Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 32, 2008, 261–312

FILIPPOV–WAŻEWSKI THEOREMS AND STRUCTURE OF SOLUTION SETS FOR FIRST ORDER IMPULSIVE SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS

Smaïl Djebali — Lech Górniewicz — Abdelghani Ouahab

Abstract. In this paper, we first present an impulsive version of Filippov's Theorem for first-order semilinear functional differential inclusions of the form:

$$\begin{cases} (y' - Ay) \in F(t, y_t) & \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)) & \text{for } k = 1, \dots, m, \\ y(t) = \phi(t) & \text{for } t \in [-r, 0], \end{cases}$$

where $J = [0, b]$, A is the infinitesimal generator of a C_0-semigroup on a separable Banach space E and F is a set-valued map. The functions I_k characterize the jump of the solutions at impulse points t_k ($k = 1, \dots, m$). Then the convexified problem is considered and a Filippov–Ważewski result is proved. Further to several existence results, the topological structure of solution sets — closeness and compactness — is also investigated. Some results from topological fixed point theory together with notions of measure on noncompactness are used. Finally, some geometric properties of solution sets, AR, R_δ-contractibility and acyclicity, corresponding to Aronszajn–Browder–Gupta type results, are obtained.

2000 *Mathematics Subject Classification.* 34A37, 34A60, 34K30, 34K45, 47D60, 47H09, 47H10.

Key words and phrases. Impulsive functional differential inclusions, mild solution, Filippov's theorem, relaxation, solution set, compactness, AR, R_δ, contractibility, acyclicity.

1. Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [43] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [31].

The dynamics of many processes in physics, population dynamics, biology, medicine may be subject to abrupt changes such that shocks, perturbations (see for instance [1], [40] and the references therein). These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions.

Important contributions to the study of the mathematical aspects of such equations have been undertaken in [8], [41], [49], [51] among others. Functional differential equations with impulsive effects with fixed moments have been recently addressed by Yujun and Erxin [57] and Yujun [56]. For further readings on functional differential equations, we recommend the monographs by Azbelez *et. al* [7] or by J. Hale and S. M. V. Lunel [30]. Some existence results on impulsive functional differential equations with finite or infinite delay may be found in [9], [11], [45], [46] too. During the last couple of years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensively studied (see the book by Aubin [4] and also [12], [23], [32], [33], [52] and the references therein).

Given a real separable Banach space E with norm $|\cdot|$, we will consider in this paper the impulsive problem for first-order semilinear differential inclusions

$$\tag{1.1} \begin{cases} (y' - Ay)(t) \in F(t, y_t) & \text{for a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ \Delta y_{t=t_k} = I_k(y(t_k^-)) & \text{for } k = 1, \dots, m, \\ y(t) = \phi(t) & \text{for } t \in [-r, 0], \end{cases}$$

where $0 < r < \infty$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $J = [0, b]$. $F\colon J \times \mathcal{D} \to \mathcal{P}(E)$ is a multifunction, and $\phi \in \mathcal{D}$ where

$$\begin{aligned} \mathcal{D} = \{\psi\colon [-r, 0] \to E\colon \psi \text{ is continuous everywhere except for a finite number} \\ \text{of points } \overline{t} \text{ at which } \psi(\overline{t}^-) \text{ and } \psi(\overline{t}^+) \text{ exist and satisfy } \psi(\overline{t}^-) = \psi(\overline{t})\}. \end{aligned}$$

The operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ (see Section 2), $I_k \in C(E, E)$ $(k = 1, \dots, m)$ and $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ stand for the right and the left limits of the function y at $t = t_k$, respectively. For any function y defined on $[-r, b]$ and any $t \in J$, y_t refers to the element of $\mathcal{D}$ such that

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0];$$

thus the function y_t represents the history of the state from time $t-r$ up to the present time t.

Some auxiliary results needed in this paper are gathered together in Section 2. In this work, we shall be mainly concerned with the Filippov's theorem for first order impulsive semilinear functional differential inclusions in a Banach space. This is presented and developed in Section 3. In Section 4, we discuss the relaxed problem associated to problem (1.1), that is the problem when we consider the closure of the convex hull of the right-hand side instead. This corresponds to a Filippov–Ważewski approach; we prove that the solution set of problem (1.1) is dense in that of the convexified problem. Then some topological properties of the operator solution and of the solution sets (closeness and compactness) are provided in Section 5. In addition, some existence results are obtained. Finally, Section 6 is devoted to proving some geometric properties of solution sets such that acyclicity, AR, R_δ contractibility, and R_δ-contractibility. We end the paper with some concluding remarks and a rich bibliography.

2. Preliminaries

In this section, we recall some notations, definitions, and preliminary facts which will be used throughout. Let $[0,b]$ be a interval in $\mathbb{R}$ and $C([0,b],E)$ be the Banach space of all continuous functions from $[0,b]$ into E with the norm

$$\|y\|_\infty = \sup\{|y(t)|\colon 0 \leq t \leq b\}.$$

$B(E)$ refers to the Banach space of linear bounded operators from E into E with norm

$$\|N\|_{B(E)} = \sup\{|N(y)|\colon |y| = 1\}.$$

A function $y\colon J \to E$ is called measurable provided for every open subset $U \subset E$, the set $y^{-1}(U) = \{t \in J\colon y(t) \in U\}$ is Lebesgue measurable. A measurable function $y\colon J \to E$ is Bochner integrable if $|y|$ is Lebesgue integrable. For properties of the Bochner integral, see e.g. Yosida [55]. In what follows, $L^1(J,E)$ denotes the Banach space of functions $y\colon J \to E$, which are Bochner integrable with norm

$$\|y\|_{L^1} = \int_0^b |y(t)|\,dt.$$

2.1. Multivalued analysis. Denote by $\mathcal{P}(E) = \{Y \subset E : Y \neq \emptyset\}$, $\mathcal{P}_{\rm cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ closed}\}$, $\mathcal{P}_{\rm b}(E) = \{Y \in \mathcal{P}(E) : Y \text{ bounded}\}$, $\mathcal{P}_{\rm cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ convex}\}$, $\mathcal{P}_{\rm cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ compact}\}$, and $\mathcal{P}_{\rm wkcp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ weakly compact}\}$. Consider the Hausdorff pseudo-metric distance

$$H_d\colon \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}^+ \cup \{\infty\}$$

defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{\mathrm{b,cl}}(E), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [39]). In particular, H_d satisfies the triangle inequality.

Definition 2.1. A multivalued operator $N\colon E \to \mathcal{P}_{\mathrm{cl}}(E)$ is called

(a) γ-*Lipschitz* if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y), \quad \text{for each } x, y \in E,$$

(b) a *contraction* if it is γ-Lipschitz with $\gamma < 1$.

Notice that if N is γ-Lipschitz, then for every $\gamma' > \gamma$,

$$N(x) \subset N(y) + \gamma' d(x,y) B(0,1), \quad \text{for all } x, y \in A.$$

Let (X,d) and (Y,ρ) be two metric spaces and $G\colon X \to \mathcal{P}_{\mathrm{cl}}(Y)$ be a multivalued mapping. A single-valued map $g\colon X \to Y$ is said to be a *selection of* G and we write $g \subset G$ whenever $g(x) \in G(x)$ for every $x \in X$.

G is called *upper semi-continuous* (*u.s.c.* for short) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq Y$. That is, if the set $G^{-1}(V) = \{x \in X : G(x) \cap V \neq \emptyset\}$ is closed for any closed set V in Y. Equivalently, G is u.s.c. if the set $G^{+1}(V) = \{x \in X : G(x) \subset V\}$ is open for any open set V in Y.

G is said to be *completely continuous* if it is u.s.c. and, for every bounded subset $A \subseteq X$, $G(A)$ is relatively compact, i.e. there exists a relatively compact set $K = K(A) \subset X$ such that $G(A) = \bigcup\{G(x) : x \in A\} \subset K$. G is compact if $G(X)$ is relatively compact. It is called locally compact if, for each $x \in X$, there exists $U \in \mathcal{V}(x)$ such that $G(U)$ is relatively compact.

We denote the graph of G to be the set $\mathcal{G}r(G) = \{(x,y) \in X \times Y : y \in G(x)\}$ and recall

Lemma 2.2 ([18, Proposition 1.2]). *If $G\colon X \to \mathcal{P}_{\mathrm{cl}}(Y)$ is u.s.c. then $\mathcal{G}r(G)$ is a closed subset of $X \times Y$, i.e. for every sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G has nonempty compact values, is locally compact and has a closed graph, then it is u.s.c.*

The following two lemmas are concerned with measurability of multi-functions; they will be needed in this paper. The first one is the celebrated Kuratowski–Ryll–Nardzewski selection theorem.

Lemma 2.3 (see [27, Theorem 19.7]). *Let E be a separable metric space and G a measurable multi-valued map with nonempty closed values. Then G has a measurable selection.*

Lemma 2.4 (see [58, Lemma 3.2]). *Let $G\colon [0,b] \to \mathcal{P}(E)$ be a measurable multifunction and $u\colon [0,b] \to E$ a measurable function. Then for any measurable $v\colon [0,b] \to \mathbb{R}^+$ there exists a measurable selection g of G such that for almost every $t \in [0,b]$,*

$$|u(t) - g(t)| \leq d(u(t), G(t)) + v(t).$$

Finally, for a multi-valued function $G\colon J \times \mathcal{D} \to \mathcal{P}(E)$, denote

$$\|G(t,z)\|_{\mathcal{P}} := \sup\{|v| : v \in G(t,z)\}.$$

Definition 2.5. G is called a *multi-valued Carathéodory function* if

(a) the function $t \mapsto G(t,z)$ is measurable for each $z \in \mathcal{D}$,
(b) for almost every $t \in J$, the map $z \mapsto G(t,z)$ is upper semi-continuous.

It is further an L^1-Carathéodory if it is locally integrably bounded, i.e. for each positive real number r, there exists some $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|G(t,z)\|_{\mathcal{P}} \leq h_r(t) \quad \text{for a.e. } t \in J \text{ and all } \|z\|_{\mathcal{D}} \leq r.$$

For further readings and details on multivalued analysis, we refer to the books by Andres and Górniewicz [2], Aubin and Celina [5], Aubin and Frankowska [6], Deimling [18], Górniewicz [27], Hu and Papageorgiou [36], Kamenskiĭ [38], and Tolstonogov [54].

2.2. C_0-semigroups.

Definition 2.6. A semigroup of class C_0 (or C_0-semigroup) is a one parameter family $\{T(t) : t \geq 0\} \subset B(E)$ satisfying the conditions:

(a) $T(t) \circ T(s) = T(t+s)$, for $t, s \geq 0$,
(b) $T(0) = I$.

Here I denotes the identity operator in E.

Definition 2.7. A semigroup $T(t)$ is *uniformly continuous* if

$$\lim_{t \to 0^+} \|T(t) - T(0)\|_{B(E)} = 0, \quad \text{that is if} \quad \lim_{|t-s| \to 0} \|T(t) - T(s)\|_{B(E)} = 0.$$

Definition 2.8. We say that the semigroup $\{T(t)_{t\geq 0}\}$ is *strongly continuous* (or a C_0-semigroup) if the map $t \to T(t)(x)$ is strongly continuous, for each $x \in E$, i.e.

$$\lim_{t \to 0^+} T(t)x = T(0)x, \quad \text{for all } x \in E.$$

Definition 2.9. Let $T(t)$ be a C_0-semigroup defined on E. The infinitesimal generator $A \in B(E)$ of $T(t)$ is the linear operator defined by

$$A(x) = \lim_{t \to 0^+} \frac{T(t)(x) - T(0)x}{t}, \quad \text{for } x \in D(A),$$

where $D(A) = \{x \in E : \lim_{t \to 0^+} (T(t)(x) - x)/t \text{ exists in } E\}$.

The following properties are classical (see Pazy [50], Engel and Nagel [20], Hill and Philips [35]).

Proposition 2.10. *Let $\{T(t)\}_{t \geq 0}$ be a uniformly continuous semigroup of bounded linear operators. Then there exists some constant $\omega \geq 0$ such that*

$$\|T(t)\|_{B(E)} \leq \exp(\omega t), \quad \textit{for } t \geq 0.$$

Proposition 2.11.

(a) *If $\{T(t)\}_{t \geq 0}$ is a C_0-semigroup of bounded linear operators, then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{B(E)} \leq M \exp(\omega t), \quad \textit{for } t \geq 0.$$

(b) *If A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$, then $D(A)$, the domain of A, is dense in X and A is a closed linear operator.*

2.3. Mild solutions. Let $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for problem (1.1), consider the space

$$\begin{aligned} PC = \{y\colon [0,b] \to E,\ y_k \in C(J_k, E),\ k = 0, \ldots, m,\ \text{such that} \\ y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-) = y(t_k) \text{ for } k = 1, \ldots, m\}. \end{aligned}$$

Endowed with the norm $\|y\|_{PC} = \max\{\|y_k\|_\infty,\ k = 0, \ldots, m\}$, PC is a Banach space. Moreover, if $\Omega = \{y\colon [-r,b] \to E,\ y \in PC \cap \mathcal{D}\}$, then Ω is a Banach space with the norm $\|y\|_\Omega = \max\{\|y\|_{PC}, \|y\|_{\mathcal{D}}\}$, where $\|y\|_{\mathcal{D}} = \sup_{t \in [-r,0]} |y(t)|$.

Throughout this paper, A is an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$ and the constants $M > 0$ and ω are as introduced in Proposition 2.11. A fundamental notion for the definition of solutions to problem (1.1) is given by

Definition 2.12. A function $y \in \Omega$ is said to be a *mild solution* of problem (1.1) if there exists $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ almost everywhere on J, $y(t) = \phi(t)$, $t \in [-r, 0]$ and

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)\,ds + \sum_{0 < t_k < t} T(t - t_k) I_k(y(t_k^-)).$$

3. Filippov's Theorem

Regarding differential equations and inclusions, some existence results for problem (1.1) can be found in [47]. Further results will be given subsequently in this paper. In this section, we are mainly concerned with a Filippov's result for problem (1.1). Such results are of great importance in stability and control theory. In the finite dimensional case, the problem was investigated by Filippov [21] in 1967 for first-order differential inclusions and later by Frankowska [23] in 1990 for first-order semilinear differential inclusions; see e.g. also Aubin and Cellina [5, Theorem 1, p. 120], Aubin and Frankowska [6, Theorem 10.4.1, p. 401]). When E is not necessarily separable, interesting results are given in [58]. Filippov's Theorem yields an estimate of the distance of a given solution to the solution set of a problem providing a kind of Gronwall's inequality (see also [53, Theorem 4.5, p. 91]).

3.1. Filippov's Theorem on a bounded interval. Let $\psi \in \mathcal{D}$, $g \in L^1(J,E)$ and let $x \in \Omega$ be a mild solution of the impulsive differential problem with semi-linear equation:

$$\begin{cases} x'(t) - Ax(t) = g(t) & \text{for a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ \Delta x_{t=t_k} = I_k(x(t_k^-)) & \text{for } k = 1, \dots, m, \\ x(t) = \psi(t) & \text{for } t \in [-r, 0]. \end{cases} \tag{3.1}$$

We will consider the following two assumptions

($\mathcal{H}_1$) The function $F: J \times \mathcal{D} \to \mathcal{P}_{\rm cl}(E)$ is such that
  (a) for all $y \in \mathcal{D}$ the map $t \mapsto F(t,y)$ is measurable,
  (b) the map $\gamma: t \mapsto d(g(t), F(t,x_t))$ is integrable.

($\mathcal{H}_2$) There exist a function $p \in L^1(J,\mathbb{R}^+)$ and a positive constant $\beta > 0$ such that

$$H_d(F(t,z_1),F(t,z_2)) \le p(t)\|z_1 - z_2\|_{\mathcal{D}}, \quad \text{for all } z_1, z_2 \in \mathcal{B}(x_t,\beta),$$

where $\mathcal{B}(x_t,\beta)$ is the closed ball of $\mathcal{D}$ centered in x_t with radius β.

Remark 3.1. From Assumptions ($\mathcal{H}_1$)(a) and ($\mathcal{H}_2$), it follows that the multi-function $t \mapsto F(t,x_t)$ is measurable and by Lemmas 1.4, 1.5 from [23], we deduce that $\gamma(t) = d(g(t),F(t,x_t))$ is measurable (see also Remark p. 400 in [6]).

Let $P(t) = \int_0^t p(s)\,ds$ and δ be a positive constant. Define the family of functions $(\eta_k(t))_{t\ge 0}$ $(k = 0, \dots, m)$ by

$$\eta_0(t) = Me^{\omega t_1}\delta + Me^{\omega t_1}\int_0^t [Me^{\omega t_1}H_0(s)P(s) + \gamma(s)]\,ds, \quad t \in (0,t_1]$$

where

$$H_0(t) = \delta M \exp(Me^{\omega t + P(t)}) + M\int_0^t \gamma(s)\exp(Me^{\omega t + P(t) - P(s)})\, ds,$$

and for $k = 1,\dots,m$

$$\eta_k(t) = Me^{\omega t_{k+1}}\int_{t_k}^t [Me^{\omega t}H_k(s)P(s) + \gamma(s)]\, ds, \quad t \in (t_k, t_{k+1}],$$

where

$$H_k(t) = \delta\exp(Me^{\omega(t-t_k)+P(t)}) + \int_{t_k}^t \gamma(s)\exp(Me^{\omega(t-t_k)+P(t)-P(s)})\, ds.$$

Theorem 3.2. *Let $\gamma_k := \gamma_{|_{J_k}}$ and assume that $\eta_{k-1}(t_k) \le \beta$ for $k = 1,\dots,m$. Then, for every $\phi \in \mathcal{D}$ with $\|\phi - \psi\|_{\mathcal{D}} \le \delta$, problem* (1.1) *has at least one solution y satisfying, for almost every $t \in [0,b]$, the estimates*

$$\|y_t - x_t\|_{\mathcal{D}} \le M\sum_{0<t_k<t} e^{\omega(t-t_k)}[|y(t_k) - x(t_k)| + |I_k(y(t_k)) - I_k(x(t_k))|] + \sum_{0<t_k<t}\eta_k(t),$$

and

$$|y'(t) - Ay(t) - g(t)| \le Mp(t)\sum_{0<t_k<t} e^{\omega t_{k+1}}H_k(t) + \sum_{0<t_k<t}\gamma_k(t).$$

Proof. We are going to study problem (1.1) respectively in the intervals $[-r,t_1]$, $(t_1,t_2],\dots,(t_m,b]$. The proof will be given in three steps and then continued by induction. Let $\phi \in \mathcal{D}$ be such that $\|\phi - \psi\|_{\mathcal{D}} \le \delta$.

Step 1. In this first step, we construct a sequence of function $(y_n)_{n\in\mathbb{N}}$ which will be shown to converge to some solution of problem (1.1) on the interval $[-r,t_1]$, namely to

$$\begin{cases} (y'(t) - Ay(t)) \in F(t,y_t) & \text{for } t \in J_0 = (0,t_1], \\ y(t) = \phi(t) & \text{for } t \in [-r,0]. \end{cases}$$

Let $f_0 = g$ on $[-r,t_1]$ and $y^0(t) = x(t),\ t \in [0,t_1)$, i.e.

$$y^0(t) = \begin{cases} \psi(t) & \text{for } t \in [-r,0], \\ T(t)\psi(0) + \displaystyle\int_0^t T(t-s)f_0(s)\, ds & \text{for } t \in (0,t_1], \end{cases}$$

Then define the multi-valued map $U_1\colon [0,t_1] \to \mathcal{P}(E)$ by $U_1(t) = F(t,y_t^0) \cap \mathcal{B}(g(t),\gamma(t))$. Since g and γ are measurable, Theorem III.4.1 in [16] tells us that the ball $\mathcal{B}(g(t),\gamma(t))$ is measurable. Moreover $F(t,y_t^0)$ is measurable (see Remark 3.1) and U_1 is nonempty. Indeed, since $v = 0$ is a measurable function, from Lemma 2.4, there exists a function u which is a measurable selection of $F(t,y_t^0)$ and such that

$$|u(t)-g(t)|\le d(g(t),F(t,y_t^0))=\gamma(t).$$

Then $u\in U_1(t)$, proving our claim. We deduce that the intersection multivalued operator $U_1(t)$ is measurable (see [6], [16], [27]). By Lemma 2.3 (Kuratowski–Ryll–Nardzewski selection theorem), there exists a function $t\mapsto f_1(t)$ which is a measurable selection for U_1. Consider

$$y^1(t)=\begin{cases}\phi(t) & \text{for } t\in[-r,0],\\ T(t)\phi(0)+\displaystyle\int_0^t T(t-s)f_1(s)\,ds & \text{for } t\in[0,t_1].\end{cases}$$

For each $t\in[0,t_1]$, we have by Proposition 2.11

(3.2)
$$\begin{aligned}|y^1(t)-y^0(t)| &\le Me^{\omega t}|\phi(0)-\psi(0)|+M\int_0^t e^{\omega(t-s)}|f_0(s)-f_1(s)|\,ds\\ &\le Me^{\omega t_1}\|\phi-\psi\|_{\mathcal{D}}+Me^{\omega t_1}\int_0^t|f_0(s)-f_1(s)|\,ds.\end{aligned}$$

Hence

$$\sup_{t\in[-r,t_1]}\{|y^1(t)-y^0(t)|\}\le Me^{\omega t_1}\delta+Me^{\omega t_1}\int_0^t\gamma(s)\,ds$$

and then

$$\begin{aligned}\|y_t^1-y_t^0\|_{\mathcal{D}} &= \sup_{\theta\in[-r,0]}|y_t^1(\theta)-y_t^0(\theta)|=\sup_{\theta\in[-r,0]}|y^1(t+\theta)-y^0(t+\theta)|\\ &= \sup_{\theta\in[-r+t,t]}|y^1(s)-y^0(s)|\le\eta_0(t_1)\le\beta.\end{aligned}$$

Then Lemma 1.4 in [23] tells us that $F(t,y_t^1)$ is measurable.

The ball $\mathcal{B}(f_1(t),p(t)\|y_t^1-y_t^0\|_{\mathcal{D}})$ is also measurable by Theorem III.4.1 in [16]. The set $U_2(t)=F(t,y_t^1)\cap\mathcal{B}(f_1(t),p(t)\|y_t^1-y_t^0\|_{\mathcal{D}})$ is nonempty. Indeed, since f_1 is a measurable function, Lemma 2.4 yields a measurable selection u of $F(t,y_t^1)$ such that

$$|u(t)-f_1(t)|\le d(f_1(t),F(t,y_t^1).$$

Moreover, $\|y_t^1-y_t^0\|_{\mathcal{D}}\le\eta_0(t_1)\le\beta$. Then using $(\mathcal{H}_2)$, we get

$$|u(t)-f_1(t)|\le d(f_1(t),F(t,y_t^1))\le H_d(F(t,y_t^0),F(t,y_t^1))\le p(t)\|y_t^0-y_t^1\|_{\mathcal{D}},$$

i.e. $u\in U_2(t)$, proving our claim Now, since the intersection multi-valued operator U_2 defined above is measurable (see [6], [16], [27]), there exists a measurable selection $f_2(t)\in U_2(t)$. Hence

$$|f_1(t)-f_2(t)|\le p(t)\|y_t^1-y_t^0\|_{\mathcal{D}}. \tag{3.3}$$

Define

$$y^2(t) = \begin{cases} \phi(t) & \text{for } t \in [-r, 0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s) f_2(s)\, ds & \text{for } t \in (0, t_1]. \end{cases}$$

Using (3.2) and (3.3), a simple integration by parts yields the following estimates, valid for every $t \in [-r, t_1]$,

$$\begin{aligned} |y^2(t) - y^1(t)| &\leq \int_0^t \|T(t-s)\| \cdot |f_2(s) - f_1(s)|\, ds \\ &\leq \int_0^t M e^{\omega t_1} p(s) \bigg(M e^{\omega t_1} \delta + M e^{\omega t_1} \int_0^s \gamma(u)\, du \bigg)\, ds \\ &= M^2 e^{2\omega t_1} \bigg(\delta \int_0^t p(s)\, ds + \int_0^t p(s)\, ds \int_0^s \gamma(u)\, du \bigg) \\ &\leq M^2 e^{2\omega t_1} \bigg(\delta \int_0^t p(s) e^{P(s)}\, ds + \int_0^t p(s) e^{P(s)}\, ds \int_0^s e^{-P(u)} \gamma(u)\, du \bigg) \\ &\leq M^2 e^{2\omega t_1} \bigg(\delta e^{P(t)} + \int_0^t \gamma(s) e^{P(t) - P(s)}\, ds \bigg). \end{aligned}$$

Let $U_3(t) = F(t, y_t^2) \cap \mathcal{B}(f_2(t), p(t)\|y_t^2 - y_t^1\|_{\mathcal{D}})$. Arguing as for U_2, we can prove that U_3 is a measurable multi-valued map with nonempty values; so there exists a measurable selection $f_3(t) \in U_3(t)$. This allows us to define

$$y^3(t) = \begin{cases} \phi(t) & \text{for } t \in [-r, 0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s) f_3(s)\, ds & \text{for } t \in (0, t_1]. \end{cases}$$

For $t \in [0, t_1]$, we have

$$|y^3(t) - y^2(t)| \leq M e^{\omega t_1} \int_0^t |f_2(s) - f_3(s)|\, ds \leq M e^{\omega t_1} \int_0^t p(s) \|y_s^2 - y_s^1\|_{\mathcal{D}}\, ds.$$

However

$$\|y_s^2 - y_s^1\|_{\mathcal{D}} = \sup_{\theta \in [-r, 0]} |y_s^2(\theta) - y_s^1(\theta)| = \sup_{\theta \in [-r, 0]} |y^2(s+\theta) - y^1(s+\theta)|,$$

and from the estimates above, for $\theta \in [-r, 0]$, we have

$$\begin{aligned} |y^2(s+\theta) - y^1(s+\theta)| &\leq M^2 e^{2\omega t_1} \bigg(\delta e^{P(\theta+s)} + \int_0^{\theta+s} \gamma(u) e^{P(\theta+s) - P(u)}\, du \bigg) \\ &\leq M^2 e^{2\omega t_1} \bigg(\delta e^{P(s)} + \int_0^s \gamma(u) e^{P(\theta+s) - P(u)}\, du \bigg). \end{aligned}$$

Performing an integration by parts, we obtain, since P is a nondecreasing function, the following estimates

$$\begin{aligned}|y^3(t)-y^2(t)| &\le \frac{M^3e^{3\omega t_1}}{2}\int_0^t 2p(s)\biggl(\delta e^{2P(s)}+\int_0^s \gamma(u)e^{P(s)-P(u)}\,du\biggr)\,ds\\
&\le \frac{M^3e^{3\omega t_1}}{2}\biggl(\delta e^{2P(t)}+\int_0^t 2p(s)\,ds\int_0^s \gamma(u)e^{2(P(s)-P(u))}\,du\biggr)\\
&\le \frac{M^3e^{3\omega t_1}}{2}\biggl(\delta e^{2P(t)}+\int_0^t (e^{2P(s)})'\,ds\int_0^s \gamma(u)e^{-2P(u))}\,du\biggr)\\
&\le \frac{M^3e^{3\omega t_1}}{2}\biggl(\delta e^{2P(t)}+e^{2P(t)}\int_0^t \gamma(s)e^{-2P(s)}\,ds-\int_0^t\gamma(s)\,ds\biggr)\\
&\le \frac{M^3e^{3\omega t_1}}{2}\biggl(\delta e^{2P(t)}+\int_0^t \gamma(s)e^{2(P(t)-P(s))}\,ds\biggr),\end{aligned}$$

for $t\in[-r,t_1]$. Let $U_4(t)=F(t,y_t^3)\cap\mathcal{B}(f_3(t),p(t)\|y_t^3-y_t^2\|_{\mathcal{D}})$. Then, arguing again as for U_1, U_2, U_3, we show that U_4 is a measurable multi-valued map with nonempty values and that there exists a measurable selection $f_4(t)$ in $U_4(t)$. Define

$$y^4(t)=\begin{cases}\phi(t) & \text{for } t\in[-r,0],\\[4pt] T(t)\phi(0)+\displaystyle\int_0^t T(t-s)f_4(s)\,ds & \text{for } t\in(0,t_1].\end{cases}$$

For $t\in[0,t_1]$, we have

$$\begin{aligned}&|y^4(t)-y^3(t)|\le Me^{\omega t_1}\int_0^t |f_4(s)-f_3(s)|\,ds\le Me^{\omega t_1}\int_0^t p(s)\|y_s^3-y_s^2\|_{\mathcal{D}}\,ds\\
&\le \frac{M^4e^{4\omega t_1}}{2}\int_0^t p(s)\biggl(\delta e^{2P(s)}+\int_0^s \gamma(s)e^{2(P(s)-P(u))}\,du\biggr)\,ds\\
&\le \frac{M^4e^{4\omega t_1}}{6}\delta\int_0^t 3p(s)e^{3P(s)}\,ds+\frac{M^4e^{4\omega t_1}}{6}\int_0^t 3p(s)e^{3P(s)}\,ds\int_0^s\gamma(s)e^{-3P(u)}\,du\\
&\le \frac{M^4e^{4\omega t_1}}{6}\biggl(\delta e^{3P(t)}+\int_0^t \gamma(s)e^{3(P(t)-P(s))}\,ds\biggr).\end{aligned}$$

Repeating the process for $n=0,1,\ldots$, we arrive at the following bound

$$|y^n(t)-y^{n-1}(t)|\le \frac{M^ne^{n\omega t_1}}{(n-1)!}\int_0^t \gamma(s)e^{(n-1)(P(t)-P(s))}\,ds+\frac{M^ne^{n\omega t_1}}{(n-1)!}\delta e^{(n-1)P(t)},\tag{3.4}$$

for $t\in[-r,t_1]$. By induction, suppose that (3.4) holds for some n and check it for $n+1$. Let $U_{n+1}(t)=F(t,y^n(t))\cap\mathcal{B}(f_n,p(t)\|y_t^n-y_t^{n-1}\|_{\mathcal{D}})$. As in the above arguing, U_{n+1} is a nonempty measurable set, then has a measurable selection

$f_{n+1}(t) \in U_{n+1}(t)$; this allows us to define for $n \in \mathbb{N}$

$$
y^{n+1}(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s) f_{n+1}(s)\, ds & \text{for } t \in (0,t_1]. \end{cases} \tag{3.5}
$$

Therefore, for almost every $t \in [0,t_1]$, we have

$$
\begin{aligned}
|y^{n+1}(t) - y^n(t)| &\leq M e^{\omega t_1} \int_0^t |f_{n+1}(s) - f_n(s)|\, ds \\
&\leq \frac{M^{n+1} e^{(n+1)\omega t_1}}{(n-1)!} \int_0^t p(s) \|y_s^n - y_s^{n-1}\|_{\mathcal{D}}\, ds \\
&\leq \frac{M^{n+1} e^{(n+1)\omega t_1}}{(n-1)!} \int_0^t p(s)\, ds \bigg(\delta e^{(n-1)P(s)} + \int_0^s \gamma(u) e^{(n-1)(P(s)-P(u))}\, du \bigg) \\
&\leq \frac{M^{n+1} e^{(n+1)\omega t_1}}{n!} \int_0^t \delta n p(s) e^{nP(s)}\, ds \\
&\quad + \frac{M^{n+1} e^{(n+1)\omega t_1}}{n!} \int_0^t n p(s) e^{nP(s)}\, ds \int_0^s \gamma(u) e^{-nP(u)}\, du.
\end{aligned}
$$

Again, an integration by parts yields

$$
\begin{aligned}
|y^{n+1}(t) - y^n(t)| \leq \frac{M^{(n+1)} e^{(n+1)\omega t_1}}{n!} \int_0^t \gamma(s) e^{n(P(t)-P(s))}\, ds& \\
+ \frac{M^{(n+1)} e^{(n+1)\omega t_1}}{n!} \delta e^{nP(t)}.&
\end{aligned}
$$

Consequently, (3.4) holds true for all $n \in \mathbb{N}$. We infer that $\{y^n\}$ is a Cauchy sequence in Ω_1, converging uniformly to a limit function $y \in \Omega_1$, where $\Omega_1 = C([0,t_1],E) \cap \mathcal{D}$. Moreover, from the definition of $\{U_n\}$, we have

$$
|f_{n+1}(t) - f_n(t)| \leq p(t) \|y_t^n - y_t^{n-1}\|_{\mathcal{D}}, \quad \text{for a.e. } t \in [0,t_1].
$$

Hence, for almost every $t \in [0,t_1]$, $\{f_n(t)\}$ is also a Cauchy sequence in E and then converges almost everywhere to some measurable function $f(\,\cdot\,)$ in E. In addition, since $f_0 = g$, we have, for almost every $t \in [0,t_1]$

$$
\begin{aligned}
|f_n(t)| &\leq \sum_{k=1}^n p(t) |f_k(t) - f_{k-1}(t)| + |f_0(t)| \\
&\leq \sum_{k=1}^n p(t) |y^{k-1}(t) - y^{k-2}(t)| + \gamma(t) + |g(t)| \\
&\leq p(t) \sum_{k=1}^\infty |y^k(t) - y^{k-1}(t)| + \gamma(t) + |g(t)|.
\end{aligned}
$$

Hence

$$
|f_n(t)| \leq M e^{\omega t_1} H_0(t) p(t) + \gamma(t) + |g(t)|,
$$

where

$$H_0(t) := \delta M \exp(Me^{\omega t+P(t)}) + M \int_0^t \gamma(s) \exp(Me^{\omega t+P(t)-P(s)})\, ds. \tag{3.6}$$

From the Lebesgue dominated convergence theorem, we deduce that $\{f_n\}$ converges to f in $L^1([0,t_1],E)$. Passing to the limit in (3.5), we find that the function

$$y(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ y(t) = T(t)\phi(0) + \displaystyle\int_0^t T(t-s)f(s)\, ds, & \text{for } t \in (0,t_1] \end{cases}$$

is solution to problem (1.1) on $[0,t_1]$; thus $y \in S_{[0,t_1]}(\phi)$. Moreover, for almost every $t \in [0,t_1]$, we have

$$\begin{aligned} |x(t)-y(t)| &= \left| T(t)\phi(0) + \int_0^t T(t-s)g(s)\, ds - T(t)\psi(0) - \int_0^t T(t-s)f(s)\, ds \right| \\ &\leq Me^{\omega t_1}|\phi(0)-\psi(0)| + Me^{\omega t_1} \int_0^t |f(s)-f_0(s)|\, ds \\ &\leq Me^{\omega t_1}\|\phi-\psi\|_{\mathcal{D}} + Me^{\omega t_1} \int_0^t |f(s)-f_n(s)|\, ds \\ &\quad + Me^{\omega t_1} \int_0^t |f_n(s)-f_0(s)|\, ds \\ &\leq Me^{\omega t_1}\|\phi-\psi\|_{\mathcal{D}} + Me^{\omega t_1} \int_0^t |f(s)-f_n(s)|\, ds \\ &\quad + Me^{\omega t_1} \int_0^t (Me^{\omega t_1} H(s)P(s) + \gamma(s))\, ds. \end{aligned}$$

Passing to the limit as $n \to \infty$, we get

$$|x(t)-y(t)| \leq \eta_0(t) \quad \text{for a.e. } t \in [-r,t_1] \tag{3.7}$$

with

$$\eta_0(t) := Me^{\omega t_1}\delta + Me^{\omega t_1} \int_0^t (Me^{\omega t_1} H_0(s)P(s) + \gamma(s))\, ds.$$

Next, we give an estimate for $|y'(t) - Ay(t) - g(t)|$ for $t \in [0,t_1]$. We have

$$\begin{aligned} |y'(t) - Ay(t) - g(t)| &= |f(t) - f_0(t)| \leq |f_n(t) - f_0(t)| + |f_n(t) - f(t)| \\ &\leq p(t) \sum_{k=1}^{\infty} |y^{k+1}(t) - y^k(t)| + \gamma(t) + |f_n(t) - f(t)|. \end{aligned}$$

Arguing as in (3.6) and passing to the limit as $n \to \infty$, we deduce that

$$|y'(t) - Ay(t) - g(t)| \leq Me^{\omega t_1} H_0(t)p(t) + \gamma(t), \quad t \in [0,t_1].$$

The obtained solution is thus denoted by $y_1 := y_{|[-r,t_1]}$.

Step 2. Consider now problem (1.1) on the second interval $[t_1 - r, t_2]$, i.e.

$$
(3.8)\qquad \begin{cases} (y'(t) - Ay(t)) \in F(t, y_t) & \text{for a.e. } t \in (t_1, t_2], \\ y(t_1^+) = y_1(t_1) + I_1(y_1(t_1)), \\ y(t) = y_1(t) & \text{for } t \in [t_1 - r, t_1]. \end{cases}
$$

Let $f_0 = g$ and set

$$
y^0(t) = \begin{cases} y_1(t) & \text{for } t \in [t_1 - r, t_1], \\ T(t - t_1)[x(t_1) - I_1(x(t_1))] + \displaystyle\int_{t_1}^t T(t - s) f_0(s)\, ds & \text{for } t \in (t_1, t_2]. \end{cases}
$$

Notice that (3.7) allows us to use Assumption $(\mathcal{H}_2)$, apply again Lemma 1.4 in [23] and argue as in Step 1 to prove that the multi-valued map $U_1\colon [t_1, t_2] \to \mathcal{P}(E)$ defined by $U_1(t) = F(t, y_t^0) \cap \mathcal{B}(g(t), \gamma(t))$ is $U_1(t)$ is measurable. Hence, there exists a function $t \mapsto f_1(t)$ which is a measurable selection for U_1. Define

$$
y^1(t) = \begin{cases} y_1(t) & \text{for } t \in [t_1 - r, t_1], \\ T(t - t_1)[y_1(t_1) + I_1(y_1(t_1))] + \displaystyle\int_{t_1}^t T(t - s) f_1(s)\, ds & \text{for } t \in (t_1, t_2]. \end{cases}
$$

Next define the measurable multi-valued map $U_2(t) = F(t, y_t^1) \cap \mathcal{B}(f_1(t), p(t)\|y_t^1 - y_t^0\|_{\mathcal{D}})$. It has a measurable selection $f_2(t) \in U_2(t)$ by the Kuratowski–Ryll–Nardzewski selection theorem. Repeating the process of selection as in Step 1, we can define by induction a sequence of multi-valued maps $U_n(t) = F(t, y_t^{n-1}) \cap \mathcal{B}(f_{n-1}(t), p(t)\|y_t^{n-1} - y_t^{n-2}\|_{\mathcal{D}})$ where $\{f_n\} \in U_n$ and $(y^n)_{n \in \mathbb{N}}$ is as defined by

$$
y^n(t) = \begin{cases} y_1(t) & \text{for } t \in [t_1 - r, t_1], \\ T(t - t_1)[y_1(t_1) + I_1(y_1(t_1))] + \displaystyle\int_{t_1}^t T(t - s) f_n(s)\, ds & \text{for } t \in (t_1, t_2]. \end{cases}
$$

Let $\Omega_2 = \{y\colon y \in \mathcal{D} \cap C[0, t_1] \cap C(t_1, t_2] \text{ and } y(t_1^+) \ \text{ exists}\}$. As in Step 1, we can prove that the sequence $\{y^n\}$ converges to some $y \in \Omega_2$ solution to problem (3.8) such that, for almost every $t \in (t_1, t_2]$, we have

$$
\begin{aligned} |x(t) - y(t)| \le{}& \eta_0(t_1) + Me^{\omega(t - t_1)}|I_1(x(t_1)) - I_1(y_1(t_1))| \\ &+ Me^{\omega(t - t_1)}|x_1(t_1) - y_1(t_1)| + Me^{\omega t_2} \int_{t_1}^t (M^{\omega t_2} H_1(s) P(s) + \gamma(s))\, ds \end{aligned}
$$

and

$$
|y'(t) - Ay(t) - g(t)| \le Me^{\omega t_2} H_1(t) p(t) + \gamma(t),
$$

where

$$
H_1(t) := \delta \exp(Me^{\omega(t - t_1) + P(t)}) + \int_{t_1}^t \gamma(s) \exp(Me^{\omega(t - t_1) + P(t) - P(s)})\, ds.
$$

Denote the restriction $y_{|[t_1, t_2]}$ by y_2.

Step 3. We continue this process until we arrive at the function $y_{m+1} := y|_{[t_m-r,t_m]\cup(t_m,b]}$ solution of the problem

$$\begin{cases} (y'(t) - Ay(t)) \in F(t, y_t) & \text{for a.e. } t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_m) + I_m(y_{m-1}(t_m)), \\ y(t) = y_{m-1}(t) & \text{for } t \in [t_m - r, t_m]. \end{cases}$$

Then, for almost every $t \in (t_m, b]$, the following estimates are easily derived

$$\begin{aligned} |x(t) - y(t)| \leq{}& Me^{\omega(t-t_m)}[|I_m(x(t_m)) - I_m(y(t_m))| + |y_m(t_m) - x(t_m)|] \\ &+ Me^{\omega t_m}\int_{t_m}^{t}(Me^{\omega t_m}H_m(s)P(s) + \gamma(s))\,ds \end{aligned}$$

and

$$|y'(t) - Ay(t) - g(t)| \leq Me^{\omega b}H_m(t)P(t) + \gamma(t)).$$

Step 4. Summarizing, a solution y of problem (1.1) can be defined as follows

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots \\ y_{m+1}(t) & \text{if } t \in (t_m, b]. \end{cases}$$

From Steps 1–3, we have that, for almost every $t \in [-r, t_1]$,

$$|x(t) - y(t)| \leq \eta_0(t) \quad \text{and} \quad |y'(t) - Ay(t) - g(t)| \leq Me^{\omega t_1}H_0(t)p(t) + \gamma(t),$$

as well as the following estimates, valid for $t \in (t_1, b]$

$$\begin{aligned} |x(t) - y(t)| \leq \sum_{k=2}^{m+1}|x(t) - y_k(t)| \leq{}& M\sum_{0<t_k<t}e^{\omega(t-t_k)}|x(t_k) - y_k(t_k)| \\ &+ M\sum_{0<t_k<t}e^{\omega(t-t_k)}|I_k(x(t_k)) - I_k(y_k(t_k))| + \sum_{k=0}^{m}\eta_k(t). \end{aligned}$$

Similarly

$$|y'(t) - Ay(t) - g(t)| \leq Mp(t)\sum_{0<t_k<t}e^{\omega t_{k+1}}H_k(t) + \sum_{0<t_k<t}\gamma_k(t),$$

where $\gamma_k := \gamma|_{J_k}$. The proof of Theorem 3.2 is complete. □

3.2. Filippov's Theorem on the half line. We may consider Filippov's problem on the half-line

$$\begin{cases} (y' - Ay)(t) \in F(t, y_t) & \text{for a.e. } t \in \widetilde{J} \setminus \{t_1, \dots\}, \\ \Delta y_{t=t_k} = I_k(y(t_k^-)) & \text{for } k = 1, 2 \dots, \\ y(t) = \phi(t) & \text{for } t \in [-r, 0], \end{cases} \tag{3.10}$$

where $\widetilde{J}=[0,\infty)$, $0<r<\infty$, $0=t_0<t_1<\ldots<t_m<\ldots$, $\lim_{m\to\infty}t_m=\infty$, $F\colon \widetilde{J}\times\mathcal{D}\to\mathcal{P}(E)$ is a multifunction, and $\phi\in\mathcal{D}$ where

$$\begin{aligned}\mathcal{D}=g\{\psi\colon[-r,0]\to E:{}&\psi \text{ is continuous everywhere exept}\\ &\text{for a finite number of points } \overline{t} \text{ at which } \psi(\overline{t}^-) \text{ and } \psi(\overline{t}^+) \text{ exist},\\ &\psi(\overline{t}^-)=\psi(\overline{t}) \text{ and } \sup_{\theta\in[-r,0]}|\psi(\theta)|<\infty\}.\end{aligned}$$

Let x be the solution of problem (3.1) on the half-line. We will consider the following assumptions:

($\widetilde{\mathcal{H}_1}$) The function $F\colon\widetilde{J}\times\mathcal{D}\to\mathcal{P}_{\rm cl}(E)$ is such that:
(a) for all $y\in\mathcal{D}$ the map $t\mapsto F(t,y)$ is measurable,
(b) the map $t\mapsto\gamma(t)=d(g(t),F(t,x_t))\in L^1([0,\infty),\mathbb{R}_+)$.

($\widetilde{\mathcal{H}_2}$) There exist a function $p\in L^1([0,\infty),\mathbb{R}^+)$ and a positive constant $\beta>0$ such that

$$H_d(F(t,z_1),F(t,z_2))\le p(t)\|z_1-z_2\|_{\mathcal{D}},\quad \text{for all } z_1,z_2\in\mathcal{B}(x_t,\beta),$$

where $\mathcal{B}(x_t,\beta)$ is the closed ball of $\mathcal{D}$ centered in x_t with radius β.

($\widetilde{\mathcal{H}_3}$) For every $x\in E$,

$$\sum_{k=1}^{\infty}|I_k(x)|<\infty.$$

Then we can extend Filippov's Theorem to the half-line. We have

Theorem 3.3. *Let $\gamma_k:=\gamma_{|_{J_k}}$ and assume ($\widetilde{\mathcal{H}_1}$)–($\widetilde{\mathcal{H}_3}$) hold together with*

$$\lim_{k\to\infty}\sup\eta_{k-1}(t_k)\le\beta.$$

Then, for every $\phi\in\mathcal{D}$ with $\|\phi-\psi\|_{\mathcal{D}}\le\delta$, problem (3.10) has at least one solution y satisfying, for $t\in[0,\infty)$, the estimates

$$\begin{aligned}&\|y_t-x_t\|_{\mathcal{D}}\\ &\le M\sum_{0<t_k<t}e^{\omega(t-t_k)}[|y(t_k)-x(t_k)|+|I_k(y(t_k))-I_k(x(t_k))|]+\sum_{0<t_k<t}\eta_k(t),\end{aligned}$$

and

$$|y'(t)-Ay(t)-g(t)|\le Mp(t)\sum_{0<t_k<t}e^{\omega t_{k+1}}H_k(t)+\sum_{0<t_k<t}\gamma_k(t).$$

Proof. The solution will be sought in the space

$$\begin{aligned}\widetilde{PC}=\{y\colon[0,\infty)\to E:{}&y_k\in C(J_k,E),\ k=0,1,\ldots \text{ such that}\\ &y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-)=y(t_k) \text{ for } k=1,2,\ldots\},\end{aligned}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k \geq 0$. Theorem 3.2 yields estimates of y_k on each one of the bounded intervals $J_0 = [-r, t_1]$, and $J_k = [t_{k-1} - r, t_k]$, $k = 2, 3, \ldots$. Let y_0 be solution of problem (1.1) on J_0 with

$$\|x_t - y_t\|_{\mathcal{D}} \leq \eta_0(t_1) \leq \beta.$$

Then, consider the following problem

$$\begin{cases} (y'(t) - Ay(t)) \in F(t, y_t) & \text{for a.e. } t \in (t_1, t_2], \\ y(t_1^+) = y_0(t_1) + I_1(y_0(t_1)), \\ y(t) = y_0(t) & \text{for } t \in [t_1 - r, t_1]. \end{cases}$$

From Theorem 3.2, this problem has a solution y_1. We continue this process taking into account that $y_m := y|_{[t_m, b]}$ is a solution to the problem

$$\begin{cases} (y'(t) - Ay(t)) \in F(t, y_t) & \text{for a.e. } t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_m) + I_m(y_{m-1}(t_m^-)), \\ y(t) = y_{m-1}(t) & \text{for } t \in [t_{m-1} - r, t_m]. \end{cases}$$

Then a solution y of problem (3.10) may be rewritten as

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_m(t) & \text{if } t \in (t_m, t_{m+1}], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

□

4. The relaxed problem

In this section, we examine whether the solutions of the nonconvex problem are dense in those of the convexified one, that is the problem where the right-hand side is replaced by its convex hull. Such a result is important in optimal control theory whether the relaxed optimal state can be approximated by original states; the relaxed problems are generally much simpler to build. For the problem for first-order differential inclusions, we refer e.g. to [5, Theorm 2, p. 124] or [6, Theorem 10.4.4, p. 402]. More precisely, in this section, we compare trajectories of the following problem

$$\begin{cases} (y'(t) - Ay(t)) \in F(t, y_t) & \text{for a.e. } t \in J \setminus \{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k) = I_k(y(t_k^-)) & \text{for } k = 1, \ldots, m, \\ y(t) = \phi(t) & \text{for } t \in [-r, 0], \end{cases} \tag{4.1}$$

and those of the convexified problem

$$(4.2)\qquad \begin{cases} (x'(t) - Ax(t)) \in \overline{\operatorname{co}}\, F(t, x_t) & \text{for a.e. } t \in J \setminus \{t_1, \ldots, t_m\}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k^-)) & \text{for } k = 1, \ldots, m, \\ x(t) = \phi(t) & \text{for } t \in [-r, 0], \end{cases}$$

where $\overline{\operatorname{co}}\, A$ refers to the closure of the convex hull of the set A. We will need the following auxiliary results in order to prove our main relaxation theorem. The first two are concerned with measurability of multi-valued mappings. The third one is due to Mazur, 1933 while the last one is a classical fixed point theorem.

Lemma 4.1 ([34]). *Let E be a separable Banach space, $U\colon [0, b] \to \mathcal{P}_{\rm cl}(E)$ be a measurable, integrably bounded set-valued map and let $t \mapsto d(0, U(t))$ be an integrable map. Then the integral $\int_0^b U(t)\, dt$ is convex, the map $t \mapsto \operatorname{co} U(t)$ is measurable and, for every $\varepsilon > 0$, and every measurable selection u of $\overline{\operatorname{co}}\, U(t)$, there exists a measurable selection $\overline{u}$ of U such that*

$$\sup_{t\in[0,b]} \bigg| \int_0^t u(s)\, ds - \int_0^t \overline{u}(s)\, ds \bigg| \le \varepsilon$$

and

$$\overline{\int_0^b \overline{\operatorname{co}}\, U(t)\, dt} = \overline{\int_0^b U(t)\, dt} = \int_0^b \overline{\operatorname{co}}\, U(t)\, dt.$$

Lemma 4.2 ([23]). *Let E be a separable Banach space and $G\colon [0, b] \to \mathcal{P}_{\rm cl}(E)$ be a measurable, integrably bounded multifunction; then so is $s \mapsto T(b - s)G(s)$. Moreover, if $f(s) \in T(b - s)G(s)$ almost everywhere in $[0, b]$, then there exists a measurable selection $g(s) \in G(s)$ such that $f(s) = T(b - s)g(s)$ almost everywhere in $[0, b]$.*

Lemma 4.3 (Mazur's Lemma, [44, Theorem 21.4]). *Let E be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^m \alpha_{mk} x_k$, where $\alpha_{mk} > 0$ for $k = 1, \ldots, m$ and $\sum_{k=1}^m \alpha_{mk} = 1$, which converges strongly to x.*

Lemma 4.4 (Covitz–Nadler, [16]). *Let (X, d) be a complete metric space. If $N\colon X \to \mathcal{P}_{\rm cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

The following hypotheses will be assumed in this section:

($\overline{\mathcal{H}_1}$) The function $F\colon J \times \mathcal{D} \to \mathcal{P}_{\rm cl}(E)$ satisfies
- (a) for all $y \in \mathcal{D}$, the map $t \mapsto F(t, y)$ is measurable,
- (b) the map $t \mapsto d(0, F(t, 0))$ is integrable.

($\overline{\mathcal{H}_2}$) There exist a function $p \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t, z_1), F(t, z_2)) \le p(t)\|z_1 - z_2\|_{\mathcal{D}} \quad \text{for each } z_1, z_2 \in \mathcal{D}.$$

$(\overline{\mathcal{H}_3})$ there exist constants $c_k \geq 0$ such that

$$|I_k(u_1) - I_k(u_2)| \leq c_k |u_1 - u_2|, \quad \text{for each } u_1, u_2 \in E.$$

Also either E will be, in this section, a reflexive Banach or $F\colon J \times \mathcal{D} \to \mathcal{P}_{\mathrm{wkcp}}(E)$. Then our main contribution is the following

Theorem 4.5. *Assume that* $(\overline{\mathcal{H}_1})$–$(\overline{\mathcal{H}_3})$ *hold. Then problem* (4.2) *has at least one solution. In addition, for all* $\varepsilon > 0$ *and every solution* x *of problem* (4.2), *problem* (4.1) *has a solution* y *defined on* $[0,b]$ *satisfying*

$$x(t) = y(t), \quad t \in [-r,0] \quad \textit{and} \quad \|x - y\|_{PC} \leq \varepsilon.$$

In particular $S^{\mathrm{co}}_{[-r,b]}(\phi) = \overline{S_{[-r,b]}(\phi)}$ *where*

$$S^{\mathrm{co}}_{[-r,b]} = \{y : y \textit{ is a solution to } (4.2) \textit{ on } [-r,b] \textit{ and } y(t) = \phi(t),\ t \in [-r,0]\}.$$

Remark 4.6. Notice that the multi-valued map $t \mapsto \overline{\mathrm{co}}\, F(t,\,\cdot\,)$ also satisfies condition $(\overline{\mathcal{H}_2})$.

Proof. Part 1. $S^{\mathrm{co}}_{[-r,b]} \neq \emptyset$. For this, we first transform problem (4.2) into a fixed point problem and then make use of Lemma 4.4. Consider the problem on the interval $[-r,t_1]$, that is

$$\begin{cases} (y'(t) - Ay(t)) \in \overline{\mathrm{co}}\, F(t,y_t) & \text{for a.e. } t \in [0,t_1], \\ y(t) = \phi(t) & \text{for } t \in [-r,0]. \end{cases} \tag{4.3}$$

It is clear that all solutions of problem (4.3) are fixed points of the multivalued operator $N\colon \Omega([-r,t_1]) \to \mathcal{P}(\Omega[-r,t_1])$ defined by

$$N(y) := \left\{ h \in \Omega([-r,t_1]) : \right.$$
$$\left. h(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)g(s)\,ds & \text{for } t \in (0,t_1], \end{cases} \right\}$$

where $g \in S_{\overline{\mathrm{co}}F,y} = \{g \in L^1([0,t_1],E) : g(t) \in \overline{\mathrm{co}}\, F(t,y_t)$ for a.e. $t \in (0,t_1]\}$ and $\Omega([-r,t_1]) = \mathcal{D} \cap C([0,t_1],E)$. To show that N satisfies the assumptions of Lemma 4.4, the proof will be given in two steps. In Steps 3, 4, we study the problem on the intervals $(t_k,t_{k+1}]$ for $k = 1,\dots,m-1$.

Step 1. $N(y) \in P_{\mathrm{cl}}(\Omega([-r,t_1])$ for each $y \in \Omega([-r,t_1])$. Indeed, let $\{y_n\} \in N(y)$ be such that $y_n \to \widetilde{y}$ in $\Omega([-r,t_1]$, as $n \to \infty$. Then $\widetilde{y} \in \Omega([-r,t_1])$ and there exists a sequence $g_n \in S_{\overline{\mathrm{co}}\,F,y}$ such that $y_n(t) = \phi(t)$ for $t \in [-r,0]$ and

$$y_n(t) = T(t)\phi(0) + \int_0^t T(t-s)g_n(s)\,ds, \quad t \in (0,t_1].$$

Then $\{g_n\}$ is integrably bounded. Since $F(\,\cdot\,,\,\cdot\,)$ has closed values, let $w(\,\cdot\,) \in F(\,\cdot\,,0)$ be such that $|w(t)| = d(0, F(t,0))$. From $(\overline{\mathcal{H}_1})$ and $(\overline{\mathcal{H}_2})$, we infer that for almost every $t \in [0,t_1]$

$$|g_n(t)| \le |g_n(t) - w(t)| + |w(t)| \le p(t)\|y\|_{PC} + d(0, F(t,0)) := M(t),$$

for all $n \in \mathbb{N}$, that is $g_n(t) \in M(t)B(0,1)$, for a.e. $t \in [0,t_1]$.

Since $B(0,1)$ is weakly compact in the reflexive Banach space E, there exists a subsequence still denoted $\{g_n\}$ which converges weakly to g by the Dunford–Pettis theorem. By Mazur's Lemma 4.3, there exists a second subsequence which converges strongly to g in E, hence almost everywhere (see [19, p. 150]). Then the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\|g_n - g\|_{L^1} \to 0 \quad \text{and thus} \quad y_n(t) \to \widetilde{y}(t)$$

with $\widetilde{y}(t) = \phi(t)$, for almost every $t \in [-r,0]$ and

$$\widetilde{y}(t) = T(t)\phi(0) + \int_0^t T(t-s)g(s)\,ds, \quad t \in (0,t_1],$$

proving that $\widetilde{y} \in N(y)$.

Step 2. There exists $\gamma < 1$ such that $H_d(N(y), N(\overline{y})) \le \gamma\|y - \overline{y}\|_{[-r,t_1]}$ for each $y, \overline{y} \in \Omega([-r,t_1])$ where the norm $|y - \overline{y}\|_{[-r,t_1]}$ will be chosen conveniently. Indeed, let $y, \overline{y} \in \Omega([-r,t_1])$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in \overline{\text{co}}\,F(t, y_t)$ such that for each $t \in [0,t_1]$

$$h_1(t) = T(t)\phi(0) + \int_0^t T(t-s)g_1(s)\,ds.$$

Since, for each $t \in [-r,t_1]$,

$$H_d(\overline{\text{co}}\,F(t,y_t), \overline{\text{co}}\,F(t,\overline{y}_t)) \le p(t)\|y_t - \overline{y}_t\|_{\mathcal{D}},$$

then there exists some $w(t) \in \overline{\text{co}}\,F(t,\overline{y}_t)$ such that

$$|g_1(t) - w(t)| \le p(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}, \quad t \in [0,t_1].$$

Consider the multi-map $U_1\colon [0,t_1] \to \mathcal{P}(E)$ defined by

$$U_1(t) = \{w \in E : |g_1(t) - w| \le p(t)\|y_t - \overline{y}_t\|_{\mathcal{D}}\}.$$

As in the proof of Theorem 3.2, we can show that the multi-valued operator $V_1(t) = U_1(t) \cap \overline{\text{co}}\,F(t,\overline{y}_t)$ is measurable and takes nonempty values. Then there exists a function $g_2(t)$, which is a measurable selection for V_1. Thus, $g_2(t) \in \overline{\text{co}}\,F(t,\overline{y}_t)$ and

$$|g_1(t) - g_2(t)| \le p(t)\|y - \overline{y}\|_{\mathcal{D}}, \quad \text{for a.e. } t \in [0,t_1].$$

For each $t \in [0, t_1]$, let

$$h_2(t) = T(t)\phi(0) + \int_0^t T(t-s)g_2(s)\,ds.$$

Therefore, for each $t \in (0, t_1]$, we have

$$\begin{aligned}
|h_1(t) - h_2(t)| &\leq \int_0^t |g_1(s) - g_2(s)|\,ds \\
&\leq \int_0^t p(s)\|T(t-s)\|\|y_s(\theta) - \overline{y}_s(\theta)\|_{\mathcal{D}}\,ds \\
&= \int_0^t p(s)Me^{\omega(t-s)}\Big(\sup_{-r\leq\theta\leq 0}|y_s(\theta) - \overline{y}_s(\theta)|\Big)\,ds \\
&= \int_0^t Mp(s)e^{\omega t}e^{-\omega s}\Big(\sup_{-r\leq\theta\leq 0}|y(s+\theta) - \overline{y}(s+\theta)|\Big)\,ds \\
&= \int_0^t Mp(s)e^{\omega t}\Big(\sup_{s-r\leq z\leq s}|y(z) - \overline{y}(z)|\Big)\,ds \\
&\leq Me^{\omega t_1}\int_0^t p(s)e^{\tau\int_0^s p(u)du}\Big(\sup_{-r\leq z\leq t_1} e^{-\tau\int_0^z p(u)du}|y(z) - \overline{y}(z)|\Big)\,ds \\
&\leq \frac{Me^{\omega t_1}}{\tau}\int_0^t (e^{\tau\int_0^s p(u)du})'\|y - \overline{y}\|_{[-r,t_1]}\,ds.
\end{aligned}$$

Hence

$$\|h_1 - h_2\|_{[-r,t_1]} \leq \frac{Me^{\omega t_1}}{\tau}\|y - \overline{y}\|_{[-r,t_1]},$$

where

$$\|y\|_{[-r,t_1]} = \sup\{e^{-\tau\int_0^t p(s)\,ds}|y(t)| : t \in [-r, t_1],\ \tau > Me^{\omega t_1}\}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we find that

$$H_d(N(y), N(\overline{y})) \leq \frac{Me^{\omega t_1}}{\tau}\|y - \overline{y}\|_{[-r,t_1]}.$$

Then N is a contraction and hence, by Lemma 4.4, N has a fixed point y_0, which is solution to problem (4.3).

Step 2. Let $y_2 := y|_{[t_1,t_2]}$ be a possible solution to the problem

$$\begin{cases}
(y'(t) - Ay(t)) \in \overline{\mathrm{co}}\,F(t, y_t) & \text{for } t \in (t_1, t_2], \\
y(t_1^+) = y_0(t_1) + I_1(y_0(t_1^-)), \\
y(t) = y_0(t) & \text{for } t \in [t_1 - r, t_1].
\end{cases} \tag{4.4}$$

Then y_2 is a fixed point of the multivalued operator

$$N\colon \Omega([t_1 - r, t_2]) \to \mathcal{P}(\Omega([t_1 - r, t_2]))$$

defined by

$$N(y) := \left\{ h \in \Omega([t_1 - r, t_2]) : \vphantom{\int} \right.$$

$$\left. h(t) = \begin{cases} y_0(t) & \text{for } t \in [t_1 - r, t_1], \\ T(t - t_1)[y_0(t_1) + I_1(y_0(t_1))] & \\ \qquad\qquad + \displaystyle\int_{t_1}^{t} T(t - s) g(s)\, ds & \text{for } t \in (t_1, t_2], \end{cases} \right\}$$

where

$$g \in S_{\overline{\mathrm{co}}\, F, y} = \{ g \in L^1([t_1, t_2], E) : g(t) \in \overline{\mathrm{co}}\, F(t, y_t) \text{ for a.e. } t \in [t_1, t_2] \}.$$

Again, we show that N satisfies the assumptions of Lemma 4.4. Clearly, $N(y) \in \mathcal{P}_{\mathrm{cl}}(\Omega([t_1 - r, t_2]))$ for each $y \in \Omega([t_1 - r, t_2])$. It remains to show that there exists $0 < \gamma < 1$ such that

$$H_d(N(y), N(\overline{y})) \leq \gamma \| y - \overline{y} \|_{[t_1 - r, t_2]}$$

for each $y, \overline{y} \in \Omega([t_1 - r, t_2])$. For this purpose, let $y, \overline{y} \in \Omega([t_1 - r, t_2])$ and $h_1 \in N(y)$. Then there exists $g_1(t) \in \overline{\mathrm{co}}\, F(t, y_t)$ such that, for each $t \in [t_1 - r, t_2]$,

$$h_1(t) = \int_{t_1}^{t} T(t - s) g_1(s)\, ds + T(t - t_1)[y_0(t_1) + I_1(y_0(t_1))].$$

Since from $(\overline{\mathcal{H}}_2)$

$$H_d(\overline{\mathrm{co}}\, F(t, y_t), \overline{\mathrm{co}}\, F(t, \overline{y}_t)) \leq p(t) \| y_t - \overline{y}_t \|_{\mathcal{D}}, \quad t \in [t_1, t_2],$$

then there is a $w(\,\cdot\,) \in \overline{\mathrm{co}}\, F(\,\cdot\,, \overline{y}_{\,\cdot\,})$ such that

$$|g_1(t) - w(t)| \leq p(t) \| y_t - \overline{y}_t \|_{\mathcal{D}}, \quad t \in [t_1, t_2].$$

Consider the multi-valued map $U_2 \colon [t_1, t_2] \to \mathcal{P}(E)$ defined by

$$U_2(t) = \{ w \in E : |g_1(t) - w| \leq p(t) \| y_t - \overline{y}_t \|_{\mathcal{D}} \}.$$

As in the above arguments, we can show that the multivalued operator $V_2(t) = U_2(t) \cap \overline{\mathrm{co}}\, F(t, \overline{y}_t)$ is measurable with nonempty values; hence there exists $g_2(t)$ which is a measurable selection for V_2. Then $g_2(t) \in \overline{\mathrm{co}}\, F(t, \overline{y}_t)$ and

$$|g_1(t) - g_2(t)| \leq p(t) \| y_t - \overline{y}_t \|_{\mathcal{D}}, \quad \text{for a.e. } t \in [t_1, t_2].$$

For almost every $t \in [t_1, t_2]$, define

$$h_2(t) = \int_{0}^{t} T(t - s) g_2(s)\, ds + T(t - t_1)[y_0(t_1) + I_1(y_0(t_1))].$$

For some $\tau > Me^{wt_2}$, we have the estimates

$$\begin{aligned}|h_1(t) - h_2(t)| &\le \int_{t_1}^{t} \|T(t-s)\| |g_1(s) - g_2(s)|\, ds \\ &\le \int_{t_1}^{t} p(s) M e^{\omega(t-s)} \|y_{1s} - y_{2s}\|_{\mathcal{D}}\, ds \\ &\le M e^{\omega t_2} \int_{t_1}^{t} p(s) \Big(\sup_{-r\le\theta\le 0} |y_{1s}(\theta) - y_{2s}(\theta)| \Big)\, ds \\ &\le \frac{M e^{\omega t_2}}{\tau} \|y - \overline{y}\|_{[t_1,t_2]}.\end{aligned}$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we obtain

$$H_d(N(y), N(\overline{y})) \le \frac{M e^{\omega t_2}}{\tau} \|y - \overline{y}\|_{[t_1,t_2]},$$

where

$$\|y\|_{[t_1-r,t_2]} = \sup\{e^{-\tau \int_{t_1}^{t} p(s)\, ds} |y(t)| : t \in [t_1 - r, t_2]\}.$$

Therefore N is a contraction and thus, by Lemma 4.4, N has a fixed point y_2 solution of problem (4.4).

Step 3. We continue this process taking into account that $y_m := y|_{[t_m,b]}$ is a solution of the following problem

$$\begin{cases} (y'(t) - Ay(t)) \in F(t, y_t) & \text{for } t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_m) + I_m(y_{m-1}(t_m^-)), \\ y(t) = y_{m-1}(t) & \text{for } t \in [t_{m-1} - r, t_m]. \end{cases}$$

Then a solution y of problem (4.2) may be defined by

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots \\ y_m(t) & \text{if } t \in (t_m, b]. \end{cases}$$

Part 2. Let x be a solution of problem (4.2). Then, there exists $g \in S_{\overline{\mathrm{co}}\, F,x}$ such that

$$x(t) = \begin{cases} \phi(t) & \text{for } t \in [-r, 0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s) g(s)\, ds \\ \qquad + \displaystyle\sum_{0<t_k<t} T(t - t_k) I_k(x(t_k)) & \text{for } t \in [0, b], \end{cases}$$

i.e. x is a mild solution of the problem

$$\begin{cases} x'(t) - Ax(t) = g(t) & \text{for a.e. } t \in [0,b] \setminus \{t_1, \dots, t_m\}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k^-)) & \text{for } k = 1, \dots, m, \\ x(t) = \phi(t) & \text{for } t \in [-r, 0]. \end{cases}$$

Let $\varepsilon > 0$ and $\delta > 0$ be given by the relation $Me^{\omega b}\varepsilon = \delta L \sum_{k=1}^{m} R_k$ where L and R_k, for $k = 0, \dots, m$, will be defined later on. From Lemmas 4.1 and 4.2, there exists a measurable selection f_* of $t \mapsto F(t, x_t)$ such that

$$\sup_{t \in [0,b]} \left| \int_0^t T(t-s)g(s)\,ds - \int_0^t T(t-s)f_*(s)\,ds \right| \leq \delta.$$

Let

$$z(t) = \begin{cases} \phi(t) & \text{for } t \in [-r, 0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)f_*(s)\,ds & \\ \qquad\qquad + \displaystyle\sum_{0 < t_k < t} T(t-t_k)I_k(x(t_k)) & \text{for } t \in (0, b]. \end{cases}$$

Hence, for each $t \in [-r, b]$, $\|x_t - z_t\|_{\mathcal{D}} \leq \delta$.

With assumption $(\overline{\mathcal{H}_2})$, we infer that, for all $u \in \overline{\text{co}}\, F(t, z_t)$,

$$\begin{aligned} \gamma(t) := d(g(t), F(t, x_t)) &\leq d(g(t), u) + H_d(F(t, z_t), F(t, x_t)), \\ &\leq H_d(\overline{\text{co}}\, F(t, x_t), \overline{\text{co}}\, F(t, z_t)) + H_d(F(t, z_t), F(t, x_t)) \\ &\leq 2p(t)\|x_t - z_t\|_{\mathcal{D}} \leq 2\delta p(t). \end{aligned}$$

Since, under $(\overline{\mathcal{H}_1}(a))$ and $(\overline{\mathcal{H}_2})$, γ is measurable (see [6] or [23, Lemma 1.5]), by the above inequality, we deduce that $\gamma \in L^1(J, E)$. From Theorem 3.2, problem (4.1) has a solution y which satisfies

$$|y(t) - x(t)| \leq \eta_0(t), \quad t \in [0, t_1],$$

Also for $t \in (t_1, t_2]$, we have the estimates

$$|y(t) - x(t)| \leq Me^{\omega(t_2 - t_1)}(1 + c_1)\eta_0(t_1) + \eta_1(t_2) = L_1[\eta_0(t_1) + \eta_1(t_2)],$$

where $L_1 = Me^{\omega(t_2 - t_1)}(1 + c_1)$. And for $t \in (t_2, t_3]$, we have

$$\begin{aligned} |y(t) - x(t)| &\leq Me^{\omega(t_3 - t_2)}|y(t_2) - x(t_2)| + Mc_2 e^{\omega(t_3 - t_2)}|y(t_2) - x(t_2)| + \eta_3(t_3) \\ &\leq M^2 e^{\omega(t_3 - t_1)}(1 + c_1)\eta_0(t_1) + Me^{\omega(t_3 - t_2)}c_2\eta_2(t_2) + \eta_3(t_3) \\ &\leq L_2[\eta_0(t_1) + \eta_2(t_2) + \eta_2(t_3)], \end{aligned}$$

where $L_2 = M^2 e^{\omega(t_3 - t_1)}(1 + \max(c_1, c_2))$. We continue this process until we arrive at

$$|y(t) - x(t)| \leq L_k \sum_{i=0}^{k+1} \eta_i(t_i), \ t \in (t_k, t_{k+1}],$$

where $L_k = M^k e^{\omega(t_{k+1}-t_k)}(1+\max_{1\leq i\leq k} c_i)$. Thus for all $t \in [-r,b]$, it holds that

$$|y(t)-x(t)| \leq L \sum_{k=0}^{m} \eta_k(t_k) \leq L\delta \sum_{k=0}^{m} R_k$$

where

$$\eta_0(t) \leq \delta\bigg(Me^{\omega t_1} + Me^{\omega t_1}\int_0^{t_1}(MH_0P(s)+p(s))\,ds\bigg) := \delta R_0,$$

for $t \in [0,t_1]$, and

$$H_0 = \exp(Me^{\omega t_1+P(t_1)}) + \int_0^{t_1} p(s)\exp(Me^{\omega t_1+P(t_1)-P(s)})\,ds,$$

while for $k=1,\ldots,m$,

$$\eta_k(t) \leq \delta Me^{\omega t_{k+1}}\int_{t_k}^{t_{k+1}}(M\overline{H}_kP(s)+p(s))\,ds := \delta R_k,$$

where

$$\overline{H}_k = M\exp(Me^{\omega(t_{k+1}-t_k)}) + \int_{t_k}^{t_{k+1}} M2p(s)\,ds$$

and

$$L = M^m e^{\omega b}(1+\max_{1\leq i\leq m} c_i).$$

Using the definition of δ, we obtain the upper bound $\|y-x\|_\Omega \leq Me^{\omega b}\varepsilon$. Since ε is arbitrary, $\|y-x\|_\Omega \leq \varepsilon$, showing the density relation $S^{\rm co}_{[0,b]}(\phi) = \overline{S_{[0,b]}(\phi)}$. $\square$

5. Topological structure of the solution sets

5.1. Closeness of the set of solutions. Let us introduce the following hypotheses:

($\mathcal{A}_1$) For fixed y, the multi-function $t \mapsto F(t,y)$ is measurable.

($\mathcal{A}_2$) There exists $p \in L^1([0,b],\mathbb{R}^+)$ such that

$$\begin{aligned} H_d(F(t,z_1),F(t,z_2)) &\leq p(t)\|z_1-z_2\|_{\mathcal{D}} && \text{for all } z_1,z_2 \in \mathcal{D},\\ 0 < d(0,F(t,0)) &\leq p(t) && \text{for a.e. } t \in J. \end{aligned}$$

Theorem 5.1. *Under assumptions* ($\mathcal{A}_1$)–($\mathcal{A}_2$), *the operator solution* $S_{[-r,b]}$ *of problem* (1.1) *has nonempty, closed valued and a closed graph.*

Proof. Let $S_{[-r,b]}\colon \mathcal{D} \to \mathcal{P}(E)$ be the operator solution of problem (1.1) defined by

$$S_{[-r,b]}(\phi) = \{y \in \Omega\colon y \text{ solution of problem (1.1)}\}.$$

With assumptions ($\mathcal{A}_1$)–($\mathcal{A}_2$), we may use Covitz–Nadler fixed point theorem (Lemma 4.4), as in the proof of Theorem 4.5, to prove that

$$S_{[-r,b]}(\phi) \neq \emptyset \quad \text{for every } \phi \in \mathcal{D}.$$

Thus we only show the closeness of both the values and the graph of $S_{[-r,b]}$.

Step 1. $S_{[-r,b]}(\,\cdot\,) \in \mathcal{P}_{\rm cl}(E)$. For this, let $\phi \in \mathcal{D}$ and let $y_n \in S_{[-r,b]}(\phi)$, $n \in \mathbb{N}$ be a sequence which converges to some limit y_* in Ω. Then

$$y_n(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v_n(s)\,ds & \\ \qquad\qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k((y_n(t_k)) & \text{for } t \in [0,b], \end{cases}$$

where $v_n \in \{v \in L^1([0,b],E) \colon v(\,\cdot\,) \in F(\,\cdot\,,(y_{\cdot})_n)\}$. Since $F(t,\,\cdot\,)$ is p-Lipschitz and $\{y_n\}$ converges to y_*, there exists $M_* > 0$ such that $\|y_n\|_\Omega \leq M_*$ and for every ε, there exists $n_0 = n_0(\varepsilon) \geq 0$ such that, for every $n \geq n_0$

$$v_n(t) \in F(t,(y_t)_n) \subset F(t,(y_*)_t) + \varepsilon p(t)B(0,1), \quad \text{for almost every } t \in [0,b].$$

Since $F(\,\cdot\,,\,\cdot\,)$ has compact values, there exists a subsequence $v_{n_m}(\,\cdot\,)$ such that

$$v_{n_m}(\,\cdot\,) \to v(\,\cdot\,) \quad \text{as } m \to \infty,$$

and $v(t) \in F(t,(y_*)_t)$ for almost every $t \in [0,b]$. Since $F(\,\cdot\,,\,\cdot\,)$ has closed values, let $w(\,\cdot\,) \in F(\,\cdot\,,0)$ be such that $|w(t)| = d(0,F(t,0))$. Then $|w(t)| \leq p(t)$ for almost every $t \in [0,b]$ and

$$|v_{n_m}(t)| \leq |v_{n_m}(t) - w(t)| + |w(t)| \leq p(t)|(y_{n_m})_t| + p(t) \leq (1+M_*)p(t).$$

Hence

$$|y_{n_m}(t) - z(t)| \leq Me^{\omega b}\int_0^b |v_{n_m}(s) - v(s)|\,ds + Me^{\omega b}\sum_{k=1}^m |I_k((y_n(t_k)) - I_k((y_*(t_k))|$$

where

$$z(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s)\,ds & \\ \qquad\qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k((y_*(t_k)) & \text{for } t \in [0,b]. \end{cases}$$

Using the continuity of I_k and the Lebesgue dominated convergence theorem, we conclude that $y_* = z$.

Step 2. $S_{[-r,b]}$ has a closed graph. Let $\phi_n \to \phi_*$, $y_n \in S_{[-r,b]}(\phi_n)$ and $y_n \to y_*$. $y_n \in S_{[-r,b]}(\phi_n)$ means that there exists $g_n \in L^1$ such that, for each $t \in [-r,0]$, $y_n(t) = \phi_n(t)$ and for $t \in [0,b]$,

$$y_n(t) = T(t)\phi_n(0) + \int_0^t T(t-s)g_n(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)).$$

We prove that $y_* \in S_{[-r,b]}(\phi_*)$, i.e. there exists $g_* \in S_{F,y_*}$ such that for each $t \in J$

$$y_*(t) = T(t)\phi_*(0) + \int_0^t T(t-s)g_*(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)).$$

Using the fact that F has compact values and is an L^1-Carathéodory function, we may pass to a subsequence, if necessary, to get that $\{g_n\}$ converges to some limit g_* in $L^1(J,E)$. Since the functions I_k, $k = 1,\dots,m$, are continuous, we obtain the estimates

$$\begin{aligned}
&\bigg| y_*(t) - T(t)\phi_*(0) - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) - \int_0^t T(t-s)g_*(s)\,ds \bigg| \\
&\leq \bigg| \bigg(y_n(t) - T(t)\phi_n(0) - \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k^-)) - \int_0^t T(t-s)g_n(s)\,ds \bigg) \\
&\quad - \bigg(y_*(t) - T(t)\phi_*(0) - \sum_{0<t_k<t} T(t-t_k)I_k(y_*(t_k^-)) - \int_0^t T(t-s)g_*(s)\,ds \bigg) \bigg| \\
&\leq \|y_n - y_*\|_\Omega + Me^{\omega b} \sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y_*(t_k))| \\
&\quad + \|T(t)\|_{B(E)} |\phi_n(0) - \phi_*(0)| + Me^{\omega b} \int_0^b |g_n(s) - g_*(s)|\,ds.
\end{aligned}$$

The right-hand side terms tend to 0, as $n \to \infty$, proving our claim. □

5.2. Compactness of the set of solutions.

5.2.1. Auxiliary results. First, we collect some definitions and properties about measures of noncompactness in Banach spaces. More details can be found in [38].

Definition 5.1. Let E be a Banach space and $(\mathcal{A},\geq)$ a partially ordered set. A map $\beta\colon \mathcal{P}(E) \to \mathcal{A}$ is called a *measure of noncompactness on E* (MNC for short) if, for every subset $\Omega \in \mathcal{P}(E)$, it satisfies $\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega)$.

Notice that if D is dense in Ω, then $\overline{\mathrm{co}}\,\Omega = \overline{\mathrm{co}}\,D$ and hence $\beta(\Omega) = \beta(D)$.

Definition 5.2. A measure of noncompactness β is called

(a) *Monotone* if $\Omega_0, \Omega_1 \in \mathcal{P}(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$.
(b) *Nonsingular* if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$, $\Omega \in \mathcal{P}(E)$.
(c) *Invariant* with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset E$ and $\Omega \in \mathcal{P}(E)$.
(d) *Real* if $\mathcal{A} = \overline{\mathbb{R}}_+ = [0,\infty]$ and $\beta(\Omega) < \infty$ for every bounded Ω.
(e) *Regular* if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As example of an MNC, one may consider the Hausdorff measure:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Recall that a bounded set $A \subset E$ has a finite ε-net if there exits a finite subset $S \subset E$ such that $A \subset S + \varepsilon\overline{B}$ where $\overline{B}$ is a closed ball in E.

Definition 5.3. Let $\mathcal{M}$ be a closed subset of a Banach space E, $(\mathcal{A}, \geq)$ a partially ordered set and $\beta\colon \mathcal{P}(E) \to (\mathcal{A}, \geq)$ an MNC on E. A multimap $\mathcal{F}\colon \mathcal{M} \to \mathcal{P}_{\rm cp}(E)$ is said to be *β-condensing* if for every bounded $\Omega \subset \mathcal{M}$, the inequality $\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$, implies the relative compactness of Ω.

Definition 5.4. A sequence $\{v_n\}_{n\in\mathbb{N}} \subset L^1([0,b],E)$ is said to be *semi-compact* if

(a) it is integrably bounded, i.e. if there exists $\psi \in L^1([0,b],\mathbb{R}^+)$ such that

$$|v_n(t)| \leq \psi(t), \quad \text{for a.e. } t \in [0,b] \text{ and every } n \in \mathbb{N},$$

(b) the image sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in E for almost every $t \in [0,b]$.

The following result follows from the Dunford–Pettis theorem (see also [38, Proposition 4.2.1])

Lemma 5.5. *Every semi-compact sequence is weakly compact in $L^1([0,b],E)$.*

Lemma 5.6 ([38, Theorem 5.1.1]). *Let $N\colon L^1([a,b],E) \to C([a,b],E)$ be an abstract operator satisfying the following conditions:*

($\mathcal{S}_1$) *N is ξ-Lipschitz: there exists $\xi > 0$ such that for every $f,g \in L^1([a,b],E)$*

$$|Nf(t) - Ng(t)| \leq \xi \int_a^b |f(s) - g(s)|\,ds, \quad \text{for all } t \in [a,b].$$

($\mathcal{S}_2$) *N is weakly-strongly sequentially continuous on compact subsets: for any compact $K \subset E$ and any sequence $\{f_n\}_{n=1}^\infty \subset L^1([a,b],E)$ such that $\{f_n(t)\}_{n=1}^\infty \subset K$ for almost every $t \in [a,b]$, the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $N(f_n) \to N(f_0)$ as $n \to \infty$.*

Then for every semi-compact sequence $\{f_n\}_{n=1}^\infty \subset L^1([0,b],E)$, the image sequence $N(\{f_n\}_{n=1}^\infty)$ is relatively compact in $C([a,b],E)$.

Lemma 5.7 ([38, Theorem 5.2.2]). *Let an operator*

$$N\colon L^1([a,b],E) \to C([a,b],E)$$

satisfy conditions ($\mathcal{S}_1$)–($\mathcal{S}_2$) together with

($\mathcal{S}_3$) *There exits $\eta \in L^1([a,b])$ such that for every integrably bounded sequence $\{f_n\}_{n=1}^\infty$, we have $\chi(\{f_n(t)\}_{n=1}^\infty) \leq \eta(t)$ for almost every $t \in [a,b]$, where χ is the Hausdorff* MNC.

Then

$$\chi(\{N(f_n)(t)\}_{n=1}^{\infty}) \leq 2\xi \int_a^b \eta(s)\,ds, \quad \textit{for all } t \in [a,b],$$

where ξ *is the constant in* $(\mathcal{S}_1)$.

Finally, two useful properties of the fixed point set of β-condensing multimaps are the following (see [38]):

Lemma 5.8. *Let* W *be a convex closed subset of a Banach space* E *and let* $N\colon W \to \mathcal{P}_{\mathrm{cp,cv}}(W)$ *be a closed* β*-condensing multimap where* β *is a nonsingular measure of noncompactness defined on subsets of* W. *Then* $\operatorname{Fix} N \neq \emptyset$.

Lemma 5.9. *Let* W *be a closed subset of a Banach space* E *and let* $N\colon W \to \mathcal{P}_{\mathrm{cp}}(E)$ *be a closed* β*-condensing multimap where* β *is a monotone* MNC *on* E. *Then* $\operatorname{Fix} N$ *is compact.*

The following so-called nonlinear alternative of Leray and Schauder for multi-valued maps will be needed in this section.

Lemma 5.10 ([28], [27]). *Let* $(X, \|\cdot\|)$ *be a normed space and* $F\colon X \to \mathcal{P}_{\mathrm{cl,cv}}(X)$ *be a compact, u.s.c. multi-valued map. Then either one of the following conditions holds:*

(a) F *has at least one fixed point,*
(b) *the set* $\mathcal{M} := \{x \in X : x \in \lambda F(x),\ \lambda \in\,]0,1[\}$ *is unbounded.*

5.2.2. Compactness result. Let $F\colon J \times \mathcal{D} \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory multimap which satisfies some of the following assumptions:

$(\mathcal{B}_1)$ There exist a function $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\rho\colon [0,\infty) \to [0,\infty)$ such that

$$\|F(t,z)\| \leq p(t)\rho(\|z\|_{\mathcal{D}}) \quad \text{for a.e. } t \in J \text{ and each } z \in \mathcal{D},$$

with

$$\int_0^b p(s)\,ds < \int_1^{\infty} \frac{du}{\rho(u)}.$$

$(\mathcal{B}_2)$ There exist constants $\overline{c}_k > 0$ and continuous functions $\phi_k\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|I_k(x)| \leq \overline{c}_k \phi_k(|x|) \quad \text{for each } x \in E,\ k = 1, \ldots, m.$$

$(\mathcal{B}_3)$ E is a reflexive Banach space and either one of the following conditions holds:

(a) the semigroup $T(\,\cdot\,)$ is uniformly continuous,
(b) the semigroup $T(\,\cdot\,)$ is compact in E.

($\mathcal{B}_4$) There exists $\overline{p} \in L^1([0,b],\mathbb{R}^+)$ such that for every bounded subset D in $\mathcal{D}$

$$\chi(F(t,D)) \leq \overline{p}(t)\sup\{\chi(D(\theta)) : \theta \in [-r,0]\}$$

and there exist $L_k > 0$, $k = 0,\dots,m$ such that

$$q_k := 2Me^{\omega t_{k+1}} \sup_{t\in[t_k,t_{k+1}]} \int_{t_k}^{t_{k+1}} e^{-L_k(t-s)}p(s)\,ds < 1, \quad k = 0,\dots,m.$$

Here χ is the Hausdorff MNC and $D(\theta) := \{\psi(\theta),\ \psi \in \mathcal{D}\}$.

Theorem 5.11. *Assume that F satisfies either* ($\mathcal{B}_1$), ($\mathcal{B}_2$) *and* ($\mathcal{B}_3$) *or* ($\mathcal{B}_1$), ($\mathcal{B}_2$) *and* ($\mathcal{B}_4$). *Then the set of solutions for problem* (1.1) *in nonempty and compact.*

Proof. According to the hypotheses considered, the proof is split in two parts.

Part 1. Under assumptions ($\mathcal{B}_1$)–($\mathcal{B}_3$), the solutions set is nonempty and compact.

Step 1. $S_{[-r,b]}(\phi) \neq \emptyset$. Consider the operator $N\colon \Omega \to \mathcal{P}(\Omega)$ defined for $y \in \Omega$ by

$$N(y) = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s)\,ds & \\ \qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)) & \text{for } t \in [0,b], \end{cases} \right\}$$

where $v \in S_{F,y} = \{u \in L^1(J,E)\colon u \in F(t,y_t), \text{ for almost every } t \in J\}$. Clearly, fixed points of the operator N are mild solutions of problem (1.1). Since, for each $y \in \Omega$, the nonlinearity F takes convex values, the selection set $S_{F,y}$ is convex and then N has convex values. As in [10], [45], [47], we can prove that N maps bounded sets into bounded sets and there exists $M_1 > 0$ such that for every y solution of problem (1.1), we have $\|y\|_\Omega \leq M_1$. Thus we only prove that $N(\mathcal{B}_q)$ is relatively compact in Ω, where $\mathcal{B}_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$. First, $N(\mathcal{B}_q)$ is an equicontinuous set of Ω. To see this, let $0 < \tau_1 < \tau_2 \leq b$, $y \in \mathcal{B}_q$, and $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$h(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s)\,ds & \\ \qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)) & \text{for } t \in (0,b]. \end{cases}$$

Letting $d_k = \sup_{|r|\le q} \phi_k(r)$, we obtain the estimate

$$\begin{aligned}|h(\tau_2)-h(\tau_1)| \le\ & |T(\tau_2)\phi(0)-T(\tau_1)\phi(0)|\\ & + \int_0^{\tau_1} \|T(\tau_2-s)-T(\tau_1-s)\|_{B(E)} p(s)\rho(q)\,ds\\ & + \int_{\tau_1}^{\tau_2} \|T(\tau_2-s)\|_{B(E)} p(s)\rho(q)\,ds + \sum_{k=1}^{m} T(\tau_2-\tau_1)I_k(y_k)\\ & + \sum_{0<t_k<\tau_1} \overline{c_k} d_k \|T(\tau_1-t_k)-T(\tau_2-t_k)\|_{B(E)}.\end{aligned}$$

Hence

$$\begin{aligned}|h(\tau_2)-h(\tau_1)| \le\ & \|T(\tau_2-\tau_1)-\mathrm{id}\|_{B(E)}\|\phi\|_{\mathcal{D}}\\ & + Me^{\omega b}\rho(q)\|T(\tau_2-\tau_1)-\mathrm{id}\|_{B(E)} \int_0^{\tau_1} p(s)\,ds\\ & + Me^{\omega b}\rho(q)\int_{\tau_1}^{\tau_2} p(s)\,ds + Me^{\omega b}\sum_{\tau_1<t_k<\tau_2} \overline{c_k} d_k\\ & + \|T(\tau_2-\tau_1)-\mathrm{id}\|_{B(E)} \sum_{0<t_k<\tau_1} \overline{c_k} d_k.\end{aligned}$$

If $(\mathcal{B}_3)(a)$ holds, then $T(\,\cdot\,)$ is a uniformly continuous, which implies that

$$\|T(h)-\mathrm{id}\| \to 0 \quad \text{as } h\to 0^+.$$

Thus the right-hand side tends to zero as $\tau_2-\tau_1\to 0$. This proves the equicontinuity for the case where $t\neq t_i$, $i=1,\dots,m$. In the same way, we can show equicontinuity when $(\mathcal{B}_3)$(b) holds. So, it remains to examine the equicontinuity at $t=t_i$. Let

$$h_1(t) = T(t)\phi(0) + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)) \quad \text{and} \quad h_2(t) = \int_0^t T(t-s)v(s)\,ds.$$

To prove equicontinuity at $t=t_i^-$, fix $\delta_1>0$ such that $\{t_k : k\neq i\}\cap[t_i-\delta_1, t_i+\delta_1]=\emptyset$. Then

$$\begin{aligned}h_1(t_i) &= T(t_i)\phi(0) + \sum_{0<t_k<t_i} T(t_i-t_k)I_k(y(t_k))\\ &= T(t_i)\phi(0) + \sum_{k=1}^{i-1} T(t_i-t_k)I_k(y(t_k)).\end{aligned}$$

For $0<\theta<\delta_1$, we obtain the estimates

$$\begin{aligned}|h_1(t_i-\theta)-h_1(t_i)| \le\ & |(T(t_i-\theta)-T(t_i))\phi(0)|\\ & + \sum_{k=1}^{i-1} |[T(t_i-\theta-t_k)-T(t_i-t_k)]I(y(t_k^-))|\end{aligned}$$

$$\begin{aligned}&\leq Me^{\omega(t_i-\theta)}\|T(\theta)-\mathrm{id}\|_{B(E)}\|\phi\|_{\mathcal{D}}\\&\quad+\|T(\theta)-\mathrm{id}\|_{B(E)}\sum_{k=1}^{i-1}Me^{\omega(t_i-\theta-t_k)}\overline{c}_k\sup_{|r|\leq q}\phi_k(r).\end{aligned}$$

Again, the terms in the right-hand side tend to zero as $\theta\to 0$. Moreover,

$$\begin{aligned}|h_2(t_i-\theta)-h_2(t_i)|\leq M\rho(q)\|T(\theta)-\mathrm{id}\|_{B(E)}&\int_0^{t_i-\theta}e^{\omega(t_i-\theta-s)}p(s)|\,ds\\&+M\rho(q)\int_{t_i-\theta}^{t_i}Me^{\omega(t_i-s)}p(s)\,ds,\end{aligned}$$

which tends to zero as $\theta\to 0$. Now, define

$$\widehat{h}_0(t)=h(t)\quad\text{for } t\in[0,t_1]$$

and

$$\widehat{h}_i(t)=\begin{cases}h(t)&\text{for } t\in(t_i,t_{i+1}],\\h(t_i^+)&\text{for } t=t_i.\end{cases}$$

To prove equicontinuity at points $t=t_i^+$, let $\delta_2>0$ be such that $\{t_k\colon k\neq i\}\cap[t_i-\delta_2,t_i+\delta_2]=\emptyset$. Then

$$\widehat{h}(t_i)=T(t_i)\phi(0)+\int_0^{t_i}T(t_i-s)v(s)\,ds+\sum_{k=1}^{i}T(t_i-t_k)I_k(y(t_k)).$$

For $0<\theta<\delta_2$, we have the estimates

$$\begin{aligned}|\widehat{h}(t_i+\theta)-\widehat{h}(t_i)|&\leq|(T(t_i+\theta)-T(t_i))\phi(0)|\\&\quad+\rho(q)\int_0^{t_i}|[T(t_i+\theta-s)-T(t_i-s)|p(s)\,ds\\&\quad+\sum_{k=1}^{i}|[T(t_i+\theta-t_k)-T(t_i-t_k)]I(y(t_k^-))|\\&\leq Me^{\omega t_i}\|T(\theta)-\mathrm{id}\|_{B(E)}\|\phi\|_{\mathcal{D}}\\&\quad+M\rho(q)\|T(\theta)-\mathrm{id}\|_{B(E)}\int_0^{t_i}e^{\omega(t_i-s)}p(s)\,ds\\&\quad+M\|T(\theta)-\mathrm{id}\|_{B(E)}\sum_{k=1}^{i}e^{\omega(t_i-t_k)}\overline{c}_kd_k.\end{aligned}$$

The terms in the right-hand side tend to zero as $\theta\to 0$. By the Arzelá–Ascoli theorem, we conclude that $N\colon\Omega\to\mathcal{P}_{\mathrm{cp,cv}}(\Omega)$ is a completely continuous operator. Finally, the nonlinear alternative for multi-valued mappings (Lemma 5.10) implies that $S_{[-r,b]}(\phi)\neq\emptyset$.

Step 2. $S_{[-r,b]}(\phi)$ is a compact set in Ω. Let $\{y_n\}_{n\in\mathbb{N}} \subset S_{[-r,b]}(\phi)$, then there exists $v_n \in S_{F,y_n}$ such that

$$y_n(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v_n(s)\,ds & \\ \qquad\qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k)) & \text{for } t \in [0,b]. \end{cases}$$

The sequence $\{v_n(\,\cdot\,)\}_{n\in\mathbb{N}}$ is integrably bounded and E is reflexive. By the Dunford–Pettis theorem [55], there is a subsequence, still denoted $(v_n)_{n\in\mathbb{N}}$ which converges weakly to an element $v(\,\cdot\,) \in L^1$. Mazur's Lemma implies the existence of $\alpha_i^n \geq 0$, $i = n,\dots,k(n)$, such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinaisons $g_n(\,\cdot\,) = \sum_{i=1}^{k(n)} \alpha_i^n v_i(\,\cdot\,)$ converges strongly to v in L^1. Since $F(\,\cdot\,,\,\cdot\,) \in \mathcal{P}_{\mathrm{cp,cv}}(E)$ and $F(t,\,\cdot\,)$ is upper semicontinuous, for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$

$$g_n(t) \in \sum_{i=1}^{k(n)} \alpha_i^n F(t,(y_n)_t) \subset F(t,y_t) + \varepsilon \sum_{i=1}^{k(n)} \alpha_i^n \|(y_n)_t - y_t\|_{\mathcal{D}} B(0,1)$$

where

$$y(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s)\,ds & \\ \qquad\qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)) & \text{for } t \in [0,b]. \end{cases}$$

From [19], $(g_n)_{n\in\mathbb{N}}$ has a subsequence which converges almost everywhere to v. Moreover, the functions I_k, $k = 1,\dots,m$, are continuous. The Lebesgue dominated convergence theorem implies that

$$\|y_n - y\|_\Omega \leq Me^{\omega b} \int_0^b |g_n(s) - v(s)|\,ds + Me^{\omega b} \sum_{k=1}^m |I_k(y_n(t_k) - I_k(y(t_k))| \to 0,$$

as $n \to \infty$. Therefore $S_{[-r,b]}(\,\cdot\,) \in \mathcal{P}_{\mathrm{cp}}(E)$.

Part 2. Under assumptions $(\mathcal{B}_1)$, $(\mathcal{B}_2)$, $(\mathcal{B}_4)$, the set $S_{[-r,b]}$ is nonempty and compact.

Step 1. $S_{[-r,b]} \neq \emptyset$. Let $N_0\colon \mathcal{D}\cap C([0,t_1],E) \to \mathcal{P}(\mathcal{D}\cap C([0,t_1],E))$ be defined by

$$N_0(y) = \Bigg\{ h \in \mathcal{D}\cap C([0,t_1],E) :$$

$$h(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)v(s) & \text{for } t \in [0,t_1], \end{cases} \Bigg\}$$

where $v \in S_{F,y} = \{v \in L^1([0,t_1],E) : v(t) \in F(t,y_t)$ for almost every $t \in [0,t_1]\}$. Set

$$\mathcal{K}_0 := \{y \in \mathcal{D} \cap C([0,t_1],E) : \|y_t\|_{\mathcal{D}} \le a_0(t),\ t \in [0,t_1]\},$$

where

$$a_0(t) = \Gamma^{-1}\bigg(\int_0^t \widehat{M}(s)\,ds\bigg), \quad \Gamma(z) = \int_c^z \frac{du}{\rho(u)}$$

and $c = Me^{\omega t_1}\|\phi\|_{\mathcal{D}}$, $\widehat{M}(t) = Me^{\omega t_1}p(t)$, $t \in [0,t_1]$.

It is clear that $\mathcal{K}_0$ is a closed bounded convex set in $\mathcal{D} \cap C([0,t_1],E)$. If $h \in N_0(y)$, then there exists $v \in S_{F,y}$ such that $h(t) = \phi(t)$, $t \in [-r,0]$ and

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)v(s)\,ds, \quad t \in [0,t_1].$$

Claim 1. $N_0(\mathcal{K}_0) \subset \mathcal{K}_0$. We have

$$\begin{aligned}
|h(t)| &\le Me^{\omega t_1}\|\phi\|_{\mathcal{D}} + \int_0^t |T(t-s)\|v(s)|\,ds \\
&\le Me^{\omega t_1}\|\phi\|_{\mathcal{D}} + Me^{\omega t_1}\int_0^t p(s)\rho(\|y_s\|_{\mathcal{D}})\,ds \\
&\le Me^{\omega t_1}\|\phi\|_{\mathcal{D}} + \int_0^t \widehat{M}(s)\rho(a_0(s))\,ds.
\end{aligned}$$

It follows that, for each $t \in [0,t_1]$,

$$|h(t)| \le c + \int_0^t a_0'(s)\,ds = a_0(t),$$

whence our claim.

Claim 2. The multi-valued map $N_0\colon \mathcal{K}_0 \to \mathcal{P}(\mathcal{K}_0)$ has at least one fixed point. Since F is a multifunction with convex values, N_0 has convex values. Moreover N_0 has a closed graph. Indeed, let $\{y_n : n \in \mathbb{N}\} \subseteq \mathcal{K}_0$ be such that $y_n \to y_*$, $h_n \in N_0(y_n)$ and $h_n \to h_*$, as $n \to \infty$. We shall prove that $h_* \in N_0(y_*)$. $h_n \in N_0(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that for almost every $t \in [0,t_1]$,

$$h_n(t) = T(t)\phi(t) + \int_0^t T(t-s)v_n(s)\,ds.$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that for almost every $t \in [0,t_1]$, we have

$$h_*(t) = T(t)\phi(0) + \int_0^t T(t-s)v_*(s)\,ds. \tag{5.1}$$

Since $\{y_n : n \in \mathbb{N}\} \subseteq \mathcal{K}_0$ and $\{v_n : n \in \mathbb{N}\} \subseteq F(t,(y_t)_n)$, assumption $(\mathcal{B}_1)$ implies that

$$|v_n(t)| \le p(t)\rho(a_0(t_1)), \quad t \in [0,t_1].$$

In addition, the set $\{v_n(t) : n \in \mathbb{N}\}$ is relatively compact for almost every $t \in J$ because Assumption $(\mathcal{B}_4)$ both with the convergence of $\{y_n\}_{n\in\mathbb{N}}$ imply that

$$\chi(\{v_n(t) : n \in \mathbb{N}\}) \leq \chi(F(t,(y_t)_n) \leq \overline{p}(t)\chi((y_t)_n) = 0.$$

Then the sequence $\{v_n : n \in \mathbb{N}\}$ is semi-compact, hence weakly compact in $L^1([0,t_1];E)$ by Lemma 5.5, i.e. there exists $v_* \in L^1$ such that $\{v_n\}$ converges weakly to v_*. Finally, (5.1) follows from the Lebesgue dominated convergence theorem.

Claim 3. N_0 is a β-condensing operator for a suitable MNC β.

For a bounded subset $D \subset \mathcal{K}_0$, let $\operatorname{mod}_C(D)$ be the modulus of quasi-equicontinuity of the set of functions D given by

$$\operatorname{mod}_C(D) = \lim_{\delta\to 0} \sup_{x\in D} \max_{|\tau_2-\tau_1|\leq\delta} |x(\tau_1) - x(\tau_2)|.$$

It is well known (see Example 2.1.2 in [38]) that $\operatorname{mod}_C(D)$ defines an MNC in $C([a,b],E)$ which satisfies all of the properties in Definition 5.2. Given the Hausdorff MNC χ, let γ_0 be the real MNC defined on bounded subsets on $\mathcal{K}_0$ by

$$\gamma_0(D) = \sup_{t\in[0,t_1]} e^{-L_0 t}\chi(D(t)).$$

Finally, define the following MNC on bounded subsets of $\mathcal{K}_0$ by

$$\beta_0(B) = \max_{D\in\Delta(\mathcal{K}_0)} (\gamma_0(D), \operatorname{mod}_C(D)),$$

where $\Delta(\mathcal{K}_0)$ is the collection of all denumerable subsets of B. Then the MNC β is monotone, regular and nonsingular (see Example 2.1.4 in [38]). This measure is also used in [14], [15], [24] in the discussion of semilinear evolution differential inclusions when E is not necessarily separable.

To show that N_0 is β-condensing, let $B \subset \mathcal{K}_0$ be a bounded set in $\mathcal{K}_0$ such that

$$\beta_0(B) \leq \beta_0(N_0(B)). \tag{5.2}$$

We will show that B is relatively compact. Let $\{y_n : n \in \mathbb{N}\} \subset B$ and let $N_0 = \widetilde{L_0} \circ S_F$, where $S_F\colon \mathcal{D} \cap C([0,t_1],E) \to L^1([0,t_1],E)$ is defined by

$$S_F(y) = S_{F,y} = \{v \in L^1([0,t_1],E) : v(t) \in F(t,y_t) \text{ a.e. } t \in [0,t_1]\}$$

and $\widetilde{L_0}\colon L^1([0,t_1],E) \to \mathcal{D} \cap C([0,t_1],E)$ is defined by

$$\widetilde{L_0}(v)(t) = \int_0^t T(t-s)v(s)\,ds, \quad t \in [0,t_1].$$

Then

$$
\begin{aligned}
|\widetilde{L_0}v_1(t)-\widetilde{L_0}v_2(t)| &\le \int_0^t \|T(t-s)\|\cdot|v_1(s)-v_2(s)|\,ds\\
&\le Me^{wt_1}\int_0^t |v_1(s)-v_2(s)|\,ds.
\end{aligned}
$$

Moreover, each element h_n in $N_0(y_n)$ can be represented as

$$
h_n = T(\,\cdot\,)\phi(0)+\widetilde{L_0}(v_n),\quad v_n\in S_F(y_n).
$$

Using (5.2), we infer that $\beta_0(\{h_n : n\in\mathbb{N}\})\ge\beta_0(\{y_n : n\in\mathbb{N}\})$. From ssumption ($\mathcal{B}4$), it holds that for almost every $t\in[0,t_1]$,

$$
\begin{aligned}
\chi(\{v_n(t):n\in\mathbb{N}\}) &\le \chi(F(t,\{(y_t)_n)\}_{n=1}^{\infty})\\
&\le \overline{p}(t)\sup_{-r\le\theta\le 0}\chi(\{(y_t)_n(\theta)\}_{n=1}^{\infty})\le \overline{p}(t)\sup_{0\le s\le t}\chi(\{y_n(s)\}_{n=1}^{\infty})\\
&\le e^{L_0t}\overline{p}(t)\sup_{0\le s\le t}e^{-L_0s}\chi(\{y_n(s)\}_{n=1}^{\infty})\le e^{L_0t}\overline{p}(t)\gamma_0(\{y_n\}_{n=1}^{\infty}).
\end{aligned}
$$

From Lemmas 5.6 and 5.7, we deduce that

$$
e^{-L_0t}\chi(\{\widetilde{L_0}(v_n)(t)\}_{n=1}^{\infty})\le\gamma_0(\{y_n\}_{n=1}^{\infty})\sup_{t\in[0,t_1]}2Me^{wt_1}\int_0^t e^{-L_0(t-s)}\overline{p}(s)\,ds.
$$

Therefore

$$
\gamma_0(\{y_n\}_{n=1}^{\infty})\le\gamma_0(\{h_n\}_{n=1}^{\infty})=\sup_{t\in[0,t_1]}e^{-L_0t}\chi(\{h_n(t)\}_{n=1}^{\infty}\le q_0\gamma_0(\{y_n\}_{n=1}^{\infty}).
$$

Since $0<q_0<1$, we infer that

$$
\gamma_0(\{y_n\}_{n=1}^{\infty})=0. \tag{5.3}
$$

Next, we show that $\mod_C(B)=0$ i.e. the set B is equicontinuous. This is equivalent to show that for every $\{h_n\}\subset N_0(B)$ satisfies this property. Given a sequence $\{h_n\}$, there exist sequences $\{y_n\}\subset B$ and $\{v_n\}\subset S_{F,y_n}$ such that

$$
h_n = T(\,\cdot\,)\phi(0)+\widetilde{L_0}(v_n).
$$

Back to (5.3), we infer that $\{y_n\}$ satisfies the equality

$$
\chi(\{y_n(t)\})=0,\quad \text{for all } t\in[0,t_1].
$$

Assumption ($\mathcal{B}_4$) in turn implies that

$$
\chi(\{v_n(t)\})=0\quad\text{for a.e. } t\in[0,t_1].
$$

From ($\mathcal{B}_1$), the sequence $\{v_n\}$ is integrably bounded, hence semi-compact. Arguing as in Part 1, Step 2, we deduce that, up to a subsequence, $\{h_n\}$ is relatively compact. Therefore $\beta_0(\{h_n\}_{n=1}^{\infty})=0$ which implies that $\beta_0(\{y_n\}_{n=1}^{\infty})=0$. We have proved that B is relatively compact and so the map N_0 is β-condensing.

From Lemma 5.8, we deduce that N_0 has at least point fixe denoted y_0. Moreover since $\operatorname{Fix} N_0$ is bounded, by Lemma 5.9, $\operatorname{Fix} N_0$ is compact.

Step 2. $S_{[-r,b]}$ is compact. Let

$$C_1 = \{y \in C((t_1,t_2],E) : y(t_1^+) \text{ exists}\}, \quad C^* = \mathcal{D} \cap C([0,t_1],E) \cap C_1,$$
$$\mathcal{K}_1 := \{z \in C^* : \|y_t\|_\infty \le a_1(t),\ t \in [t_1,t_2]\},$$

where

$$a_1(t) = \Gamma^{-1}\bigg(\int_{t_1}^t \widehat{M}(s)\,ds\bigg) \quad \text{and} \quad \Gamma(z) = \int_c^z \frac{du}{\rho(u)}.$$

Define the operator $N_1 \colon C^* \to \mathcal{P}(C^*)$ by $Ny = \{h\}$ where

$$h(t) = \begin{cases} y_0(t) & \text{for } t \in [-r,t_1], \\ \displaystyle\int_{t_1}^t T(t-s)v(s)\,ds + T(t-t_1)[y_0(t_1) + I_1(y_0(t_1^-))] & \text{for } t \in [t_1,t_2], \end{cases}$$

and $v \in S_{F,y} = \{v \in L^1([t_1,t_2],E) : v(t) \in F(t,y_t),\ \text{for a.e. } t \in [t_1,t_2]\}$. We can easily check that $N_1(\mathcal{K}_1) \subset (\mathcal{K}_1)$. Thus we only prove that N_1 is a β-condensing operator. For a bounded subset $B \subset \mathcal{K}_1$, let $mod_C(B)$ be the modulus of quasi-equicontinuous of the set of functions B, γ_1 be the real MNC defined on bounded subset on $\overline{U}$ by

$$\gamma_1(B) = \sup_{t \in [t_1,t_2]} e^{-L_1 t}\chi(B(t)),$$

and β_1 the MNC defined on $\mathcal{K}_1$ by

$$\beta_1(B) = \max_{\Delta(\mathcal{K}_1)} (\gamma_1(B), mod_C(B)),$$

where $\Delta(\mathcal{K}_1)$ is the collection of all denumerable subsets of B. Let $B \subset \mathcal{K}_1$ be a bounded set in $\mathcal{K}_1$ such that $\beta_1(B) \le \beta_1(N_1(B))$.

We will show that B is relatively compact. It is clear that h_n has the representation:

$$h_n(t) = \widetilde{L_1}(v_n(t)) + T(t-t_1)[y_0(t_1) + I_1(y_0(t_1))],$$

where $\widetilde{L_1}$ is as defined in Step 1, Claim 3. Then, we have the estimates

$$\begin{aligned} \chi(\{v_n(t) : n \in \mathbb{N}\}) &\le \chi(F(t, \{(y_n)_n\}_{n=1}^\infty + T(t-t_1)[y_0(t_1) + I_1(y_0(t_1))] \\ &\le \chi(F(t, \{(y_n)_s\}_{n=1}^\infty)) + Me^{\omega(t_2-t_1)}\chi(I_1(y_0(t_1))) \\ &\le \overline{p}(t) \sup_{-r \le \theta \le 0} \chi(\{(y_n)_t(\theta)\}_{n=1}^\infty)) \\ &\le \overline{p}(t) \sup_{t-r \le s \le t} \chi(\{(y_t)_n(s)\}_{n=1}^\infty)) \\ &\le e^{L_1 t}\overline{p}(t) \sup_{t-r \le s \le t} e^{-L_1 s}\chi(\{(y_n)_s\}_{n=1}^\infty)) \\ &\le e^{L_1 t}\overline{p}(t)\gamma_1(\{y_n\}_{n=1}^\infty). \end{aligned}$$

From Lemmas 5.6 and 5.7, we deduce that

$$e^{-L_1 t}\chi(\{\widetilde{L_1}(v_n)(t)\}_{n=1}^{\infty}) \leq \gamma_1(\{y_n\}_{n=1}^{\infty}) \sup_{t\in[t_1,t_2]} 2Me^{\omega t_2}\int_0^t e^{-L_1(t-s)}\overline{p}(s)\,ds.$$

Therefore

$$\gamma_1(\{y_n\}_{n=1}^{\infty}) \leq \gamma_1(\{h_n\}_{n=1}^{\infty}) = \sup_{t\in[t_1,t_2]} e^{-L_1 t}\chi(\{h_n(t)\}_{n=1}^{\infty} \leq q_1\gamma_1(\{y_n\}_{n=1}^{\infty}).$$

Since $0 < q_1 < 1$, it follows that $\gamma_1(\{y_n\}_{n=1}^{\infty}) = 0$.

By the same argument used in Step 1, we can show that $\mathrm{mod}_C(\{y_n\}_{n=1}^{\infty}) = 0$ and then $\beta_1(\{y_n\}_{n=1}^{\infty}) = 0$. Finally, N_1 is β-condensing and from Lemma 5.8, we deduce that N_1 has a fixed point y_1 in $\mathcal{K}_1$ denoted by y_1. As in Step 1, we can prove that $\mathrm{Fix}\, N_1$ is a compact set.

Step 3. We continue this process taking into account that $y_m := y|_{[t_m-r,b]}$ is a solution of the problem

$$\begin{cases} (y'(t) - Ay(t)) \in F(t, y_t) & \text{for a.e. } t \in (t_m, b], \\ y(t_m^+) = y_{m-1}(t_{m-1}) + I_m(y_{m-1}(t_m^-)), & \\ y(t) = y_{m-1}(t) & \text{for } t \in [t_m - r, b]. \end{cases}$$

A solution y of problem (1.1) is ultimately defined by

$$y(t) = \begin{cases} y_0(t) & \text{if } t \in [-r, t_1], \\ y_2(t) & \text{if } t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots & \\ y_m(t) & \text{if } t \in (t_m, t_{m+1}]. \end{cases}$$

To sum up, we obtain that $S_{[-r,b]}(\phi) = \bigcap_{k=0}^{k=m} \mathrm{Fix}\, N_k$, hence $\emptyset \neq S_{[-r,b]}(\,\cdot\,) \in \mathcal{P}_{\mathrm{cp}}(PC)$. This completes the proof of Theorem 5.11. □

6. Geometric structure of solution sets

6.1. Background in geometric topology. First, we start with some elementary notions and notations from algebraic topology. For details, we recommend [12], [25]–[28], [37], [42]. In what follows (X, d) and (Y, d') stand for two metric spaces.

Definition 6.1. A set $A \in \mathcal{P}(X)$ is called a *contractible space provided* if there exists a continuous homotopy $h\colon A \times [0,1] \to A$ and $x_0 \in A$ such that

(a) $h(x, 0) = x$, for every $x \in A$,
(b) $h(x, 1) = x_0$, for every $x \in A$,

i.e. if the identity map $A \to A$ is homotopic to a constant map (A is homotopically equivalent to a point).

Note that if $A \in \mathcal{P}_{\mathrm{cv,cl}}(X)$, then A is contractible. Also the class of contractible sets is much larger than the class of closed convex sets.

Definition 6.2. A compact nonempty space X is called an R_δ*-set provided* if there exists a decreasing sequence of compact nonempty contractible spaces $\{X_n\}$ such that $X = \bigcap_{n=1}^{\infty} X_n$.

Definition 6.3. A space X is called an *absolute retract* (in short $X \in \mathrm{AR}$) provided that for every space Y, every closed subset $B \subseteq Y$ and any continuous map $f: B \to X$ if there exists a continuous extension $\widetilde{f}: Y \to X$ of f over Y, i.e. $\widetilde{f}(x) = f(x)$ for every $x \in B$. In other words, for every space Y and for any embedding $f: X \to Y$, the set $f(X)$ is a retract of Y.

From [2, Proposition 2.15], if $X \in \mathrm{AR}$, then it is a contractible space. Also, define

Definition 6.4. A space A is *closed acyclic* if

(a) $H_0(A) = \mathbb{Q}$,
(b) $H_n(A) = 0$, for every $n > 0$,

where $H_* = \{H_n\}_{n \geq 0}$ is the Čech-homology functor with compact carriers and coefficients in the field of rationals $\mathbb{Q}$. In other words, a space A is acyclic if the map $j: \{p\} \to X$, $j(p) = x_0 \in A$, induces an isomorphism $j_*: H_*(\{p\}) \to H_*(A)$.

Definition 6.5. An u.s.c. map $F: X \to \mathcal{P}(Y)$ is called *acyclic* if for each $x \in X$, the image set $F(x)$ is compact and acyclic.

From the continuity of Čech-homology functors, we have:

Lemma 6.6. *Let X be a compact metric space. Then X is an acyclic space and its structure corresponds to one of the following type:*

(a) *X is convex,*
(b) *X is contractible,*
(c) *X is* AR*,*
(d) *X is an R_δ-set.*

The next definitions were introduced in [26]

Definition 6.7. A metric space X is called acyclically contractible if there exists an acyclic homotopy $\Pi: X \times [0,1] \to \mathcal{P}(X)$ such that

(a) $x_0 \in \Pi(x,1)$, for every $x \in X$ and for some $x_0 \in X$,
(b) $x \in \Pi(x,0)$, for every $x \in X$.

Notice that any contractible space and any acyclic, compact metric space are acyclically contractible (see [2, Theorem 19]). Also from [27], any acyclically contractible space is acyclic.

Definition 6.8. A metric space X is called *R_δ-contractible* if there exists a multivalued homotopy $\Pi: X \times [0,1] \to \mathcal{P}(X)$ which is u.s.c. and satisfies

(a) $x \in \Pi(x,1)$, for every $x \in X$,
(b) $\Pi(x,0) = B$ for every $x \in X$ and for some $B \subset X$,
(c) $\Pi(x,\alpha)$ is an R_δ-set, for every $\alpha \in [0,1]$ and $x \in X$.

Next, we present a result about the topological structure of the set of solutions of some nonlinear functional equations due to N. Aronszajn and developed by F. Browder and Ch. P. Gupta in [13] (see also [2, Theorem 1.2]).

Theorem 6.9. *Let X be a space, $(E, \|\cdot\|)$ a Banach space and $f: X \to E$ a proper map i.e. f is continuous and for every compact $K \subset E$, the set $f^{-1}(K)$ is compact. Assume further that for each $\varepsilon > 0$ a proper map $f_\varepsilon: X \to E$ is given and the following two conditions are satisfied:*

(a) *$\|f_\varepsilon(x) - f(x)\| < \varepsilon$, for every $x \in X$,*
(b) *for every $\varepsilon > 0$ and $u \in E$ in a neighbourhood of the origin such that $\|u\| \leq \varepsilon$, the equation $f_\varepsilon(x) = u$ has exactly one solution x_k.*

Then the set $S = f^{-1}(0)$ is an R_δ-set.

The following Lasota–Yorke Approximation Theorem (see [27]) will be needed in this section.

Lemma 6.10. *Let E be a normed space, X be a metric space and $f: X \to E$ be a continuous map. Then, for each $\varepsilon > 0$, there is a locally Lipschitz map $f_\varepsilon: X \to E$ such that*

$$\|f(x) - f_\varepsilon(x)\| < \varepsilon, \quad \text{for every } x \in X.$$

6.2. Application. Consider the first-order impulsive single-valued problem:

$$\begin{cases} y'(t) - Ay(t) = f(t, y_t) & \text{for a.e. } t \in J = [t_0, b] \setminus \{t_1, \ldots, t_m\}, \\ \Delta y|_{t=t_k} = I_k(y(t_k^-)) & \text{for } k = 1, \ldots, m, \\ y(t) = \phi(t) & \text{for } t \in [-r, t_0], \end{cases} \tag{6.1}$$

where $f: J \times \mathcal{D} \to E$ is a given function $t_0 < \ldots < t_m < t_{m+1} = b$, $\phi \in \mathcal{D}$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, respectively.

Denote by $S(f, \phi)$ the set of all solutions of problem (6.1). We are in a position to state and prove an Aronsajn-type result for this problem. First, we list two assumptions:

($\mathcal{C}_1$) $f: J \times \mathcal{D} \to E$ is an L^1-Carathéodory function.

($\mathcal{C}_2$) There exist a function $p \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\rho : [0, \infty) \to [0, \infty)$ such that

$$\|f(t, x)\| \le p(t)\rho(\|x\|_{\mathcal{D}}) \quad \text{for a.e. } t \in J \text{ and each } x \in \mathcal{D}$$

with

$$\int_0^b p(s)\, ds < \int_{Me^{\omega b}\|\phi\|_{\mathcal{D}}}^{\infty} \frac{du}{\rho(u)}.$$

Then, our first result in this section is

Theorem 6.11. *Assume that Assumptions* ($\mathcal{C}_1$)–($\mathcal{C}_2$) *hold together with either* ($\mathcal{B}_2$), ($\mathcal{B}_3$) *or* ($\mathcal{B}_2$), ($\mathcal{B}_4$). *Then the set* $S(f, \phi)$ *is an* R_δ, *hence an acyclic space.*

Proof. Let $F : \Omega \to \Omega$ be defined by:

$$F(y)(t) = \begin{cases} \phi(t) & \text{for } t \in [-r, t_0], \\ T(t - t_0)\phi(t_0) + \displaystyle\int_{t_0}^{t} T(t - s) f(s, y_s)\, ds & \\ \qquad\qquad + \displaystyle\sum_{t_0 < t_k < t} T(t - t_k) I_k(y(t_k)) & \text{for } t \in (t_0, b]. \end{cases}$$

Thus $\operatorname{Fix} F = S(f, \phi)$. From Theorem 5.11, we know that $S(f, \phi) \ne \emptyset$ and there exists $\overline{M} > 0$ such that

$$\|y\|_\Omega \le \overline{M}, \quad \text{for every } y \in S(f, \phi).$$

Define

$$\widetilde{f}(t, y_t) = \begin{cases} f(t, y_t) & \text{if } \|y_t\|_{\mathcal{D}} \le \overline{M}, \\ f\left(t, \dfrac{\overline{M} y_t}{\|y_t\|_{\mathcal{D}}}\right) & \text{if } \|y_t\|_{\mathcal{D}} \ge \overline{M}. \end{cases}$$

Since f is L^1-Carathéodory, the function $\widetilde{f}$ is Carathéodory and is integrably bounded by ($\mathcal{C}_2$). So there exists $h \in L^1(J, \mathbb{R}^+)$ such that

$$\|\widetilde{f}(t, x)\| \le h(t), \quad \text{for a.e. } t \text{ and all } x \in \mathcal{D}. \tag{6.2}$$

Consider the modified problem

$$\begin{cases} y'(t) - Ay(t) = \widetilde{f}(t, y_t) & \text{for a.e. } t \in J \setminus \{t_1, \dots, t_m\}, \\ \Delta y|_{t = t_k} = I_k(y(t_k^-)) & \text{for } k = 1, \dots, m, \\ y(t) = \phi(t) & \text{for } t \in [-r, t_0]. \end{cases}$$

We can easily prove that $S(f,\phi) = S(\widetilde{f},\phi) = \operatorname{Fix}\widetilde{F}$, where $\widetilde{F}\colon\Omega \to \Omega$ is as defined by

$$\widetilde{F}(y)(t) = \begin{cases} \phi(t) & \text{for } t \in [-r,t_0], \\ T(t-t_0)\phi(t_0) + \displaystyle\int_{t_0}^{t} T(t-s)\widetilde{f}(s,y_s)\,ds & \\ \qquad\qquad + \displaystyle\sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)) & \text{for } t \in [t_0,b]. \end{cases}$$

By the inequality (6.2) and the continuity of I_k, we deduce that

$$\|\widetilde{F}(y)\|_\Omega \le Me^{\omega b}\|\phi\|_{\mathcal{D}} + Me^{\omega b}\|h\|_{L^1} + Me^{\omega b}\sum_{k=1}^{m}\overline{c}_k\phi_k(\overline{M}) := R.$$

Then $\widetilde{F}$ is uniformly bounded. As in Theorem 5.1, we can prove that $\widetilde{F}\colon\Omega\to\Omega$ is compact which allows us to define the compact perturbation of the identity $\widetilde{G}(y) = y - \widetilde{F}(y)$ which is a proper map. From the compactness of $\widetilde{F}$ and the Lasota–Yorke approximation theorem, we can easily prove that all conditions of Theorem 6.9 are met. Therefore the solution set $S(\widetilde{f},\phi) = \widetilde{G}^{-1}(0)$ is an R_δ set, hence an acyclic space by Lemma 6.6. □

6.3. σ-selectionable multivalued maps. The following definitions and the result can be found in [27], [29] (see also [5, p. 86]). Let (X,d) and (Y,d') be two metric spaces.

Definition 6.12. We say that a map $F\colon X \to \mathcal{P}(Y)$ is *σ-Ca-selectionable* if there exists a decreasing sequence of compact valued u.s.c. maps $F_n\colon X \to Y$ satisfying:

(a) F_n has a Carathédory selection, for all $n \ge 0$ (F_n are called *Ca-selectionable*),
(b) $F(x) = \bigcap_{n\ge 0} F_n(x)$, for all $x \in X$.

Definition 6.12. A single-valued map $f\colon[0,a]\times X \to Y$ is said to be *measurable-locally-Lipschitz* (mLL) if $f(\,\cdot\,,x)$ is measurable for every $x \in X$ and for every $x \in X$, there exists a neighbourhood V_x of $x \in X$ and an integrabe function $L_x\colon[0,a]\to[0,\infty)$ such that

$$d'(f(t,x_1),f(t,x_2)) \le L_x(t)d(x_1,x_2) \quad \text{for every } t\in[0,a] \text{ and } x_1,x_2 \in V_x.$$

Definition 6.13. A multi-valued mapping $F\colon[0,a]\times X \to \mathcal{P}(Y)$ is *mLL-selectionable* if it has an mLL-selection.

Definition 6.14. We say that a multivalued map $\phi\colon[0,a]\times\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ with closed values is upper-Scorza–Dragoni if, given $\delta > 0$, there exists a closed subset $A_\delta \subset [0,1]$ such that the measure $\mu([0,a]\setminus A_\delta) \le \delta$ and the restriction ϕ_δ of ϕ to $A_\delta \times \mathbb{R}^n$ is u.s.c.

Theorem 6.15 (see [27, Theorem 19.19]). *Let E, E_1 be two separable Banach spaces and let $F\colon [a,b]\times E\to \mathcal{P}_{\mathrm{cp,cv}}(E_1)$ be an upper-Scorza–Dragoni map. Then F is σ-Ca-selectionable, the maps $F_n\colon [a,b]\times E\to \mathcal{P}(E_1)$ $(n\in\mathbb{N})$ are almost upper semicontinuous and we have*

$$F_n(t,e)\subset \overline{\mathrm{conv}}\bigg(\bigcup_{x\in E} F_n(t,x)\bigg).$$

Moreover, if F is integrably bounded, then F is σ-mLL-selectionable.

Let $S_{[-r,b]}(\phi)$ denote the set of all solutions of problem (1.1). Now, we are in position to state and prove another characterization of the geometric structure of $S_{[-r,b]}(\phi)$.

Theorem 6.16. *Let $F\colon J\times\mathcal{D}\to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory and an mLL-selectionable multi-valued map which satisfies conditions $(\mathcal{B}_1)$, $(\mathcal{B}_2)$ and $(\overline{\mathcal{H}}_3)$ with $\sum_{k=1}^{k=m} c_k<1$. Then, for every $\phi\in\mathcal{D}$, the set $S_{[-r,b]}(\phi)$ is contractible.*

Proof. Let $f\subset F$ be a measurable, locally Lipschitz selection and consider the single-valued problem

$$\begin{cases} y'(t)-Ay(t)=f(t,y_t) & \text{for a.e. } t\in J\setminus\{t_1,\dots,t_m\},\\ y(t_k^+)-y(t_k)=I_k(y(t_k^-)) & \text{for } k=1,\dots,t_m,\\ y(t)=\phi(t) & \text{for } t\in[-r,0]. \end{cases}\tag{6.3}$$

As in [11] or in [45], [46, Theorems 3.3 and 3.5], we can prove that problem (6.3) has exactly one solution for every $\phi\in\mathcal{D}$. Define the homotopy $h\colon S_{[-r,b]}(\phi)\times [0,1]\to S_{[-r,b]}(\phi)$ by

$$h(y,\alpha)(t)=\begin{cases} y(t) & \text{for } -r\le t\le \alpha b,\\ \overline{x}(t) & \text{for } \alpha b<t\le b, \end{cases}$$

where $\overline{x}=S_{[-r,b]}(f,\phi)$ is the unique solution of problem (6.3). In particular,

$$h(y,\alpha)=\begin{cases} y & \text{for } \alpha=1,\\ \overline{x} & \text{for } \alpha=0. \end{cases}$$

To show that h is a continuous homotopy, let $(y_n,\alpha_n)\in S_{[-r,b]}(\phi)\times[0,1]$ be such that $(y_n,\alpha_n)\to(y,\alpha)$, as $n\to\infty$. We shall prove that $h(y_n,\alpha_n)\to h(y,\alpha)$. We have

$$h(y_n,\alpha_n)(t)=\begin{cases} y_n(t) & \text{for } t\in[-r,\alpha_n b],\\ \overline{x}(t) & \text{for } t\in(\alpha_n b,b]. \end{cases}$$

Three cases may occur.

Case 1. If $\lim_{n\to\infty}\alpha_n=0$, then

$$h(y,0)(t)=\begin{cases} \phi(t) & \text{for } t\in[-r,0],\\ \overline{x}(t) & \text{for } t\in(0,b]. \end{cases}$$

Hence

$$\|h(y_n,\alpha_n) - h(y,\alpha)\|_\Omega \leq \|y_n - \phi\|_\Omega + \|y_n - \overline{x}\|_{[0,\alpha_n b]},$$

which tends to 0 as $n \to \infty$ for $y_n \equiv \phi$ on $[-r,0]$. The case when $\lim_{n\to\infty} \alpha_n = 1$ is treated similarly.

Case 2. If $\alpha_n \neq 0$ and $0 < \lim_{n\to\infty} \alpha_n = \alpha < 1$,] then we may distinguish between two sub-cases:

(a) If $t \in [-r, \alpha b]$, then

$$\text{if } y_n(t) = \phi(t) \text{ then } h(y_n,\alpha_n)(t) = h(\phi,\alpha_n)(t), \text{ for all } t \in [-r,0].$$

Furthermore, $y_n \in S_{[-r,b]}(\phi)$ implies the existence of $v_n \in S_{F,y_n}$ such that for $t \in [0, \alpha_n b]$

$$y_n(t) = T(t)\phi(t) - \int_0^t T(t-s)v_n(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y_n(t_k)).$$

$F(t,\,\cdot\,)$ being u.s.c., for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \geq 0$ such that for any $n \geq n_0$, we have

$$v_n(t) \in F(t,(y_n)_t) \subset F(t,y_t) + \varepsilon B(0,1), \quad \text{for a.e. } t \in [0,\alpha b].$$

In addition $F(\,\cdot\,,\,\cdot\,)$ has compact values; then there exists a subsequence $v_{n_m}(\,\cdot\,)$ such that $v_{n_m}(\,\cdot\,)$ converges to a limit $v(\,\cdot\,)$ satisfying

$$v(t) \in F(t,y_t) + \varepsilon B(0,1), \quad \text{for all } \varepsilon > 0.$$

Therefore $v(t) \in F(t,y_t)$ for amost every $t \in [0,\alpha b]$. Now $\{y_n\}$ converges to y; then some $R > 0$ exits and satisfies $\|y_n\|_\Omega \leq R$. Then assumption $(\mathcal{B}1)$ implies that

$$|v_{n_m}(t)| \leq p(t)\rho(R), \quad \text{for a.e. } t \in [0,b].$$

By the Lebesgue dominated convergence theorem, $v \in L^1([0,b],E)$, hence $v \in S_{F,y}$. Using the continuity of I_k, we deduce that for $t \in [0,b]$

$$y(t) = T(t)\phi(t) - \int_0^t T(t-s)v(s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k)).$$

(b) If $t \in (\alpha_n b, b]$, then $h(y_n,\alpha_n)(t) = h(y,\alpha)(t) = \overline{x}(t))$. Thus

$$\|h(y_n,\alpha_n) - h(y,\alpha)\|_\Omega \to 0, \quad \text{as } n \to \infty.$$

Therefore h is a continuous function, proving that $S_{[-r,b]}(\phi)$ is contractible to the point $\overline{x} = S_{[-r,b]}(f,\phi)$. $\square$

A further precise result is given by

THEOREM 6.17. *Let $F: J \times \mathcal{D} \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory and a Ca-selectionable multi-valued map. Further to the assumption in Theorem* 6.16*, let either condition* ($\mathcal{B}_3$) *or* ($\mathcal{B}_4$) *be satisfied. Then the solution set $S_{[-r,b]}(\phi)$ is R_δ-contractible and acyclic.*

PROOF. Replace the singlevalued homotopy $h: S_{[-r,b]} \times [0,1] \to S_{[-r,b]}$ in Theorem 6.16 by the multivalued homotopy $\Pi: S_{[-r,b]}(\phi) \times [0,1] \to \mathcal{P}(S_{[-r,b]}(\phi))$ defined by

$$\Pi(x,\alpha) = \{y \in S(f, \alpha b, x)\},$$

where $f \subset F$ and $S(f,\alpha b, x)$ is the solution set of the following problem

$$\begin{cases} (y' - Ay)(t) = f(t, y_t) & \text{for a.e. } t \in [\alpha b, b] \setminus \{t_1, \dots t_m\}, \\ y(t_k^+) - y(t_k) = I_k(y(t_k^-)) & \text{for } k = 1, \dots, m, \\ y(t) = x(t) & \text{for } t \in [-r, \alpha b]. \end{cases}$$

From the definition of Π, $\Pi(x,0) = S(f,0,x)$ and $x \in \Pi(x,1)$ for every $x \in S_{[-r,b](\phi)}$. It remains to prove that $\Pi(\,\cdot\,,\,\cdot\,)$ is u.s.c. Since $\Pi(\,\cdot\,,\,\cdot\,)$ has nonempty compact values, we only check (see Lemma 2.2) that Π is locally compact and has a closed graph. Finally, we show that $\Pi(x,\alpha)$ is an R_δ-set for each x, α. This will be performed in three steps.

Step 1. Π is locally compact. We argue in two sub-steps.

(a) The multivalued map $\widetilde{S}: [0,b] \times \mathcal{D} \to \mathcal{P}(\Omega)$ defined by

$$\widetilde{S}(t,\phi) = S(f,t,\phi)$$

is u.s.c. Here $S(f,u,\phi)$ refers to the solution set of the problem

$$\begin{cases} (y' - Ay)(t) = f(t, y_t) & \text{for a.e. } t \in [u, b] \setminus \{t_1, \dots t_m\}, \\ y(t_k^+) - y(t_k) = I_k(y(t_k^-)) & \text{for } k = 1, \dots, m, \\ y(t) = \phi(t) & \text{for } t \in [-r, u]. \end{cases}$$

On the contrary, assume that $\widetilde{S}$ is not u.s.c. at some point (t_0, ϕ_0). Then there exists an open neighbourhood U of $\widetilde{S}(t_0,\phi_0)$ in Ω such that for every open neighbourhood V at (t_0,ϕ_0) in the metric space $[0,b] \times \mathcal{D}$, there exists $(t_1,\phi_1) \in V$ such that $\widetilde{S}(t_1,\phi_1) \not\subset U$. Let $V_n = \{(t,\phi) \in [0,b] \times \mathcal{D}: d((t,\phi),(t_0,\phi_0)) < 1/n\}$, for each $n = 1,2,\dots,$ where d denotes the product metric in $[0,b] \times \mathcal{D}$. Then for each $n = 1,2,\dots,$ we get some $(t_n,\phi_n) \in V_n$ and $y_n \in \widetilde{S}(t_n,\phi_n)$ such that $y_n \notin U$. Define the maps $G_{t_0,\phi_0}, F_{t_0,\phi_0}: \Omega \to \Omega$ by

$$F_{t_0,\phi_0}(x)(t) = \begin{cases} \phi_0(t) & \text{for } t \in [-r, t_0], \\ T(t-t_0)\phi_0(t_0) + \displaystyle\int_{t_0}^t T(t-s) f(s, x_s)\, ds & \\ \qquad\qquad + \displaystyle\sum_{t_0 < t_k < t} T(t-t_k) I_k(x(t_k)) & \text{for } t \in [t_0, b], \end{cases}$$

and the compact perturbation of the identity

$$G_{t_0,\phi_0}(x) = x - F_{t_0,\phi_0}(x), \quad \text{for } t \in [0,b] \text{ and } x \in \Omega.$$

By a simple calculation, for $x \in \Omega$, $t, t_0 \in [0,b]$ and $\phi_0 \in \mathcal{D}$, we have

$$F_{t_0,\phi_0}(x)(t) = T(t-t_0)\phi_0(t_0) - F_{t_0,0}(x)(t_0) + F_{t_0,0}(x)(t).$$

Since

$$G_{t_0,\phi_0}(x) = x - F_{t_0,\phi_0}(x)$$

we have

$$\begin{aligned} G_{t_0,\phi_0}(x)(t) &= x(t) - F_{t_0,\phi_0}(x)(t) \\ &= x(t) - [T(t-t_0)\phi_0(t_0) - F_{t_0,0}(x)(t_0) + F_{t_0,0}(x)(t)] \\ &= -T(t-t_0)\phi_0(t_0) + x(t) - F_{t_0,0}(x)(t) + F_{t_0,0}(x)(t_0) \\ &= -T(t-t_0)\phi_0(t_0) + F_{t_0,0}(x)(t_0) + G_{t_0,0}(x)(t). \end{aligned}$$

Thus

$$G_{t_0,\phi_0}(x)(t) = -T(t-t_0)\phi_0(t) + F_{t_0,0}(x)(t_0) + G_{t_0,0}(x)(t).$$

Since F_{t_0,ϕ_0} is a compact map (see Theorem 3.3 in [46]), the compact perturbation of the identity G_{t_0,ϕ_0} is proper. Moreover, $y_n \in \widetilde{S}(t_n,\phi_n)$. Then

$$y_n(t) = \begin{cases} \phi_n(t) & \text{for } t \in [-r, t_n], \\ T(t-t_n)\phi_n(t_n) + \displaystyle\int_{t_n}^t T(t-s) f(s,(y_n)_s)\,ds & \\ \qquad\qquad + \displaystyle\sum_{t_n<t_k<t} T(t-t_k) I_k(y_n(t_k)) & \text{for } t \in [t_n, b]. \end{cases}$$

It follows that

$$0 = G_{t_n,\phi_n}(y_n)(t) = -T(t-t_n)\phi_n(t_n) + F_{t_n,0}(y_n)(t_n) + G_{t_n,0}(y_n)(t)$$

and

$$G_{t_0,\phi_0}(y_n)(t) = -T(t-t_0)\phi_0(t_0) + F_{t_0,0}(y_n)(t_0) + G_{t_0,0}(y_n)(t).$$

Then, we obtain by substraction the successive estimates

$$\begin{aligned} &\|G_{t_0,\phi_0}(y_n)(t)\| \\ &\quad \le \|T(t-t_n)\phi_n(t_n) - T(t-t_0)\phi_0(t_0)\| + \|G_{t_n,0}(y_n)(t_n) - G_{t_0,0}(y_n)(t_0)\| \\ &\quad = \|T(t-t_n)\phi_n(t_n) - T(t-t_0)\phi_0(t_0)\| + \|F_{t_n,0}(y_n)(t) - F_{t_0,0}(y_n)(t)\| \\ &\quad \le \|T(t-t_n)\phi_n(t_n) - T(t-t_n)\phi_0(t_0)\| + \|T(t-t_n)\phi_n(t_0) - T(t-t_0)\phi_0(t_0)\| \\ &\qquad + \bigg\| \int_{t_n}^t T(t-s) f(s,(y_n)_s)\,ds - \int_{t_0}^t T(t-s) f(s,(y_n)_s)\,ds \bigg\| \\ &\qquad + \bigg\| \sum_{t_0<t_k<t_n} T(t-t_k) I_k(y_n(t_k)) \bigg\| \\ &\quad \le M e^{\omega b} \|\phi_n(t_n) - \phi_0(t_0)\| + \|T(t-t_n)\phi_n(t_0) - T(t-t_0)\phi_0(t_0)\| \end{aligned}$$

$$+\int_{t_0}^{t_n} \|T(t-s)\|\|f(s,(y_n)_s)\|\,ds + Me^{\omega b}\sum_{t_0<t_k<t_n}\|I_k(y_n(t_k))\|$$
$$\le Me^{\omega b}\|\phi_n-\phi_n\|_{\mathcal{D}} + \|T(t-t_n)\phi_n(t_0)-T(t-t_0)\phi_0(t_0)\|$$
$$+ Me^{\omega b}\int_{t_0}^{t_n}\|f(s,(y_n)_s)\|\,ds + Me^{\omega b}\sum_{t_0<t_k<t_n}\|I_k(y_n(t_k))\|.$$

In addition, $\|y_n\|\le R$ and Assumption $(\mathcal{B}1)$ implies that

$$\|G_{t_0,\phi_0}(y_n)(t)\| \le Me^{\omega b}\|\phi_n-\phi_0\|_{\mathcal{D}} + \|T(t-t_n)\phi_0(t_0)-T(t-t_0)\phi_0(t_0)\|$$
$$+ Me^{\omega b}\int_{t_0}^{t_n} p(s)\rho(R)\,ds + Me^{\omega b}\sum_{t_0<t_k<t_n}\overline{c}_k\phi_k(R).$$

$\lim_{n\to\infty}\phi_n=\phi_0$ and $\lim_{n\to\infty}t_n=t_0$ imply that $\lim_{n\to\infty}G_{t_0,\phi_0}(y_n)(t)=0$. The set $A=\overline{\{G_{t_0,\phi_0}(y_n)\}}$ is a compact set and so is $G^{-1}_{t_0,\phi_0}(A)$ because G is proper. It is clear that $\{y_n\}\subset G^{-1}_{t_0,\phi_0}(A)$. Without loss of generality, we may assume that $\lim_{n\to\infty}y_n=y_0$, hence $y_0\in\widetilde{S}(t_0,\phi_0)\subset U$ but this is a contradiction to the assumption that $y_n\notin U$ for each n.

(b) Π is locally compact. For some $r>0$, Let

$$B\times I=\{(x,\alpha)\in S_{[-r,b]}\times[0,1]:\|x\|_\Omega\le r\}$$

and $\{y_n\}\in\Pi(B\times I)$; then there exists $(\phi_n,\alpha_n)\in B\times I$ such that

$$y_n(t)=\begin{cases}\phi_n(t) & \text{for } -r\le t\le\alpha_n b,\\ z_n(t) & \text{for } \alpha b<t\le b,\ z_n\in S(f,\alpha_n b,\phi_n).\end{cases}$$

Since $S_{[-r,b]}$ is compact, there exist subsequences of $\{\phi_n\}$ and $\{\alpha_n\}$ which converge to x and α, respectively. $\widetilde{S}$ u.s.c. implies that for every $\varepsilon>0$ there exists $n_0=n(\varepsilon)$ such that $z_n(t)\in\widetilde{S}(t,x)=S(f,\alpha b,x)$, for any $n\ge n_0$. Hence there exists a subsequence of $\{z_n\}\in S(f,\alpha b,x)$. By the compactness of $S(f,\alpha b,x)$, there exists z such that the subsequence $\{z_n\}$ converges to $z\in S(f,\alpha b,x)$. Therefore Π is locally compact.

Step 2. Π has a closed graph. Let $(x_n,\alpha_n)\to(x_*,\alpha)$, $h_n\in\Pi(x_n,\alpha_n)$ and $h_n\to h_*$ as $n\to\infty$. We shall prove that $h_*\in\Pi(x_*,\alpha)$. $h_n\in\Pi(y_n,\alpha_n)$ means that there exists $z_n\in S(f,\alpha_n b,\phi_n)$ such that for each $t\in J$

$$h_n(t)=\begin{cases}\phi_n(t) & \text{for } -r\le t\le\alpha_n b,\\ z_n(t) & \text{for } \alpha_n b<t\le b.\end{cases}$$

We must prove that there exists $z_*\in S(f,\alpha b,x_*)$ such that for each $t\in J$

$$h_*(t)=\begin{cases}x_*(t) & \text{for } -r\le t\le\alpha b,\\ z_*(t) & \text{for } \alpha b<t\le b.\end{cases}$$

Clearly $(\alpha_n b, \phi_n) \to (\alpha, x_*)$ as $n \to \infty$ and we can easily show that there exists a subsequence $\{z_n\}$ converging to some limit z_*. The cases $\alpha = 0$ or $\alpha = 1$ can be treated as in the proof of Theorem 6.16. From the above arguing, we find that $z_* \in S(f, \alpha b, x)$, proving our claim.

Step 3. We claim that $\Pi(x,\alpha)$ is an R_δ-set for each fixed $\alpha \in [0,1]$ and $x \in S_{[-r,b]}$. Clearly $\Pi(x,\alpha) = S_{J_0 \cup J}(x)$ where $J_0 = [-r, \alpha b]$ and $J = [\alpha b, b]$. Since F is σ-Ca-selectionable, there exists a decreasing sequence of multivalued maps $F_k\colon [0,b] \times \mathcal{D} \to \mathcal{P}(E) (k \in \mathbb{N})$ which have Carathéodory selections and satisfy

$$F_{k+1}(t,u) \subset F_k(t,u) \quad \text{for almost all } t \in [0,b],\ u \in \mathcal{D}$$

and

$$F(t,u) = \bigcap_{k=0}^{\infty} F_k(t,u), \quad u \in \mathcal{D}.$$

Then

$$\Pi(x,\alpha) = \bigcap_{k=0}^{\infty} S_{[-r,b]}(F_k, x).$$

When either condition $(\mathcal{B}_3)$ or $(\mathcal{B}_4)$ is satisfied, Theorem 5.11 implies that $\Pi(x,\alpha)$ and $S_{[-r,b]}(F_k, x)$ are compact sets. Moreover from Theorem 6.16, the sets $S_{[-r,b]}(F_k, x)$ are contractible sets. Therefore $\Pi(x,\alpha)$ is an R_δ-set.

Conclusion. As a consequence, all properties in Definition 6.8 are met. Therefore, the set $S_{[-r,b]}(\phi)$ is R_δ-contractible, ending the proof of the theorem. □

Next, more results regarding the topological structure of the solution sets are derived.

Theorem 6.18. *Let $F\colon J \times \mathcal{D} \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory and a σ-Ca-selectionable multi-valued map. Assume that all conditions of Theorem* 6.16 *are satisfied. Then the solution set $S_{[-r,b]}(\phi)$ is an R_δ-set.*

Proof. Since F is σ-Ca-selectionable, there exists a decreasing sequence of multivalued maps $F_k\colon [0,b] \times \mathcal{D} \to \mathcal{P}(E)$ $(k \in \mathbb{N})$ which have Carathéodory selections such that

$$F_{k+1}(t,u) \subset F_k(t,x) \quad \text{for almost all } t \in [0,b],\ x \in \mathcal{D}$$

and

$$F(t,x) = \bigcap_{k=0}^{\infty} F_k(t,x), \quad x \in \mathcal{D}.$$

Then

$$S_{[-r,b]}(F,\phi) = \bigcap_{k=0}^{\infty} S_{[-r,b]}(F_k, \phi).$$

From Theorem 6.17, the set $S_{[-r,b]}(F_k,\phi)$ is contractible for each $k\in\mathbb{N}$. Hence $S_{[-r,b]}(F,\phi)$ is an R_δ-set. □

Theorem 6.19. *Let $F\colon J\times\mathcal{D}\to\mathcal{P}_{\mathrm{cp,cv}}(E)$ be a Carathéodory and a σ-mLL-selectionable map. Assume that all conditions of Theorem 6.16 are fulfilled. Then the solution set $S_{[-r,b]}(\phi)$ is an R_δ-set.*

Proof. It is enough to prove that F is a σ-mLL-selectionable and then apply Theorem 6.16. □

Theorem 6.20. *Let $F\colon J\times\mathcal{D}\to\mathcal{P}_{\mathrm{cp,cv}}(E)$ be an upper-Scorza–Dragoni. Assume that all conditions of Theorem 6.16 are satisfied. Then the solution set $S_{[-r,b]}(\phi)$ is an R_δ.*

Proof. Since F is upper-Scorza–Dragoni, then from Theorem 6.15, F is a $\sigma-Ca$-selection map. Therefore $S_{[-r,b]}(F,\phi)$ is an R_δ-set. □

7. Concluding remarks

In this paper, we investigated problem (1.1) under various assumptions on the multi-valued hand-side nonlinearity and we obtained a number of new results regarding existence of solutions. We first proved Filippov's and Filippov-Ważewski results to semilinear impulsive differential inclusions providing extensions of similar results obtained in [5], [6], [21], [23], [48], [58]. The main assumption on the nonlinearity are the Carathéodory and the Lipschitz conditions with respect to the Hausdorf distance in generalized metric spaces. This allowed us to prove also closeness of the solutions set. Then, Nagumo–Bernstein type growth conditions were assumed and the compactness of the set of solutions is proved.

In 1976, Lasry and Robert [42] proved that, if the nonlinearity F is compact, convex valued, u.s.c. and bounded, then the set of all solutions for fisrt-order differential inclusions with right-hand side F is a compact and acyclic set. In 1986, Górniewicz [26] discussed the topological structure of the set of solutions (contractibility and acyclic contractibility) when F is an ML- or σ-selectionable.

When the multi-valued nonlinearity is further $\sigma-Ca$ or σ-mLL selectionable, based on Aronszajn type results, we investigated the geometric properties of the solutions set, proving that it enjoys AR, R_δ, R_δ-contractibility, contractibility and acyclicity. An application to a single-valued problem was given.

We hope this paper can make a contribution in the domain of impulsive semi-linear differential inclusions, widely studied in the recent literature.

Acknowledgements. This work was revised and completed while the authors met in Mostaganem, Algeria for the "3rd ADEs-ODEs School, May 24–29, 2008". They would like to thank the organizers for their warm hospitality. The first and the third authors acknowledge constant support from their Laboratories in Algiers and Sidi-Bel-Abbès Universities, respectively.

References

[1] Z. Agur, L. Cojocaru, G. Mazaur, R. M. Anderson and Y. L. Danon, *Pulse mass measles vaccination across age cohorts*, Proc. Nat. Acad. Sci. USA **90** (1993), 11698–11702.

[2] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer Academic Publishers, Dordrecht, 2003.

[3] N. Aronszajn, *Le correspondant topologique de l'unicité dans la théorie des équations différentielles*, Ann. Math. **43** (1942), 730–738.

[4] J. P. Aubin, *Impulse Differential Inclusions and Hybrid Systems: a Viability Approach*, Lecture Notes, Université Paris, Dauphine, 2002.

[5] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer–Verlag, Berlin–Heidelberg–New York, 1984.

[6] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[7] N. B. Azbelev, V. P. Maximov and L. F. Rakhmatulina, *Introduction to the Functional Differential Equations*, Nauka, Moscow, 1991. (in Russian)

[8] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[9] G. Ballinger and X. Z. Liu, *Existence and uniqueness results for impulsive delay differential equations*, Dynam. Contin. Discrete Impuls. Systems **5** (1999), 579–591.

[10] M. Benchohra, L. Górniewicz, S. K. Ntouyas and A. Ouahab, *Controllability results for impulsive functional differential inclusions*, Rep. Math. Phys. **54** (2004), 211–227.

[11] M. Benchohra and A. Ouahab, *Some uniqueness results for impulsive semilinear neutral functional differential equations*, Georgian Math. J. **9** (2002), 423–430.

[12] R. Bielawski, L. Górniewicz and S. Plaskacz, *Topological approach to differential inclusions on closed sets of* $\mathbb{R}^n$, Dynamics Reported **1** (1992), 225–250.

[13] F. E. Browder and G. P. Gupta, *Topological degree and nonlinear mappings of analytic type in Banach spaces*, J. Math. Anal. Appl. **26** (1969), 390–402.

[14] T. Cardinali and P. Rubbioni, *Mild solutions for impulsive semilinear evolution differential inclusions*, J. Appl. Funct. Anal. **1** (2006), 303–325.

[15] ———, *Impulsive semilinear differential inclusions: topological structure of the solutions set and solutions on noncompact domains*, Nonlinear Anal. **69** (2008), 73–84.

[16] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math., vol. 580, Springer–Verlag, Berlin–Heidelberg–New York, 1977.

[17] F. C. Clarke, *Optimization and Nonsmooth Analysis*, Springer–Verlag, Wiley–Interscience, New York, 1983.

[18] K. Deimling, *Multivalued Differential Equations*, De Gruyter, Berlin-New York, 1992.

[19] N. Dunford and J. T. Schwartz, *Linear Operators,* Part I. *General Theory*, Wiley–Interscience, New York, 1967.

[20] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer–Verlag, New York, 2000.

[21] A. H. Filippov, *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control Optim. **5** (1967), 609–621.

[22] H. Frankowska, *Set-valued Analysis and Control Theory*, Centre de Recherche de Mathématique de la Décision, Université Paris-Dauphine, 1992.

[23] ———, *A priori estimates for operational differential inclusions*, J. Differential Equations **84** (1990), 100–128.

[24] C. Gori, V. Obukhovskiĭ, M. Ragni and P. Rubbioni, *Existence and continuous dependence results for semilinear functional differential inclusions with infinite delay,* Nonlinear Anal. **51** (2002), 765–782.

[25] L. Górniewicz, *Homological methods in fixed point theory of multivalued maps*, Dissertationes Math. **129** (1976), 1–71.

[26] ______, *On the solution sets of differential inclusions*, J. Math. Anal. Appl. **113** (1986), 235–244.

[27] ______, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications, vol. 495, Kluwer Academic Publishers, Dordrecht, 1999.

[28] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer–Verlag, New York, 2003.

[29] G. Haddad and J. M. Lasry, *Periodic solutions of functional differential inclusions and fixed points of σ-selectionable correspondences*, J. Math. Anal. Appl. **96** (1983), 295–312.

[30] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer–Verlag, New York, 1993.

[31] A. Halanay and D. Wexler, *Teoria Calitativa a systeme cu Impulduri*, Editura Republicii Socialiste Romania, Bucharest, 1968.

[32] J. Henderson and A. Ouahab, *Extrapolation spaces and controllability of impulsive semilinear functional differential inclusions with infinite delay in Fréchet spaces*, Appl. Anal. **85** (2006), 1255–1270.

[33] J. Henderson and A. Ouahab, *Local and global existence and uniqueness results for second and higher order impulsive functional differential equations with infinite delay*, Austral. J. Math. Anal. Appl. **4** (2007), 149–182.

[34] F. Hiai and H. Umegaki, *Integrals conditional expectations and martingales of multivalued functions*, J. Multivariate Anal. **7** (1977), 149–182.

[35] R. Hille and R. S. Philips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc., Providence, RI, 1957.

[36] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Volume I: *Theory*, Kluwer, Dordrecht, 1997.

[37] D. M. Hyman, *On decreasing sequences of compact absolute retracts*, Fund. Math. **64** (1969), 91–97.

[38] M. Kamenskiĭ, V. Obukhovskiĭ and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, Nonlinear Anal. Appl., vol. 7, Walter de Gruyter, Berlin–Newyork, 2001.

[39] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[40] E. Kruger-Thiemr, *Formal theory of drug dosage regiments*, J. Theoret. Biol. **13** (1966), 212–235.

[41] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[42] J. M. Lasry and R. Robert, *Analyse Non Linéaire Multivoque*, Publ. No. 7611, Centre de Recherche de Mathématique de la Décision, Université de Dauphine, Paris, 1–190.

[43] V. D. Milman and A. A. Myshkis, *On the stability of motion in the presence of impulses*, Siberian Math. J. **1** (1960), 233–237. (in Russian)

[44] J. Musielak, *Introduction to Functional Analysis*, PWN, Warszawa, 1976. (in Polish)

[45] A. Ouahab, *Local and global existence and uniqueness results for impulsive functional differential equations with multiple delay*, J. Math. Anal. Appl. **323** (2006), 456–472.

[46] ______, *Existence and uniqueness results for impulsive functional differential equations with scalar multiple delay and infinite delay*, Nonlinear Anal. **67** (2007), 1027–1041.

[47] ______, *Some contributions in impulsives differential equations and inclusions with fixed and variable times*, Ph. D. Dissertation (2006), University of Sidi-Bel-Abbès, Algeria.

[48] N. S. Papageorgiou, *A relaxation theorem for differential inclusions in Banach spaces*, Tôhoku Math. J. **39** (1987), 505–517.

[49] S. G. Pandit and S. G. Deo, *Differential Systems Involving Impulses*, Lecture Notes in Mathematics, vol. 954, Springer–Verlag, 1982.

[50] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[51] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[52] A. Sghir, *On the solution set of second order delay differential inclusions in Banach space*, Ann. Math. Blaise Pascal **7** (2000), 65–79.

[53] G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*, Graduate Studies in Math., vol. 41, Amer. Math. Soc., Providence, Rhode Island, 2002.

[54] A. A. Tolstonogov, *Differential Inclusions in a Banach Space*, Kluwer, Dordrecht, 2000.

[55] K. Yosida, *Functional Analysis*, 6-th edition, Springer–Verlag, Berlin, 1980.

[56] D. Yujun, *Periodic boundary value problems for functional differential equations with impulses*, J. Math. Anal. Appl. **210** (1997), 170–181.

[57] D. Yujun and Z. Erxin, *An application of coincidence degree continuation theorem in existence of solutions of impulsive differential equations*, J. Math. Anal. Appl. **197** (1996), 875–889.

[58] J. Zhu, *On the solution set of differential inclusions in Banach space*, J. Differential Equations **93** (1991), 213–237.

Manuscript received April 30, 2008

Smaïl Djebali
Laboratory of EDP&HM
Department of Mathematics, E.N.S.
P.B. 92, 16050 Kouba, ALGERIA

E-mail address: djebali@ens-kouba.dz

Lech Górniewicz
Faculty of Mathematics and Computer Science
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, POLAND

E-mail address: gorn@mat.uni.torun.pl

Abdelghani Ouahab
Department of Mathematics
Sidi-Bel-Abbès University
P.B. 89, 22000. Sidi-Bel-Abbès, ALGERIA

E-mail address: agh_ouahab@yahoo.fr