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LOCAL MILD SOLUTIONS AND IMPULSIVE MILD SOLUTIONS FOR SEMILINEAR CAUCHY PROBLEMS INVOLVING LOWER SCORZA–DRAGONI MULTIFUNCTIONS

Tiziana Cardinali — Francesco Portigiani — Paola Rubbioni

Abstract. In this note we investigate in Banach spaces the existence of mild solutions for initial problems, also in presence of impulses, governed by semilinear differential inclusions where the non-linear part is a Scorza–Dragoni multifunction. All the results are obtained via a generalization of *Artstein–Prikry selection theorem* that we obtain in the first part of the paper.

1. Introduction

In the last years the study of the existence of local and global mild solutions for Cauchy problems involving semilinear differential inclusions has been extensively developed. In the literature we first encounter semilinear differential inclusions where the linear part is given by an operator generating a C_0-semigroup (see the monograph [12]). Later the linear part is given by a family of operators generating an evolution system (see e.g. [2], [5], [15], [16]). Such research is applied in various fields, like engineering and physics (see e.g. [12] and examples therein). Moreover, we point out that the study of the existence of global solutions is of special importance since it permits to provide solutions in presence of

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impulses too. We recall that impulsive problems find wide applicability in the modellization of many real phenomena. About impulsive problems we refer to [6], [7] and [10].

In this paper we first obtain the existence of local mild solutions for semilinear differential inclusions of the type $x' \in A(t)x + F(t,x)$, for $t \in [0,b]$, where $\{A(t)\}_{t\in[0,b]}$ is a family of linear operators in a Banach space E and $F\colon [0,b]\times E \to \mathcal{P}(E)$ is a lower Scorza–Dragoni multifunction.

Our results extend in a broad sense the analogous ones in [3] even in the setting of multifunctions which take values on $\mathcal{P}_{fc}(E)$, as shown by Remark 3.9.

In the second part, we continue the study of the above semilinear differential inclusion and we achieve both the existence of global mild solutions and the existence of mild solutions in the impulsive case. In Remark 4.5 we compare these results with the ones in [4].

According to the aim of obtaining our existence results we provide, in a preliminary way, a new Carathèodory selection theorem for lower Scorza–Dragoni multifunctions in abstract spaces (see Theorem 3.1). We recall that from paper [8] an important number of works has been devoted to the existence of such selections. From the others we mention the paper [1], where the authors furnish an interesting answer to the open problem posed in [13] (cf. [1, Theorem 3.2]).

Our selection theorem is a generalization to a larger class of multifunctions of the mentioned result proved in [1] (see Remark 3.2).

2. Preliminaries

Let $(Y,\mathcal{T})$ be a topological space or a linear topological space.

We will use the following notations:

$$\begin{aligned}
\mathcal{P}(Y) &= \{H \subset Y : H \neq \emptyset\};\\
\mathcal{P}_c(Y) &= \{H \in \mathcal{P}(Y) : H \text{ convex}\};\\
\mathcal{P}_f(Y) &= \{H \in \mathcal{P}(Y) : H \text{ closed}\};\\
\mathcal{P}_k(Y) &= \{H \in \mathcal{P}(Y) : H \text{ compact}\};\\
\mathcal{P}_{fc}(Y) &= \mathcal{P}_f(Y) \cap \mathcal{P}_c(Y); \quad \text{etc.}
\end{aligned}$$

Moreover, it will be useful to consider the following family introduced by Michael in [14]

$$\mathcal{D}(Y) = \{H \in \mathcal{P}_c(Y) : H \supset I(\overline{H})\} \tag{2.1}$$

where $I(\overline{H}) = \{x \in \overline{H} : x \notin S,\ S \text{ supporting set for } \overline{H}\}$ and a *supporting set* for $\overline{H}$ is a proper closed and convex subset S of the closed and convex set $\overline{H}$ which satisfies the property:

- for every segment $[x_1, x_2] \subset \overline{H}$ such that $]x_1, x_2[\cap S \neq \emptyset$, then the whole segment $[x_1, x_2]$ is contained in S.

For this family the following chain inclusion holds:

$$\mathcal{P}_{fc}(Y) \subset \mathcal{D}(Y) \subset \mathcal{P}_c(Y) .$$

Let $(X, \mathcal{S}_X)$ be a measurable space; a multifunction $F: X \to \mathcal{P}(Y)$ is *measurable* if $F^-(A) \in \mathcal{S}_X$, $A \in \mathcal{T}$, where $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$.

Let $(X, \mathcal{S}_X, \mu)$ and (Y, d) be a measure space and a metric space respectively; a multifunction $F: X \to \mathcal{P}_k(Y)$ is said to be *strongly measurable* if there exists a sequence of step multifunctions $(F_n)_{n\in\mathbb{N}}$, $F_n: X \to \mathcal{P}_k(Y)$, such that

$$h(F_n(x), F(x)) \xrightarrow{n\to\infty} 0, \quad \mu\text{-a.e. in } X$$

where h is the Hausdorff metric.

If X and Y are Hausdorff topological spaces, we introduce the following definitions for multifunctions $F : X \to \mathcal{P}(Y)$ (see e.g. [11], [12]).

A multifunction $F: X \to \mathcal{P}(Y)$ is said to be:

- *upper semicontinuous* at $x_0 \in X$ if, for every open set $\Omega \subseteq Y$ with $F(x_0) \subseteq \Omega$, there exists a neighbourhood V of x_0 such that $F(x) \subseteq \Omega$ for every $x \in V$;
- *lower semicontinuous* at $x_0 \in X$ if, for every open set $\Omega \subseteq Y$ with $F(x_0) \cap \Omega \neq \emptyset$, there exists a neighbourhood V of x_0 such that $F(x) \cap \Omega \neq \emptyset$ for every $x \in V$.

Now, let T be a Hausdorff topological space, $\mathcal{B}(T)$ be the Borel σ-algebra on T and μ a Radon measure on T (i.e. $\mu: \mathcal{B}(T) \to \mathbb{R}_0^+$ for which $\mu(C) = sup\{\mu(K) : K \subset C,\ K \text{ compact}\}$); we recall the following property for multifunctions which will play an essential role throughout the paper (see [1]).

A multifunction $F: T \times X \to \mathcal{P}(Y)$ verifies the *lower Scorza–Dragoni property* if

(l-SD) for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset T$ such that $\mu(T \setminus K_\varepsilon) < \varepsilon$ and $F_{|K_\varepsilon \times X}$ is lower semicontinuous.

Moreover, we will use a property introduced in [1], that we present it in the form which follows.

A multifunction $F: T \times X \to \mathcal{P}(Y)$ is a *Michael map* if it verifies property

(M) for every closed set $Z \subset T \times X$ such that $F_{|Z}$ is lower semicontinuous, there exists a continuous selection of F on Z (i.e. there exists a continuous function $f: Z \to Y$ such that $f(t, x) \in F(t, x)$, $(t, x) \in Z$).

Moreover, we recall that a multifunction $F: T \times X \to \mathcal{P}(Y)$ has a *Carathèodory selection* (see e.g. [1]) if there exists a function $f: T \times X \to Y$ such that

(i) for every $t \in T$, $f(t, \cdot)$ is continuous on X;

(ii) for every $x \in X$, $f(\,\cdot\,, x)$ is measurable;
(iii) for μ-a.e. $t \in T$ and every $x \in X$, $f(t,x) \in F(t,x)$.

If Y is a normed space, a multifunction $F: T \times X \to \mathcal{P}(Y)$ is said to be

- (DK)-*almost lower semicontinuous* at $(t_0, x_0) \in T \times X$ if for every $\varepsilon > 0$ there exists a neighbourhood U of (t_0, x_0) such that $\bigcap_{(t,x)\in U}\{F(t,x) + \varepsilon B_1(0)\} \neq \emptyset$, where $B_1(0)$ is the open unit ball in the space Y;
- *weak lower semicontinuous* at $(t_0, x_0) \in T \times X$ if for every $\varepsilon > 0$ and every neighbourhood V of (t_0, x_0) there is a point $(t', x') \in V$ such that for every $z \in F(t', x')$ there is a neighbourhood U_z of (t_0, x_0) such that $z \in \bigcap_{(t,x)\in U_z}\{F(t,x) + \varepsilon B_1(0)\}$.

From now on, E will denote a Banach space and $[a,b]$ will denote an interval of the real line endowed with the usual Lebesgue measure.

By the symbol $C([a,b];E)$ ($L^1([a,b];E)$) we will denote the space of all the continuous (Bochner integrable) functions $x: [a,b] \to E$; for the sake of simplicity, we will put $L^1_+([a,b])$ instead of $L^1([a,b];\mathbb{R}^+)$.

Given the multifunction $G: [a,b] \to \mathcal{P}(E)$, we will consider the set $\mathcal{S}^1_G = \{g \in L^1([a,b];E) : g(t) \in G(t), \text{a.e. } t \in [a,b]\}$.

Fixed $x_0 \in E$, we will deal with the following semilinear Cauchy problem

$$\text{(P)} \qquad \begin{cases} x' \in A(t)x + F(t,x) & \text{for } t \in [0,b], \\ x(0) = x_0 \end{cases}$$

where $F: [0,b] \times E \to \mathcal{P}(E)$ is a given multifunction.

On the family $\{A(t)\}_{t\in[0,b]}$ we will consider the following assumption

(A) $\{A(t)\}_{t\in[0,b]}$ is a family of linear operators $A(t): D(A) \subseteq E \to E$, with $D(A)$ not depending on t and dense in E, generating an *evolution system* $\{T(t,s)\}_{(t,s)\in\Delta}$, where $\Delta = \{(t,s) : 0 \leq s \leq t \leq b\}$.

Let us recall that a two parameter family $\{T(t,s)\}_{(t,s)\in\Delta}$, $T(t,s): E \to E$ bounded linear operator, is an *evolution system* if

(j) $T(t,t) = I$, $t \in [0,b]$; $T(t,r)T(r,s) = T(t,s)$, $0 \leq s \leq r \leq t \leq b$;
(jj) $(t,s) \mapsto T(t,s)$ is strongly continuous on Δ (i.e. the map $(t,s) \mapsto T(t,s)x$ is continuous on Δ, for every $x \in E$).

A function $x \in C([0,h];E)$, $0 < h \leq b$, is said to be a *mild solution for* (P) if

$$x(t) = T(t,0)x_0 + \int_0^t T(t,s)f(s)\,ds, \quad t \in [0,h]$$

where $f \in \mathcal{S}^1_{F(\,\cdot\,,x(\,\cdot\,))}$.

Moreover, let us consider the following impulsive Cauchy problem, governed by the same semilinear differential inclusion.

$$\text{(IP)}\qquad \begin{cases} x' \in A(t)x + F(t,x) & \text{for } t \in [0,b] \setminus \{t_1,\dots,t_m\},\\ x(t_k^+) = x(t_k) + I_k(x(t_k)) & \text{for } k = 1,\dots,m,\\ x(0) = x_0, \end{cases}$$

where $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$; $I_k\colon E \to E$, $k = 1,\dots,m$ are given functions and $x(t^+) = \lim_{s\to t^+} x(s)$.

In order to define a mild solution for (IP), we introduce the intervals $J_0 = [0,t_1]$, $J_k =]t_k, t_{k+1}]$, $k = 1,\dots,m$, and the set

$$C_p([0,b];E) = \{x\colon [0,b] \to E \text{ such that } x_{|J_k} \in C(J_k,E), k = 0,\dots,m,\\ \text{there exists } x(t_k^+) \in E,\ k = 1,\cdots,m\}.$$

It is easy to check that $(C_p([0,b];E), \|\cdot\|_{C_p})$ is a Banach space, endowed with the norm

$$\|x\|_{C_p} = \max\{\|x^k\|_{C(\overline{J_k};E)},\ k = 0\dots,m\}$$

where $x^0 = x_{|J_0}$ and, for $k \in \{1,\dots,m\}$, x^k is the function defined as

$$x^k(t) = \begin{cases} x(t) & \text{for } t \in J_k,\\ x(t_k^+) & \text{for } t = t_k. \end{cases}$$

A function $x \in C_p([0,b];E)$ is said to be a *mild solution for* (IP) if

$$x(t) = T(t,0)x_0 + \sum_{0<t_k<t} T(t,t_k)I_k(x(t_k)) + \int_0^t T(t,s)f(s)\,ds, \qquad t \in [0,b]$$

where $f \in \mathcal{S}^1_{F(\,\cdot\,,x(\,\cdot\,))}$.

In the sequel, on the multifunction $F\colon [0,b] \times E \to \mathcal{P}(E)$ which appears in (P) and (IP), we will use case by case some of the properties

(F1) for every bounded $\Omega \subset E$ there exists a function $\mu_\Omega \in L^1_+([0,b])$ such that, for every $x \in \Omega$,

$$\|F(t,x)\| \le \mu_\Omega(t), \qquad \text{a.e. } t \in [0,b];$$

(F1)$'$ there exists a function $\alpha \in L^1_+([0,b])$ such that, for every $x \in E$,

$$\|F(t,x)\| \le \alpha(t)(1 + \|x\|), \quad \text{a.e. } t \in [0,b];$$

(F2) there exists a function $k \in L^1_+([0,b])$ such that, for every bounded $D \subset E$,

$$\chi(F(t,D)) \le k(t)\chi(D), \quad \text{a.e. } t \in [0,b],$$

where χ is the Hausdorff measure of noncompactness (see e.g. [12]).

3. Existence of Carathèodory selections and of local mild solutions for (P)

The following selection theorem will play a crucial role in this paper.

Theorem 3.1. *Let T, X and Y be Hausdorff topological spaces and μ be Radon measure on T. If $F: T\times X \to \mathcal{P}(Y)$ is a multifunction satisfying properties* (l-SD) *and* (M)*, then F has a Carathèodory selection.*

Proof. Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. From property (l-SD) on F we deduce that for every $j \in \mathbb{N}$ there exists a compact $K_j \subset T$ such that

$$\mu(T \setminus K_j) < \varepsilon_j \tag{3.1}$$

and F is lower semicontinuous on $K_j \times X$.

By (M), for each closed set $K_j \times X$ we claim that there exists a continuous function $f_j: K_j \times X \to Y$ such that

$$f_j(t,x) \in F(t,x), \quad (t,x) \in K_j \times X. \tag{3.2}$$

Put $K = \bigcup_{j\in\mathbb{N}} K_j$, we denote

$$M_1 = K_1, \quad M_j = K_j \setminus \bigcup_{i<j} K_i, \quad j = 2,3,\ldots$$

Obviously $M_i \cap M_j = \emptyset$ for $i \neq j$ and

$$K = \bigcup_{j\in\mathbb{N}} M_j. \tag{3.3}$$

Now, fixed $\overline{y} \in Y$, let us denote by f the function $f: T \times X \to Y$ defined by

$$f(t,x) = \begin{cases} f_j(t,x) & \text{for } (t,x) \in M_j \times X,\ j \in \mathbb{N}, \\ \overline{y} & \text{otherwise.} \end{cases}$$

We prove that f is a Carathèodory selection of F.

First of all, it is easy to see that for every $t \in T$ the function $f(t,\cdot\,)$ is continuous on X. Next, we go to show that, fixed $x \in X$, the function $f(\,\cdot\,,x)$ is measurable on T.

Given an open set $A \subset Y$, we have

$$f^-(A,x) = [f^-(A,x) \cap K] \cup [f^-(A,x) \cap K^c],$$

where $K^c = T \setminus K$. Let us observe that the set $f^-(A,x) \cap K^c$ is equal to K^c if $\overline{y} \in A$, while it is the empty set if $\overline{y} \notin A$. So, in both cases, the set $f^-(A,x)\cap K^c$ is measurable.

On the other hand, from (3.3) and taking into account the definition of f, it follows that

$$f^-(A,x) \cap K = \bigcup_{j\in\mathbb{N}} [f_j^-(A,x) \cap M_j]. \tag{3.4}$$

Now, $f_j^-(A,x)$ is open in K_j; from the measurability of K_j it is easily deducted the measurability of $f_j^-(A,x)$. Moreover, M_j is measurable and so, by (3.4), $f^-(A,x) \cap K$ is measurable too.

Finally, taking into account (3.2) and (3.3), we can state that

$$f(t,x) \in F(t,x), \quad (t,x) \in K \times X.$$

Now, μ is monotone and (3.1) holds; then

$$0 \le \mu(K^c) \le \mu(T \setminus K_j) < \varepsilon_j, \quad j \in \mathbb{N},$$

hence $\mu(K^c) = 0$. Therefore, we can conclude that f is a Carathèodory selection of the multifunction F. □

Remark 3.2. Let us notice that our Theorem 3.1 strictly contains the selection theorem proved by Artstein and Prikry ([1, Theorem 3.2]). In fact, in that theorem the authors consider measurable multifunctions lower semicontinuous with respect to the second variable; by Theorem 2.1 in [1] those multifunctions verify the hypotheses of our selection theorem. Moreover, our framework is more general than the one in [1].

Anyway, even if we consider the more restrictive framework of [1], we can provide an example of a multifunction satisfying all the assumptions of our selection theorem, but which does not verify anyone of the hypotheses required on the multifunction of Theorem 3.2 in [1].

Example 3.3. We consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$, where λ is the Lebesgue measure, and (cf. [9, Esempio 20]) a non-measurable set A, $\overline{A} = [0,1] \times [0,1]$, such that

$$\begin{aligned} \operatorname{card}\{x \in [0,1] : (t,x) \in A\} &= 1 \quad \text{for } t \in [0,1], \\ \operatorname{card}\{t \in [0,1] : (t,x) \in A\} &= 1 \quad \text{for } x \in [0,1], \\ (t,t) &\in A \quad \text{for } t \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}). \end{aligned}$$

Let $F\colon [0,1] \times [0,1] \to \mathcal{P}([0,1])$ be the multifunction defined by

$$F(t,x) = \begin{cases} [0,1] & \text{for } (t,x) = (0,0), \\ \{0\} & \text{for } (t,x) \in A \setminus \{(0,0)\}, \\ [0,1/2] & \text{otherwise.} \end{cases}$$

First of all, it is easy to check that F verifies both properties (l-SD) and (M). On the other hand, F does not satisfy the whole set of assumptions of Theorem 3.2

in [1]; in fact the multifunction $F(0,\,\cdot\,)$ is not lower semicontinuous at $x = 0$ and F is not measurable on $[0,1] \times [0,1]$.

Thanks to previous theorem, we state and prove the following two results on the existence of local mild solutions for (P).

Theorem 3.4. *Let E be a separable Banach space. We suppose that family $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and that $F\colon[0,b]\times E \to \mathcal{P}(E)$ verifies properties* (l-SD)*,* (M)*,* (F1) *and* (F2)*. Then there exists at least one local mild solution for* (P)*.*

Proof. From Theorem 3.1 the multifunction F has a Carathèodory selection $f\colon[0,b]\times E \to E$.

Let us consider the multifunction $G\colon[0,b]\times E \to \mathcal{P}_{kc}(E)$ defined as

$$G(t,x) = \{f(t,x)\}, \quad (t,x) \in [0,b] \times E \tag{3.5}$$

and the associated semilinear Cauchy problem

$$\begin{cases} x' \in A(t)x + G(t,x) & \text{for } t \in [0,b], \\ x(0) = x_0. \end{cases} \tag{3.6}$$

Let us show that G fulfills all the hypotheses of Theorem 3 in [5].

For each $x \in E$, being $G(\,\cdot\,,x)$ measurable with compact values in the separable Banach space E, from Theorem 1.3.1 in [12] we have that $G(\,\cdot\,,x)$ is strongly measurable. Moreover, for each $t \in [0,b]$, $f(t,\,\cdot\,)$ is continuous and so $G(t,\,\cdot\,)$ is upper semicontinuous in E.

Now we prove that G fulfills (F1). Fixed a bounded set $\Omega \subset E$, by (F1) on F, for every $x \in \Omega$, we get

$$\|G(t,x)\| \le \|F(t,x)\| \le \mu_\Omega(t), \quad \text{a.e. } t \in [0,b].$$

Finally, let k be the function presented in (F2) relatively to F. Then, for every bounded set $D \subset E$, taking into account the monotonicity of the measure of noncompactness χ, we obtain the following inequality:

$$\chi(G(t,D)) \le \chi(F(t,D)) \le k(t)\chi(D), \quad \text{a.e. } t \in [0,b]$$

which allows us to conclude that also G satisfies (F2).

By applying Theorem 3 in [5], the existence of at least one local mild solution for problem (3.6) is deduced, and this solutions is a local mild solution for (P) too. □

Theorem 3.5. *Let E be a finite dimensional Banach space. We suppose that $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and that $F\colon[0,b]\times E \to \mathcal{P}(E)$ verifies* *properties* (l-SD)*,* (M) *and* (F1)*. Then there exists at least one local mild solution for* (P)*.*

Proof. By proceeding as in Theorem 3.4, we can say that there exists a Carathèodory selection f of F. Then we can consider the multifunction G defined by (3.5), which is strongly measurable in the first variable, upper semi-continuous in the second one and has property (F1).

Now, let D be a bounded set of E. Since D and $f(t, D)$ are relatively compact, by means of (3.5) we can write

$$\chi(G(t, D)) = \chi(f(t, D)) = 0 = \chi(D), \quad t \in [0, b].$$

Hence we can conclude that G verifies (F2).

Therefore, by Theorem 3 of [5], we deduce the existence of at least one local mild solution for problem in (3.6), and so for (P) as well. □

Restricting our considerations to multifunctions which assume values in the family $\mathcal{D}(E)$ (see (2.1)), from Theorem 3.4 we can deduce the following result.

Corollary 3.6. *Let E be a separable Banach space. We suppose that $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and that $F\colon [0, b] \times E \to \mathcal{D}(E)$ verifies properties* (l-SD)*,* (F1) *and* (F2)*. Then there exists at least one local mild solution for* (P)*.*

Proof. First of all, let us consider a closed set $Z \subset [0, b] \times E$ such that the multifunction $F_{|Z}$ is lower semicontinuous. The multifunction $F_{|Z}$ satisfies all the assumptions of Theorem 3.1′′′ in [14], and so it has a continuous selection. Hence F verifies (M). Then we can apply Theorem 3.4 and the proof is concluded. □

Now, as a consequence of Theorem 3.5, we are in position to provide the following result.

Corollary 3.7. *Let E be a finite dimensional Banach space. We suppose that $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and that $F\colon [0, b] \times E \to \mathcal{P}_c(E)$ verifies properties* (l-SD) *and* (F1)*. Then there exists at least one local mild solution for* (P)*.*

Proof. First of all, let us note that in the case when the Banach space is finite dimensional, one has $\mathcal{P}_c(E) = \mathcal{D}(E)$ (see [14, p. 372]). Then, by following the outline of the proof of Corollary 3.6 and by applying Theorem 3.5 instead of Theorem 3.4, we achieve the thesis. □

Remark 3.8. We note that by requiring in Theorems 3.4 and 3.5 and in Corollaries 3.6 and 3.7 the lower semicontinuity in the second variable and the global measurability instead of property (l-SD), thanks to Theorem 2.1 in [1] we obtain results which are analogous to others existing in the literature for upper

Carathèodory-type multifunctions (see e.g. [5]). Whereas, it is not possible to proceed in the same way if one assumes on the multifunction the lower semicontinuity in the second variable and the measurability only in the first variable. In fact, these two assumptions do not guarantee property (l-SD) nor the existence of a Carathèodory selection for the multifunction (see [1, §4]).

Remark 3.9. Our local existence results extend in a broad sense the theorems proved in [3]. This claim is a consequence of the fact that $\mathcal{D}(E)$ strictly contains $\mathcal{P}_{fc}(E)$. In fact, it is obvious if E is a finite dimensional space since $\mathcal{D}(E) = \mathcal{P}_c(E)$, as observed above. Whereas, if E is infinite dimensional, we prove the strict inclusion by means of the following example.

Example 3.10. Let $E = C([a,b];\mathbb{R})$. The set $\Gamma = \{x \in C([a,b];\mathbb{R}) : x(s) = ms,\ m > 0\}$ is a one dimensional linear subspace of $C([a,b];\mathbb{R})$. Therefore Γ is an element of $\mathcal{D}(C([a,b];\mathbb{R}))$ (see [14, §5]). On the other hand, of course, Γ is not closed.

Anyway, even in the setting of multifunctions taking values in $\mathcal{P}_{fc}(E)$, we can say that there exist multifunctions which verify the hypotheses of our existence results but do not satisfy the whole set of assumptions required in Theorem 3.1 or 3.2 or 3.3 in [3]. This fact is obvious for Theorem 3.1 in [3] since there the values of the multifunction are compact too. In the following example we show a multifunction having all the properties required in our propositions but not (DK)-almost lower semicontinuous (see §2), hypothesis assumed in Theorem 3.3 of [3]; by means of Remark 3.5 in [4], this multifunction is neither weak lower semicontinuous (see §2), hypothesis assumed in Theorem 3.2 of [3].

Example 3.11. Let $F\colon [0,1] \times \mathbb{R} \to \mathcal{P}_{fc}(\mathbb{R})$ be defined as

$$F(t,x) = \begin{cases} \{1\} & \text{if } (t,x) = (1/2,1), \\ \{0\} & \text{otherwise.} \end{cases}$$

It is easy to verify that F satisfies all the properties required in our existence results.

On the other hand, fixed $(1/2,1) \in [0,1] \times \mathbb{R}$, in correspondence to $\overline{\varepsilon} = 1/3$ we have that for every neighbourhood U of the point $(1/2,1)$ it is

$$\bigcap_{(t,x)\in U} \{F(t,x) + \overline{\varepsilon} B_1(0)\} = \left]\frac{2}{3}, \frac{4}{3}\right[\cap \left]-\frac{1}{3}, \frac{1}{3}\right[= \emptyset.$$

Therefore, F is not (DK)-almost lower semicontinuous.

4. Existence of global mild solutions for (P) and applications to the impulsive problem (IP)

In this section, we strengthen (F1) by means of (F1)$'$ so that we are able to prove the following results.

THEOREM 4.1. *Let E be a separable Banach space. We assume that family $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and that $F\colon[0,b]\times E\to\mathcal{P}(E)$ verifies properties* (l-SD)*,* (M)*,* (F1)$'$ *and* (F2)*. Then*

(a) *there exists at least one global mild solution for* (P)*;*
(b) *there exists at least one mild solution for* (IP)*.*

PROOF. First we observe that, by proceeding as in Theorem 3.4, we can define the multifunction $G\colon[0,b]\times E\to\mathcal{P}_{kc}(E)$ as in (3.5). From (F1)$'$ on F we can write

$$\|G(t,x)\|\le\alpha(t)(1+\|x\|),\quad \text{a.e. } t\in[0,b],$$

so we have that G verifies (F1)$'$. On the other hand, G satisfies also the other assumptions required on the multifunction of Theorem 4 in [5]. Then, by applying this proposition, we can say that there exists at least one global mild solution for problem (3.6); this is a solution for (P) too. Thus the first thesis is proved.

Now, in correspondence to G, we consider the impulsive problem

$$\begin{cases} x'\in A(t)x+G(t,x) & \text{for } t\in[0,b]\setminus\{t_1,\dots,t_m\},\\ x(t_k^+)=x(t_k)+I_k(x(t_k)) & \text{for } k=1,\dots,m,\\ x(0)=x_0. \end{cases}\tag{4.1}$$

From Theorem 2 in [6], it admits at least one mild solution, which obviously solves (IP) as well. □

THEOREM 4.2. *Let E be a finite dimensional Banach space. We assume that $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and that $F\colon[0,b]\times E\to\mathcal{P}(E)$ verifies properties* (l-SD)*,* (M) *and* (F1)$'$*. Then*

(a) *there exists at least one global mild solution for* (P)*;*
(b) *there exists at least one mild solution for* (IP)*.*

PROOF. By proceeding as in Theorem 3.5, we apply Theorem 4 in [5] and Theorem 2 in [6] to problems (3.6) and (4.1) respectively, where G is the same multifunction used above. Hence we achieve both the theses. □

In analogy with the previous section, we can now formulate the following corollaries.

Corollary 4.3. *Let E be a separable Banach space. If $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and $F\colon[0,b]\times E\to\mathcal{D}(E)$ verifies properties* (l-SD), (F1)$'$ *and* (F2), *then*

(a) *there exists at least one global mild solution for* (P)*;*
(b) *there exists at least one mild solution for* (IP).

Corollary 4.4. *Let E be a finite dimensional Banach space. If $\{A(t)\}_{t\in[0,b]}$ satisfies property* (A) *and $F\colon[0,b]\times E\to\mathcal{P}_c(E)$ verifies properties* (l-SD) *and* (F1)$'$*, then*

(a) *there exists at least one global mild solution for* (P)*;*
(b) *there exists at least one mild solution for* (IP).

Remark 4.5. The same reasonings developed in Remark 3.9 allow us to claim that the results presented in this section extend in a broad sense the analogous ones in [4].

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Tiziana Cardinali
Department of Mathematics and Computer Science
University of Perugia
Via L. Vanvitelli 1
06123 Perugia, ITALY

E-mail address: tiziana@dipmat.unipg.it

Francesco Portigiani
Department of Pure and Applied Mathematics
University of Modena and Reggio Emilia
Via Campi, 213
41100 Modena, ITALY

E-mail address: francesco.portigiani@unimore.it

Paola Rubbioni
INFM and Department of Mathematics and Computer Science
University of Perugia
Via L. Vanvitelli 1
06123 Perugia, ITALY

E-mail address: rubbioni@dipmat.unipg.it