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ARONSZAJN TYPE RESULTS FOR VOLTERRA EQUATIONS AND INCLUSIONS

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Abstract. This paper discusses the topological structure of the set of solutions for a variety of Volterra equations and inclusions. Our results rely on the existence of a maximal solution for an appropriate ordinary differential equation.

1. Introduction

This paper looks at the structure of the set of solutions for various Volterra equations and inclusions. The results are new and extend previously known results in the literature (see [3], [6], [7], [9], [11] and the references therein). In Section 2 we discuss the abstract Volterra equation

$$\begin{cases} y'(t) = V(y)(t) & \text{a.e. } t \in [0,T], \\ y(0) = x_0 \in \mathbb{R}^n \end{cases}$$

(here $T > 0$ and V is an abstract Volterra operator), and in Section 3 we discuss the differential inclusion

$$\begin{cases} y'(t) \in F(t, y(t)) & \text{a.e. } t \in [0,T], \\ y(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1.2}$$

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and the integral inclusion

$$y(t) \in h(t) + \int_0^t k(t,s)F(s,y(s))\,ds \quad \text{for } t \in [0,T]. \tag{1.3}$$

For (1.2) and (1.3) we have $F\colon [0,T] \times \mathbb{R}^n \to CK(\mathbb{R}^n)$ where $CK(\mathbb{R}^n)$ denotes the family of nonempty, convex, compact subsets of $\mathbb{R}^n$. In addition for (1.3) we have $h\colon [0,T] \to \mathbb{R}^n$ and the matrix valued function $k\colon \{(s,t) : 0 \le s \le t \le T\} \to L_{n\times n}[0,T]$. For (1.1)–(1.3) we will show that the solution set is a continuum (in the appropriate space) if our nonlinearity (V in (1.1) and F in (1.2) and (1.3)) is bounded by a L^1-Carathéodory function g and if the ordinary differential equation

$$\begin{cases} v'(t) = ag(t,v(t)) & \text{a.e. } t \in [0,T], \\ v(0) = a_0 \end{cases}$$

has a maximal solution (here $a = 1$ and $a_0 = |x_0|$ for (1.1) and (1.2) whereas $a = \sup_{t\in[0,T]} k(t)$ and $a_0 = |h|_0$ for (1.3)). Recall a function $g\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function if

(a) the map $t \mapsto g(t,y)$ is measurable for all $y \in \mathbb{R}$,
(b) the map $y \mapsto g(t,y)$ is continuous for a.e. $t \in [0,T]$, and
(c) for any $r > 0$ there exists $\mu_r \in L^1[0,T]$ such that $|y| \le r$ implies $|g(t,y)| \le \mu_r(t)$ for a.e. $t \in [0,T]$.

The analysis in Section 2 makes use of the following well known results from the literature.

Theorem 1.1 (Banach, Alaoglu [2], [10]). *The unit ball in the dual of a normed space is compact in the weak* topology.*

Theorem 1.2 ([10]). *The unit ball in the dual of a normed space X is metrizable in the weak* topology if and only if X is separable.*

Theorem 1.3 (Eberlein, Šmulian [5]). *Suppose K is weakly closed in a Banach space E. Then the following are equivalent:*

(a) *K is weakly compact,*
(b) *K is weakly sequentially compact.*

Theorem 1.4 ([4]). *For every $g \in L^1[0,T]$ the set*

$$\{f \in L^1([0,T],\mathbb{R}^n) : |f(t)| \le g(t) \text{ for a.e. } t \in [0,T]\}$$

is compact in the weak topology of $L^1([0,T],\mathbb{R}^n)$.

If we supply $L^\infty([0,T],\mathbb{R}^n)$ with the weak* topology then we will let

$$A([0,T],\mathbb{R}^n) = \{f \in C([0,T],\mathbb{R}^n) : f' \in L^\infty([0,T],\mathbb{R}^n)\}.$$

If we supply $L^1([0,T],\mathbb{R}^n)$ with the weak topology then we let

$$A^1([0,T],\mathbb{R}^n)=\{f\in C([0,T],\mathbb{R}^n): f'\in L^1([0,T],\mathbb{R}^n)\}.$$

2. Solution sets for abstract Volterra operators

Let $S_V(x_0;\mathbb{R}^n)$ denote the solution set of (1.1) (of course solutions to (1.1) are sought in $AC([0,T],\mathbb{R}^n)$). In [9] we established the following result using a well known result from the literature (see [6], [11]).

Theorem 2.1. *Suppose the following conditions hold:*

(2.1) *V is an abstract Volterra operator i.e. if $x(t)=y(t)$ for $t\in[0,\varepsilon]$, $\varepsilon\le T$, then $V(x)(t)=V(y)(t)$ for a.e. $t\in[0,\varepsilon]$,*

(2.2) *$V\colon C([0,T],\mathbb{R}^n)\to L^1([0,T],\mathbb{R}^n)$ is a continuous operator, and*

(2.3) *there exists $\mu\in L^1[0,T]$ such that for any $y\in C([0,T],\mathbb{R}^n)$ we have $|V(y)(t)|\le\mu(t)$ for a.e. $t\in[0,T]$.*

Then $S_V(x_0;\mathbb{R}^n)$ is a nonempty compact connected set in $C([0,T],\mathbb{R}^n)$ (in fact $S_V(x_0;\mathbb{R}^n)$ is a R_δ set).

It is possible to discuss the solution set in $A([0,T],\mathbb{R}^n)$ if we replace (2.3) with a stronger condition.

Theorem 2.2. *Suppose* (2.1) *and* (2.2) *hold and in addition assume the following condition is satisfied:*

(2.4) *there exists a constant $M>0$ such that for any $y\in C([0,T],\mathbb{R}^n)$ we have $|V(y)(t)|\le\mu(t)$ for a.e. $t\in[0,T]$.*

Then $S_V(x_0;\mathbb{R}^n)$ is a nonempty compact connected set in $A([0,T],\mathbb{R}^n)$.

Proof. We first show $S_V(x_0;\mathbb{R}^n)$ is a compact set in $A([0,T],\mathbb{R}^n)$. Let $\{y_\alpha\}_{\alpha\in\Lambda}$ be a Moore–Smith sequence in $A([0,T],\mathbb{R}^n)$ with $y_\alpha\in S_V(x_0;\mathbb{R}^n)$ for each $\alpha\in\Lambda$. From (2.4) we know that $w_\alpha(t)=y'_\alpha(t)/M$ belongs to the unit ball of $L^\infty([0,T],\mathbb{R}^n)$. Now Theorems 1.1 and 1.2 guarantee that there is a subsequence N of Λ with y'_m/M (respectively y'_m) converging weak* to a $w\in L^\infty([0,T],\mathbb{R}^n)$ (respectively Mw) as $m\to\infty$ in N. Lets look at $\{y_m\}_{m\in N}$. Note $y_m\in S_V(x_0;\mathbb{R}^n)$ and $S_V(x_0;\mathbb{R}^n)$ is a compact subset of $C([0,T],\mathbb{R}^n)$ by Theorem 2.1.Thus there exists $y\in C([0,T],\mathbb{R}^n)\cap S_V(x_0;\mathbb{R}^n)$ and a subsequence N_0 of N with $y_m\to y$ in $C([0,T],\mathbb{R}^n)$ as $m\to\infty$ in N_0. Also from above we have y'_m converging weak* to Mw as $m\to\infty$ in N_0. Now [1, p. 14] guarantees that y'_m converges weakly in $L^1([0,T],\mathbb{R}^n)$ to Mw as $m\to\infty$ in N_0. Also since

$$y_m(t)=y_m(0)+\int_0^t y'_m(s)\,ds$$

we have immeditaely that

$$y(t) = y(0) + \int_0^t Mw(s)\,ds,$$

so $y' = Mw$ a.e. As a result $y_m \to y$ in $C([0,T],\mathbb{R}^n)$ and y'_m converging weak* to y' as $m \to \infty$ in N_0. As a result $S_V(x_0;\mathbb{R}^n)$ is a compact subset of $A([0,T],\mathbb{R}^n)$.

Next we show $S_V(x_0;\mathbb{R}^n)$ is a connected set in $A([0,T],\mathbb{R}^n)$. We argue by contradiction. Suppose

$$S_V(x_0;\mathbb{R}^n) = A \cup B$$

where A and B are nonempty closed disjoint subsets of $A([0,T],\mathbb{R}^n)$. In particular A and B are nonempty disjoint subsets of $C([0,T],\mathbb{R}^n)$, so if we show A and B are closed subsets of $C([0,T],\mathbb{R}^n)$ then we have a contradiction since $S_V(x_0;\mathbb{R}^n)$ is a connected subset of $C([0,T],\mathbb{R}^n)$ by Theorem 2.1. Let $\{y_m\}_1^\infty \subseteq A$ with $y_m \to y$ in $C([0,T],\mathbb{R}^n)$. As above there exists a subsequence N of $\{1,2,\dots\}$ with $w_m = y'_m/M$ converging weak* to a $w \in L^\infty([0,T],\mathbb{R}^n)$ as $m \to \infty$ in N and so [1, p. 14] guarantees that y'_m converges weakly in $L^1([0,T],\mathbb{R}^n)$ to Mw as $m \to \infty$ in N. Also as above $y' = Mw$ a.e. As a result $y_m \to y$ in $C([0,T],\mathbb{R}^n)$ and y'_m converging weak* to y' as $m \to \infty$ in N. Thus since A is a closed subset of $A([0,T],\mathbb{R}^n)$ we have that $y \in A$. Thus A (and similarly B) is closed in $C([0,T],\mathbb{R}^n)$. As a result $S_V(x_0;\mathbb{R}^n)$ is a connected subset of $A([0,T],\mathbb{R}^n)$. □

Essentially the same reasoning as in Theorem 2.2 establishes the next result.

Theorem 2.3. *Suppose* (2.1)–(2.3) *hold. Then* $S_V(x_0;\mathbb{R}^n)$ *is a nonempty compact connected set in* $A^1([0,T],\mathbb{R}^n)$.

Proof. The result follows as in Theorem 2.2 (except here we use Theorems 1.3 and 1.4) with $w_\alpha(t) = y'_\alpha(t)$. □

We next remove the "global" boundedness assumption (2.3) and (2.4). First we establish general existence principles. Assume (2.1) and (2.2) hold. In addition suppose *one* of the following conditions hold:

(2.5) for each $r > 0$ there exists $M_r > 0$ such that for any $y \in C([0,T],\mathbb{R}^n)$ with $|y|_0 = \sup_{t\in[0,T]} |y(t)| \le r$ we have $|V(y)(t)| \le M_r$ for a.e. $t \in [0,T]$

or

(2.6) for each $r > 0$ there exists $\mu_r \in L^1[0,T]$ such that for any $y \in C([0,T],\mathbb{R}^n)$ with $|y|_0 \le r$ we have $|V(y)(t)| \le \mu_r(t)$ for a.e. $t \in [0,T]$.

For our general existence principles we also assume the following condition is satsified:

(2.7) there exists $M_0 > |x_0|$ with $|y|_0 < M_0$ for any possible solution y to (1.1).

Let $\varepsilon > 0$ be given and let $\tau_\varepsilon\colon \mathbb{R}^n \to [0,1]$ be the Urysohn function for

$$(\overline{B}(0,M_0), \mathbb{R}^n \setminus B(0, M_0+\varepsilon))$$

such that $\tau_\varepsilon(x) = 1$ if $|x| \le M_0$ and $\tau_\varepsilon(x) = 0$ if $|x| \ge M_0 + \varepsilon$. Let $V_\varepsilon(x) = \tau_\varepsilon(x)V(x)$ and consider the problem

$$\begin{cases} y'(t) = V_\varepsilon(y(t)) \quad \text{a.e. } t \in [0,T], \\ y(0) = x_0. \end{cases} \tag{2.8}$$

Let $S_{V_\varepsilon}(x_0;\mathbb{R}^n)$ denote the solution set of (2.8).

Theorem 2.4. *Suppose* (2.1)*,* (2.2)*,* (2.5) *and* (2.7) *hold. Let* $\varepsilon > 0$ *be given and assume*

$$|w|_0 < M_0 \quad \textit{for any possible solution } w \textit{ to } (2.8). \tag{2.9}$$

Then $S_V(x_0;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $A([0,T],\mathbb{R}^n)$*.*

Proof. Notice (2.7) and (2.9) imply $S_V(x_0;\mathbb{R}^n) = S_{V_\varepsilon}(x_0;\mathbb{R}^n)$. It is easy to see that V_ε satisfies (2.1), (2.2) and (2.4) (with V replaced by V_ε). Now Theorem 2.2 implies $S_{V_\varepsilon}(x_0;\mathbb{R}^n)$ is a nonempty compact connected subset of $A([0,T],\mathbb{R}^n)$. □

Combining Theorem 2.3 with the argument in Theorem 2.4 immediately yields our next result.

Theorem 2.5. *Suppose* (2.1)*,* (2.2)*,* (2.6) *and* (2.7) *hold. Let* $\varepsilon > 0$ *be given and assume* (2.9) *is satisfied. Then* $S_V(x_0;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $A^1([0,T],\mathbb{R}^n)$*.*

Remark 2.6. In Theorems 2.4 and 2.5 notice that $S_V(x_0;\mathbb{R}^n)$ is a R_δ subset of $C([0,T],\mathbb{R}^n)$.

These existence principles now enable us to discuss the structure of the solution set to (1.1) in a very general setting.

Theorem 2.7. *Suppose* (2.1) *and* (2.2) *hold. In addition assume the following conditions are satisfied:*

(2.10) *there exists a* L^1*-Carathéodory function* $g\colon [0,T]\times[0,\infty) \to [0,\infty)$ *such that for any* $y \in C([0,T],\mathbb{R}^n)$ *we have* $|V(y)(t)| \le g(t,|y(t)|)$ *for almost every* $t \in [0,T]$*,*

and

(2.11) *the problem*

$$\begin{cases} v'(t) = g(t,v(t)) \quad \textit{a.e. } t \in [0,T], \\ v(0) = |x_0|, \end{cases}$$

has a maximal solution $r(t)$ on $[0,T]$.

Then $S_V(x_0;\mathbb{R}^n)$ is a nonempty compact connected subset of $A^1([0,T],\mathbb{R}^n)$.

Proof. Let $\varepsilon > 0$ be given and $M_0 = \sup_{t\in[0,T]} r(t) + 1 = r(T) + 1$. We will show any possible solution u of (1.1) satisfies $|u|_0 < M_0$ and any possible solution y of (2.8) satisfies $|y|_0 < M_0$. If this is true then Theorem 2.5 guarantees the result. Suppose u is a possible solution of (1.1). Let $t \in [0,T]$ and we will show $|u(t)| < M_0$. If $|u(t)| \le |x_0|$ we are finished so it remains to discuss the case when $|u(t)| > |x_0|$. In this case there exists $a \in [0,t)$ with

$$|u(s)| > |x_0| \quad \text{for } s \in (a,t] \text{ and } |u(a)| = |x_0|.$$

Also

$$|u(s)|' \le |u'(s)| = |Vu(s)| \le g(s,|u(s)|) \quad \text{a.e. on } (a,t)$$

so

$$\begin{cases} |u(s)|' \le g(s,|u(s)|) & \text{a.e. on } (a,t), \\ |u(a)| = |x_0|. \end{cases}$$

Now a standard comparison theorem for ordinary differential equations in the real case [8, Theorem 1.10.2] guarantees that $|u(s)| \le r(s)$ for $s \in [a,t]$. In particular $|u(t)| \le r(t)$. As a result $|u|_0 < M_0$. Next suppose y is a possible solution of (2.8). Let $t \in [0,T]$ and assume $|y(t)| > |x_0|$. Then there exists $a \in [0,t)$ with

$$|y(s)| > |x_0| \quad \text{for } s \in (a,t] \text{ and } |y(a)| = |x_0|.$$

Also since

$$|y(s)|' \le |y'(s)| = |\tau_\varepsilon(y(s))V(y)(s)| \le |V(y)(s)| \le g(s,|y(s)|) \quad \text{a.e. on } (a,t)$$

we have as above $|y(s)| \le r(s)$ for $s \in [a,t]$. In particular $|y(t)| \le r(t)$. As a result $|y|_0 < M_0$. □

Remark 2.8. In Theorem 2.7 notice that $S_V(x_0;\mathbb{R}^n)$ is a R_δ subset of $C([0,T],\mathbb{R}^n)$.

Corollary 2.9. *Suppose* (2.1) *and* (2.2) *hold. In addition assume the following conditions are satisfied:*

(2.12) *there exists $\alpha \in L^1[0,T]$ and a continuous function $g\colon[0,\infty) \to (0,\infty)$ such that $|V(y)(t)| \le \alpha(t)g(|y(t)|)$ for almost every $t \in [0,T]$ and all $y \in C([0,T],\mathbb{R}^n)$,*

and

$$\int_0^T \alpha(s)\,ds < \int_{|x_0|}^\infty \frac{dx}{g(x)}. \tag{2.13}$$

Then $S_V(x_0;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $A^1([0,T],\mathbb{R}^n)$.

Proof. Let

$$r(t) = I^{-1}\bigg(\int_0^t \alpha(s)\,ds\bigg) \quad \text{for } t\in[0,1] \text{ with } I(z)=\int_0^z \frac{ds}{g(s)}.$$

Now (2.14) guarantees that r is well defined. The result is immediate from Theorem 2.7 once we show

$$\begin{cases} v'(t)=\alpha(t)g(v(t)) & \text{a.e. } t\in[0,T],\\ v(0)=|x_0| \end{cases} \tag{2.14}$$

has a maximal solution given by $r(t)$. Let y be a solution of (2.14), so

$$\frac{y'(t)}{g(y(t))}=\alpha(t)$$

and so integration from 0 to t gives $y(t)=r(t)$. □

Remark 2.10. One could obtain an analogue of Theorem 2.7 (with the solution set in $A([0,T],\mathbb{R}^n)$) if we use Theorem 2.4 instead of Theorem 2.5 in the proof of Theorem 2.7. We leave the obvious details to the reader.

3. Solution sets for differential and integral inclusions

In this section we first discuss the differential inclusion (1.2). We let $S(x_0;\mathbb{R}^n)$ denote the solution set of (1.2). The following result can be found in [3] and [6].

Theorem 3.1. *Suppose the following conditions hold:*

(3.1) $x\mapsto F(t,x)$ *is upper semicontinuous for a.e.* $t\in[0,T]$,
(3.2) $t\mapsto F(t,x)$ *is measurable for every* $x\in\mathbb{R}^n$,

and

(3.3) *there exists* $\mu\in L^1[0,T]$ *with* $|F(t,x)|\le\mu(t)$ *for a.e.* $t\in[0,T]$ *and every* $x\in\mathbb{R}^n$.

Then $S(x_0;\mathbb{R}^n)$ *is a nonempty compact connected set in* $C([0,T],\mathbb{R}^n)$ (*in fact* $S(x_0;\mathbb{R}^n)$ *is a* R_δ *set*).

Essentially the same reasoning as in Section 2 immediately yield the following results.

Theorem 3.2. *Suppose* (3.1) *and* (3.2) *hold and in addition assume the following condition is satisfied:*

(3.4) *there exists* $M>0$ *with* $|F(t,x)|\le M$ *for a.e.* $t\in[0,T]$ *and every* $x\in\mathbb{R}^n$.

Then $S(x_0;\mathbb{R}^n)$ *is a nonempty compact connected set in* $A([0,T],\mathbb{R}^n)$.

Theorem 3.3. *Suppose* (3.1)–(3.3) *hold. Then* $S(x_0;\mathbb{R}^n)$ *is a nonempty compact connected set in* $A^1([0,T],\mathbb{R}^n)$.

Next suppose (3.1) and (3.2) hold. In addition assume *one* of the following conditions hold:

(3.5) for each $r>0$ there exists $M_r>0$ with $|F(t,x)|\le M_r$ for a.e. $t\in[0,T]$ and every $x\in\mathbb{R}^n$ with $|x|\le r$

or

(3.6) for each $r>0$ there exists $\mu_r\in L^1[0,T]$ with $|F(t,x)|\le\mu_r(t)$ for a.e. $t\in[0,T]$ and every $x\in\mathbb{R}^n$ with $|x|\le r$.

For our general existence principles we also assume the following condition is satisfied:

(3.7) there exists $M_0>|x_0|$ with $|y|_0<M$ for any possible solution y to (1.2).

Let $\varepsilon>0$ be given and let τ_ε be as in Section 2. Let $\widetilde{F}(t,x)=\tau_\varepsilon(x)F(t,x)$ and consider the problem

$$\begin{cases} y'(t)\in\widetilde{F}(t,y(t)) & \text{a.e. } t\in[0,T],\\ y(0)=x_0. \end{cases}\tag{3.8}$$

Let $S_\varepsilon(x_0;\mathbb{R}^n)$ denote the solution set of (3.8).

Theorem 3.4. *Suppose* (3.1), (3.2), (3.5) *and* (3.7) *hold. Let* $\varepsilon>0$ *be given and assume*

$$|w|_0<M_0 \text{ for any possible solution } w \text{ to } (3.8).\tag{3.9}$$

Then $S(x_0;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $A([0,T],\mathbb{R}^n)$.

Theorem 3.5. *Suppose* (3.1), (3.2), (3.6) *and* (3.7) *hold. Let* $\varepsilon>0$ *be given and assume* (3.9) *holds. Then* $S(x_0;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $A^1([0,T],\mathbb{R}^n)$.

Theorem 3.6. *Suppose* (3.1) *and* (3.2) *hold. In addition assume the following conditions are satisfied:*

(3.10) *there exists a* L^1*-Carathéodory function* $g\colon[0,T]\times[0,\infty)\to[0,\infty)$ *such that* $|F(t,x)|\le g(t,|x|)$ *for a.e.* $t\in[0,T]$ *and all* $x\in\mathbb{R}^n$,

and

(3.11) *the problem*

$$\begin{cases} v'(t)=g(t,v(t)) & \text{a.e. } t\in[0,T],\\ v(0)=|x_0| \end{cases}$$

has a maximal solution $r(t)$ *on* $[0,T]$.

Then $S(x_0;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $A^1([0,T],\mathbb{R}^n)$.

Remark 3.7. In Theorems 3.4–3.6 notice that $S(x_0;\mathbb{R}^n)$ is a R_δ subset of $C([0,T],\mathbb{R}^n)$.

It is possible to extend these results to the integral inclusion (1.3) (however an extra condition on g will be needed in (3.10)). Let $S(h;\mathbb{R}^n)$ denote the solution set of (1.3). The following result can be found in [3].

Theorem 3.8. *Suppose* (3.1)–(3.3) *hold and in addition assume the following conditions are satisfied:*

(3.12) $$h \in C([0,T],\mathbb{R}^n)$$

(3.13) *for each* $t \in [0,T]$, $k(t,s)$ *is measurable on* $[0,t]$ *and* $k(t) = \operatorname{esssup} |k(t,s)|$, $0 \le s \le t$, *is bounded on* $[0,T]$,

and

(3.14) *the map* $t \mapsto k_t$ *is continuous from* $[0,T]$ *to* $L^\infty([0,T], L^1_{n\times n}[0,T])$, *here* $k_t(s) = k(t,s)$.

Then $S(h;\mathbb{R}^n)$ *is a nonempty compact connected set* $C([0,T],\mathbb{R}^n)$.

Next suppose (3.1) and (3.2) hold. In addition assume the following conditions are satisfied:

(3.15) for each $r > 0$ there exists $\mu_r \in L^1[0,T]$ with $|F(t,x)| \le \mu_r(t)$ for a.e. $t \in [0,T]$ and every $x \in \mathbb{R}^n$ with $|x| \le r$,

and

(3.16) there exists $M_0 > |h|_0$ with $|y|_0 < M$ for any possible solution y to (1.3).

Let $\varepsilon > 0$ be given and let τ_ε and $\widetilde{F}$ be as before. Consider the problem

(3.17) $$y(t) \in h(t) + \int_0^t k(t,s)\widetilde{F}(s,y(s))\,ds \quad \text{for } t \in [0,T]$$

and let $S_\varepsilon(h;\mathbb{R}^n)$ denote the solution set of (3.17). Essentially the same reasoning as in Section 2 immediately yields the following result.

Theorem 3.9. *Suppose* (3.1), (3.2), (3.12)–(3.16) *hold. Let* $\varepsilon > 0$ *be given and assume*

(3.18) $$|w|_0 < M_0 \text{ for any possible solution } w \text{ to } (3.17).$$

Then $S(h;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $C([0,T],\mathbb{R}^n)$.

Theorem 3.10. *Suppose* (3.1), (3.2), (3.12)–(3.14) *hold. In addition suppose the following conditions are satisfied:*

(3.19) *there exists a* L^1*-Carathéodory function* $g\colon [0,T]\times[0,\infty)\to[0,\infty)$ *such that* $|F(t,x)|\le g(t,|x|)$ *for a.e.* $t\in[0,T]$ *and all* $x\in\mathbb{R}^n$,

(3.20) $g(t,x)$ *is nondecreasing in* x *for a.e.* $t\in[0,T]$,

and

(3.21) *the problem*

$$\begin{cases} v'(t) = (\sup_{t\in[0,T]} k(t))g(t,v(t)) & \text{a.e. } t\in[0,T],\\ v(0)=|h|_0, \end{cases}$$

has a maximal solution $r(t)$ *on* $[0,T]$.

Then $S(h;\mathbb{R}^n)$ *is a nonempty compact connected subset of* $C([0,T],\mathbb{R}^n)$.

Proof. We will apply Theorem 3.9 with $\varepsilon>0$ and $M_0=\sup_{t\in[0,T]} r(t)+1$. Let u be a possible solution of (1.3). Then

$$|u(t)|\le |h|_0 + (\sup_{t\in[0,T]} k(t))\int_0^t g(s,|u(s)|)\,ds \equiv v(t)$$

for $t\in[0,T]$. Now (3.20) implies

$$v'(t) = (\sup_{t\in[0,T]} k(t))g(t,|u(t)|) \le (\sup_{t\in[0,T]} k(t))g(t,v(t))$$

almost everywhere. So

$$\begin{cases} v'(t)\le (\sup_{t\in[0,T]} k(t))g(t,v(t)) & \text{for a.e. } t\in[0,T],\\ v(0)=|h|_0. \end{cases}$$

Now [8, Theorem 1.10.2] guarantees that $v(t)\le r(t)$ for $t\in[0,T]$, so $|u(t)|<M_0$ for $t\in[0,T]$. A similar argument guarantees that $|y(t)|<M_0$, $t\in[0,T]$, for any possible solution y of (3.17). □

References

[1] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer–Verlag, New York, 1984.

[2] J. Conway, *A Course in Functional Analysis*, Springer–Verlag, New York, 1990.

[3] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, New York, 1991.

[4] R. Datko, *On the integration of set valued mappings in Banach spaces*, Fund. Math. **78** (1973), 205–208.

[5] N. Dunford and J. T. Schwartz, *Linear operators, Part* 1, Interscience Publishers, New York, 1958.

[6] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Maps,*, Kluwer Acad. Publishers, Dordrecht, 1999.

[7] R. Kannan and D. O'Regan, *A note on the solution set of integral inclusions*, J. Integral Equations Appl. **12** (2000), 85–94.

[8] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, vol. I, Academic Press, New York, 1969.

[9] D. O'Regan, *A note on the topological structure of the solution set of abstract Volterra equations*, Proc. Roy. Irish Acad. Sect. A **99** (1999), 67–74.

[10] C. Swartz, *An introduction to functional analysis*, Marcel Dekker, New York, 1992.

[11] S. Szufla, *Set of fixed points of nonlinear mappings in function spaces*, Funkcial. Ekvac. **22** (1979), 121–126.

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