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ON THE COHOMOLOGY OF AN ISOLATING BLOCK AND ITS INVARIANT PART

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Abstract. We give a sufficient condition for the existence of an isolating block B for an isolated invariant set S such that the inclusion induced map in cohomology $H^*(B) \to H^*(S)$ is an isomorphism. We discuss the Easton's result concerning the special case of flows on a 3-manifold. We prove that if S is an isolated invariant set for a flow on a 3-manifold and S is of finite type, then each isolating neighbourhood of S contains an isolating block B such that B and B^- are manifolds with boundary and the inclusion induced map in cohomology is an isomorphism.

1. Introduction

Let us consider a continuous flow φ on a locally compact metric space X with an isolated invariant set S. Assume that $H^*(S)$ is of finite type, where H^* is the Aleksander-Spanier cohomology functor. We will discuss the problem of the existence of an isolating block B for S such that the inclusion induced map $H^*(B) \to H^*(S)$ is an isomorphism. Our main result (Theorem 3.1) gives a partial answer to this question.

In the case of a flow on n-dimensional topological manifold M and a stationary point $S \subset M$ being the isolated invariant set, the natural question is if S has an isolating block that is homeomorphic to the n-dimensional compact ball $\mathbb{D}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. This problem was first studied by Easton (see [9]) for flows in dimension 3. The main result of the paper of Easton (Theorem 1

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in [9]) says that if φ is a smooth flow on $\mathbb{R}^3$ and an isolated invariant set S is respectively a torus, a simple closed curve or a point, then there exists an isolating block B for S which is respectively homeomorphic to $S \times \mathbb{D}^1$, $S \times \mathbb{D}^2$, $S \times \mathbb{D}^3$ (where $\mathbb{D}^k$ is the unit disk in $\mathbb{R}^k$). Unfortunately, it seems that there is an essential gap in the proof of the above result presented in [9]. In this note we would like to point out the mistakes in the proof of Easton. We adapt the idea given in [9] to present the positive solution for the question in dimension 3. On the other hand, one can use Bing's example of the dog-bone space X that is not locally homeomorphic to $\mathbb{R}^3$ but $X \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$ (see [1]), to show that the answer in dimension 4 is negative.

We will mention that the similar problems can be posed also in the case of discrete Conley index theory. It was proved in [17] that a fixed point of the planar homeomorphism being an isolated invariant set has an isolating block (in the sense of discrete Conley index theory, see [14]) homeomorphic to a compact ball.

First, we very briefly recall known facts concerning the isolating blocks. The Conley index theory is based on the following

Theorem 1.1. *An invariant set S is isolated if and only if there exists an isolating block B such that $S = \operatorname{inv} B$. Moreover, each isolating neighbourhood N of S contains such a block.*

This result was first proved in the case of a smooth flow φ (i.e. of C^∞-class) on a smooth manifold (see [8], [24]). In the smooth case one can prove that an isolating block B can be chosen as a smooth submanifold with corners such that B^- and B^+ are smooth submanifolds with boundary. The proof in the purely topological case (i.e. a continuous flow on locally compact metric space) was given in [5]. In more general context of semiflows on arbitrary metric spaces the existence of an isolating block for those invariant sets which admits strongly φ-admissible isolating neighbourhood was given by Rybakowski in [19].

In the context of a flow on a locally compact, metric ENR (euclidean neighbourhood retract) the natural question that arises is the existence of an isolating block B such that B and the exit set B^- are both ENRs. In general the answer to this question is negative. Borsuk in [2] constructed the example of a compact ENR $X \subset \mathbb{R}^3$ such that each proper 2-dimensional subset of X is not an ENR. One can use it to construct a simple counterexample to the above question. Another counterexample was presented in [18]. In the case of flows on $\mathbb{R}^n$ this question is, at present, far from being solved for $n \geq 4$. Some affirmative answers for flows on topological manifolds can be summarized in the following

Theorem 1.2. *Let φ be a continuous flow on a topological n-manifold. If $n = 2$ or 3 then in each isolating neighbourhood of an isolated invariant set S*

there exists an isolating block B such that B is an ENR *and B^- is a $n-1$-dimensional manifold with boundary.*

The case $n=2$ was given by Srzednicki in [21]. Later the result was extended to the case of flows on 3-dimensional manifolds in [18]. Since Ruchała's thesis are available in Polish, so for the convenience of the reader we present the idea of his proof in the Appendix.

2. Isolating blocks and the Conley index

Let X be a locally compact metric space. A *flow* on X is a continuous map $\phi\colon \mathbb{R}\times X\to X$ such that

$$\phi(0,x) = x, \qquad x\in X \tag{2.1}$$

$$\phi(s+t,x) = \phi(t,\phi(s,x)), \quad s,t\in\mathbb{R},\ x\in X. \tag{2.2}$$

In the sequel we frequently use the following notation: we write $\phi_t(x)$ instead of $\phi(x,t)$ and if $W\subset X$ and $J\subset\mathbb{R}$ then we write $\phi(J,W)$ instead of $\phi(J\times W)$. The sets

$$\phi(x) := \phi(\mathbb{R},x), \quad \phi^+(x) := \phi([0,\infty),x), \quad \phi^-(x) := \phi((-\infty,0],x),$$

are called, respectively, the *trajectory*, the *positive semitrajectory* and the *negative semitrajectory* of x. For $x\in W\subset X$, by $\phi(x,W)$ we denote the component of $\phi(x)\cap W$ which contains x. This is the *orbit segment* of x in W. Similarly, $\phi^{\pm}(x,W)$ are those components of $\phi^{\pm}(x)\cap W$ which contain x. If $D\subset W$, then

$$\phi(D,W) := \bigcup_{x\in D}\phi(x,W).$$

The sets

$$\omega(x) := \bigcap_{t\geq 0}\overline{\phi([t,\infty),x)} \quad\text{and}\quad \alpha(x) := \bigcap_{t\leq 0}\overline{\phi((-\infty,t],x)}$$

are called, respectively, the *ω-limit set* and the *α-limit set* of x. A set $W\subset X$ is called *invariant* if $\phi(\mathbb{R},W)=W$.

For $W\subset X$ we define the sets

$$\begin{aligned}
\operatorname{Inv}^{\pm}(W) &:= \{x\in W : \phi^{\pm}(x)\subset W\},\\
\operatorname{Inv}(W) &:= \operatorname{Inv}^-(W)\cap\operatorname{Inv}^+(W),\\
W^- &= \{x\in W : \text{for all } t>0 \text{ there exists } s\in[0,t] \text{ such that } \phi_s(x)\notin W\},\\
W^+ &= \{x\in W : \text{for all } t<0 \text{ there exists } s\in[-t,0] \text{ such that } \phi_s(x)\notin W\},
\end{aligned}$$

and functions $\sigma^{\pm}\colon W\to[0,\infty]$ by

$$\begin{aligned}
\sigma^+(x) &= \sup\{t\geq 0 : \phi([0,t],x)\subset W\},\\
\sigma^-(x) &= \sup\{t\geq 0 : \phi([-t,0],x)\subset W\}.
\end{aligned}$$

The sets $\mathrm{Inv}^{\pm}(W)$ are called, respectively, the *positive invariant part* and the *negative invariant part* of W, and $\mathrm{Inv}(W)$ is the *invariant part* of W. Moreover, W^- is called the *exit set* and W^+ is the *entrance set* of W. The function σ^+ is called the *escape-time function* of W. It follows by the Wa.zewski Retract Theorem that if W and W^- are compact then σ^+ is continuous (see [6], [15], [22]).

An invariant set S is *isolated* if there exists a compact neighbourhood N of S such that

- $S = \mathrm{inv}(N)$,
- $S \subset \mathrm{int}\, N$.

Such N is called an *isolating neighbourhood for S*. We also say that N *isolates S*. In particular, if $x \in \partial N$ then $\phi(x) \not\subset N$.

The Conley index theory is based on the existence of some special isolating neighbourhoods for an isolated invariant set S called *isolating blocks*. In order to define an isolating block we need some auxiliary notions. For $\Sigma \subset X$ and $\delta > 0$, we define a map $\phi_\delta \colon (-\delta, \delta) \times \Sigma \to X$ by

$$\phi_\delta(t, x) = \phi(t, x).$$

If ϕ_δ is a homeomorphism onto its image, then $\phi((-\delta,\delta),\Sigma)$ is a *collar* of Σ (with respect to ϕ). In this case, Σ is a strong deformation retract of $\phi((-\delta,\delta),\Sigma)$. If ϕ_δ is a homeomorphism with open range, then Σ is a *local section* (w.r.t. ϕ).

Let $B \subset X$ be a compact set, and let $\Sigma^{\pm}$ be local sections with disjoint closures i.e. such that

$$\overline{\Sigma^+} \cap \overline{\Sigma^-} = \emptyset.$$

Let $\delta > 0$ be such that

$$\phi((-\delta,\delta),\Sigma^+), \quad \phi((-\delta,\delta),\Sigma^-)$$

are disjoint collars of Σ^+, Σ^-.

Definition 2.1. We call B an *isolating block* if

(a) $(\overline{\Sigma^{\pm}} \setminus \Sigma^{\pm}) \cap B = \emptyset$,
(b) $\phi((-\delta,\delta),\Sigma^+) \cap B = \phi([0,\delta),\Sigma^+ \cap B)$
(c) $\phi((-\delta,\delta),\Sigma^-) \cap B = \phi((-\delta,0],\Sigma^- \cap B)$
(d) for each $x \in \partial B \setminus (\Sigma^+ \cup \Sigma^-)$ there exist real numbers $\varepsilon_1 < 0 < \varepsilon_2$, such that

$$\phi(\varepsilon_1, x) \in \Sigma^+, \quad \phi(\varepsilon_2, x) \in \Sigma^-, \quad \phi([\varepsilon_1,\varepsilon_2], x) \subset \partial B.$$

Let B be an isolating block. It follows that

$$B^- = \Sigma^- \cap B, \quad B^+ = \Sigma^+ \cap B,$$

and $S = \mathrm{Inv}(B)$ is an isolated invariant set. We put

$$A = \mathrm{Inv}^+(B) \cup \mathrm{Inv}^-(B), \quad a^\pm = \mathrm{Inv}^\pm(B) \cap B^\pm.$$

It follows that $a^\pm \subset \mathrm{int}_{\partial B}(B^\pm)$ (see [6], [5]).

We will finish this section with the well-known results concerning the cohomology of an isolated invariant set S. Let H^* be the Alexander–Spanier (or Čech) cohomology functor. *The cohomology Conley index $CH(S)$ of S is defined* by

$$CH(S) = H^*(B, B^-),$$

where B is an isolating block for S. One can check that $CH(S)$ is independent (up to an isomorphism) on the choice of an isolating block B for S (see [5]).

Consider the following commutative diagram of homomorphisms of $\mathbb{Q}$-vector spaces with exact rows. Let g_i (for $i = 1, \ldots, 5$) denotes a vertical homomorphism $A_i \to B_i$.

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_4 \arrow[r] \arrow[d] & A_5 \arrow[d] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

In the sequel we will use the following version of "Five Lemma" (see [20]).

Lemma 2.2. *If g_2, g_4 are monomorphisms and g_1 is an epimorphism, then g_3 is a monomorphism. If g_2, g_4 are epimorphisms and g_5 is a monomorphism, then g_3 is an epimorphism.*

The following results are due to Churchill (see Propositions 4.6 and Lemma 4.3 in [5] or Propositions 7.2, 7.3 in [22]).

Proposition 2.3. *The inclusion $(A, a^-) \hookrightarrow (B, B^-)$ induces an isomorphism $H^*(B, B^-) \to H^*(A, a^-)$.*

Proposition 2.4. *The inclusion $S \hookrightarrow A$ induces an isomorphism $H^*(A) \to H^*(S)$.*

The proofs of the above results are based on the continuity of the Čech cohomology.

3. Main result

The aim of this section is to discuss the existence of an isolating block for an isolated invariant set S with the same cohomology type. Let us begin with a remark concerning the simplest case when B is an isolating block for S such that $B^- = \emptyset$ (or $B^+ = \emptyset$). Then S is an asymptotically stable set and by Theorem 7.2 in [22] the inclusion $S \hookrightarrow B$ induces an isomorphism $H^*(B) \to H^*(S)$. Actually, one can prove that $S \hookrightarrow B$ is a shape equivalence (see [22]).

Remark 3.1. If the origin $\{0\}$ is an isolated invariant set for the continuous flow on $\mathbb{R}^2$, then there exists an isolating block for $\{0\}$ homeomorphic to a disc. Indeed, by Theorem 1.2 there exists an isolating B for $\{0\}$ such that B is a topological manifold with boundary and B^- is its submanifold with boundary. Without loss of generality we can assume that B is a closed ball $\mathbb{D}^2$ of radius 1 centered at the origin with removed a finite union of open, small discs K_i such that $\overline{K_i} \cap \overline{K_j} = \emptyset$ and $\{0\} \notin \bigcup_{i=1}^n \overline{K_i}$. Moreover, we can assume that there is no isolating blocks for $\{0\}$ of this form with the smaller number of K_i. Let $x \in \partial K_i$ for some $i = 1, \ldots, n$. Suppose that $\sigma^+(x) < \infty$ and $\phi(\sigma^+(x), x) \in \mathbb{S}^1$. Then we can remove from B the part of the trajectory $\phi([0, \sigma^+(x)], x)$ of x together with the small neighbourhood and we get an isolating block with a smaller number of holes. So if $x \in \partial K_i$ and $\sigma^+(x) < \infty$ then $\phi(\sigma^+(x), x) \notin \mathbb{S}^1$. By reversing of time we get that if $\sigma^-(x) < \infty$ then $\phi(-\sigma^-(x), x) \notin \mathbb{S}^1$. It is easy to check that $\mathbb{D}^2$ is an isolating block and if $S = \operatorname{inv} \mathbb{D}^2$ then

$$\bigcup_{i=1}^{n} K_i \subset S. \tag{3.1}$$

In particular $\operatorname{int} S \neq \emptyset$. It follows by (3.1) that $\partial S \subset B$. Since ∂S is invariant, nonempty and different from $\{0\}$, we get a contradiction.

Lemma 3.2. *If B is an isolating block for S, then*

(a) *$H^*(B) \to H^*(S)$ is an isomorphism if and only if $H^*(B^-) \to H^*(a^-)$ is an isomorphism,*
(b) *if $CH^q(S) = 0$ and $H^q(B^-) \to H^q(a^-)$ is a monomorphism, then $H^q(B) \to H^q(S)$ is also a monomorphism,*

where all homomorphisms are induced by inclusions.

Proof. Since the inclusion $S \hookrightarrow B$ factors to $S \hookrightarrow A \hookrightarrow B$, so it follows by Proposition 2.4 that $H^*(B) \to H^*(S)$ is a monomorphism (an isomorphism) if and only if $H^*(B) \to H^*(A)$ is a monomorphism (an isomorphism).

Consider the following commutative diagram (all homomorphisms are induced by inclusions)

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(B,B^-) \arrow[r] \arrow[d] & H^q(B) \arrow[r] \arrow[d] & H^q(B^-) \arrow[r] \arrow[d] & H^{q+1}(B,B^-) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^q(A,a^-) \arrow[r] & H^q(A) \arrow[r] & H^q(a^-) \arrow[r] & H^{q+1}(A,a^-) \arrow[r] & \cdots
\end{tikzcd}$$

The rows are the long exact sequences for the pairs (A, a^-) (bottom row) and (B, B^-) (top row). The part (a) follows by the Five Lemma, Propositions 2.3 and 2.4.

For the proof of (b) it is sufficient to consider the following part of the above diagram

$$\begin{tikzcd} 0 = H^q(B,B^-) \arrow[r] & H^q(B) \arrow[r] \arrow[d] & H^q(B^-) \arrow[d] \\ & H^q(A) \arrow[r] & H^q(a^-) \end{tikzcd}$$

Indeed, since the composition $H^q(B) \to H^q(A) \to H^q(a^-)$ is a monomorphism, so $H^q(B) \to H^q(A)$ is a monomorphism and consequently, $H^q(B) \to H^q(S)$ is a monomorphism by Proposition 2.4. □

Lemma 3.3. *If B is an isolating block for S such that $H^*(B) \to H^*(S)$ is a monomorphism, then $H^*(B^-) \to H^*(a^-)$ is a monomorphism.*

Proof. It follows directly by the commutative diagram in the proof of Lemma 3.2 and the Five Lemma. □

Definition 3.4. Let B be an isolating block for a flow on a locally compact metric space X. We say that $a^\pm$ is co-stable in $B^\pm$ if there exists a compact neighbourhood $Y \subset \mathrm{int}_{\partial B}(B^\pm)$ of $a^\pm$ such that the inclusion induced map $H^*(Y) \to H^*(a^\pm)$ is an isomorphism.

Example 3.5. Let $S = \{z = (0,x) \in \mathbb{R}^2 : x = 1/n \vee x = 0\}$. There is a continuous function $g\colon \mathbb{R}^2 \to [0,\infty)$ such that

$$S = \{z \in \mathbb{R}^2 : g(z) = 0\}.$$

Let ϕ be a flow generated by the equation

$$z' = g(z)e_1 = g(z)(1,0). \tag{3.2}$$

In particular, the set S consists of stationary points of (3.2). Observe that the set of bounded trajectories of (3.2) is equal to S, so S is an isolated invariant set. One can check that $B = [-1,1] \times [-2,2]$ is an isolating block for S with the exit and entrance sets given by $B^\mp = \{\pm 1\} \times [-2,2]$. Moreover, $a^\mp = \{(\pm 1, x) : (0,x) \in K\}$.

Since the sets $a^\pm$ have infinitely many connected components, hence $H^0(a^\pm)$ are of infinite dimensions, so one can check that they are not co-stable in $B^\pm$.

On the other hand, if $B^\pm$ are manifolds with boundaries and $a^\pm$ are finite sets, then $a^\pm$ are co-stable in $B^\pm$ (compare the example given in Figure 1).

Theorem 3.6. *If B is an isolating block for S and a^- is co-stable in B^-, then there is an isolating block $W \subset B$ for S such that the inclusion induced map $H^*(W) \to H^*(S)$ is an isomorphism. Moreover, a^+ is also co-stable in B^+.*

Proof. There exists a neighbourhood $Y \subset \mathrm{int}\, B^-$ of a^- such that the inclusion induced map

$$H^*(Y) \to H^*(a^-)$$

is an isomorphism. Let $D = B^- \setminus Y$. We define $W = \overline{B \setminus \phi(D, B)}$, and say that W is obtained from B by shaving D. It follows by Proposition 3.11 in [15] that W is an isolating block for S and $W^- = Y$. By Lemma 3.2 we get that $H^*(W) \to H^*(S)$ is an isomorphism.

It follows that $a^+ \subset W^+ \subset \operatorname{int}(B^+)$ and $H^*(W) \to H^*(S)$ is an isomorphism, so again by Lemma 3.2, $H^*(W^+) \to H^*(a^+)$ is an isomorphism, hence a^+ is co-stable in B^+. □

4. Blocks in 3D — Easton's result

In this section we present some results concerning flows near an isolated invariant set in dimension 3. The main result in [9] is the following

Theorem 4.1 ([E, Theorem 1]). *Let V be a C^1 vector field on an orientable smooth 3-manifold which generates a flow φ and suppose that S is an isolated invariant set of φ. Then there exists an isolating block B for S such that the homomorphism $H^*(B) \to H^*(S)$ induced by the inclusion is injective.*

Let us mention that in Easton's paper the isolating blocks without sliding on the boundary (i.e. $\partial B = B^- \cup B^+$), were considered. The proof presented in [9] consists of the following two steps:

(1) there exists an isolating block for S such that

$$H^*(B^-) \to H^*(a^-)$$

is a monomorphism,

(2) if B is an isolating block such that $H^*(B^-) \to H^*(a^-)$ is a monomorphism, then $H^*(B) \to H^*(S)$ is also a monomorphism.

The proof of the first step is correct and is based on the following lemma (see [9] or [12])

Lemma 4.2. *Let M be a compact, connected, orientable 2-manifold with boundary and let C be a closed subset of the interior of M. Then given any neighbourhood U of C in M, there exists a compact manifold with boundary N such that $C \subset N \subset U$ and such that $H^*(N) \to H^*(C)$ is an injection. If, in addition, $H^*(C)$ is finitely generated, then N can be chosen so that $H^*(N) \to H^*(C)$ is an isomorphism.*

Let us observe that by our Lemma 3.3 the injectivity of $H^*(B^-) \to H^*(a^-)$ is a necessary condition for $H^*(B) \to H^*(S)$ to be a monomorphism, but it is not a sufficient condition. The simple example of the planar flow given in Figure 1 shows that the second step is false. To obtain the example in dimension 3 one can add one stable direction at the stationary point.

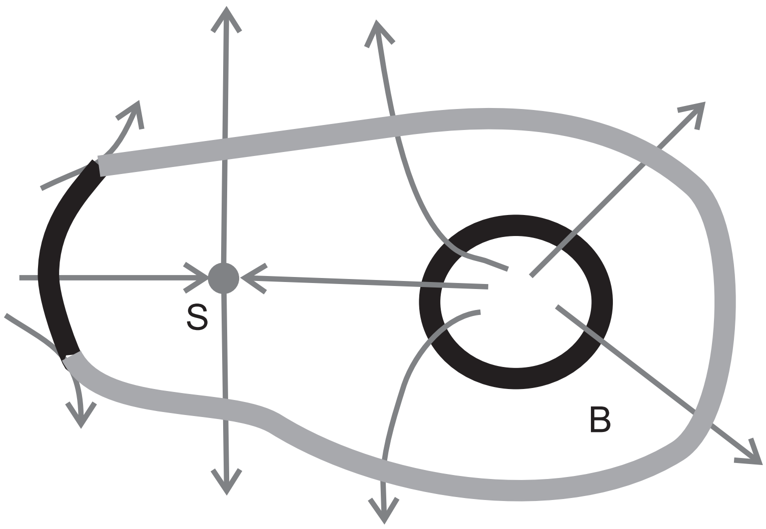

FIGURE 1. The annulus B is an isolating block for the stationary point S. Since B^- is contractible and a^- consists of two points, so $H^*(B^-) \to H^*(a^-)$ is a monomorphism, however $H^*(B) \to H^*(S)$ has a non-trivial kernel.

The Easton's proof of the step (2) is based on the existence of the following exact sequence in homology

$$\cdots \longrightarrow H_n(B^- \setminus a^-, \tau) \longrightarrow H_n(B, \partial B) \longrightarrow H_n(B, B \setminus S) \longrightarrow \cdots$$

where $\tau = B^+ \cap B^-$, the last arrow is a homomorphism of degree -1 and the other are induced by inclusions (see the sequence (B) in [9, p. 335]). One can check that for the flow in Figure 1 we get

$$H_1(B^- \setminus a^-, \tau) \cong H_1(B, B \setminus S) = 0, \quad H_1(B, \partial B) \cong \mathbb{Z}$$

which contradicts the exactness of the above sequence. $\square$

The main result of this section is the following theorem obtained by a modification of Easton's arguments.

THEOREM 4.3. *If S is an isolated invariant set for a flow on a 3-manifold and $H^*(S)$ is finitely generated, then in each isolating neighbourhood N of S there exists an isolating block B for S such that B is a topological 3-manifold with boundary and $H^*(B) \to H^*(S)$ is an isomorphism. In particular, S has the cohomology type of a compact manifold.*

PROOF. By Theorem 1.2 there exists an isolating block $W \subset N$ for S such that W is an ENR and W^- is a 2-dimensional manifold with boundary. It follows that $CH(S)$ is finitely generated, so by Propositions 2.3 and 2.4 and the long exact sequence of the pair (A, a^-) we get that $H^*(a^-)$ is also finitely generated.

By Lemma 4.2, set a^- is co-stable in W^-, hence by Theorem 3.6, for each open neighbourhood U of a^- in W^- there exists an isolating block $B \subset W$ for S such that $a^- \subset B^- \subset U$ and $H^*(B) \to H^*(S)$ is an isomorphism. Moreover, the sets $B^\pm$ are 2-manifolds. One can easily check that then ∂B is also a 2-manifold and B is a 3-manifold with boundary. $\square$

Corollary 4.4. *If the origin $S = \{0\}$ is an isolated invariant set for the continuous flow on $\mathbb{R}^3$ then there exists an isolating block B for S being a 3-dimensional homological ball.*

5. Contractibility of $B^\pm$ in ∂B

Lemma 5.1. *If B is an isolating block such that for some $p, q \geq 1$ the inclusion induced maps*

$$H^p(\partial B) \to H^p(B^+), \quad H^q(\partial B) \to H^q(B^-)$$

are trivial (for example $B^\pm$ are contractible in ∂B), then the cup product

$$\cup \colon H^p(\partial B) \oplus H^q(\partial B) \to H^{p+q}(\partial B)$$

is trivial.

Proof. By the exactness of the sequence of the pair (B, B^+)

$$\cdots \longrightarrow H^p(\partial B, B^+) \longrightarrow H^p(\partial B) \longrightarrow H^p(B^+) \longrightarrow \cdots$$

the homomorphism $H^p(\partial B, B^+) \to H^p(\partial B)$ is an epimorphism. In the same way $H^q(\partial B, B^-) \to H^q(\partial B)$ is an epimorphism. Since $B^\pm$ is a strong deformation retract of $\partial B \setminus B^\mp$, so the homomorphisms

$$H^p(\partial B, \partial B \setminus B^-) \to H^p(\partial B), \quad H^q(\partial B, \partial B \setminus B^+) \to H^q(\partial B)$$

are epimorphisms. Consider a commutative diagram

$$\begin{tikzcd}
\cup \colon H^p(\partial B, \partial B \setminus B^-) \oplus H^q(\partial B, \partial B \setminus B^+) \arrow[r] \arrow[d] & H^{p+q}(\partial B, (\partial B \setminus B^-) \cup (\partial B \setminus B^+)) = 0 \arrow[d] \\
\cup \colon H^p(\partial B) \oplus H^q(\partial B) \arrow[r] & H^{p+q}(\partial B)
\end{tikzcd}$$

Since the first vertical arrow is an epimorphism, so the cup-product in the bottom row has to be trivial. $\square$

Example 5.2. If $S \subset \mathbb{R}^3$ is a periodic orbit which is an isolated invariant set, then there is an isolating block B for S homeomorphic to the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$. Since the cup-product

$$\cup \colon H^1(\partial B) \oplus H^1(\partial B) \to H^2(\partial B)$$

is non-trivial, so either $H^1(\partial B) \to H^1(B^+)$ or $H^1(\partial B) \to H^1(B^-)$ is non-trivial. Observe that if $H^1(\partial B) \to H^1(B^\pm)$ is non-trivial, then $H^1(\partial B) \to H^1(a^\pm)$ is non-trivial. In particular, a^+ or a^- is not contractible in ∂B.

Lemma 5.3. *If B^+ is contractible in ∂B, then for $p \geq 1$ and $q \in \mathbb{Z}$ the cup product*

$$\cup\colon H^q(\partial B, B^-) \oplus H^q(\partial B) \to H^{p+q}(\partial B, B^-)$$

is trivial.

Proof. Since for $p \geq 1$, homomorphism $H^p(\partial B) \to H^p(B^+)$ is an epimorphism and B^+ is a strong deformation retract of $\partial B \setminus B^-$, so

$$H^p(\partial B, \partial B \setminus B^-) \to H^p(\partial B)$$

is an epimorphism, so the result follows by the commutativity of the diagram

$$\begin{array}{ccc} \cup\colon H^p(\partial B, B^-) \oplus H^q(\partial B, \partial B \setminus B^-) & \longrightarrow & H^{p+q}(\partial B, \partial B) = 0 \\ \downarrow & & \downarrow \\ \cup\colon H^p(\partial B, B^-) \oplus H^q(\partial B) & \longrightarrow & H^{p+q}(\partial B, B^-) \end{array}$$

□

Let $B \subset \mathbb{R}^{n+1}$ be an isolating block and let B be a $n+1$-manifold with boundary. Observe that B^+ is a strong deformation retract of $\partial B \setminus B^-$ (in particular B^+ is an ENR). Then by duality

$$H^i(\partial B, B^-) \cong H_{n-i}(\partial B \setminus B^-) \cong H_{n-i}(B^+).$$

Lemma 5.4. *Assume that B^+ is contractible in ∂B. If $H_i(B^+) \neq 0$ for some $i \geq 1$, then $H^{n-i-1}(B^-) \neq 0$.*

Proof. Since B^+ is contractible in ∂B, so $H_i(B^+) \to H_i(\partial B)$ is trivial. By the long exact sequence

$$\cdots \longrightarrow H_{i+1}(\partial B, B^+) \longrightarrow H_i(B^+) \to H_i(\partial B) \longrightarrow \cdots$$

of the pair $(\partial B, B^+)$ we get that

$$H_{i+1}(\partial B, B^+) \to H_i(B^+)$$

is an epimorphism. In particular, $H_{i+1}(\partial B, B^+) \neq 0$. Since

$$H_{i+1}(\partial B, B^+) \cong H_{i+1}(\partial B, \partial B \setminus B^-),$$

and by duality

$$H_{i+1}(\partial B, \partial B \setminus B^-) \cong H^{n-i-1}(B^-),$$

so

$$H_{i+1}(\partial B, B^+) \cong H^{n-i-1}(B^-).$$ □

Corollary 5.5. *If B^+ is contractible in ∂B, then for $i \geq 1$,*

$$\dim H_{n-i-1}(B^-) = \dim H^i(B^+) + \dim H^{i+1}(\partial B).$$

Proof. If $i \geq 1$ then $H^i(\partial B) \to H^i(B^+)$ and $H^{i+1}(\partial B) \to H^{i+1}(B^+)$ are trivial, so by the long exact sequence of the pair $(\partial B, B^+)$ we get the following exact sequence

$$0 \to H^i(B^+) \longrightarrow H^{i+1}(\partial B, B^+) \longrightarrow H^{i+1}(\partial B) \longrightarrow 0,$$

hence

$$\dim H^{i+1}(\partial B, B^+) = \dim H^i(B^+) + \dim H^{i+1}(\partial B).$$

By duality,

$$\dim H_{n-i-1}(\partial B \setminus B^+) = \dim H_{n-i-1}(B^-) = \dim H^i(B^+) + \dim H^{i+1}(\partial B). \quad \square$$

6. Appendix. Ruchała's proof of Theorem 1.2

In this section we present the Ruchała's proof of Theorem 1.2 (see [18]).

The idea of Ruchała's proof of Theorem 1.2. Let N be an isolating neighbourhood of an isolated invariant set S for a flow φ on 2 (respectively 3) dimensional manifold. By Theorem 1.1 there exists an isolating block $B \subset N$ for S. First we show that $\dim \Sigma^- = 1$ (respectively, 2). We can assume that Σ^- is a separable metric space. We will use the following result: if A, B are compact metric spaces and $\dim A = 1$, then (see [10])

$$\dim (A \times B) = \dim A + \dim B.$$

Let $x \in \Sigma^-$. There are compact neighbourhoods, P of x in Σ^- and J of 0 in $\mathbb{R}$ such that $J \times P$ is homeomorphic to $\varphi(J \times P)$. As $\varphi(J \times P)$ contains a neighbourhood of x homeomorphic to $\mathbb{R}^2$ (respectively $\mathbb{R}^3$) and $\dim J = 1$, so $\dim P = 1$ (resp. 2). It follows by Theorem 3 (or Theorem 13) in [3] that Σ^- as a topological divisor of $\mathbb{R}^n$ of dimension less or equal to 2, is a 1 (respectively, 2) dimensional manifold.

Since $B^- \subset \Sigma^-$ is compact, there exists a finite number of connected components of $\varphi((-\varepsilon,\varepsilon) \times \Sigma^-)$ that cover B^-.

Let $n = 2$. From [11] we know, that a separable connected space locally homeomorphic to $\mathbb{R}$ is homeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$, so we can take a neighbourhood W of B^- in Σ^- consisting of closed intervals or circles, small enough that

(∗) for all $x \in W \setminus B^-$there exists $t_x > 0$ such that

$$\varphi(t_x, x) \in \Sigma^+, \ \varphi((0,t_x), x) \cap \Sigma^+ = \emptyset.$$

The proof is finished by taking the new block

$$C = B \cup \bigcup_{x \in W \setminus B^-} \varphi([0, t_x], x)$$

with $C^- = W$.

Let $n = 3$. It follows from [16] that Σ^- has a triangulation. By a barycentric subdivision of some triangulation of Σ^-, we choose a finite number of triangles (closed) such that B^- is contained in the union W of them. We can assume that the condition $(*)$ holds. If a point $w \in W$ is not in the interior of some triangle, then it has to be on the edge. If w is not a vertex then it has a neighbourhood homeomorphic to a plane or a half-plane. Assume that w is a vertex. If w separates a neighbourhood in W, then one can choose a point (not a vertex) on each edge with a vertex w, remove triangles with vertices in these points and w and obtain a neighbourhood of B^- with the number of such a vertices w reduced by 1. After a finite number of steps we are able to remove all vertices that separate neighbourhood in W. If w does not separate a neighbourhood in W, then it is easy to check that w has a neighbourhood homeomorphic to a plane or a half-plane. We finish the proof by taking a block C like in the case $n = 2$. □

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