon)^\perp \rightarrow (T_{U_P} Z^\varepsilon)^\perp$ denote the operator defined by setting $(L_{\varepsilon, Q} v | w) = D^2 f_\varepsilon(U_P)[v, w]$.

LEMMA 3.1. *There exists $C > 0$ such that for ε small enough one has that*

$$(3.1) \quad |(L_{\varepsilon, Q} v | v)| \geq C \|v\|^2, \quad \text{for all } v \in (T_{U_P} Z^\varepsilon)^\perp.$$

PROOF. By (2.2), if we set $\alpha(Q) = V(Q)^{1/(p-1)}$ and $\beta(Q) = \sqrt{V(Q)/J(Q)}$, we have that $U^Q(x) = \alpha(Q) \overline{U}(\beta(Q)x)$. Therefore, we have:

$$\begin{aligned} \partial_{P_i} U^Q(x - P) &= \partial_{P_i} [\alpha(\varepsilon P) \overline{U}(\beta(\varepsilon P)(x - P))] \\ &= \varepsilon \partial_{P_i} \alpha(\varepsilon P) U^Q(\beta(\varepsilon P)(x - P)) \\ &\quad + \varepsilon \alpha(\varepsilon P) \partial_{P_i} \beta(\varepsilon P) \nabla U^Q(\beta(\varepsilon P)(x - P)) \cdot (x - P) \\ &\quad - \alpha(\varepsilon P) \beta(\varepsilon P) (\partial_{x_i} U^Q)(\beta(\varepsilon P)(x - P)). \end{aligned}$$

Hence

$$(3.2) \quad \partial_{P_i} U^Q(x - P) = -\partial_{x_i} U^Q(x - P) + O(\varepsilon).$$

For simplicity, we can assume that $Q = \varepsilon P$ is the origin \mathcal{O} .

Following [10], without loss of generality, we assume that $Q = \varepsilon P$ is the origin \mathcal{O} , x_N is the tangent plane of $\partial\Omega$ at Q and $\nu(Q) = (0, \dots, 0, -1)$. We also assume that part of $\partial\Omega$ is given by $x_N = \psi(x') = (1/2) \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3)$ for $|x'| < \mu$, where μ is some constant depending only on Ω . Then for $|y'| < \mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N = \Psi(y') = \varepsilon^{-1} \psi(\varepsilon y') = (\varepsilon/2) \sum_{i=1}^{N-1} \lambda_i y_i^2 + O(\varepsilon^2 |y'|^3)$.

We recall that $T_{U^\mathcal{O}} Z^\varepsilon = \text{span}_{H^1(\Omega_\varepsilon)} \{\partial_{P_1} U^\mathcal{O}, \dots, \partial_{P_{N-1}} U^\mathcal{O}\}$. We set

$$\begin{aligned} \mathcal{V}_\varepsilon &= \text{span}_{H^1(\Omega_\varepsilon)} \{U^\mathcal{O}, \partial_{x_1} U^\mathcal{O}, \dots, \partial_{x_{N-1}} U^\mathcal{O}\}, \\ \mathcal{V}_+ &= \text{span}_{H^1(\mathbb{R}_+^N)} \{U^\mathcal{O}, \partial_{x_1} U^\mathcal{O}, \dots, \partial_{x_{N-1}} U^\mathcal{O}\}. \end{aligned}$$

By (3.2) it suffices to prove (3.1) for all $v \in \text{span}\{U^\mathcal{O}, \phi\}$, where ϕ is orthogonal to \mathcal{V}_ε . Precisely we shall prove that there exist $C_1, C_2 > 0$ such that, for all

$\varepsilon > 0$ small enough, one has:

$$(3.3) \quad (L_{\varepsilon, \mathcal{O}} U^{\mathcal{O}} | U^{\mathcal{O}}) \leq -C_1 < 0.$$

$$(3.4) \quad (L_{\varepsilon, \mathcal{O}} \phi | \phi) \geq C_2 \|\phi\|^2.$$

The proof of (3.3) follows easily from the fact that $U^{\mathcal{O}}$ is a Mountain Pass critical point of $F^{\mathcal{O}}$ and so from the fact that there exists $c_0 > 0$ such that, for all $\varepsilon > 0$ small enough, one finds:

$$D^2 F^{\mathcal{O}}(U^{\mathcal{O}})[U^{\mathcal{O}}, U^{\mathcal{O}}] < -c_0 < 0.$$

Indeed, arguing as in the proof of Lemma 2.1 (see (2.8) and (2.9)) and by (2.10) and (2.12), we have:

$$\begin{aligned} (L_{\varepsilon, \mathcal{O}} U^{\mathcal{O}} | U^{\mathcal{O}}) &= \int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla U^{\mathcal{O}}|^2 + \int_{\Omega_\varepsilon} V(\varepsilon x) (U^{\mathcal{O}})^2 - p \int_{\Omega_\varepsilon} (U^{\mathcal{O}})^{p+1} \\ &= J(\mathcal{O}) \int_{\Omega_\varepsilon} |\nabla U^{\mathcal{O}}|^2 + V(\mathcal{O}) \int_{\Omega_\varepsilon} (U^{\mathcal{O}})^2 - p \int_{\Omega_\varepsilon} (U^{\mathcal{O}})^{p+1} + O(\varepsilon) \\ &= D^2 F^{\mathcal{O}}(U^{\mathcal{O}})[U^{\mathcal{O}}, U^{\mathcal{O}}] + O(\varepsilon) < -c_0 + O(\varepsilon) < -C_1. \end{aligned}$$

Let us prove (3.4). As before, the fact that $U^{\mathcal{O}}$ is a Mountain Pass critical point of $F^{\mathcal{O}}$ implies that

$$(3.5) \quad D^2 F^{\mathcal{O}}(U^{\mathcal{O}})[\tilde{\phi}, \tilde{\phi}] > c_1 \|\tilde{\phi}\|_+^2 \quad \text{for all } \tilde{\phi} \perp \mathcal{V}_+.$$

Let us consider a smooth function $\chi_1: \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \chi_1(x) &= 1 && \text{for } |x| \leq \varepsilon^{-1/8}, \\ \chi_1(x) &= 0 && \text{for } |x| \geq 2\varepsilon^{-1/8}, \\ |\nabla \chi_1(x)| &\leq 2\varepsilon^{1/8} && \text{for } \varepsilon^{-1/8} \leq |x| \leq 2\varepsilon^{-1/8}. \end{aligned}$$

We also set $\chi_2(x) = 1 - \chi_1(x)$. Given $\phi \perp \mathcal{V}_\varepsilon$, let us consider the functions

$$\phi_i(x) = \chi_i(x)\phi(x), \quad i = 1, 2.$$

If $Q \neq \mathcal{O}$, then we would take

$$\phi_i(x) = \chi_i(x - P)\phi(x), \quad i = 1, 2.$$

With calculations similar to those of [3], we have

$$(3.6) \quad \|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2 \underbrace{\int_{\mathbb{R}^N} \chi_1 \chi_2 (\phi^2 + |\nabla \phi|^2)}_{I_\phi} + O(\varepsilon^{1/8}) \|\phi\|^2.$$

We need to evaluate the three terms in the equation below:

$$(3.7) \quad (L_{\varepsilon, \mathcal{O}} \phi | \phi) = (L_{\varepsilon, \mathcal{O}} \phi_1 | \phi_1) + (L_{\varepsilon, \mathcal{O}} \phi_2 | \phi_2) + 2(L_{\varepsilon, \mathcal{O}} \phi_1 | \phi_2).$$

Let us start with $(L_{\varepsilon, \mathcal{O}} \phi_1 | \phi_1)$. Let $\eta = \eta_\varepsilon$ a smooth cutoff function satisfying

$$\begin{aligned} \eta(y) &= 1 && \text{for } |y| \leq \varepsilon^{-1/4}, \\ \eta(y) &= 0 && \text{for } |y| \geq 2\varepsilon^{-1/4}, \\ |\nabla \eta(y)| &\leq 2\varepsilon^{1/4} && \text{for } \varepsilon^{-1/4} \leq |y| \leq 2\varepsilon^{-1/4}. \end{aligned}$$

Now we will straighten $\partial\Omega_\varepsilon$ in the following way: let $\Phi: \mathbb{R}_+^N \cap B_{\varepsilon^{-1/2}} \rightarrow \Omega_\varepsilon$ be a function so defined:

$$\Phi(y', y_N) = (y', y_N + \Psi(y')).$$

We observe that

$$D\Phi(y) = \left(\begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \hline & & \nabla_{y'} \Psi(y') & 1 \end{array} \right).$$

Let us defined $\widetilde{\phi}_1 \in H^1(\mathbb{R}_+^N)$ as

$$\widetilde{\phi}_1(y) = \begin{cases} \phi_1(\Phi(y))\eta(y) & \text{if } |y| \leq \varepsilon^{-1/2}, \\ 0 & \text{if } |y| > \varepsilon^{-1/2}. \end{cases}$$

We get

$$\begin{aligned} \int_{\mathbb{R}_+^N} |\nabla \widetilde{\phi}_1|^2 &= \int_{\mathbb{R}_+^N \cap B_{2\varepsilon^{-1/4}}} |\nabla[\phi_1(\Phi(y))]|^2 dy \\ &= \int_{\mathbb{R}_+^N \cap B_{2\varepsilon^{-1/4}}} \sum_{i=1}^{N-1} \left| \frac{\partial \phi_1}{\partial x_i}(\Phi) + \varepsilon \lambda_i y_i \frac{\partial \phi_1}{\partial x_N}(\Phi) \right|^2 + \left| \frac{\partial \phi_1}{\partial x_N}(\Phi) \right|^2 + o(\varepsilon) \|\phi\|^2 \\ &= \int_{\mathbb{R}_+^N \cap B_{2\varepsilon^{-1/4}}} |(\nabla \phi_1)(\Phi)|^2 + O(\varepsilon^{7/8}) \|\phi\|^2 \\ &= \int_{\Omega_\varepsilon} |\nabla \phi_1|^2 + O(\varepsilon^{7/8}) \|\phi\|^2. \end{aligned}$$

Analogously, we have

$$\int_{\mathbb{R}_+^N} |\widetilde{\phi}_1|^2 = \int_{\Omega_\varepsilon} |\phi_1|^2, \quad \text{and so} \quad \|\widetilde{\phi}_1\|_+^2 = \|\phi_1\|^2 + O(\varepsilon^{7/8}) \|\phi\|^2.$$

Let us now evaluate $(L_{\varepsilon, \mathcal{O}} \phi_1 | \phi_1)$:

$$\begin{aligned} (L_{\varepsilon, \mathcal{O}} \phi_1 | \phi_1) &= \int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla \phi_1|^2 + \int_{\Omega_\varepsilon} V(\varepsilon x) \phi_1^2 - p \int_{\Omega_\varepsilon} (U^\mathcal{O})^{p-1} \phi_1^2 \\ &= J(\mathcal{O}) \int_{\Omega_\varepsilon} |\nabla \phi_1|^2 + V(\mathcal{O}) \int_{\Omega_\varepsilon} \phi_1^2 - p \int_{\Omega_\varepsilon} (U^\mathcal{O})^{p-1} \phi_1^2 \\ &\quad + \varepsilon \int_{\Omega_\varepsilon} J'(\mathcal{O})[x] |\nabla \phi_1|^2 + \varepsilon \int_{\Omega_\varepsilon} V'(\mathcal{O})[x] \phi_1^2 + o(\varepsilon) \|\phi\|^2 \end{aligned}$$

$$\begin{aligned}
&= J(\mathcal{O}) \int_{\Omega_\varepsilon} |\nabla \phi_1|^2 + V(\mathcal{O}) \int_{\Omega_\varepsilon} \phi_1^2 - p \int_{\Omega_\varepsilon} (U^\mathcal{O})^{p-1} \phi_1^2 + O(\varepsilon^{7/8}) \|\phi\|^2 \\
&= J(\mathcal{O}) \int_{\mathbb{R}_+^N} |\nabla \widetilde{\phi}_1|^2 + V(\mathcal{O}) \int_{\mathbb{R}_+^N} \widetilde{\phi}_1^2 \\
&\quad - p \int_{\mathbb{R}_+^N} [U^\mathcal{O}(\Phi)]^{p-1} \widetilde{\phi}_1^2 + O(\varepsilon^{7/8}) \|\phi\|^2 \\
&= D^2 F^\mathcal{O}(U^\mathcal{O})[\widetilde{\phi}_1, \widetilde{\phi}_1] \\
&\quad - p \int_{\mathbb{R}_+^N} ([U^\mathcal{O}(\Phi)]^{p-1} - (U^\mathcal{O})^{p-1}) \widetilde{\phi}_1^2 + O(\varepsilon^{7/8}) \|\phi\|^2.
\end{aligned}$$

We have

$$\begin{aligned}
\left| \int_{\mathbb{R}_+^N} ([U^\mathcal{O}(\Phi)]^{p-1} - (U^\mathcal{O})^{p-1}) \widetilde{\phi}_1^2 \right| &\leq C \int_{\mathbb{R}_+^N} |\Psi(y')| \widetilde{\phi}_1^2 \\
&= O(\varepsilon^{3/4}) \|\widetilde{\phi}_1\|^2 = O(\varepsilon^{3/4}) \|\phi\|^2.
\end{aligned}$$

Therefore, we have that

$$(3.8) \quad (L_{\varepsilon, \mathcal{O}} \phi_1 | \phi_1) = D^2 F^\mathcal{O}(U^\mathcal{O})[\widetilde{\phi}_1, \widetilde{\phi}_1] + O(\varepsilon^{3/4}) \|\phi\|^2.$$

We can write $\widetilde{\phi}_1 = \xi + \zeta$, where $\xi \in \mathcal{V}_+$ and $\zeta \perp \mathcal{V}_+$. More precisely

$$\xi = (\widetilde{\phi}_1 | U^\mathcal{O})_+ U^\mathcal{O} \|U^\mathcal{O}\|_+^{-2} + \sum_{i=1}^{N-1} (\widetilde{\phi}_1 | \partial_{P_i} U^\mathcal{O})_+ \partial_{P_i} U^\mathcal{O} \|\partial_{P_i} U^\mathcal{O}\|_+^{-2}.$$

Let us calculate $(\widetilde{\phi}_1 | U^\mathcal{O})_+$.

$$\begin{aligned}
(\widetilde{\phi}_1 | U^\mathcal{O})_+ &= \int_{\mathbb{R}_+^N} \nabla \widetilde{\phi}_1 \cdot \nabla U^\mathcal{O} + \int_{\mathbb{R}_+^N} \widetilde{\phi}_1 U^\mathcal{O} \\
&= \int_{\mathbb{R}_+^N \cap B_{2\varepsilon^{-1/4}}} \nabla [\phi_1(\Phi(y))] \cdot \nabla U^\mathcal{O} + \int_{\mathbb{R}_+^N \cap B_{2\varepsilon^{-1/4}}} \phi_1(\Phi(y)) U^\mathcal{O} \\
&= \int_{\mathbb{R}_+^N \cap B_{2\varepsilon^{-1/4}}} [(\nabla \phi_1)(\Phi) \cdot \nabla U^\mathcal{O} + \phi_1(\Phi) U^\mathcal{O}] \\
&\quad + \varepsilon \sum_{i=1}^{N-1} \int_{\mathbb{R}_+^N \cap B_{2\varepsilon^{-1/4}}} \lambda_i y_i \frac{\partial \phi_1}{\partial x_N}(\Phi) \frac{\partial U^\mathcal{O}}{\partial x_i} \\
&= \int_{\Omega_\varepsilon} \nabla \phi_1 \cdot \nabla U^\mathcal{O}(\Phi^{-1}) + \int_{\Omega_\varepsilon} \phi_1 U^\mathcal{O}(\Phi^{-1}) + O(\varepsilon^{7/8}) \|\phi\|^2 \\
&= \int_{\Omega_\varepsilon} \nabla \phi_1 \cdot \nabla U^\mathcal{O} + \int_{\Omega_\varepsilon} \phi_1 U^\mathcal{O} + O(\varepsilon^{3/4}) \|\phi\| = O(\varepsilon^{3/4}) \|\phi\|.
\end{aligned}$$

In an analogous way, we can prove also that $(\widetilde{\phi}_1 | \partial_{P_i} U^\mathcal{O})_+ = O(\varepsilon^{3/4}) \|\phi\|$, and so

$$(3.9) \quad \|\xi\|_+ = O(\varepsilon^{3/4}) \|\phi\|,$$

$$(3.10) \quad \|\zeta\|_+ = \|\phi_1\| + O(\varepsilon^{3/4}) \|\phi\|.$$

Let us estimate $D^2F^\mathcal{O}(U^\mathcal{O})[\widetilde{\phi}_1, \widetilde{\phi}_1]$. We get:

$$(3.11) \quad \begin{aligned} D^2F^\mathcal{O}(U^\mathcal{O})[\widetilde{\phi}_1, \widetilde{\phi}_1] \\ = D^2F^\mathcal{O}(U^\mathcal{O})[\zeta, \zeta] + 2D^2F^\mathcal{O}(U^\mathcal{O})[\zeta, \xi] + D^2F^\mathcal{O}(U^\mathcal{O})[\xi, \xi]. \end{aligned}$$

By (3.5) and (3.10), we know that

$$D^2F^\mathcal{O}(U^\mathcal{O})[\zeta, \zeta] > c_1\|\zeta\|_+^2 = c_1\|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2,$$

while, by (3.9) and straightforward calculations, we have

$$\begin{aligned} D^2F^\mathcal{O}(U^\mathcal{O})[\zeta, \xi] &= O(\varepsilon^{3/4})\|\phi\|^2, \\ D^2F^\mathcal{O}(U^\mathcal{O})[\xi, \xi] &= O(\varepsilon^{3/2})\|\phi\|^2. \end{aligned}$$

By these estimates, (3.11) and (3.8), we can say that

$$(3.12) \quad (L_{\varepsilon, \mathcal{O}}\phi_1|\phi_1) > c_1\|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2.$$

Using the definition of χ_i and the exponential decay of $U^\mathcal{O}$, we easily get

$$(3.13) \quad (L_{\varepsilon, \mathcal{O}}\phi_2|\phi_2) \geq c_2\|\phi_2\|^2 + o(\varepsilon)\|\phi\|^2,$$

$$(3.14) \quad (L_{\varepsilon, \mathcal{O}}\phi_1|\phi_2) \geq c_3I_\phi + O(\varepsilon^{1/8})\|\phi\|^2,$$

where I_ϕ is defined in (3.6). Therefore by (3.7), (3.12)–(3.14) and recalling (3.6) we get

$$(L_{\varepsilon, \mathcal{O}}\phi|\phi) \geq c_4\|\phi\|^2 + O(\varepsilon^{1/8})\|\phi\|^2.$$

This completes the proof of the lemma. \square

4. The finite dimensional reduction

LEMMA 4.1. *For $\varepsilon > 0$ small enough, there exists a unique $w = w(\varepsilon, Q) \in (T_{U_P}Z^\varepsilon)^\perp$ such that $\nabla f_\varepsilon(U_P + w) \in T_{U_P}Z$. Such a $w(\varepsilon, Q)$ is of class C^2 , resp. $C^{1,p-1}$, with respect to Q , provided that $p \geq 2$, resp. $1 < p < 2$. Moreover, the functional $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_{Q/\varepsilon} + w(\varepsilon, Q))$ has the same regularity of w and satisfies:*

$$\nabla \mathcal{A}_\varepsilon(Q_0) = 0 \Leftrightarrow \nabla f_\varepsilon(U_{Q_0/\varepsilon} + w(\varepsilon, Q_0)) = 0.$$

PROOF. Let $\mathcal{P} = \mathcal{P}_{\varepsilon, Q}$ denote the projection onto $(T_{U_P}Z^\varepsilon)^\perp$. We want to find a solution $w \in (T_{U_P}Z^\varepsilon)^\perp$ of the equation $\mathcal{P}\nabla f_\varepsilon(U_P + w) = 0$. One has that $\nabla f_\varepsilon(U_P + w) = \nabla f_\varepsilon(U_P) + D^2f_\varepsilon(U_P)[w] + R(U_P, w)$ with $\|R(U_P, w)\| = o(\|w\|)$, uniformly with respect to U_P . Therefore, our equation is:

$$(4.1) \quad L_{\varepsilon, Q}w + \mathcal{P}\nabla f_\varepsilon(U_P) + \mathcal{P}R(U_P, w) = 0.$$

According to Lemma 3.1, this is equivalent to

$$w = N_{\varepsilon, Q}(w), \quad \text{where } N_{\varepsilon, Q}(w) = -L_{\varepsilon, Q}(\mathcal{P}\nabla f_\varepsilon(U_P) + \mathcal{P}R(U_P, w)).$$

By (2.5) it follows that

$$(4.2) \quad \|N_{\varepsilon, Q}(w)\| = O(\varepsilon) + o(\|w\|).$$

Then one readily checks that $N_{\varepsilon, Q}$ is a contraction on some ball in $(T_{U_P}Z^\varepsilon)^\perp$ provided that $\varepsilon > 0$ is small enough. Then there exists a unique w such that $w = N_{\varepsilon, Q}(w)$. Let us point out that we cannot use the Implicit Function Theorem to find $w(\varepsilon, Q)$, because the map $(\varepsilon, u) \mapsto \mathcal{P}\nabla f_\varepsilon(u)$ fails to be C^2 . However, fixed $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(Q, w) \mapsto \mathcal{P}\nabla f_\varepsilon(U_P + w)$. Then, in particular, the function $w(\varepsilon, Q)$ turns out to be of class C^1 with respect to Q . Finally, it is a standard argument, see [1] and [2], to check that the critical points of $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w)$ give rise to critical points of f_ε . \square

REMARK 4.2. From (4.2) it immediately follows that:

$$(4.3) \quad \|w\| = O(\varepsilon).$$

For future references, it is convenient to estimate the derivative $\partial_{P_i} w$.

LEMMA 4.3. *If $\gamma = \min\{1, p-1\}$, then, for $i = 1, \dots, N-1$, one has that:*

$$(4.4) \quad \|\partial_{P_i} w\| = O(\varepsilon^\gamma).$$

PROOF. We will set $h(U_P, w) = (U_P + w)^p - U_P^p - pU_P^{p-1}w$. With these notations, and recalling that $L_{\varepsilon, Q}w = -\operatorname{div}(J(\varepsilon x)\nabla w) + V(\varepsilon x)w - pU_P^{p-1}w$, it follows that, for all $v \in (T_{U_P}Z^\varepsilon)^\perp$, since w satisfies (4.1), then:

$$\begin{aligned} & \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla U_P \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)U_P v - \int_{\Omega_\varepsilon} U_P^p v \\ & + \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla w \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)w v - p \int_{\Omega_\varepsilon} U_P^{p-1}w v - \int_{\Omega_\varepsilon} h(U_P, w)v = 0. \end{aligned}$$

Hence $\partial_{P_i} w$ verifies:

$$(4.5) \quad \begin{aligned} & \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla(\partial_{P_i} U_P) \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)(\partial_{P_i} U_P)v - p \int_{\Omega_\varepsilon} U_P^{p-1}(\partial_{P_i} U_P)v \\ & + \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla(\partial_{P_i} w) \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x)(\partial_{P_i} w)v - p \int_{\Omega_\varepsilon} U_P^{p-1}(\partial_{P_i} w)v \\ & - p(p-1) \int_{\Omega_\varepsilon} U_P^{p-2}(\partial_{P_i} U_P)w v - \int_{\Omega_\varepsilon} [h_{U_P}(\partial_{P_i} U_P) + h_w(\partial_{P_i} w)]v = 0. \end{aligned}$$

Let us set $L' = L_{\varepsilon, Q} - h_w$. Then (4.5) can be written as

$$(4.6) \quad \begin{aligned} (L'(\partial_{P_i} w)|v) & = p(p-1) \int_{\Omega_\varepsilon} U_P^{p-2}(\partial_{P_i} U_P)w v + \int_{\Omega_\varepsilon} h_{U_P}(\partial_{P_i} U_P)v \\ & - \int_{\Omega_\varepsilon} J(\varepsilon x)\nabla(\partial_{P_i} U_P) \cdot \nabla v - \int_{\Omega_\varepsilon} V(\varepsilon x)(\partial_{P_i} U_P)v + p \int_{\Omega_\varepsilon} U_P^{p-1}(\partial_{P_i} U_P)v. \end{aligned}$$

It is easy to see that

$$(4.7) \quad \left| p(p-1) \int_{\Omega_\varepsilon} U_P^{p-2} (\partial_{P_i} U_P) w v \right| \leq c_1 \|w\| \|v\|$$

and, if $\gamma = \min\{1, p-1\}$,

$$(4.8) \quad \left| \int_{\Omega_\varepsilon} h_{U_P} (\partial_{P_i} U_P) v \right| \leq c_2 \|w\|^\gamma \|v\|.$$

Let us study the second line of (4.6). We recall that often we will write U instead of U^Q . Reasoning as in the proof of Lemma 2.1 (see (2.8) and (2.9)), we infer:

$$\begin{aligned} I &\equiv \int_{\Omega_\varepsilon} J(\varepsilon x) \nabla(\partial_{P_i} U_P) \cdot \nabla v + \int_{\Omega_\varepsilon} V(\varepsilon x) (\partial_{P_i} U_P) v - p \int_{\Omega_\varepsilon} U_P^{p-1} (\partial_{P_i} U_P) v \\ &= \int_{(\Omega-Q)/\varepsilon} J(Q) \nabla(\partial_{P_i} U) \cdot \nabla v_{-P} + \int_{(\Omega-Q)/\varepsilon} V(Q) (\partial_{P_i} U) v_{-P} \\ &\quad + \varepsilon \int_{\Omega_\varepsilon} J'(Q) [x-P] \nabla(\partial_{P_i} U_P) \cdot \nabla v + \varepsilon \int_{\Omega_\varepsilon} V'(Q) [x-P] (\partial_{P_i} U_P) v \\ &\quad - p \int_{\Omega_\varepsilon} U_P^{p-1} (\partial_{P_i} U_P) v + O(\varepsilon) \|v\|. \end{aligned}$$

Suppose, for simplicity, Q coincides with the origin \mathcal{O} and that part of $\partial\Omega$ is given by $x_N = \psi(x') = (1/2) \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3)$ for $|x'| < \mu$, where μ is some constant depending only on Ω . Then for $|y'| < \mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N = \Psi(y') = \varepsilon^{-1} \psi(\varepsilon y') = (\varepsilon/2) \sum_{i=1}^{N-1} \lambda_i y_i^2 + O(\varepsilon^2 |y'|^3)$.

Since by (3.2) $\partial_{P_i} U_P = -\partial_{x_i} U_P + O(\varepsilon)$, by integration by parts, we get:

$$\begin{aligned} \varepsilon \int_{\Omega_\varepsilon} J'(Q) [x-P] \nabla(\partial_{P_i} U_P) \cdot \nabla v &= \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} J(Q) \nabla U_P \cdot \nabla v + O(\varepsilon) \|v\|, \\ \varepsilon \int_{\Omega_\varepsilon} V'(Q) [x-P] (\partial_{P_i} U_P) v &= \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} V(Q) U_P v + O(\varepsilon) \|v\|. \end{aligned}$$

Hence

$$\begin{aligned} I &= \int_{\Omega_\varepsilon} J(Q) \nabla(\partial_{P_i} U_P) \cdot \nabla v + \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} J(Q) \nabla U_P \cdot \nabla v \\ &\quad + \int_{\Omega_\varepsilon} V(Q) (\partial_{P_i} U_P) v + \varepsilon \int_{\Omega_\varepsilon} \partial_{Q_i} V(Q) U_P v \\ &\quad - p \int_{\Omega_\varepsilon} U_P^{p-1} (\partial_{P_i} U_P) v + O(\varepsilon) \|v\|. \end{aligned}$$

Being $U = U^Q$ solution of (2.3), we have that

$$-J(Q) \Delta(\partial_{P_i} U) - \varepsilon \partial_{Q_i} J(Q) \Delta U + V(Q) (\partial_{P_i} U) + \varepsilon \partial_{Q_i} V(Q) U - p U^{p-1} (\partial_{P_i} U) = 0$$

and so

$$I = J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial}{\partial \nu} (\partial_{P_i} U_P) v + \varepsilon \partial_{Q_i} J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} v + O(\varepsilon) \|v\|.$$

Arguing again as in the proof of Lemma 2.1 (see (2.7)), we can prove that

$$\left| J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial}{\partial\nu} (\partial_{P_i} U_P) v + \varepsilon \partial_{Q_i} J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial\nu} v \right| = O(\varepsilon) \|v\|.$$

Hence

$$(4.9) \quad I = O(\varepsilon^{3/4}) \|v\|.$$

Putting together (4.6)–(4.9), we find

$$|(L'(\partial w_i)|v)| = (c_3 \|w\|^\gamma + O(\varepsilon)) \|v\|.$$

Since $h_w \rightarrow 0$ as $w \rightarrow 0$, the operator L' , likewise L , is invertible for $\varepsilon > 0$ small and therefore one finds

$$\|\partial_{P_i} w\| \leq c_4 \|w\|^\gamma + O(\varepsilon).$$

Finally, by Remark 4.2, the Lemma follows. \square

5. The finite dimensional functional

THEOREM 5.1. *Let $Q \in \partial\Omega$ and $P = Q/\varepsilon \in \partial\Omega_\varepsilon$. Suppose (J) and (V). Then, for ε sufficiently small, we get:*

$$(5.1) \quad \mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w(\varepsilon, Q)) = c_0 \Gamma(Q) + \varepsilon \Sigma(Q) + o(\varepsilon),$$

where Γ is the auxiliary functions introduced in (1.3),

$$c_0 \equiv \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}_+^N} \bar{U}^{p+1},$$

and $\Sigma: \partial\Omega \rightarrow \mathbb{R}$ is so defined:

$$(5.2) \quad \Sigma(Q) \equiv \frac{1}{2} \int_{\mathbb{R}_+^-(Q)} J'(Q)[x] |\nabla U^Q|^2 dx + \frac{1}{2} \int_{\mathbb{R}_+^-(Q)} V'(Q)[x] (U^Q)^2 dx \\ - \frac{1}{2} \bar{B}^Q J(Q) H(Q) - \left(\frac{1}{2} - \frac{1}{p+1} \right) \bar{A}^Q H(Q),$$

with

$$\bar{A}^Q \equiv \frac{1}{2} \int_{\mathbb{R}^{N-1}} [U^Q(x', 0)]^{p+1} |x'|^2 dx', \\ \bar{B}^Q \equiv \frac{(N-1)}{4} \int_{\mathbb{R}^{N-1}} [U^Q(x', 0)]^2 dx.$$

Moreover, for all $i = 1, \dots, N-1$, we get:

$$(5.3) \quad \partial_{P_i} \mathcal{A}_\varepsilon(Q) = \varepsilon c_0 \partial_{Q_i} \Gamma(Q) + o(\varepsilon).$$

PROOF. In the sequel, to be short, we will often write w instead of $w(\varepsilon, Q)$. It is always understood that ε is taken in such a way that all the results discussed previously hold.

First of all, reasoning as in the proofs of (2.13) and (2.14) and by (4.3), we can observe that

$$(5.4) \quad \int_{\Omega_\varepsilon} J(\varepsilon x) \nabla U_P \cdot \nabla w = J(Q) \int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + o(\varepsilon),$$

$$(5.5) \quad \int_{\Omega_\varepsilon} V(\varepsilon x) U_P w = V(Q) \int_{\Omega_\varepsilon} U_P w + o(\varepsilon).$$

We have:

$$\begin{aligned} \mathcal{A}_\varepsilon(Q) &= f_\varepsilon(U_P + w(\varepsilon, Q)) \\ &= \frac{1}{2} \int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla(U_P + w)|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} V(\varepsilon x) (U_P + w)^2 \\ &\quad - \frac{1}{p+1} \int_{\Omega_\varepsilon} (U_P + w)^{p+1} \end{aligned}$$

by (4.3)

$$\begin{aligned} &= \frac{1}{2} \int_{\Omega_\varepsilon} J(\varepsilon x) |\nabla U_P|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} V(\varepsilon x) U_P^2 - \frac{1}{2} \int_{\Omega_\varepsilon} U_P^{p+1} \\ &\quad + \int_{\Omega_\varepsilon} J(\varepsilon x) \nabla U_P \cdot \nabla w + \int_{\Omega_\varepsilon} V(\varepsilon x) U_P w \\ &\quad - \int_{\Omega_\varepsilon} U_P^p w + \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\Omega_\varepsilon} U_P^{p+1} \\ &\quad - \frac{1}{p+1} \int_{\Omega_\varepsilon} [(U_P + w)^{p+1} - U_P^{p+1} - (p+1)U_P^p w] + o(\varepsilon) \end{aligned}$$

by (2.12)–(2.14), (5.4) and (5.5) and with our notations

$$\begin{aligned} &= \frac{1}{2} \int_{\mathbb{R}_+^N} U^{p+1} - \frac{\varepsilon}{2} \overline{A}^Q H(Q) - \frac{\varepsilon}{2} \overline{B}^Q J(Q) H(Q) + \frac{\varepsilon}{2} \int_{\mathbb{R}_+^-(Q)} J'(Q) [x] |\nabla U|^2 \\ &\quad + \frac{\varepsilon}{2} \int_{\mathbb{R}_+^-(Q)} V'(Q) [x] U^2 - \frac{1}{2} \int_{\mathbb{R}_+^N} U^{p+1} + \frac{\varepsilon}{2} \overline{A}^Q H(Q) \\ &\quad + J(Q) \int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + V(Q) \int_{\Omega_\varepsilon} U_P w - \int_{\Omega_\varepsilon} U_P^p w \\ &\quad + \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}_+^N} U^{p+1} - \varepsilon \left(\frac{1}{2} - \frac{1}{p+1} \right) \overline{A}^Q H(Q) + o(\varepsilon). \end{aligned}$$

From the fact that U is solution of (2.3), we infer

$$\begin{aligned} &J(Q) \int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla w + V(Q) \int_{\Omega_\varepsilon} U_P w - \int_{\Omega_\varepsilon} U_P^p w \\ &= \int_{\Omega_\varepsilon} [-J(Q) \Delta U_P + V(Q) U_P - U_P^p] w + J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} w \\ &= J(Q) \int_{\partial\Omega_\varepsilon} \frac{\partial U_P}{\partial \nu} w = o(\varepsilon). \end{aligned}$$

By these considerations we can say that

$$\begin{aligned} \mathcal{A}_\varepsilon(Q) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}_+^N} U^{p+1} \\ &\quad + \varepsilon \left[\frac{1}{2} \int_{\mathbb{R}_{\nu(Q)}^-} J'(Q)[x] |\nabla U|^2 + \frac{1}{2} \int_{\mathbb{R}_{\nu(Q)}^-} V'(Q)[x] U^2 \right. \\ &\quad \left. - \frac{1}{2} \overline{B}^Q J(Q) H(Q) - \left(\frac{1}{2} - \frac{1}{p+1}\right) \overline{A}^Q H(Q) \right] + o(\varepsilon). \end{aligned}$$

Now the conclusion of the first part of the theorem follows observing that, since by (2.2)

$$U^Q(x) = V(Q)^{1/(p-1)} \overline{U}(x \sqrt{V(Q)/J(Q)}),$$

then

$$\int_{\mathbb{R}_+^N} U^{p+1} = V(Q)^{(p+1)/(p-1) - N/2} J(Q)^{N/2} \int_{\mathbb{R}_+^N} \overline{U}^{p+1}.$$

Let us prove now the estimate on the derivatives of \mathcal{A}_ε . First of all, we observe that by (2.5) and by (4.4), we infer that

$$|\nabla f_\varepsilon(U_P)[\partial_{P_i} w]| = O(\varepsilon^{1+\gamma}),$$

and so, by (4.3) and (4.4), we have:

$$\begin{aligned} \partial_{P_i} \mathcal{A}_\varepsilon(Q) &= \nabla f_\varepsilon(U_P + w)[\partial_{P_i} U_P + \partial_{P_i} w] = \nabla f_\varepsilon(U_P + w)[\partial_{P_i} U_P] + O(\varepsilon^{1+\gamma}) \\ &= \nabla f_\varepsilon(U_P)[\partial_{P_i} U_P] + D^2 f_\varepsilon(U_P)[w, \partial_{P_i} U_P] \\ &\quad + (\nabla f_\varepsilon(U_P + w) - \nabla f_\varepsilon(U_P) - D^2 f_\varepsilon(U_P)[w])[\partial_{P_i} U_P] + O(\varepsilon^{1+\gamma}). \end{aligned}$$

But

$$\|\nabla f_\varepsilon(U_P + w) - \nabla f_\varepsilon(U_P) - D^2 f_\varepsilon(U_P)[w]\| = o(\|w\|) = o(\varepsilon)$$

and, moreover, by (4.1) also $D^2 f_\varepsilon(U_P)[w, \partial_{P_i} U_P] = O(\varepsilon^{1+\gamma})$, therefore

$$(5.6) \quad \partial_{P_i} \mathcal{A}_\varepsilon(Q) = \nabla f_\varepsilon(U_P)[\partial_{P_i} U_P] + O(\varepsilon^{1+\gamma}).$$

Let us calculate $\nabla f_\varepsilon(U_P)[\partial_{P_i} U_P]$.

$$\begin{aligned} &\nabla f_\varepsilon(U_P) [\partial_{P_i} U_P] \\ &= \int_{\Omega_\varepsilon} J(\varepsilon x) \nabla U_P \cdot \nabla(\partial_{P_i} U_P) + \int_{\Omega_\varepsilon} V(\varepsilon x) U_P (\partial_{P_i} U_P) - \int_{\Omega_\varepsilon} U_P^p (\partial_{P_i} U_P) \\ &= J(Q) \int_{\Omega_\varepsilon} \nabla U_P \cdot \nabla(\partial_{P_i} U_P) + V(Q) \int_{\Omega_\varepsilon} U_P (\partial_{P_i} U_P) \\ &\quad + \varepsilon \int_{\mathbb{R}_{\nu(Q)}^-} J'(Q)[x] \nabla U \cdot \nabla(\partial_{P_i} U) + \varepsilon \int_{\mathbb{R}_{\nu(Q)}^-} V'(Q)[x] U (\partial_{P_i} U) \\ &\quad - \int_{\Omega_\varepsilon} U_P^p (\partial_{P_i} U_P) + o(\varepsilon). \end{aligned}$$

Suppose, for simplicity, Q coincides with the origin \mathcal{O} and that part of $\partial\Omega$ is given by $x_N = \psi(x') = (1/2) \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3)$ for $|x'| < \mu$, where μ is some

constant depending only on Ω . Then for $|y'| < \mu/\varepsilon$, the corresponding part of $\partial\Omega_\varepsilon$ is given by $y_N = \Psi(y') = \varepsilon^{-1}\psi(\varepsilon y') = (\varepsilon/2) \sum_{i=1}^{N-1} \lambda_i y_i^2 + O(\varepsilon^2|y'|^3)$.

Since by (3.2) $\partial_{P_i} U_P = -\partial_{x_i} U_P + O(\varepsilon)$, by integration by parts, we get:

$$\begin{aligned} \int_{\mathbb{R}^-_{\nu(Q)}} J'(Q)[x] \nabla U \cdot \nabla(\partial_{P_i} U) &= \frac{1}{2} \int_{\mathbb{R}^-_{\nu(Q)}} \partial_{Q_i} J(Q) |\nabla U|^2, \\ \int_{\mathbb{R}^-_{\nu(Q)}} V'(Q)[x] U(\partial_{P_i} U) &= \frac{1}{2} \int_{\mathbb{R}^-_{\nu(Q)}} \partial_{Q_i} V(Q) U^2. \end{aligned}$$

Therefore we infer

$$\begin{aligned} \nabla f_\varepsilon(U_P)[\partial_{P_i} U_P] &= \frac{1}{2} \partial_{P_i} \left[J(Q) \int_{\Omega_\varepsilon} |\nabla U_P|^2 + V(Q) \int_{\Omega_\varepsilon} U_P^2 \right] \\ &\quad - \int_{\Omega_\varepsilon} U_P^p(\partial_{P_i} U_P) + o(\varepsilon), \end{aligned}$$

and so, by (2.15) and (2.16),

$$\nabla f_\varepsilon(U_P)[\partial_{P_i} U_P] = \varepsilon \left[\left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^N_+} \bar{U}^{p+1} \right] \partial_{Q_i} \Gamma(Q) = \varepsilon c_0 \partial_{Q_i} \Gamma(Q) + o(\varepsilon).$$

By this equation and by (5.6), (5.3) follows immediately. \square

REMARK 5.2. Let us observe that by (5.1) and (5.3), for ε sufficiently small, we have

$$(5.7) \quad \|\mathcal{A}_\varepsilon - c_0 \Gamma\|_{C^1(\partial\Omega)} = O(\varepsilon).$$

REMARK 5.3. By (2.2), it is easy to see that, if J and V are constant on the boundary $\partial\Omega$, then $\bar{\Sigma}$, defined in (1.4), coincides with Σ , defined in (5.2) with the following definitions:

$$\begin{aligned} C_J &\equiv J|_{\partial\Omega}, & C_V &\equiv V|_{\partial\Omega}, \\ k_1 &\equiv \frac{(C_V)^{(p+1)/(p-1)}}{2C_J}, & k_2 &\equiv \sqrt{C_V/C_J}, \\ k_3 &\equiv \frac{(C_V)^{2/(p-1)}}{2}, & k_4 &\equiv -\frac{1}{2} \bar{B} C_J - \left(\frac{1}{2} - \frac{1}{p+1} \right) \bar{A}, \end{aligned}$$

where

$$\begin{aligned} \bar{A} &\equiv \frac{(C_V)^{(p+1)/(p-1)}}{2} \int_{\mathbb{R}^{N-1}} [\bar{U}(x' \sqrt{C_V/C_J}, 0)]^{p+1} |x'|^2 dx', \\ \bar{B} &\equiv \frac{(N-1)(C_V)^{2/(p-1)}}{4} \int_{\mathbb{R}^{N-1}} [\bar{U}(x' \sqrt{C_V/C_J}, 0)]^2 dx'. \end{aligned}$$

6. Proofs of Theorems 1.1 and 1.2

In this section we will state and prove two multiplicity results for (1.1) whose Theorem 1.1 is a particular case. Finally we will prove also Theorem 1.2.

Let us start introducing a topological invariant related to Conley theory.

DEFINITION 6.1. Let M be a subset of \mathbb{R}^N , $M \neq \emptyset$. The *cup long* $l(M)$ of M is defined by

$$l(M) = 1 + \sup\{k \in \mathbb{N} \mid \exists \alpha_1, \dots, \alpha_k \in \check{H}^*(M) \setminus 1, \alpha_1 \cup \dots \cup \alpha_k \neq 0\}.$$

If no such class exists, we set $l(M) = 1$. Here $\check{H}^*(M)$ is the Alexander cohomology of M with real coefficients and \cup denotes the cup product.

Let us recall Theorem 6.4 in Chapter II of [4].

THEOREM 6.2. *Let N a Hilbert–Riemannian manifold. Let $g \in C^2(N)$ and let $M \subset N$ be a smooth compact nondegenerate manifold of critical points of g . Let U be a neighbourhood of M and let $h \in C^1(N)$. Then, if $\|g - h\|_{C^1(\bar{U})}$ is sufficiently small, the function g possesses at least $l(M)$ critical points in U .*

Let us suppose that Γ has a smooth manifold of critical points M . We say that M is nondegenerate (for Γ) if every $x \in M$ is a nondegenerate critical point of $\Gamma|_{M^\perp}$. The Morse index of M is, by definition, the Morse index of any $x \in M$, as critical point of $\Gamma|_{M^\perp}$.

We now can state our first multiplicity result.

THEOREM 6.3. *Let (J) and (V) hold and suppose Γ has a nondegenerate smooth manifold of critical points $M \subset \partial\Omega$. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then (1.1) has at least $l(M)$ solutions that concentrate near points of M .*

PROOF. Fix a δ -neighbourhood M_δ of M such that the only critical points of Γ in M_δ are those in M . We will take $U = M_\delta$.

For ε sufficiently small, by (5.7) and Theorem 6.2, \mathcal{A}_ε possesses at least $l(M)$ critical points, which are solutions of (2.1) by Lemma 4.1. Let $Q_\varepsilon \in M$ be one of these critical points, then $u_\varepsilon^{Q_\varepsilon} = U_{Q_\varepsilon/\varepsilon} + w(\varepsilon, Q_\varepsilon)$ is a solution of (2.1). Therefore

$$u_\varepsilon^{Q_\varepsilon}(x/\varepsilon) \simeq U_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = U^{Q_\varepsilon}\left(\frac{x - Q_\varepsilon}{\varepsilon}\right)$$

is a solution of (1.1). □

Moreover, when we deal with local minima (resp. maxima) of Γ , the preceding results can be improved because the number of positive solutions of (1.1) can be estimated by means of the category and M does not need to be a manifold.

THEOREM 6.4. *Let (J) and (V) hold and suppose Γ has a compact set $X \subset \partial\Omega$ where Γ achieves a strict local minimum (resp. maximum), in the sense that there exist $\delta > 0$ and a δ -neighbourhood $X_\delta \subset \partial\Omega$ of X such that*

$$b \equiv \inf\{\Gamma(Q) : Q \in \partial X_\delta\} > a \equiv \Gamma|_X, \quad (\text{resp. } \sup\{\Gamma(Q) : Q \in \partial X_\delta\} < \Gamma|_X).$$

Then there exists $\varepsilon_0 > 0$ such that (1.1) has at least $\text{cat}(X, X_\delta)$ solutions that concentrate near points of X_δ , provided $\varepsilon \in (0, \varepsilon_0)$. Here $\text{cat}(X, X_\delta)$ denotes the Lusternik–Schnirelman category of X with respect to X_δ .

PROOF. We will treat only the case of minima, being the other one similar. We set $Y = \{Q \in X_\delta : \mathcal{A}_\varepsilon(Q) \leq c_0(a+b)/2\}$. By (5.1) it follows that there exists $\varepsilon_0 > 0$ such that

$$(6.1) \quad X \subset Y \subset X_\delta,$$

provided $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $Q \in \partial X_\delta$ then $\Gamma(Q) \geq b$ and hence

$$\mathcal{A}_\varepsilon(Q) \geq c_0\Gamma(Q) + O(\varepsilon) \geq c_0b + O(\varepsilon).$$

On the other side, if $Q \in Y$ then $\mathcal{A}_\varepsilon(Q) \leq c_0(a+b)/2$. Hence, for ε small, Y cannot meet ∂X_δ and this readily implies that Y is compact. Then \mathcal{A}_ε possesses at least $\text{cat}(Y, X_\delta)$ critical points in X_δ . Using (6.1) and the properties of the category one gets

$$\text{cat}(Y, Y) \geq \text{cat}(X, X_\delta),$$

and the result follows. \square

REMARK 6.5. Let us observe that the (a) of Theorem 1.1 is a particular case of Theorem 6.3 while the (b) of Theorem 1.1 is a particular case of Theorem 6.4.

Let us now prove Theorem 1.2.

PROOF OF THEOREM 1.2. Let Q be a minimum point of $\bar{\Sigma}$ (the other case is similar) and let $\Lambda \subset \partial\Omega$ be a compact neighbourhood of Q such that

$$\min_{\Lambda} \bar{\Sigma} < \min_{\partial\Lambda} \bar{\Sigma}.$$

By (5.1) and Remark 5.3, it is easy to see that for ε sufficiently small, there results:

$$\min_{\Lambda} \mathcal{A}_\varepsilon < \min_{\partial\Lambda} \mathcal{A}_\varepsilon.$$

Hence, \mathcal{A}_ε possesses a critical point Q_ε in Λ . By Lemma 4.1 we have that $u_{\varepsilon, Q_\varepsilon} = U_{Q_\varepsilon/\varepsilon} + w(\varepsilon, Q_\varepsilon)$ is a critical point of f_ε and so a solution of problem (2.1). Therefore

$$u_{\varepsilon, Q_\varepsilon}(x/\varepsilon) \simeq U_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = U^{Q_\varepsilon} \left(\frac{x - Q_\varepsilon}{\varepsilon} \right)$$

is a solution of (1.1). \square

REFERENCES

- [1] A. AMBROSETTI AND M. BADIÀLE, *Variational perturbative methods and bifurcation of bound states from the essential spectrum*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1131–1161.
- [2] A. AMBROSETTI, M. BADIÀLE AND S. CINGOLANI, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal. **159** (2001), 253–271.
- [3] A. AMBROSETTI, A. MALCHIODI AND S. SECCHI, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Rational Mech. Anal. **159** (2001), 253–271.
- [4] K. C. CHANG, *Infinite Dimensional Morse Theory and Multiple Solutions Problems*, Birkhäuser, 1993.
- [5] E. N. DANCER AND S. YAN, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific J. Math. **189** (1999), 241–262.
- [6] M. DEL PINO, P. FELMER AND J. WEI, *On the role of mean curvature in some singularly perturbed Neumann problems*, SIAM J. Math. Anal. **31** (1999), 63–79.
- [7] M. GROSSI, A. PISTOIA AND J. WEI, *Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory*, Cal. Var. Partial Differential Equations **11** (2000), 143–175.
- [8] C. GUI, *Multipeak solutions for a semilinear Neumann problem*, Duke Math. J. **84** (1996), 739–769.
- [9] C. GUI AND J. WEI, *On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems*, Canad. J. Math. **52** (2000), 522–538.
- [10] Y. Y. LI, *On a singularly perturbed equation with Neumann boundary condition*, Comm. Partial Differential Equations **23** (1998), 487–545.
- [11] W. M. NI AND I. TAKAGI, *On the shape of the least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), 819–851.
- [12] W. M. NI AND I. TAKAGI, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. **70** (1993), 247–281.
- [13] A. POMPONIO AND S. SECCHI, *On a class of singularly perturbed elliptic equations in divergence form: existence and multiplicity results*, Preprint SISSA, 36/2003/M (2003).
- [14] J. WEI, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Differential Equations **134** (1997), 104–133.

Manuscript received January 24, 2004

ALESSIO POMPONIO
SISSA
via Beirut 2/4
I-34014 Trieste, ITALY

E-mail address: pomponio@sissa.it