

**DEGREE COMPUTATIONS  
FOR POSITIVELY HOMOGENEOUS  
DIFFERENTIAL EQUATIONS**

CHRISTIAN FABRY — PATRICK HABETS

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ABSTRACT. We study  $2\pi$ -periodic solutions of

$$u'' + f(t, u) = 0$$

using positively homogeneous asymptotic approximations of this equation near zero and infinity. Our main results concern the degree of  $I - P$ , where  $P$  is the Poincaré map associated to these approximations. We indicate classes of problems, some with degree 1 and others with degree different from 1. Considering results based on first order approximations, we work out examples of equations for which the degree is the negative of any integer.

### 1. Introduction

The idea to study a boundary value problem associated to the scalar equation

$$(1.1) \quad u'' + f(t, u) = 0$$

assuming the nonlinearity to be asymptotically positively homogeneous goes back at least to J. Leray in 1933. As it is noted in [19], J. Leray has considered (see [15, I-7]) an integral equation which, in a particular case, is equivalent to the periodic

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problem associated with (1.1). His main assumption can be reinterpreted as

$$\lim_{u \rightarrow \pm\infty} \frac{f(t, u)}{u} = p,$$

where  $p \in \mathbb{R}$  does not belong to the spectrum of the linear problem

$$u'' + \lambda u = 0, \quad u(0) = 0, \quad u(\pi) = 0.$$

More explicitly, the periodic problem was considered in 1967 by W. S. Loud (see [16]). Since then, a large variety of results of this type has been worked out (see e.g. [13], [18]). An interesting generalization is due to J. Mawhin and J. Ward in [20], (see also [17]). Working such a periodic problem, these authors consider an asymptotic condition

$$q(t) \leq \liminf_{u \rightarrow \pm\infty} \frac{f(t, u)}{u} \leq \limsup_{u \rightarrow \pm\infty} \frac{f(t, u)}{u} \leq Q(t),$$

where  $q(t)$  and  $Q(t)$  are so that the quotient  $f(t, u)/u$  “avoids” the spectrum of the eigenvalue problem

$$u'' + \lambda u = 0, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

A major breakthrough in this direction is due to E. N. Dancer ([3]) in 1977 for Dirichlet problem and to S. Fučík ([8]) in 1980 for other problems. These authors assume the nonlinearity to have different asymptotic behaviour at plus and minus infinity

$$\lim_{u \rightarrow -\infty} \frac{f(t, u)}{u} = \nu, \quad \lim_{u \rightarrow +\infty} \frac{f(t, u)}{u} = \mu.$$

Here the existence of a solution depends upon the position of the pair  $(\mu, \nu)$  with respect to a set of points which are since then called the Fučík spectrum. Extensions of this approach can be found among other works in [2], [10], [7].

A proof of the above results can be based on the computation of a degree associated with a corresponding asymptotic equation. Consider for example the periodic problem

$$(1.2) \quad u'' + f(t, u) = 0, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

and assume

$$(1.3) \quad p_-(t) = \lim_{u \rightarrow -\infty} \frac{f(t, u)}{u}, \quad p_+(t) = \lim_{u \rightarrow +\infty} \frac{f(t, u)}{u}.$$

A possible approach considers the asymptotic equation

$$(1.4) \quad u'' + p_+(t)u^+ - p_-(t)u^- = 0,$$

where  $u^+ = \max\{u, 0\}$ ,  $u^- = \max\{-u, 0\}$ . Let  $u(t; u_0, u_1)$  be the solution of the Cauchy problem

$$u'' + p_+(t)u^+ - p_-(t)u^- = 0 \quad u(0) = u_0, \quad u'(0) = u_1.$$

and

$$P: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (u_0, u_1) \mapsto (u(2\pi; u_0, u_1), u'(2\pi; u_0, u_1))$$

the corresponding Poincaré operator. The main problem is then to compute the Brouwer degree  $d_B(I - P, B_R)$  of  $I - P$  with respect to the disk  $B_R$  of center 0 and radius  $R$ . This can be done using the area preserving property of the Poincaré operator. Notice also that this degree is the same with respect to any disk centered at 0 since equation (1.4) is positively homogeneous. Using the invariance property with respect to an homotopy, and if  $R > 0$  is large enough, it is also the degree of the Poincaré operator for the nonlinear problem. Hence, if this degree is non-zero, it implies existence of a solution of (1.2).

A further class of existence results supposes different asymptotic behaviours at infinity and near zero. This idea was used in 1964 by M. A. Krasnosel'skiĭ considering compressed cones (see [12, p. 138 and Theorem 7.5, p. 246]). In such cases, the Poincaré operator is different at infinity and near zero. If the corresponding degrees are different, it is easy to find  $0 < r < R$  so that for the nonlinear problem the degree of the Poincaré operator with respect to a set  $B_R \setminus \overline{B_r}$  is non-zero. Existence of a solution of (1.2) follows. Several results have been obtained along this direction (see e.g. [5], [11]). In Section 5, we present such a theorem.

This last section is based on degree computations. To this end, we introduce in Section 2 a function  $\Theta$  which associates to the angular coordinate  $\theta$  of a point  $x = (u_0, u_1)$  in the phase plane the angular coordinate  $\Theta(\theta)$  of  $Px$ . We relate then the degree  $d_B(I - P, B_1)$  to the number of zeros of the function  $\Delta(\theta) = \Theta(\theta) - \theta \pmod{2\pi}$ . In Section 4, we recall on one hand conditions on  $p_+$ ,  $p_-$  in (1.4), due to Dong [4], under which the degree is equal to 1. On the other hand, we elaborate alternative conditions ensuring that this degree is different from 1. This is the main result of the paper and the key to prove results as in Section 5.

As shown in Section 2, the degree  $d_B(I - P, B_1)$ , when it is defined, is less or equal to 1. For the problem with constant and positive coefficients

$$(1.5) \quad u'' + \alpha u^+ - \beta u^- = 0, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

this degree can only be equal to 1. It is  $-1$  if  $\alpha$  and  $\beta$  are negative, and 0 if the product  $\alpha\beta$  is negative. For linear problems with variable coefficients the degree can take the values  $\pm 1$ . This is the case for the problem

$$u'' + (\delta + \cos t)u = 0, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where  $\delta \in \mathbb{R}$  is a small parameter. Degrees different from 0 or  $\pm 1$  appear in [6] for the problem

$$u'' + \alpha u^+ - \beta u^- = p(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where  $p \in L^1(0, 2\pi)$ . By the type of argument used in the present paper, it is shown there that, provided that it is defined, the degree  $d_B(I - P, B_R)$ , with respect to balls  $B_R$  of sufficiently large radius  $R$ , is also less or equal to 1 and can take arbitrary large negative values. In Section 3, we give examples of equations (1.4) with positive coefficients for which the degree is the negative of any integer.

## 2. The Poincaré map in polar coordinates

The computation of the degree will rely on a description of the map  $P$  in polar coordinates. For that purpose, let us introduce polar coordinates  $u = r \cos \theta$ ,  $u' = r \sin \theta$ . This transforms (1.4) into

$$(2.1) \quad \begin{aligned} r' &= r \sin \theta [\cos \theta - p_+(t)(\cos \theta)^+ + p_-(t)(\cos \theta)^-], \\ \theta' &= -\sin^2 \theta - \cos \theta [p_+(t)(\cos \theta)^+ - p_-(t)(\cos \theta)^-]. \end{aligned}$$

The equations (1.4) and (2.1) are equivalent if one excludes the trivial solution. We denote by  $(r(t; \theta_0), \theta(t; \theta_0))$  the unique solution of (2.1) satisfying the initial conditions  $r(0; \theta_0) = 1$ ,  $\theta(0; \theta_0) = \theta_0$  and consider the functions

$$(2.2) \quad \begin{aligned} R: \mathbb{R} &\rightarrow \mathbb{R}^+ \setminus \{0\}, & \theta_0 &\mapsto r(2\pi; \theta_0), \\ \Theta: \mathbb{R} &\rightarrow \mathbb{R}, & \theta_0 &\mapsto \theta(2\pi; \theta_0). \end{aligned}$$

Taking into account the property of positive homogeneity of (1.4), the action of the Poincaré map  $P$ , for the period  $2\pi$ , associated to that equation, can be described by

$$P(r \cos \theta_0, r \sin \theta_0) = rR(\theta_0)(\cos \Theta(\theta_0), \sin \Theta(\theta_0)).$$

LEMMA 2.1. *Let  $p_+, p_- \in L^1(0, 2\pi)$ . Then the functions  $R$  and  $\Theta$  defined by (2.2) are of class  $\mathcal{C}^1$ .*

PROOF. This lemma follows repeating the argument used to prove Lemma 2.2 in [14].  $\square$

The following property follows from simple arguments which can also be found in [4] and [9].

LEMMA 2.2. *Let  $p_+, p_- \in L^1(0, 2\pi)$  and define  $R, \Theta$  by (2.2). Then for any  $\theta_0 \in \mathbb{R}$  we have*

$$(2.3) \quad R^2(\theta_0)\Theta'(\theta_0) = 1.$$

PROOF. The proof is an immediate consequence of the property of area conservation for the map  $P$  (see for example Theorem 2 in Section 16 of [1]). Using polar coordinates, this conservation of area implies that

$$R(\theta_0)J(r, \theta_0) = 1,$$

where  $J(r, \theta_0)$  is the Jacobian of the function

$$(r, \theta_0) \mapsto (rR(\theta_0), \Theta(\theta_0)),$$

representing the change of variables  $P$  in polar coordinates. Computing the Jacobian leads to (2.3).  $\square$

Let us assume that  $P$  has no fixed point on the circle  $\partial B_1$ . The Brouwer degree  $d_B(I - P, B_1)$  of  $I - P$  with respect to the unit disk  $B_1$  can then be defined as the number of turns made by the nonzero vector  $x - Px$ , or equivalently  $Px - x$ , around the origin as  $x \in \mathbb{R}^2$  makes one turn along the circle  $\partial B_1$ . By convention, the number of turns is counted positively if  $x$  and  $x - Px$  turn in the same direction, so that  $d_B(I, B_1) = 1$ .

To compute this degree, let us consider the closed curve  $\Gamma$  parametrized by  $Px(s) - x(s)$ , where  $x(s) = (\cos s, \sin s)$  and  $s \in [0, 2\pi]$ . If the degree makes sense, this curve does not go through the origin and, using Lemma 2.1, we can define the argument  $\varphi(s)$  of  $Px(s) - x(s)$  as a continuous function which is periodic modulo  $2\pi$ , i.e.  $\varphi(2\pi) = \varphi(0) + 2k\pi$ . The number  $k$  is the number of turns made by  $\Gamma$  around the origin, i.e. the degree we want to compute.

Consider now  $s$  such that  $\varphi(s) = s \pmod{2\pi}$ . In this case,  $\Theta(s) = \varphi(s) = s \pmod{2\pi}$  and  $R(s) > 1$ . We deduce then from (2.3) that

$$\Theta'(s) - 1 < 0.$$

On the other hand, we compute from

$$\tan(\varphi(s) - s) = \frac{R(s) \sin(\Theta(s) - s)}{R(s) \cos(\Theta(s) - s) - 1}$$

that, in case  $\Theta(s) = \varphi(s) = s \pmod{2\pi}$ ,

$$\varphi'(s) - 1 = \frac{R(s)}{R(s) - 1} (\Theta'(s) - 1) < 0.$$

This means that the graph of  $\varphi$  intersects downwards the lines  $y = s + 2n\pi$ , where  $n \in \mathbb{Z}$ . From this remark, it is easy to see that the degree of  $I - P$  with respect to  $B_1$  is

$$d_B(I - P, B_1) = 1 - z^-,$$

where  $z^-$  is the number of crossing of the graph of  $\varphi$  with the set of lines  $y = s + 2n\pi$ ,  $n \in \mathbb{Z}$ , in the interval  $[0, 2\pi)$ .

The number  $z^-$  can be computed from the zeros of  $\Delta(s) = \Theta(s) - s \pmod{2\pi}$ . Such zeros correspond either to the zeros of  $\varphi(s) - s \pmod{2\pi}$  or to the zeros of  $\varphi(s) + \pi - s \pmod{2\pi}$ . In the first case,  $R(s) > 1$  and we deduce from (2.3) that  $\Theta'(s) < 1$ . In the second case,  $R(s) < 1$  and  $\Theta'(s) > 1$ . It follows that  $z^-$  is exactly the number of crossing, for  $s \in [0, 2\pi)$ , of the graph of  $\Delta(s) = \Theta(s) - s$  with the levels  $2n\pi$ ,  $n \in \mathbb{Z}$ , so that  $\Delta$  has a negative derivative.

Moreover, as the equation (2.1) is periodic in  $\theta$ , we can write  $\theta(t; s + 2\pi) = \theta(t; s) + 2\pi$ , which implies that  $\Delta(s)$  is  $2\pi$ -periodic. Hence, the number  $z^+$  of points  $s \in [0, 2\pi)$  such that the function  $\Delta(s)$ , crosses a value  $2n\pi$ ,  $n \in \mathbb{Z}$ , with a positive slope equals the number of points such that the function  $\Delta(s)$ , crosses a value  $2n\pi$ ,  $n \in \mathbb{Z}$ , with a negative slope. Hence, we also have  $d_B(I - P, B_1) = 1 - z^+$ .

We can still observe, using Lemma 2.2, that  $Px \neq x$  for any  $x \in \partial B_1$  if and only if the function  $\Delta$  does not cross a level  $2n\pi$ ,  $n \in \mathbb{Z}$ , with a vanishing derivative.

We have thus proved the following proposition.

**PROPOSITION 2.3.** *Let  $p_+, p_- \in L^1(0, 2\pi)$  and define  $P$  to be the Poincaré map associated to equation (1.4). Let  $\Theta$  be defined by (2.2) and assume that the function  $\Delta(s) = \Theta(s) - s$  does not cross levels  $2n\pi$ ,  $n \in \mathbb{Z}$ , with a vanishing derivative. Then, the Brouwer degree of  $I - P$  with respect to the disk  $B_1$  is defined and*

$$d_B(I - P, B_1) = 1 - z^- = 1 - z^+,$$

where  $z^-$  (resp.  $z^+$ ) is the number of crossings of the graph of  $\Delta$  with the levels  $2n\pi$ ,  $n \in \mathbb{Z}$ , in the interval  $[0, 2\pi)$ , with negative (resp. positive) derivatives.

**REMARK.** Let  $0 \leq s_1 < s_2 < 2\pi$ . It follows then from uniqueness of solutions of the Cauchy problem that, for all  $t \in [0, 2\pi]$ ,  $\theta(t; s_1) < \theta(t; s_2) < \theta(t; s_1) + 2\pi = \theta(t; s_1 + 2\pi)$ . This implies

$$-2\pi < \Delta(s_2) - \Delta(s_1) = \theta(2\pi; s_2) - \theta(2\pi; s_1) + s_1 - s_2 < 2\pi.$$

Hence, the function  $\Delta$  can cross only one of the levels  $2n\pi$  with  $n \in \mathbb{Z}$ .

### 3. Computing $z^+$ , $z^-$ from a first order approximation

Using a first order approximation, the degree  $d_B(I - P, B_1)$  can be explicitly computed for equations which are perturbations of linear equations. Consider for instance the equation

$$(3.1) \quad u'' + u + q_+(t)u^+ - q_-(t)u^- = 0,$$

together with  $2\pi$ -periodic boundary conditions. Based on a restriction of the  $L^1$ -norm of  $q_+$ ,  $q_-$  the following result holds, the  $L^1$ -norm used being  $\|q\|_{L^1} = \int_0^{2\pi} |q(t)| dt$ .

THEOREM 3.1. *Let  $q_+, q_- \in L^1(0, 2\pi)$ . Define*

$$F_0: \theta_0 \mapsto \int_0^{2\pi} \cos(\theta_0 - t)[q_+(t)(\cos(\theta_0 - t))^+ - q_-(t)(\cos(\theta_0 - t))^-] dt,$$

where  $q_+$  and  $q_-$  are extended to  $\mathbb{R}$  by  $2\pi$ -periodicity, and assume this function has  $2z$  zeros in  $[0, 2\pi)$ , with  $z \neq 0$ , all zeros being simple. Then, provided that

$$(3.2) \quad 3(\|q_+\|_{L^1} + \|q_-\|_{L^1})^2 < |F_0(\theta_0)| + |F_0'(\theta_0)|,$$

for all  $\theta_0 \in [0, 2\pi)$ , we have  $d_B(I - P, B_1) = 1 - z$ , where  $P$  is the Poincaré map for the period  $2\pi$  associated to (3.1).

PROOF. We use Proposition 2.3 with  $p_+ = 1 + q_+$ ,  $p_- = 1 + q_-$ , and will compute the number of crossings of the graph of  $\Delta: \theta_0 \mapsto \Theta(\theta_0) - \theta_0$  with the level  $-2\pi$  using (2.1),  $\Theta$  being defined as before by (2.2). Equation (2.1) gives

$$\theta' = -1 - \cos \theta [q_+(t)(\cos \theta)^+ - q_-(t)(\cos \theta)^-].$$

We will use a homotopy and consider the equation

$$\theta' = -1 - \lambda \cos \theta [q_+(t)(\cos \theta)^+ - q_-(t)(\cos \theta)^-],$$

with  $\lambda \in [0, 1]$ . Its solution, for the initial condition  $\theta(0) = \theta_0$  will be denoted by  $\theta_\lambda(t; \theta_0)$ . It is immediate that

$$(3.3) \quad |\theta_\lambda(t; \theta_0) - (\theta_0 - t)| \leq \lambda(\|q_+\|_{L^1} + \|q_-\|_{L^1}), \quad \text{for } t \in [0, 2\pi].$$

By analogy to the definition of  $\Delta$ , we introduce the function

$$\Delta_\lambda(\theta_0) = \theta_\lambda(2\pi; \theta_0) - \theta_0$$

and compute

$$(3.4) \quad \Delta_\lambda(\theta_0) + 2\pi = -\lambda \int_0^{2\pi} \cos \theta_\lambda [q_+(t)(\cos \theta_\lambda)^+ - q_-(t)(\cos \theta_\lambda)^-] dt,$$

where  $\theta_\lambda$  stands for  $\theta_\lambda(t; \theta_0)$ . On the other hand, the derivative  $\partial \theta_\lambda(t; \theta_0) / \partial \theta_0$  is a solution of the variational equation

$$\eta' = 2\lambda \sin \theta_\lambda [q_+(t)(\cos \theta_\lambda)^+ - q_-(t)(\cos \theta_\lambda)^-] \eta.$$

Consequently,

$$(3.5) \quad \begin{aligned} \Delta_\lambda'(\theta_0) &= \frac{\partial \theta_\lambda(2\pi; \theta_0)}{\partial \theta_0} - 1 \\ &= \exp \left\{ 2\lambda \int_0^{2\pi} \sin \theta_\lambda [q_+(t)(\cos \theta_\lambda)^+ - q_-(t)(\cos \theta_\lambda)^-] dt \right\} - 1. \end{aligned}$$

It is clear from (3.3) that  $\lim_{\lambda \rightarrow 0} \theta_\lambda(t; \theta_0) = \theta_0 - t$ , uniformly for  $t \in [0, 2\pi]$ ,  $\theta_0 \in [0, 2\pi]$ .

It follows now from (3.4), (3.5) that

$$\lim_{\lambda \rightarrow 0} \frac{\Delta_\lambda(\theta_0) + 2\pi}{\lambda} = -F_0(\theta_0), \quad \lim_{\lambda \rightarrow 0} \frac{\Delta'_\lambda(\theta_0)}{\lambda} = -F'_0(\theta_0).$$

Consequently, for  $\lambda$  sufficiently small,  $\Delta_\lambda(\cdot) + 2\pi$  has the same number of zeros in  $[0, 2\pi)$  than  $F_0$ .

Letting  $\lambda$  vary from 0 to 1, the number of zeros of  $\Delta_\lambda(\cdot) + 2\pi$  will remain unchanged, unless  $\Delta_\lambda(\cdot) + 2\pi$  has a multiple zero. But, looking at (3.5), we see that this occurs if and only if, for some  $\theta_0$ , we have  $\Delta_\lambda(\theta_0) + 2\pi = 0$  and

$$G_\lambda(\theta_0) = 2\lambda \int_0^{2\pi} \sin \theta_\lambda [q_+(t)(\cos \theta_\lambda)^+ - q_-(t)(\cos \theta_\lambda)^-] dt = 0.$$

However, using (3.3), we deduce from (3.4), (3.5) that

$$\begin{aligned} |\Delta_\lambda(\theta_0) + 2\pi + \lambda F_0(\theta_0)| &\leq \lambda^2 (\|q_+\|_{L^1} + \|q_-\|_{L^1})^2, \\ |G_\lambda(\theta_0) + \lambda F'_0(\theta_0)| &\leq 2\lambda^2 (\|q_+\|_{L^1} + \|q_-\|_{L^1})^2. \end{aligned}$$

It then results from condition (3.2) that  $\Delta_\lambda(\cdot) + 2\pi$  and  $G_\lambda$  cannot vanish simultaneously for  $\lambda \in (0, 1]$ . The result then follows, taking into account the observation made in the preceding section that the graph of  $\Delta$  can cross only one of the levels  $2n\pi$  ( $n \in \mathbb{Z}$ ).  $\square$

EXAMPLE. Take  $q_+(t) = \varepsilon \cos kt$ , with  $k \in \mathbb{N}$ ,  $\varepsilon \neq 0$ ,  $q_-(t) = 0$ . One computes that

$$F(\theta_0) = -\frac{4\varepsilon}{k^3 - 4k} \cos(k\theta_0) \sin\left(\frac{k\pi}{2}\right).$$

If  $k$  is odd,  $F_0$  has  $2k$  zeros in  $[0, 2\pi)$ . On the other hand, condition (3.2) is fulfilled if

$$|\varepsilon| < \frac{1}{16|k^3 - 4k|}.$$

In that case, Theorem 3.1 applies and, for  $k$  odd,  $d_B(I - P, B_1) = 1 - k$ . When  $k$  is even, the first order approximation used here does not allow the computation of the degree for the above choice of  $q_+$ ,  $q_-$ . However, with more complicated coefficients, any odd value of the degree can be obtained. Take for instance

$$q_+(t) = \varepsilon[(k+4)\cos(k+2)t + (k-2)\cos kt] + \varepsilon^2, \quad q_-(t) = \varepsilon^2.$$

The evaluation of  $F_0$  gives

$$\begin{aligned} F_0(\theta_0) &= \frac{4\varepsilon \sin(k\pi/2)}{k(k+2)} [\cos(k+2)\theta_0 - \cos k\theta_0] + \pi\varepsilon^2 \\ &= -\frac{8\varepsilon \sin(k\pi/2)}{k(k+2)} \sin \theta_0 \sin((k+1)\theta_0) + \pi\varepsilon^2. \end{aligned}$$

It can then be seen that, for  $|\varepsilon|$  small enough,  $\varepsilon \neq 0$ ,  $F_0$  has  $2k - 2$  zeros in  $[0, 2\pi)$ . In that case,  $d_B(I - P, B_1) = 2 - k$ , an odd number.



#### 4. A computation of $z^+$ , $z^-$ based on comparisons

In this section, we again deduce the degree  $d_B(I - P, B_1)$  from computations of  $z^+$  and  $z^-$ . These values are obtained by comparison arguments, for instance by comparing equation (1.4) to equations with piecewise constant coefficients. The first result is a particular case of a result given in [4].

**THEOREM 4.1.** *Let  $p_+, p_- \in L^\infty(0, 2\pi)$ . Assume that, for some  $n \in \mathbb{N}$ , there exist*

$$0 = t_0 < t_1 < \dots < t_n = 2\pi, \quad 0 = s_0 < s_1 < \dots < s_{n+1} = 2\pi$$

and positive numbers  $\lambda_i^+, \lambda_i^-$  for  $i = 1, \dots, n$  and  $\gamma_j^+, \gamma_j^-$  for  $j = 1, \dots, n+1$ , such that

$$(4.1) \quad \left( \frac{1}{\sqrt{\lambda_i^+}} + \frac{1}{\sqrt{\lambda_i^-}} \right) \frac{1}{t_i - t_{i-1}} < \frac{1}{\pi} < \left( \frac{1}{\sqrt{\gamma_j^+}} + \frac{1}{\sqrt{\gamma_j^-}} \right) \frac{1}{s_j - s_{j-1}}$$

holds for  $i = 1, \dots, n$  and  $j = 1, \dots, n+1$ . Assume further

$$(4.2) \quad p_+(t) \geq \lambda_i^+, \quad p_-(t) \geq \lambda_i^- \quad \text{for a.e. } t \in (t_{i-1}, t_i) \text{ and } i = 1, \dots, n$$

and

$$p_+(t) \leq \gamma_j^+, \quad p_-(t) \leq \gamma_j^- \quad \text{for a.e. } t \in (s_{j-1}, s_j) \text{ and } j = 1, \dots, n+1.$$

Then,

$$d_B(I - P, B_1) = 1,$$

where  $P$  is the Poincaré map associated to (1.4) and  $B_1$  the unit disk with center at the origin.

Since the argument of the proof will be used in the sequel, we reproduce it here.

**PROOF.** The proof consists in showing that, for all  $\theta_0 \in [0, 2\pi]$ , the inequalities

$$(4.3) \quad 2n\pi < \theta_0 - \Theta(\theta_0) < 2(n+1)\pi$$

hold. We then clearly have  $z^+ = z^- = 0$  and, by Proposition 2.3, it follows that  $d_B(I - P, B_1) = 1$ .

Let  $\theta(t)$  be the solution of (2.1) with initial condition  $\theta(0) = \theta_0$  and consider the first inequality in (4.3). Using (4.2), it follows from (2.1) that

$$-\theta'(t) \geq \sin^2 \theta(t) + \lambda_i^+ ((\cos \theta(t))^+)^2 + \lambda_i^- ((\cos \theta(t))^-)^2$$

for  $t \in (t_{i-1}, t_i)$ ,  $i = 1, \dots, n$  and hence, for any  $\theta_0 \in \mathbb{R}$ ,

$$-\int_{\theta(t_{i-1})}^{\theta(t_i)} \frac{d\theta}{\sin^2 \theta + \lambda_i^+ ((\cos \theta)^+)^2 + \lambda_i^- ((\cos \theta)^-)^2} \geq t_i - t_{i-1}.$$

Since, for any  $\alpha \in \mathbb{R}$ ,

$$\int_{\alpha}^{\alpha+2\pi} \frac{d\theta}{\sin^2 \theta + \lambda_i^+ ((\cos \theta)^+)^2 + \lambda_i^- ((\cos \theta)^-)^2} = \frac{\pi}{\sqrt{\lambda_i^+}} + \frac{\pi}{\sqrt{\lambda_i^-}},$$

it follows from (4.1) that

$$\theta(t_{i-1}) - \theta(t_i) > 2\pi \quad \text{for } i = 1, \dots, n.$$

Summing these inequalities, we see that

$$\theta_0 - \Theta(\theta_0) = \theta(0) - \theta(2\pi) > 2n\pi.$$

The second inequality in (4.3) is proved using the same argument.  $\square$

Using the same idea, it is possible to give conditions under which the degree of  $I - P$  is different from 1.

**THEOREM 4.2.** *Let  $p_+, p_- \in L^1(0, 2\pi)$ . Assume that for some  $n \in \mathbb{N}$ ,  $n \geq 2$ , there exist  $0 = t_0 < t_1 < \dots < t_n = 2\pi$ ,  $0 = s_0 < s_1 < \dots < s_n = 2\pi$ , positive numbers  $\lambda_i^+$ ,  $\lambda_i^-$  for  $i = 1, \dots, n$ ,  $\gamma_j^+$ ,  $\gamma_j^-$  for  $j = 1, \dots, n$  and some indices  $i^*, j^* \in 1, \dots, n-1$  such that*

$$(4.4) \quad \begin{aligned} \sqrt{\lambda_{i^*}^+} (t_{i^*} - t_{i^*-1}) &\geq \pi, & \sqrt{\lambda_{i^*+1}^-} (t_{i^*+1} - t_{i^*}) &\geq \pi, \\ \sqrt{\gamma_{j^*}^-} (s_{j^*} - s_{j^*-1}) &\leq \pi, & \sqrt{\gamma_{j^*+1}^+} (s_{j^*+1} - s_{j^*}) &\leq \pi, \end{aligned}$$

and

$$\left( \frac{1}{\sqrt{\lambda_i^+}} + \frac{1}{\sqrt{\lambda_i^-}} \right) \frac{1}{t_i - t_{i-1}} < \frac{1}{\pi} < \left( \frac{1}{\sqrt{\gamma_j^+}} + \frac{1}{\sqrt{\gamma_j^-}} \right) \frac{1}{s_j - s_{j-1}}$$

holds for  $i = 1, \dots, n$ ,  $i \neq i^*$ ,  $i \neq i^* + 1$  and  $j = 1, \dots, n$ ,  $j \neq j^*$ ,  $j \neq j^* + 1$ .

Assume further

$$\begin{aligned} p_+(t) &\geq \lambda_i^+ && \text{for a.e. } t \in (t_{i-1}, t_i) \text{ and } i = 1, \dots, n, \quad i \neq i^* + 1, \\ p_-(t) &\geq \lambda_i^- && \text{for a.e. } t \in (t_{i-1}, t_i) \text{ and } i = 1, \dots, n, \quad i \neq i^*, \\ p_+(t) &\leq \gamma_j^+ && \text{for a.e. } t \in (s_{j-1}, s_j) \text{ and } j = 1, \dots, n, \quad j \neq j^*, \\ p_-(t) &\leq \gamma_j^- && \text{for a.e. } t \in (s_{j-1}, s_j) \text{ and } j = 1, \dots, n, \quad j \neq j^* + 1. \end{aligned}$$

Then, provided that equation (1.4) does not have a nontrivial  $2\pi$ -periodic solution, we have

$$d_B(I - P, B_1) \leq 0,$$

where  $P$  is the Poincaré map associated to (1.4) and  $B_1$  the unit disk with center at the origin.

PROOF. Notice first that, if (1.4) does not have a nontrivial  $2\pi$ -periodic solution, the degree  $d_B(I - P, B_1)$  is well-defined. According to Proposition 2.3, it will be different from 1 if we can prove that  $z^+ = z^- \neq 0$ .

Let  $\theta(t)$  be any solution of (2.1). Using the arguments of Theorem 4.1, we have

$$\theta(t_{i-1}) - \theta(t_i) \geq 2\pi \quad \text{for } i = 1, \dots, n, \quad i \neq i^*, \quad i \neq i^* + 1$$

and

$$\theta(s_{j-1}) - \theta(s_j) \leq 2\pi \quad \text{for } j = 1, \dots, n, \quad j \neq j^*, \quad j \neq j^* + 1.$$

This means that for all  $\theta_0 \in [0, 2\pi]$ ,

$$2(n-2)\pi < \theta_0 - \Theta(\theta_0) - (\theta(t_{i^*-1}) - \theta(t_{i^*+1}))$$

and

$$\theta_0 - \Theta(\theta_0) - (\theta(s_{i^*-1}) - \theta(s_{i^*+1})) < 2(n-2)\pi.$$

We will show that it is possible to find two distinct solutions of (2.1), one being such that

$$(4.5) \quad \theta(t_{i^*-1}) - \theta(t_{i^*+1}) \geq 2\pi,$$

the other one such that

$$\theta(s_{i^*-1}) - \theta(s_{i^*+1}) \leq 2\pi.$$

This means that the function  $\Delta$  takes the value  $2(n-1)\pi$  and since (1.4) has no  $2\pi$ -periodic solution it does not have a zero derivative at such a point. Hence, we have  $z^+ = z^- \neq 0$ .

To find the first solution, consider the initial condition  $\theta(t_{i^*}) = \pi/2$ . Arguing as in Theorem 4.1 and using (4.4), we have

$$-\int_{\theta(t_{i^*-1})}^{\pi/2} \frac{d\theta}{\sin^2 \theta + \lambda_{i^*}^+ ((\cos \theta)^+)^2} \geq t_{i^*} - t_{i^*-1} \geq \frac{\pi}{\sqrt{\lambda_{i^*}^+}}.$$

As

$$-\int_{\theta(t_{i^*-1})}^{\pi/2} \frac{d\theta}{\sin^2 \theta + \lambda_{i^*}^+ (\cos \theta)^2} = \frac{\theta(t_{i^*-1}) - \pi/2}{\sqrt{\lambda_{i^*}^+}},$$

it follows from (4.4) that  $\theta(t_{i^*-1}) - \pi/2 \geq \pi$ . Similarly, considering the interval  $[t_{i^*}, t_{i^*+1}]$ , we obtain  $\pi/2 - \theta(t_{i^*+1}) \geq \pi$  and (4.5) follows.

To find the second solution we proceed analogously choosing  $\theta(s_{j^*-1}) = 3\pi/2$ .  $\square$

In the case  $n = 2$ , the conditions simplify considerably. In this case, we must have  $i^* = j^* = 1$  and we take  $\lambda_1^+ = \pi^2/t_1^2$ ,  $\lambda_2^- = \pi^2/(2\pi - t_1)^2$ ,  $\gamma_1^- = \pi^2/s_1^2$  and

$\gamma_2^+ = \pi^2/(2\pi - s_1)^2$ . Such a result can be generalized in order to avoid bounds on  $p_+$  and  $p_-$ .

**THEOREM 4.3.** *Let  $p_+, p_- \in L^1(0, 2\pi)$  be nonnegative functions. Assume that there exist numbers  $s_1, t_1 \in (0, 2\pi)$ , such that the eigenvalue problems*

$$\begin{aligned} u'' + \lambda p_-(t)u &= 0, & u(0) &= 0, & u(t_1) &= 0, \\ u'' + \lambda p_+(t)u &= 0, & u(t_1) &= 0, & u(2\pi) &= 0, \end{aligned}$$

have first eigenvalues  $\lambda_1 \leq 1$  and the eigenvalue problems

$$\begin{aligned} u'' + \mu p_+(t)u &= 0, & u(0) &= 0, & u(s_1) &= 0, \\ u'' + \mu p_-(t)u &= 0, & u(s_1) &= 0, & u(2\pi) &= 0, \end{aligned}$$

have first eigenvalues  $\mu_1 \geq 1$ . Then, provided that equation (1.4) does not have a nontrivial  $2\pi$ -periodic solution, we have  $d_B(I - P, B_1) \neq 1$ , where  $P$  is the Poincaré map associated to (1.4) and  $B_1$  the unit disk with center at the origin.

**PROOF.** Using a Sturm–Liouville comparison argument, we can prove that the solution of the Cauchy problem

$$u'' + p_+(t)u_+ - p_-(t)u_- = 0, \quad u(t_1) = 0, \quad u'(t_1) = 1,$$

is such that  $u(t)$  has zeros in both the intervals  $[0, t_1]$  and  $(t_1, 2\pi]$ . Hence  $\theta(t)$ , the corresponding solution of (2.1), verifies  $\theta(2\pi) - \theta(0) \geq 2\pi$ . In a similar way, we prove that the Cauchy problem

$$u'' + p_+(t)u_+ - p_-(t)u_- = 0, \quad u(s_1) = 0, \quad u'(s_1) = -1,$$

is such that  $u(t)(t - s_1) \leq 0$  for all  $t \in [0, 2\pi]$ . Therefore the corresponding solution  $\theta(t)$  verifies  $\theta(2\pi) - \theta(0) \leq 2\pi$ . The claim follows then as in Theorem 4.2.  $\square$

**REMARK.** Notice that the assumptions on the eigenvalue problems hold if

$$\begin{aligned} p_+(t) &\geq \pi^2/t_1^2 && \text{for a.e. } t \in (0, t_1), \\ p_-(t) &\geq \pi^2/(2\pi - t_1)^2 && \text{for a.e. } t \in (t_1, 2\pi), \\ p_-(t) &\leq \pi^2/s_1^2 && \text{for a.e. } t \in (0, s_1), \\ p_+(t) &\leq \pi^2/(2\pi - s_1)^2 && \text{for a.e. } t \in (s_1, 2\pi). \end{aligned}$$

In particular these conditions are clearly verified (with  $t_1 = s_1 = \pi$ ) if  $p_+, p_-$  are such that

$$\begin{aligned} p_-(t) &\leq 1 \leq p_+(t) && \text{for a.e. } t \in (0, \pi), \\ p_+(t) &\leq 1 \leq p_-(t) && \text{for a.e. } t \in (\pi, 2\pi). \end{aligned}$$

However, without further conditions on  $p_+$ ,  $p_-$ , it is not excluded that equation (1.4) admits nontrivial  $2\pi$ -periodic solutions. The following example provides conditions excluding that possibility.

EXAMPLE. Consider the equation

$$(4.6) \quad u'' + p_+(t)u^+ - u^- = 0,$$

where  $p_+ \in L^\infty(0, 2\pi)$ . Assume that there exist  $t_1 \leq s_1 \leq \pi$  such that

$$p_+ \geq \left(\frac{\pi}{t_1}\right)^2 \quad \text{for a.e. } t \in (0, t_1), \quad p_+ \leq \left(\frac{\pi}{2\pi - s_1}\right)^2 \quad \text{for a.e. } t \in (s_1, 2\pi).$$

It is easy to see now from Theorem 4.3, that the Brouwer degree  $d_B(I - P, B_1)$  is different from 1, if we can show that equation (4.6) has no nontrivial  $2\pi$ -periodic solution. To this end we assume that there exists some  $r_1 \in [t_1, s_1]$  such that

$$p_+ \geq 1 \quad \text{for a.e. } t \in (0, r_1), \quad \text{and} \quad p_+ \leq 1 \quad \text{for a.e. } t \in (r_1, 2\pi),$$

By contradiction, let us denote by  $u$  such a nontrivial  $2\pi$ -periodic solution and extend  $u$  and  $p_+$  by periodicity. Notice first that  $u$  cannot remain always strictly positive and that, on intervals on which it is negative, it must be of the form  $c \sin t$ , for some  $c < 0$ . Hence,  $u$  is negative on intervals of length  $\pi$  and, being  $2\pi$ -periodic, it must also be positive on intervals of length  $\pi$ . More precisely, a number  $\tau \in [0, 2\pi)$  must exist, such that the problem

$$\begin{aligned} u'' + p_+(t)u^+ - u^- &= 0, \\ u(\tau) = u(\tau + \pi) &= 0, \quad u'(\tau) = -u'(\tau + \pi) \end{aligned}$$

has a solution which is positive on  $(\tau, \tau + \pi)$ . Taking into account that  $p_+ - 1$  changes sign at the point  $r_1$ , we distinguish three cases:

- (1)  $p_+ - 1$  is of constant sign on  $(\tau, \tau + \pi)$ . We multiply then (4.6) by  $\sin(t - \tau)$  and integrate over  $(\tau, \tau + \pi)$ , which gives

$$\int_{\tau}^{\tau+\pi} (p_+(t) - 1)u^+(t) \sin(t - \tau) dt = 0.$$

A contradiction is obtained since the integrand is of constant sign on  $(\tau, \tau + \pi)$ .

- (2)  $r_1 \in (\tau, \tau + \pi)$ . We multiply then (4.6) by  $\sin(t - r_1)$  and integrate over  $(\tau, \tau + \pi)$ . An integration by parts again leads to a contradiction, taking into account the fact that  $u'(\tau) \sin(\tau - r_1) = u'(\tau + \pi) \sin(\tau + \pi - r_1)$ .
- (3)  $2\pi \in (\tau, \tau + \pi)$ . The same argument works, multiplying (4.6) by  $\sin t$ .

Consequently, under the hypotheses listed above for  $p_+$ , we have

$$d_B(I - P, B_1) \neq 1.$$

REMARK. Notice that, under the conditions imposed on  $p_+$ ,  $p_-$  in Theorem 4.3, if equation (1.4) has a nontrivial  $2\pi$ -periodic solution, then, for any sufficiently small perturbations (in the  $L^1$  sense) of  $p_+$ ,  $p_-$ , for which nontrivial  $2\pi$ -periodic solutions do not exist, the degree  $d_B(I - P, B_1)$  related to the perturbed equation will be different from 1. This follows from the fact that, provided that the perturbation is small enough, the function  $\Delta(\theta_0) = \Theta(\theta_0) - \theta_0$  will still cross the level  $2\pi$ . The slopes at the points of crossing cannot be equal to 0, otherwise, as observed earlier, equation (1.4) would have a nontrivial periodic solution. Hence, there is at least one point  $\theta^* \in [0, 2\pi)$  such that  $\Delta(\theta^*) = 2\pi$ ,  $\Delta'(\theta^*) \neq 0$  and it then follows from Proposition 2.3 that  $d_B(I - P, B_1) < 1$ .

### 5. Asymptotically positively homogeneous equations

Using the above theorems, it is possible to give various existence conditions, based on degree arguments, for nontrivial  $2\pi$ -periodic solutions of

$$(5.1) \quad u'' + f(t, u) = 0,$$

when  $f$  is asymptotically positively homogeneous in  $u$  for  $u \rightarrow \pm\infty$ .

For instance, following an idea recalled in the introduction, if  $p_+$ ,  $p_-$  are given by (1.3), and if the Brouwer degree  $d_B(I - P, B_1)$  associated to (1.4) can be shown to be different from 0, the existence of a solution of (5.1) can be deduced.

Another type of result assumes that  $f(t, 0) = 0$ , for all  $t \in \mathbb{R}$ ,  $f$  being asymptotically positively homogeneous in  $u$  when  $u \rightarrow 0_\pm$  and  $u \rightarrow \pm\infty$ . The idea is then to build conditions such that the positively homogeneous approximation of (5.1) for  $u \rightarrow 0_\pm$  leads to a degree 1 for the map  $I - P$ , whereas the positively homogeneous approximation of (5.1) for  $u \rightarrow \pm\infty$  leads to a degree different from 1 (or vice versa). The existence of nontrivial  $2\pi$ -periodic solutions then follows from the excision property of the degree.

As an example, we present the following result.

**THEOREM 5.1.** *Let  $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $(t, u) \mapsto f(t, u)$  satisfy  $L^1$ -Carathéodory conditions. Assume that  $f(t, 0) = 0$ , for a.e.  $t \in \mathbb{R}$ , and that there exists a  $L^1$ -function  $F$  such that*

$$\left| \frac{f(t, u)}{u} \right| \leq F(t) \quad \text{for all } u \in \mathbb{R}.$$

Let

$$\begin{aligned} q_{\infty, \pm}(t) &= \liminf_{u \rightarrow \pm\infty} \frac{f(t, u)}{u}, & Q_{\infty, \pm}(t) &= \limsup_{u \rightarrow \pm\infty} \frac{f(t, u)}{u}, \\ q_{0, \pm}(t) &= \liminf_{u \rightarrow 0_\pm} \frac{f(t, u)}{u}, & Q_{0, \pm}(t) &= \limsup_{u \rightarrow 0_\pm} \frac{f(t, u)}{u}. \end{aligned}$$

Assume that

- (A<sub>1</sub>) the conditions of Theorem 4.2 are satisfied for any function  $p_+$  between  $q_{\infty,+}$  and  $Q_{\infty,+}$ , and for any function  $p_-$  between  $q_{\infty,-}$  and  $Q_{\infty,-}$ ; assume also that, for such functions  $p_+, p_-$ , equation (1.4) has no nontrivial  $2\pi$ -periodic solution;
- (A<sub>2</sub>) the conditions of Theorem 4.1 are satisfied for any function  $p_+$  between  $q_{0,+}$  and  $Q_{0,+}$ , and for any function  $p_-$  between  $q_{0,-}$  and  $Q_{0,-}$ ; assume also that, for such functions  $p_+, p_-$ , equation (1.4) has no nontrivial  $2\pi$ -periodic solution.

Then, equation (5.1) has a nontrivial  $2\pi$ -periodic solution.

Notice that the conditions of Theorem 4.2 in (A<sub>2</sub>) can be replaced by the requirement that  $f(t, u)$  is asymptotically linear in  $u$  for  $u \rightarrow 0_{\pm}$ , since a linear equation (1.4) leads to a degree 1 for  $I - P$ , provided that there is no nontrivial  $2\pi$ -periodic solution.

Keeping this in mind, the following example can be given.

EXAMPLE. Consider the equation

$$(5.2) \quad u'' + u + \frac{u}{1+u^2} + (p_+(t) - 1) \frac{(u^+)^2}{1+|u|} = 0,$$

where  $p_+$  is as in the example of the previous section. Then, the above theorem applies. Indeed, for  $u \rightarrow 0$ , equation (5.2) is asymptotic to the linear equation

$$u'' + 2u = 0,$$

which has no nontrivial  $2\pi$ -periodic solution, whereas for  $u \rightarrow \pm\infty$ , it is asymptotic to

$$u'' + p_+(t)u^+ - u^- = 0,$$

for which we have shown above that the degree of  $I - P$  is different from 1. Hence, equation (5.2) has a nontrivial  $2\pi$ -periodic solution.

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CHRISTIAN FABRY AND PATRICK HABETS  
 Institut de Mathématique Pures et Appliquée  
 Université Catholique de Louvain  
 Chemin du Cyclotron 2  
 B-1348 Louvain-la-Neuve, BELGIUM  
*E-mail address:* fabry@math.ucl.ac.be