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A PROOF OF THE CONTINUATION PROPERTY OF THE CONLEY INDEX OVER A PHASE SPACE

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Abstract. We prove the continuation property of the Conley index over a phase space for discrete semidynamical systems.

1. Introduction

In [4] the so-called Conley index over a base was defined for an isolated invariant set and a flow. Its analogue for more complicated discrete semidynamical systems was defined in [5]. Its main advantage over the previously defined indices is that it detects how an isolated invariant set is situated in a phase space of a given system. Some applications of this index have been presented in [6].

Theorem 5.3 in [5] shows a continuation property of the index which is fundamental for applications. The theorem is followed by a proof. However, the proof is based on Theorem 3.2 from [2] which shows the existence of an index pair which is stable under small perturbations of the system. The problem is that the authors of [2] base on a slightly different definition of an index pair than the one from [7] and [5]. In [2] an index pair is, in fact, a triple including an isolating neighbourhood. If we assume that the first element of an index pair is equal to an isolating neighbourhood, then each index pair in the sense of definition from [2] is also an index pair in the sense of definition from [7]. Unfortunately,

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we cannot be sure that the triple whose existence is proved in [2] satisfies this assumption.

Therefore, it seems reasonable to present an alternative proof of the continuation property, which is the aim of this paper.

In Sections 2, 3, and 4 we recall some definitions from [7] and [5]. In the last section we present a proof of the continuation property.

2. Isolated invariant sets

Let X be a locally compact metric space, $f\colon X \to X$ — a continuous map which generates a semidynamical system.

For an arbitrary set $N \subseteq X$ we define the set

$$\operatorname{Inv} N = \{x \in N : \exists \{x_k\}_{k\in\mathbb{Z}} \subseteq N,\ x_0 = x \text{ and } f(x_k) = x_{k+1} \text{ for } k \in \mathbb{Z}\},$$

which will be called an invariant part of N. A set $N \subseteq X$ is called an invariant set when $N = \operatorname{Inv} N$. A compact set $N \subseteq X$ is called an isolating neighborhood for $S := \operatorname{Inv} N$ if $S \subseteq \operatorname{int}(N)$. The set S is called an isolated invariant set.

Fix an isolated invariant set S.

DEFINITION 2.1. A pair $P = (P_1, P_2)$ of compact subsets of X, is called an *index pair for S* if and only if

(a) $S = \operatorname{Inv} \operatorname{cl}(P_1 \setminus P_2) \subseteq \operatorname{int}(P_1 \setminus P_2)$,
(b) $f(P_2) \cap P_1 \subseteq P_2$,
(c) $f(P_1 \setminus P_2) \subseteq P_1$.

An immediate consequence of the above definition is

LEMMA 2.2. *If P is an index pair for S, then:*

(a) *if $x \in P_1$ and there exists $k \in \mathbb{N}$ such that $f^k(x) \notin P_1 \setminus P_2$, then there exists $l \in \mathbb{N}$ such that $l \leq k$ and $f^l(x) \in P_2$,*
(b) *if $x \in P_2$ and there exists $k \in \mathbb{N}$ such that $f^1(x), \dots, f^k(x) \in P_1$, then $f^1(x), \dots, f^k(x) \in P_2$.*

Let $\Lambda \subseteq \mathbb{R}$ be a compact segment, $h\colon X \times \Lambda \to X \times \Lambda$ a continuous map satisfying $h(X \times \lambda) \subseteq X \times \lambda$, for each $\lambda \in \Lambda$. For $\lambda \in \Lambda$ we define a map $h_\lambda\colon X \to X$, satisfying $h(x, \lambda) = (h_\lambda(x), \lambda)$, for each $x \in X$ and $\lambda \in \Lambda$. By π_X we denote a projection $\pi_X\colon X \times \Lambda \to X$. For a given set $R \subset X \times \Lambda$ and $\lambda \in \Lambda$, by R_λ we denote the set $\{x \in X : (x, \lambda) \in R\}$.

Now we give a lemma which follows immediately from Lemma 4.2 from [7].

LEMMA 2.3. *If Q is an isolated invariant set for h, (P_1, P_2) is an index pair for Q, then there exists $\varepsilon > 0$ and $n \in \mathbb{N}$ such that the following implication* *holds: if*

$$\alpha_1, \dots, \alpha_{2n}, \beta_1, \dots, \beta_{n+1}, \gamma, \delta_1, \dots, \delta_{2n} \in \Lambda,$$
$$\operatorname{diam}\{\alpha_1, \dots, \alpha_{2n}, \beta_1, \dots, \beta_{n+1}, \gamma, \delta_1, \dots, \delta_{2n}\} < \varepsilon,$$
$$x \in \pi_X(P_1),$$
$$((h_{\alpha_i} \circ \dots \circ h_{\alpha_1})(x), \delta_i) \in P_1 \setminus P_2 \quad \textit{for all } i \in \{1, \dots, 2n\}$$

then

$$((h_{\beta_n} \circ \dots \circ h_{\beta_1})(x), \gamma), ((h_{\beta_{n+1}} \circ \dots \circ h_{\beta_1})(x), \gamma) \in P_1 \setminus P_2.$$

Assume $f, g: X \to X$ are continuous maps, S and T are isolated invariant sets for f and g, respectively. We say that S and T are related by continuation ($(f, S) \simeq (g, T)$), if there exists a continuous map $h: X \times [0,1] \to X \times [0,1]$, an isolated invariant set Q for h, and maps $h_\lambda: X \to X$, $\lambda \in [0,1]$, such that $h_0 = f$, $h_1 = g$, $S = Q_0$, $T = Q_1$, $h(x, \lambda) = (h_\lambda(x), \lambda) \in X \times [0,1]$, for all $x \in X$ and $\lambda \in [0,1]$.

3. M-equivalence

For a given topological space X we define the category of spaces over a base X (objects and morphisms), denoted by $\mathcal{S}B(X)$.

Definition 3.1.

$$\begin{aligned}\operatorname{Ob}(\mathcal{S}B(X)) = \{(U, r, s) :\ & U \text{ is a topological space},\\ & r: U \to X,\ s: X \to U \text{ are continuous, such that } r \circ s = \mathrm{id}_X\},\end{aligned}$$

$$\begin{aligned}\operatorname{Mor}_{\mathcal{S}B(X)}((U, r, s), (U', r', s')) = \{(F, f):\ & F: U \to U',\ f: X \to X\\ & \text{are continuous, such that } F \circ s = s' \circ f \text{ and } r' \circ F = f \circ r\}.\end{aligned}$$

For two morphisms in $\mathcal{S}B(X)$ we define the relation $\simeq_*$ of homotopy:

Definition 3.2. Let $(F, f), (F', f') \in \operatorname{Mor}_{\mathcal{S}B(X)}((U, r, s), (U', r', s'))$. $(F, f) \simeq_* (F', f')$ if and only if there exists continuous $H: U \times \mathbb{I} \to U'$ and $h: X \times \mathbb{I} \to X$ such that

$$H \circ (s \times \mathrm{id}_\mathbb{I}) = s' \circ h, \quad r' \circ H = h \circ (r \times \mathrm{id}_\mathbb{I}),$$
$$H(\,\cdot\,, 0) = F, \quad H(\,\cdot\,, 1) = F', \quad h(\,\cdot\,, 0) = f, \quad h(\,\cdot\,, 1) = f'.$$

A pair (H, h) will be called a *homotopy joining* (F, f) *with* (F', f').

According to [3] we define the category $\mathit{ENDO}(\mathcal{K})$ of endomorphisms over the category $\mathcal{K}$ as follows:

$$\begin{aligned}\operatorname{Ob}(\mathit{ENDO}(\mathcal{K})) &= \{(X, e) : X \in \operatorname{Ob}(\mathcal{K}) \text{ and } e \in \operatorname{Mor}_\mathcal{K}(X, X)\},\\ \operatorname{Mor}_{\mathit{ENDO}(\mathcal{K})}((X, e), (X', e')) &= \{\varphi \in \operatorname{Mor}_\mathcal{K}(X, X') : \varphi \circ e = e' \circ \varphi\}.\end{aligned}$$

Fix $(U,r,s),(U',r',s') \in \mathrm{Ob}(\mathcal{S}B(X))$ and two morphisms $(F,f) \in \mathrm{Mor}_{\mathcal{S}B(X)}((U,r,s),(U,r,s))$ and $(F',f') \in \mathrm{Mor}_{\mathcal{S}B(X)}((U',r',s'),(U',r',s'))$.

Definition 3.3. Two objects $((U,r,s),(F,f))$ and $((U',r',s'),(F',f'))$ in $ENDO(\mathcal{S}B(X))$ are *M-equivalent over a base* X, if $f \simeq f'$ and there exist $m,n \in \mathbb{N}$ and continuous maps $\Phi: U \to U'$, $\Psi: U' \to U$, $\varphi,\psi: X \to X$, such that $\varphi \simeq f^m$, $\psi \simeq f'^n$ and there exists a $k \in \mathbb{N}$ such that

$$\Phi \circ s = s' \circ \varphi, \tag{3.1}$$
$$\Psi \circ s' = s \circ \psi, \tag{3.2}$$
$$r' \circ \Phi = \varphi \circ r, \tag{3.3}$$
$$r \circ \Psi = \psi \circ r', \tag{3.4}$$
$$(\Phi \circ F, \varphi \circ f) \simeq_* (F' \circ \Phi, f' \circ \varphi), \tag{3.5}$$
$$(\Psi \circ F', \psi \circ f') \simeq_* (F \circ \Psi, f \circ \psi), \tag{3.6}$$
$$(\Psi \circ \Phi \circ F^k, \psi \circ \varphi \circ f^k) \simeq_* (F^{m+n+k}, f^{m+n+k}), \tag{3.7}$$
$$(\Phi \circ \Psi \circ F'^k, \varphi \circ \psi \circ f'^k) \simeq_* (F'^{m+n+k}, f'^{m+n+k}). \tag{3.8}$$

The class of M-equivalence of $((U,r,s),\ (F,f))$ over X will be denoted by $[((U,r,s),(F,f))]_X$.

4. The Conley index over a phase space

Fix a locally compact metric space X, a continuous map $f: X \to X$, an isolated invariant set S for f, and an index pair $P = (P_1, P_2)$ for S.

We define $U(P)$ as the adjunction $P_1 \cup_{\mathrm{id}|_{P_2}} X$, i.e.

$$U(P) := X \times 0 \cup P_1 \times 1/\sim,$$

where $\sim$ denotes the minimal equivalence relation such that $(x,0) \sim (x,1)$ for each $x \in P_2$. Let $[x,q]_P$ denotes the equivalence class of (x,q) in $U(P)$.

We also define two maps $s_P: X \ni x \mapsto [x,0]_P \in U(P)$ and $r_P: U(P) \ni [x,q]_P \mapsto x \in X$.

An index space over X is a triple $(U(P), r_P, s_P)$. An index map $f_P: U(P) \to U(P)$ is given by a formula:

$$f_P([x,q]_P) := \begin{cases} [f(x),1]_P & \text{for } q = 1,\ x, f(x) \in P_1 \setminus P_2, \\ [f(x),0]_P & \text{otherwise.} \end{cases}$$

Theorem 4.1. *For any index pairs $P, P' \in IP(S)$ objects $((U(P), r_P, s_P), (f_P, f))$ and $((U(P'), r_{P'}, s_{P'}), (f_{P'}, f))$ in $ENDO(\mathcal{S}B(X))$ are M-equivalent over a phase space X.*

Definition 4.2. The Conley index $\widehat{h}_d(S,f)$ of an isolated invariant set S over a phase space X is the M-equivalence class over X of the object $((U(P),r_P, s_P),(f_P,f))$ in $ENDO(\mathcal{S}B,(X))$, for any index pair P for S:

$$\widehat{h}_d(S,f)=[((U(P),r_P,s_P),(f_P,f))]_X.$$

5. The continuation property

Theorem 5.1 (Continuation property, [5, Theorem 5.3]). *Assume $f,g\colon X\to X$ are continuous maps, S and T are isolated invariant sets for f and g, respectively. Then*

$$\textit{if } (f,S)\simeq(g,T) \textit{ then } \widehat{h}_d(S,f)=\widehat{h}_d(T,g).$$

Proof. $(f,S)\simeq(g,T)$, so there exists a continuous map $h\colon X\times[0,1]\to X\times[0,1]$, an isolated invariant set Q for h, and maps $h_\lambda\colon X\to X$, $\lambda\in[0,1]$, such that $h_0=f$, $h_1=g$, $S=Q_0$, $T=Q_1$, $h(x,\lambda)=(h_\lambda(x),\lambda)\in X\times[0,1]$, for all $x\in X$ and $\lambda\in[0,1]$. Let $P=(P_1,P_2)$ be an index pair for Q.

For $\Lambda=[0,1]$ take $\varepsilon>0$ and $n\in\mathbb{N}$ from Lemma 2.3. There exists a finite sequence $\mu_0,\dots,\mu_{k+1}\in\Lambda$ such that $0=\mu_0<\mu_1<\dots<\mu_k<\mu_{k+1}=1$ and $\mu_{i+1}-\mu_i<\varepsilon$ for each $i\in\{0,\dots,k\}$. Thus, it is enough to prove that for an arbitrary $i\in\{0,\dots,k\}$ indices $\widehat{h}_d(Q_{\mu_i},h_{\mu_i})$ and $\widehat{h}_d(Q_{\mu_{i+1}},h_{\mu_{i+1}})$ are equal.

To simplify notation assume $a:=\mu_i$, $b:=\mu_{i+1}$, $P_a=(P_{1a},P_{2a})$ and $P_b=(P_{1b},P_{2b})$. P_a and P_b are, of course, index pairs for Q_a and Q_b, respectively.

Fix $u,x\in X$ and $q\in\{0,1\}$ and define maps

$$\Phi\colon U(P_a)\to U(P_b),\qquad \Psi\colon U(P_b)\to U(P_a),\qquad \varphi,\psi\colon X\to X$$

by formulas:

$$\Phi([u,q]_{P_a})=\begin{cases}[h_b^{2n}(h_a^n(u)),1]_{P_b} & \text{if } q=1,\ h^0(u,a),\dots,h^{2n}(u,a)\in P_1\setminus P_2\\ & \text{and } h^0(h_a^n(u),b),\dots,h^{2n}(h_a^n(u),b)\in P_1\setminus P_2,\\ [h_b^{2n}(h_a^n(u)),0]_{P_b} & \text{otherwise,}\end{cases}$$

$$\Psi([u,q]_{P_b})=\begin{cases}[h_a^{2n}(h_b^n(u)),1]_{P_a} & \text{if } q=1,\ h^0(u,b),\dots,h^{2n}(u,b)\in P_1\setminus P_2,\\ & \text{and } h^0(h_b^n(u),a),\dots,h^{2n}(h_b^n(u),a)\in P_1\setminus P_2,\\ [h_a^{2n}(h_b^n(u)),0]_{P_a} & \text{otherwise,}\end{cases}$$

$$\varphi(x)=h_b^{2n}(h_a^n(x)),\qquad \psi(x)=h_a^{2n}(h_b^n(x)).$$

The maps φ and ψ are obviously continuous. We will show that Φ is continuous (the proof of the continuity of Ψ is analogous).

Take $[u_0, q_0]_{P_a} \in U(P_a)$. When $q_0 = 0$, continuity of Φ is obvious, so we may assume that $q_0 = 1$ and $(u_0, a) \in P_1$. Consider two sets

$$\begin{aligned} A, := \{[u,q]_{P_a} \in U(P_a) : & \, h^0(u,a), \dots , h^{2n}(u,a) \notin P_2, \\ & \text{and } h^0(h_a^n(u), b), \dots , h^{2n}(h_a^n(u), b) \notin P_2\}, \\ B := \{[u,q]_{P_a} \in U(P_a) : & \text{ there exists } i \in \{0, \dots , 2n\} \\ & h^i(u,a) \notin P_1 \text{ or } h^i(h_a^n(u), b) \notin P_1\}. \end{aligned}$$

Obviously, A and B are open.

If $[u,q]_{P_a} \in B$ then $\Phi([u,q]_{P_a}) = [h_b^{2n}(h_a^n(u)), 0]_{P_b}$, so Φ is continuous on B.

If $[u,1]_{P_a} \in A$, then $h^0(u,a) \in P_1 \setminus P_2$ and $h^1(u,a), \dots , h^{2n}(u,a) \notin P_2$. Part (a) of Lemma 2.2 implies that $h^1(u,a), \dots , h^{2n}(u,a) \in P_1 \setminus P_2$. From Lemma 2.3 it follows that $h^0(h_a^n(u), b) \in P_1 \setminus P_2$ and again from part (a) of Lemma 2.2 we get $h^1(h_a^n(u), b) \in P_1 \setminus P_2, \dots , h^{2n}(h_a^n(u), b) \in P_1 \setminus P_2$, which means that

$$\Phi([u,1]_{P_a}) = [h^{2n}(h_a^n(u), b), 1]_{P_b},$$

therefore Φ is continuous on A.

Assume that $[u_0, 1]_{P_a} \in U(P_a) \setminus (A \cup B)$. $[u_0, 1]_{P_a} \notin B$ so $h^0(u,a), \dots , h^{2n}(u,a) \in P_1$ and $h^0(h_a^n(u), b), \dots , h^{2n}(h_a^n(u), b) \in P_1$. If $h^0(h_a^n(u), b), \dots , h^{2n}(h_a^n(u), b) \notin P_2$, then from Lemma 2.3 it would follow that $h^{2n}(u,a) \in P_1 \setminus P_2$ and from part (b) of Lemma 2.2 we would get $h^{2n-1}(u,a), \dots , h^0(u,a) \in P_1 \setminus P_2$, which would imply that $[u_0, 1]_{P_a} \in A$. Thus, there exists $i \in \{0, \dots , 2n\}$ such that $h^i(h_a^n(u), b) \in P_2$, and from part (b) of Lemma 2.2 it follows that $h^{2n}(h_a^n(u), b) \in P_2$, which means that

$$\Phi([u_0, 1]_{P_a}) = [h_b^{2n}(h_a^n(u_0)), 0]_{P_b} = [h_b^{2n}(h_a^n(u_0)), 1]_{P_b}$$

and we get continuity of Φ at $[u_0, 1]_{P_a}$.

For a map Φ we have the following relations:

$$\begin{aligned} (\Phi \circ s_{U(P_a)})(x) = \Phi([x,0]_{P_a}) = [h_b^{2n}(h_a^n(x)), 0]_{P_b} = (s_{P_b} \circ \varphi_a)(x), \\ (r_{P_b} \circ \Phi)([u,q]_{P_a}) = h_b^{2n}(h_a^n(u)) = (\varphi_b \circ r_{P_a})([u,q]_{P_a}) \end{aligned}$$

It means that (Φ, φ) satisfy conditions (3.1) and (3.3). Similarly, one can prove that (Ψ, ψ) satisfy conditions (3.2) and (3.4).

One may easily see that compositions of maps $\Phi \circ h_{P_a}$, $h_{P_b} \circ \Phi$, $\varphi \circ h_a$ and $h_b \circ \varphi$ are given by formulas:

$$(\Phi \circ h_{P_a})([u,q]_{P_a}) \\ = \begin{cases} [(h_b^{2n} \circ h_a^{n+1})(u), 1]_{P_b} & \text{if } q = 1, \ h^0(u,a), \dots , h^{2n+1}(u,a) \in P_1 \setminus P_2 \\ & \text{and } h^0(h_a^{n+1}(u), b), \dots , h^{2n}(h_a^{n+1}(u), b) \in P_1 \setminus P_2, \\ [(h_b^{2n} \circ h_a^{n+1})(u), 0]_{P_b} & \text{otherwise,} \end{cases}$$

$$(h_{P_b} \circ \Phi)([u,q]_{P_a}) = \begin{cases} [(h_b^{2n+1} \circ h_a^n)(u), 1]_{P_b}, & \text{if } q = 1,\ h^0(u,a), \dots , h^{2n}(u,a) \in P_1 \setminus P_2 \\ & \text{and } h^0(h_a^n(u), b), \dots , h^{2n+1}(h_a^n(u), b) \in P_1 \setminus P_2, \\ [(h_b^{2n+1} \circ h_a^n)(u), 0]_{P_b} & \text{otherwise,} \end{cases}$$

$$\begin{aligned} (\varphi \circ h_a)(x) &= (h_b^{2n} \circ h_a^{n+1})(x), \\ (h_b \circ \varphi)(x) &= (h_b^{2n+1} \circ h_a^n)(x). \end{aligned}$$

Let $\omega: \mathbb{R} \ni t \mapsto (1-t)a + tb \in \mathbb{R}$ be a homotopy joining a with b. Consider the following conditions:

(A) $h^0(u,a), \dots , h^{2n}(u,a) \in P_1 \setminus P_2$,
(B_t) $h(h_a^n(u), \omega(t)) \in P_1 \setminus P_2$,
(C_t) $h^0((h_{\omega(t)} \circ h_a^n)(u), b), \dots , h^{2n}((h_{\omega(t)} \circ h_a^n)(u), b) \in P_1 \setminus P_2$,
(A') $h^0(u,a), \dots , h^{2n}(u,a) \notin P_2$,
(B'_t) $h(h_a^n(u), \omega(t)) \notin P_2$,
(C'_t) $h^0((h_{\omega(t)} \circ h_a^n)(u), b), \dots , h^{2n}((h_{\omega(t)} \circ h_a^n)(u), b) \notin P_2$,
(A'') there exists $i \in \{0, \dots , 2n\}$ such that $h^i(u,a) \notin P_1$,
(B''_t) $h(h_a^n(u), \omega(t)) \notin P_1$,
(C''_t) there exists $j \in \{0, \dots , 2n\}$ such that $Rh^j((h_{\omega(t)} \circ h_a^n)(u), b) \notin P_1$,
(A_a) $h^0(u,a), \dots , h^{2n+1}(u,a) \in P_1 \setminus P_2$,
(C_a) $h^0(h_a^{n+1}(u), b), \dots , h^{2n}(h_a^{n+1})(u), b) \in P_1 \setminus P_2$,
(C_b) $h^0(h_a^n(u), b), \dots , h^{2n}(h_a^n(u), b) \in P_1 \setminus P_2$.

Define the maps $K: U(P_a) \times \mathbb{I} \to U(P_b)$ and $k: X \times \mathbb{I} \to X$ by formulas:

$$K([u,q]_{P_a}, t) = \begin{cases} [(h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(u), 1]_{P_b} & \text{if } q = 1 \text{and (A), (B}_t\text{), (C}_t\text{)}, \\ [(h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(u), 0]_{P_b} & \text{otherwise,} \end{cases}$$

$$k(x,t) = (h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(x).$$

Obviously, k is continuous. For the proof of continuity of K fix $([u_0, q_0]_{P_a}, t_0) \in P_a \times \mathbb{I}$. If $q_0 = 0$, continuity of K is obvious, so we may assume that $q_0 = 1$ and $(u_0, a) \in P_1$. Consider two sets

$$\begin{aligned} M &:= \{([u,q]_{P_a}, t) \in P_a \times \mathbb{I} : (\mathrm{A}') \text{ and } (\mathrm{B}'_t) \text{ and } (\mathrm{C}'_t)\}, \\ N &:= \{([u,q]_{P_a}, t) \in P_a \times \mathbb{I} : (\mathrm{A}'') \text{ or } (\mathrm{B}''_t) \text{ or } (\mathrm{C}''_t)\}. \end{aligned}$$

Sets M and N are open. If $([u,q]_{P_a}, t) \in N$, then

$$K(([u,q]_{P_a}, t)) = [(h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(u), 0]_{P_b},$$

so K is continuous on N. If $([u,1]_{P_a}, t) \in M$, then $h^0(u,a) \in P_1 \setminus P_2$ and from part (a) of Lemma 2.2 we get condition (A).

Lemma 2.3 implies condition (B_t) and $h^0((h_{\omega(t)} \circ h_a^n)(u), b) \in P_1 \setminus P_2$.

Now, again from part (a) of Lemma 2.2 we get condition (C_t). It means that

$$K(([u,1]_{P_a},t)) = [(h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(u),1]_{P_b}$$

so K is continuous on M.

Assume that $([u_0,1]_{P_a},t_0) \in (U(P_a)\times \mathbb{I}) \setminus (M \cup N)$. $([u_0,1]_{P_a},t_0) \notin N$, so $h^0(u,a),\dots,h^{2n}(u,a) \in P_1, h(h_a^n(u),\omega(t)) \in P_1$ and $h^0((h_{\omega(t)} \circ h_a^n)(u),b),\dots,$ $h^{2n}((h_{\omega(t)} \circ h_a^n)(u),b) \in P_1$.

Suppose condition (C'_t) would be satisfied. Then, from Lemma 2.3, condition (B'_t) would follow and we would have $h^{2n}(u,a) \notin P_2$. From part (b) of Lemma 2.2 we would get $h^{2n-1}(u,a),\dots,h^0(u,a) \in P_1 \setminus P_2$, which would mean that $([u_0,1]_{P_a},t_0) \in M$. Thus, there exists $i \in \{0,\dots,2n\}$ such that $h^i((h_{\omega(t)} \circ h_a^n)(u),b) \in P_2$, so part (b) of Lemma 2.2 implies that $h^{2n}((h_{\omega(t)}\circ h_a^n)(u),b) \in P_2$, which means that

$$K(([u_0,1]_{P_a},t_0)) = [(h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(u),0]_{P_b} = [(h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(u),1]_{P_b}$$

and we get continuity of K at $([u_0,1]_{P_a},t_0)$.

We have

$$\begin{aligned}(K \circ (s_{P_a} \times \mathrm{id}_{\mathbb{I}}))(x,t) &= K([x,0]_{P_a},t)\\ &= [(h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(x),0]_{P_b} = (s_{P_b} \circ k)(x,t),\\ (r_{P_b} \circ k)([u,q]_{P_a},t) &= (h_b^{2n} \circ h_{\omega(t)} \circ h_a^n)(u)\\ &= k(u,t) = (k \circ (r_{P_a} \times \mathrm{id}_{\mathbb{I}}))([u,q]_{P_a},t).\end{aligned}$$

Moreover, notice that condition (A_a) implies (A) and (B_0), condition (C_0) is equivalent to (C_a), while by Lemma 2.3 condition (A_a) follows from (A) and (C_0). This means that $K(\,\cdot\,,0) = \Phi \circ h_{P_a}$.

Similarly, we check that condition (C_b) implies (B_1) and (C_1), while by Lemma 2.3 condition (C) follows from (A) and (C_1). Thus, $K(\,\cdot\,,1) = h_{P_b} \circ \Phi$. Obviously, $k(\,\cdot\,,0) = \varphi \circ h_a$ and $k(\,\cdot\,,1) = h_b \circ \varphi$, hence (K,k) is a homotopy joining $(\Phi,\varphi) \circ (h_{P_a},h_a)$ with $(h_{P_b},h_b) \circ (\Phi,\varphi)$. This means that (Φ,φ) satisfy condition (3.5). Similarly, one can prove that (Ψ,ψ) satisfy condition (3.6).

Consider the following conditions:

(X) $h^0(u,a),\dots,h^{2n}(u,a) \in P_1 \setminus P_2$,
(Y_t) $h^0(h_a^n(u),\omega(t)),\dots,h^{4n}(h_a^n(u),\omega(t)) \in P_1 \setminus P_2$,
(Z_t) $h^0((h_{\omega(t)}^{3n} \circ h_a^n)(u),a),\dots,h^{2n}((h_{\omega(t)}^{3n} \circ h_a^n)(u),a) \in P_1 \setminus P_2$,
(X') $h^0(u,a),\dots,h^{2n}(u,a) \notin P_2$,
(Y'_t) $h^0(h_a^n(u),\omega(t)),\dots,h^{4n}(h_a^n(u),\omega(t)) \notin P_2$,
(Z'_t) $h^0((h_{\omega(t)}^{3n} \circ h_a^n)(u),a),\dots,h^{2n}((h_{\omega(t)}^{3n} \circ h_a^n)(u),a) \notin P_2$,
(X'') there exists $i \in \{0,\dots,2n\}$ such that $h^i(u,a) \notin P_1$,
(Y''_t) there exists $j \in \{0,\dots,4n\}$ such that $h^j(h_a^n(u),\omega(t)) \notin P_1$,
(Z''_t) there exists $k \in \{0,\dots,2n\}$ such that $h^k((h_{\omega(t)}^{3n} \circ h_a^n)(u),a) \notin P_1$,

(X_a) $h^0(u,a),\dots,h^{6n}(u,a)\in P_1\setminus P_2$,

(Y_b) $h^0(h^n_a(u),b),\dots,h^{4n}(h^n_a(u),b)\in P_1\setminus P_2$,

(Z_b) $h^0((h^{3n}_b\circ h^n_a)(u),a),\dots,h^{2n}((h^{3n}_b\circ h^n_a)(u),a)\in P_1\setminus P_2$.

The formulas for compositions of maps $\Psi\circ\Phi$ and $\psi\circ\varphi$ are as follows:

$$(\Psi\circ\Phi)([u,q]_{P_a})=\begin{cases}[(h^{2n}_a\circ h^{3n}_b\circ h^n_a)(u),1]_{P_a} & \text{if } q=1 \text{ and (X), (Y}_b\text{), (Z}_b\text{)},\\ [(h^{2n}_a\circ h^{3n}_b\circ h^n_a)(u),0]_{P_a} & \text{otherwise,}\end{cases}$$

$$(\psi\circ\varphi)(x)=(h^{2n}_a\circ h^{3n}_b\circ h^n_a)(x).$$

Define the maps $L\colon U(P_a)\times\mathbb{I}\to U(P_a)$ and $l\colon X\times\mathbb{I}\to X$ by formulas:

$$L([u,q]_{P_a},t)=\begin{cases}[(h^{2n}_a\circ h^{3n}_{\omega(t)}\circ h^n_a)(u),1]_{P_a} & \text{if } q=1 \text{ and (X), (Y}_t\text{), (Z}_t\text{)},\\ [(h^{2n}_a\circ h^{3n}_{\omega(t)}\circ h^n_a)(u),0]_{P_a} & \text{otherwise,}\end{cases}$$

$$l(x,t)=(h^{2n}_a\circ h^{3n}_{\omega(t)}\circ h^n_a)(x).$$

Obviously, l is continuous.

For the proof of continuity of L fix $([u_0,q_0]_{P_a},t_0)\in P_a\times\mathbb{I}$. If $q_0=0$, continuity of L is obvious, so we can assume that $q_0=1$ and $(u_0,a)\in P_1$. Consider two sets

$$\begin{aligned}V&:=\{([u,q]_{P_a},t)\in P_a\times\mathbb{I}:(\mathrm{X}')\text{ and }(\mathrm{Y}'_t)\text{ and }(\mathrm{Z}'_t)\},\\ W&:=\{([u,q]_{P_a},t)\in P_a\times\mathbb{I}:(\mathrm{X}'')\text{ or }(\mathrm{Y}''_t)\text{ or }(\mathrm{Z}''_t)\}.\end{aligned}$$

Sets V and W are open.

If $([u,q]_{P_a},t)\in W$, then $L(([u,q]_{P_a},t))=[(h^{2n}_a\circ h^{3n}_{\omega(t)}\circ h^n_a)(u),0]_{P_a}$, hence L is continuous on W.

If $([u,1]_{P_a},t)\in V$, then $h^0(u,a)\in P_1\setminus P_2$ and from part (a) of Lemma 2.2 we get (X).

Lemma 2.3 implies that $h^0(h^n_a(u),\omega(t))\in P_1\setminus P_2$ and again from part (a) of Lemma 2.2 we get condition (Y_t), while from Lemma 2.3 it follows that $h^0((h^{3n}_{\omega(t)}\circ h^n_a)(u),a)\in P_1\setminus P_2$.

Now, once again from part (a) of Lemma 2.2 we get condition (Z_t). It means that $L(([u,1]_{P_a},t))=[(h^{2n}_a\circ h^{3n}_{\omega(t)}\circ h^n_a)(u),1]_{P_a}$, so L is continuous on U.

Assume that $([u_0,1]_{P_a},t_0)\in(U(P_a)\times\mathbb{I})\setminus(U\cup V)$. $([u_0,1]_{P_a},t_0)\notin U$, which means that $h^0(u,a),\dots,h^{2n}(u,a)\in P_1, h^0(h^n_a(u),\omega(t)),\dots,h^{4n}(h^n_a(u),\omega(t))\in P_1$ and $h^0((h^{3n}_{\omega(t)}\circ h^n_a)(u),a),\dots,h^{2n}((h^{3n}_{\omega(t)}\circ h^n_a)(u),a)\in P_1$.

Assume condition (Z'_t). Then, from Lemma 2.3 we would get $h^{4n}(h^n_a(u),\omega(t))\in P_1\setminus P_2$. Part (b) of Lemma 2.2 would imply that $h^{4n-1}(h^n_a(u),\omega(t)),\dots,h^0(h^n_a(u),\omega(t))$ and (Y'_t).

Now, again from Lemma 2.3 we would get $h^{2n}(u,a)\notin P_2$. From part (b) of Lemma 2.2 it would follow that $h^{2n-1}(u,a),\dots,h^0(u,a)\in P_1\setminus P_2$, which would mean that $([u_0,1]_{P_a},t_0)\in U$. Thus, there exists $i\in\{0,\dots,2n\}$ such that

$h^i((h^{3n}_{\omega(t)} \circ h^n_a)(u), a) \in P_2$, so part (b) of Lemma 2.2 implies that $h^{2n}((h^{3n}_{\omega(t)} \circ h^n_a)(u), b) \in P_2$, which means that

$$L(([u_0, 1]_{P_a}, t_0)) = [(h^{2n}_a \circ h^{3n}_{\omega(t)} \circ h^n_a)(u), 0]_{P_a} = [(h^{2n}_a \circ h^{3n}_{\omega(t)} \circ h^n_a)(u), 1]_{P_a}$$

and we have proved the continuity of L at $([u_0, 1]_{P_a}, t_0)$.

Now, we have

$$\begin{aligned}(L \circ (s_{P_a} \times \mathrm{id}_{\mathbb{I}}))(x,t) &= L([x,0]_{P_a}, t) \\ &= [(h^{2n}_a \circ h^{3n}_{\omega(t)} \circ h^n_a)(x), 0]_{P_a} = (s_{P_a} \circ l)(x,t), \\ (r_{P_a} \circ L)([u,q]_{P_a}, t) &= (h^{2n}_a \circ h^{3n}_{\omega(t)} \circ h^n_a)(u) \\ &= l(u,t) = (l \circ (r_{P_a} \times \mathrm{id}_{\mathbb{I}}))([u,q]_{P_a}, t).\end{aligned}$$

We know that

$$h^{6n}_{P_a}([u,q]_{P_a}) = \begin{cases} [h^{6n}_a(u), 1]_{P_a} & \text{if } q = 1 \text{ and } (\mathrm{X}_a), \\ [h^{6n}_a(u), 0]_{P_a} & \text{otherwise.} \end{cases}$$

As

$$(\mathrm{X}) \text{ and } (\mathrm{Y}_0) \text{ and } (\mathrm{Z}_0) \text{ if and only if } (\mathrm{X}_a)$$

and

$$(\mathrm{X}) \text{ and } (\mathrm{Y}_1) \text{ and } (\mathrm{Z}_1) \text{ if and only if } (\mathrm{X}) \text{ and } (\mathrm{Y}_b) \text{ and } (\mathrm{Z}_b),$$

it follows that

$$L(\,\cdot\,, 0) = h^{6n}_{P_a}, \quad L(\,\cdot\,, 1) = \Psi \circ \Phi, \quad l(\,\cdot\,, 0) = h^{6n}_a, \quad l(\,\cdot\,, 1) = \psi \circ \varphi.$$

This means that condition (3.7) is satisfied. Similarly, one can prove condition (3.8). Naturally, $h_a \simeq h_b$, as well as $\varphi \simeq h^{3n}_a$, and $\psi \simeq h^{3n}_b$. This way we have proved that

$$[(U(P_a), r_{P_a}, s_{P_a}), (h_{P_a}, h_a)]_X = [(U(P_b), r_{P_b}, s_{P_b}), (h_{P_b}, h_b)]_X. \qquad \square$$

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