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A BIFURCATION RESULT OF BÖHME–MARINO TYPE FOR QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We study a variational bifurcation problem of Böhme-Marino type associated with nonsmooth functional. The existence of two branches of bifurcation is proved.

1. Introduction

Consider the quasilinear eigenvalue problem

$$(1.1)\qquad \begin{cases} -\displaystyle\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) \\ \qquad +\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u - g(x,u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and a_{ij}, g satisfy suitable assumptions that will be specified later.

If $g(x,0) = 0$, it is natural to study the bifurcation problem from the trivial branch of solutions $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$. Since (1.1) is formally the Euler equation

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of the functional $F_\lambda\colon H^1_0(\Omega)\to\mathbb{R}$ defined as

$$F_\lambda(u)=\frac{1}{2}\int_\Omega\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju\,dx-\int_\Omega G(x,u)\,dx-\frac{1}{2}\,\lambda\int_\Omega u^2\,dx,$$

where $G(x,s)=\int_0^s g(x,t)\,dt$, it is natural to expect the well known results typical of bifurcation for potential operators (see e.g. [18], [20]).

However, the feature that the coefficients a_{ij} are dependent on u causes a lack of differentiability, hence the impossibility to apply standard techniques. More precisely, it is well known (see e.g. [5], [10], [21]) that, under natural growth conditions on a_{ij} and g, the functional F_λ is continuous on $H^1_0(\Omega)$, but not locally Lipschitz, unless the a_{ij}'s are independent of u or $n=1$.

In the previous paper [6], Rabinowitz's theorem [19] has been extended to (1.1). Here we are interested in the other basic description of bifurcation branches, namely Böhme–Marino theorem [2], [16]. As in [6], a key ingredient in our proof is the nonsmooth critical point theory developed independently in [9], [11] and in [12], [13]. However, while in [6] the key point was a finite dimensional reduction of (1.1), here the eigenvalue problem is directly treated in the infinite dimensional setting. This allows weaker differentiability assumptions on a_{ij}. More precisely, hypothesis (a.2) is weaker than the corresponding assumption in [6].

Let us recall that, while the classical Böhme–Marino theorem requires the functional to be of class C^2, various extensions have been considered in the literature. In particular the case in which the functional is of class $C^{1,1}$ or even C^1 has been treated in [17] and [15], respectively, while the case of variational inequalities involving the Laplace operator has been considered in [1]. However, the techniques used in these papers cannot be applied to (1.1).

The main result. Let Ω be a bounded open subset of $\mathbb{R}^n$ and $a_{ij}\colon\Omega\times\mathbb{R}\to\mathbb{R}$ $(1\le i,j\le n)$ be such that

$$\begin{cases}\text{for all } s\in\mathbb{R}, & a_{ij}(x,s) \text{ is measurable with respect to } x,\\ \text{for a.e. } x\in\Omega, & a_{ij}(x,s) \text{ is of class } C^1 \text{ with respect to } s.\end{cases}$$

Suppose also that:

(a.1) for almost every $x\in\Omega$, for all $s\in\mathbb{R}$ and all $1\le i,j\le n$,

$$a_{ij}(x,s)=a_{ji}(x,s);$$

(a.2) there exists a continuous function $\alpha\colon\mathbb{R}\to[0,\infty[$ such that, for almost every $x\in\Omega$, for all $s\in\mathbb{R}$ and all $1\le i,j\le n$,

$$|a_{ij}(x,s)|\le\alpha(s),\qquad |D_sa_{ij}(x,s)|\le\alpha(s);$$

(a.3) there exists a continuous function $\nu\colon \mathbb{R} \to]0,\infty[$ such that, for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \geq \nu(s)\sum_{i=1}^{n}\xi_i^2;$$

(a.4) for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \geq 0.$$

Finally, let $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that

$$\begin{cases} \text{for all } s \in \mathbb{R}, & g(x,s) \text{ is measurable with respect to } x, \\ \text{for a.e. } x \in \Omega, & g(x,s) \text{ is of class } C^1 \text{ with respect to } s. \end{cases}$$

Suppose also that:

(g.1) for a.e. $x \in \Omega$, $g(x,0) = 0$;

(g.2) there exists a continuous function $\beta : \mathbb{R} \to [0,\infty[$ such that, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,

$$|D_s g(x,s)| \leq \beta(s).$$

Consider the problem

$$\begin{cases} (\lambda,u) \in \mathbb{R} \times (H_0^1(\Omega) \cap L^\infty(\Omega)), \\ \displaystyle\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_jv\,dx + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju\,v\,dx \\ \qquad\qquad - \displaystyle\int_\Omega g(x,u)v\,dx = \lambda\int_\Omega uv\,dx \quad \text{for all } v \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{1.2}$$

Remark 1.1. By assumption (g.1), $(\lambda,0)$ is a solution of (1.1) for all $\lambda \in \mathbb{R}$.

Definition 1.2. A real number μ is said to be a bifurcation value of (1.2) if there exists a sequence (λ_h,u_h) of solutions of (1.2) with $u_h \neq 0$ such that $\lambda_h \to \mu$ and $u_h \to 0$ strongly in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$.

Let us introduce the linear operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ such that

$$\langle Au,v\rangle = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,0)D_iuD_jv\,dx - \int_\Omega D_s g(x,0)uv\,dx.$$

A real number μ is said to be an eigenvalue of A if the equation $Au = \mu u$ admits a nontrivial solution u.

Proposition 1.3. *If μ is a bifurcation value of* (1.2)*, then μ is an eigenvalue of A.*

Let us state the main result of the paper.

Theorem 1.4. *Suppose that μ is an eigenvalue of A. Then μ is a bifurcation value of* (1.2). *Moreover, there exists $\varrho_0 > 0$ such that:*

(a) *for each $\varrho \in \,]0, \varrho_0]$, there exist at least two solutions $(\lambda_k(\varrho), u_k(\varrho))$, $k = 1, 2$, of* (1.2) *with $u_1(\varrho) \neq u_2(\varrho)$ and*

$$\int_\Omega |u_k(\varrho)|^2 \, dx = \varrho^2;$$

(b) *as $\varrho \to 0$, we have $\lambda_k(\varrho) \to \mu$ and $u_k(\varrho) \to 0$ strongly in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$.*

Proposition 1.3 and Theorem 1.4 will be proved in the last section. In the next section we recall the tools of nonsmooth critical point theory we need, while in Section 3 we prove Proposition 1.3 and Theorem 1.4 in a particular case, more suitable for a direct variational approach.

2. Recall of nonsmooth analysis

In this section we recall from [4], [7], [9], [11] some notions and results of nonsmooth critical point theory we shall use to describe the variational nature of problem (1.2).

Let X denote a metric space endowed with the metric d and $f\colon X \to \mathbb{R} \cup \{\infty\}$ a function. We also consider the space $X \times \mathbb{R}$ endowed with the metric

$$d((u, s), (v, t)) = (d(u, v)^2 + (s - t)^2)^{1/2}.$$

Set $\operatorname{epi}(f) = \{(u, s) \in X \times \mathbb{R} : f(u) \leq s\}$ and, for every $c \in \mathbb{R}$, $f^c = \{u \in X : f(u) \leq c\}$. Finally, we denote by $B_r(u)$ the open ball of center u and radius r.

The next definition is taken from [4, Definition 2.1]. For an equivalent approach, see [9], [11] and, when f is continuous, [13].

Definition 2.1. For every $u \in X$ with $f(u) < \infty$, we denote by $|df|(u)$ the supremum of the σ's in $[0, \infty[$ such that there exist $\delta > 0$ and a continuous map

$$H\colon (B_\delta(u, f(u)) \cap \operatorname{epi}(f)) \times [0, \delta] \to X$$

satisfying

$$d(H((v, s), t), v) \leq t, \quad f(H((v, s), t)) \leq s - \sigma t,$$

whenever $(v, s) \in B_\delta(u, f(u)) \cap \operatorname{epi}(f)$ and $t \in [0, \delta]$. The extended real number $|df|(u)$ is called the *weak slope* of f at u.

Definition 2.2. A point $u \in X$ with $f(u) < \infty$ is said to be (*lower*) *critical* for f, if $|df|(u) = 0$. A real number c is said to be a (*lower*) *critical value* for f, if there exists $u \in X$ such that $f(u) = c$ and $|df|(u) = 0$. For every $c \in \mathbb{R}$, we set $K_c = \{u \in X : f(u) = c, \ |df|(u) = 0\}$.

Definition 2.3. Given $c \in \mathbb{R}$, we say that f satisfies $(\mathrm{PS})_c$, i.e. the Palais–Smale condition at level c, if from every sequence (u_h) in X, with $f(u_h) \to c$ and $|df|(u_h) \to 0$ as $h \to \infty$, it is possible to extract a subsequence (u_{h_k}) converging in X.

Definition 2.4. Let Y be a closed subset of X. For every closed subset A of X, we denote by $\mathrm{cat}_{X,Y} A$ the least integer $n \geq 0$ such that A can be covered by $n+1$ open subsets $U_0, \dots, U_n$ of X with the following properties:

(a) there exists a deformation $K\colon X \times [0,1] \to X$ such that $K(Y \times [0,1]) \subset Y$ and $K(U_0 \times \{1\}) \subset Y$ (if $Y = \emptyset$, we mean that U_0 must be empty);

(b) for $1 \leq h \leq n$, each U_h is contractible in X.

If no such integer n exists, we set $\mathrm{cat}_{X,Y} A = \infty$. Finally, to shorten notations, we put $\mathrm{cat}_X A = \mathrm{cat}_{X,\emptyset} A$.

For the next result, we refer the reader to [7, Theorem 1.4.9].

Theorem 2.5. *Assume that X is complete and that $f\colon X \to \mathbb{R}$ is continuous. Let $-\infty < a < b < \infty$ and let us suppose that, for every $c \in [a,b]$, the function f satisfies $(\mathrm{PS})_c$. If $\mathrm{cat}_{X,f^a} f^b \geq k$ with $k \in \mathbb{N}$, then there exist $a \leq c_1 \leq \ldots \leq c_k \leq b$ such that each c_n is a critical value of f. Moreover, if $c_m = \ldots = c_n$ for some $m < n$, we have $\mathrm{cat}_X K_{c_m} \geq n - m + 1$.*

Definition 2.6. The metric space X is said to be *weakly locally contractible*, if every $u \in X$ admits a neighbourhood U contractible in X.

For the next result, see [7, Theorem 1.4.11].

Proposition 2.7. *Assume that X is weakly locally contractible and let A be a closed subset of X. Then A contains at least $\mathrm{cat}_X A$ elements.*

Finally, we recall from [4] some notions and results which will help in the evaluation of the weak slope. Assume now that X is a Banach space.

Definition 2.8. Let $u \in X$ with $f(u) < \infty$. For every $v \in X$ and $\varepsilon > 0$, let $f^0_\varepsilon(u;v)$ be the infimum of the r's in $\mathbb{R}$ such that there exist $\delta > 0$ and a continuous map

$$V\colon (B_\delta(u, f(u)) \cap \mathrm{epi}(f)) \times \,]0,\delta] \to B_\varepsilon(v)$$

satisfying

$$f(z + tV((z,s),t)) \leq s + rt,$$

whenever $(z,s) \in B_\delta(u,f(u)) \cap \mathrm{epi}(f)$ and $t \in \,]0,\delta]$ (we agree that $\inf \emptyset = \infty$). Let also

$$f^0(u;v) = \sup_{\varepsilon > 0} f^0_\varepsilon(u;v).$$

We say that $f^0(u;v)$ is the *generalized directional derivative* of f at u with respect to v.

Definition 2.9. For every $u \in X$ with $f(u) < \infty$, we put

$$\partial f(u) = \{w \in X^* : \langle w, v\rangle \leq f^0(u; v) \text{ for all } v \in X\}.$$

The set $\partial f(u)$ is called the *subdifferential* of f at u.

For the next result, we refer the reader to [4, Theorem 4.13].

Theorem 2.10. *For every $u \in X$ with $f(u) < \infty$, we have*

$$\text{if } |df|(u) < \infty \text{ then } \partial f(u) \neq \emptyset,$$
$$\text{if } |df|(u) < \infty \text{ then } |df|(u) \geq \min\{\|w\| : w \in \partial f(u)\}.$$

In particular, if $|df|(u) = 0$, we have $0 \in \partial f(u)$.

We end the section with a Lagrange multiplier theorem. If $C \subset X$, we denote by I_C the indicator function of C, namely

$$I_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{if } u \in X \setminus C. \end{cases}$$

Definition 2.11. Let $u \in X$ with $f(u) < \infty$. For every $v \in X$ and $\varepsilon > 0$ let $\overline{f}^0_\varepsilon(u; v)$ be the infimum of the r's in $\mathbb{R}$ such that there exist $\delta > 0$ and a continuous map

$$H \colon (B_\delta(u, f(u)) \cap \operatorname{epi}(f)) \times [0, \delta] \to E$$

satisfying $H((z, s), 0) = z$,

$$\left\| \frac{H((z,s), t_1) - H((z,s), t_2)}{t_1 - t_2} - v \right\| < \varepsilon,$$
$$f(H((z,s), t)) \leq s + rt$$

whenever $(z, s) \in B_\delta(u, f(u)) \cap \operatorname{epi}(f)$ and $t, t_1, t_2 \in [0, \delta]$ with $t_1 \neq t_2$ (we agree that $\inf \emptyset = \infty$). Let also

$$\overline{f}^0(u; v) = \sup_{\varepsilon > 0} \overline{f}^0_\varepsilon(u; v).$$

Theorem 2.12. *Let U be an open subset of X with ∂U of class C^1, let $u \in \partial U$ with $f(u) < \infty$ and let $\nu(u) \in X^* \setminus \{0\}$ be an outer normal vector to U at u. Then the following facts hold:*

(a) *if there exist $v_-, v_+ \in X$ such that $\langle \nu(u), v_-\rangle < 0 < \langle \nu(u), v_+\rangle$ and $\overline{f}^0(u; v_\pm) < \infty$, we have*

$$(f + I_{\partial U})^0(u; v) \leq f^0(u; v) \quad \textit{for every } v \in X \textit{ with } \langle \nu(u), v\rangle = 0,$$
$$\partial(f + I_{\partial U})(u) \subset \partial f(u) + \{\lambda \nu(u) : \lambda \in \mathbb{R}\};$$

(b) *if there exists* $v_0 \in X$ *such that* $\langle \nu(u), v_0 \rangle < 0$ *and* $f^0(u; v_0) < \infty$, *we have*

$$(f + I_{\overline{U}})^0(u; v) \leq f^0(u; v) \quad \textit{for every } v \in X \textit{ with } \langle \nu(u), v \rangle \leq 0,$$
$$\partial(f + I_{\overline{U}})(u) \subset \partial f(u) + \{\eta\nu(u) : \eta \geq 0\}.$$

Proof. For assertion (a) we refer the reader to [4, Corollary 5.9]. Assertion (b) is a particular case of [4, Corollary 5.4]. □

3. The case with uniform bounds

Throughout this section, we consider the particular case in which a_{ij} and g satisfy (a.1), (a.4), (g.1) and the estimates

$$\text{(a.2}'\text{)} \qquad |a_{ij}(x,s)| \leq \alpha, \quad |D_s a_{ij}(x,s)| \leq \alpha,$$

$$\text{(a.3}'\text{)} \qquad \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \geq \nu \sum_{i=1}^{n} \xi_i^2,$$

$$\text{(g.2}'\text{)} \qquad |D_s g(x,s)| \leq \beta,$$

for some some constants $\alpha, \beta \geq 0$ and $\nu > 0$.

Proposition 3.1. *The assertion of Proposition* 1.3 *holds under these more restrictive assumptions.*

Theorem 3.2. *The assertion of Theorem* 1.4 *holds under these more restrictive assumptions.*

The section will be devoted to the proofs of Proposition 3.1 and Theorem 3.2.

First of all, define the continuous functionals $f, f_\varrho \colon H_0^1(\Omega) \to \mathbb{R}$ ($\varrho > 0$) by

$$f(u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u \, dx - 2\int_\Omega G(x,u)\, dx, \quad f_\varrho(u) = \frac{f(\varrho u)}{\varrho^2},$$

where $G(x,s) = \int_0^s g(x,t)\, dt$, and the smooth quadratic form $f_0 : H_0^1(\Omega) \to \mathbb{R}$ by

$$f_0(u) = \langle Au, u \rangle = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,0) D_i u D_j u \, dx - \int_\Omega D_s g(x,0) u^2 \, dx.$$

By Definition 2.1, it is easy to verify that $|df_\varrho|(u) = \frac{1}{\varrho}|df|(\varrho u)$. Moreover, by (a.2$'$) and (g.2$'$) the functionals f and f_ϱ are differentiable at any $u \in H_0^1(\Omega)$ with respect to any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Let μ be an eigenvalue of A, let V_0 be the associated eigenspace and let

$$V = \left\{ v \in H_0^1(\Omega) : \int_\Omega vw \, dx = 0, \text{ for all } w \in V_0 \right\}.$$

Let us decompose V as $V_+ \oplus V_-$, where V_+ is the closed subspace of $H_0^1(\Omega)$ spanned by the eigenvectors associated to the eigenvalues λ_j with $\lambda_j > \mu$ and V_- is the subspace of $H_0^1(\Omega)$ spanned by the eigenvectors associated to the eigenvalues λ_j with $\lambda_j < \mu$. Let us denote by P_0, P_- and P_+ the orthogonal projections, with respect to the scalar product of $L^2(\Omega)$, on V_0, V_- and V_+, respectively. Let us recall that the decomposition $H_0^1(\Omega) = V_- \oplus V_0 \oplus V_+$ is orthogonal both with respect to the scalar product of $L^2(\Omega)$ and with respect to the bilinear form $\langle Au, v\rangle$. Moreover, $V_- \oplus V_0$ is finite dimensional and contained in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

We also set

$$S = \left\{u \in H_0^1(\Omega) : \int_\Omega |u|^2\,dx = 1\right\}, \quad M = \left\{u \in S : \int_\Omega |P_0 u|^2\,dx \geq \frac{1}{4}\right\},$$

and denote by $\widetilde{f}_\varrho$ ($\varrho \geq 0$) the restriction of f_ϱ to M. Clearly, M is a submanifold with boundary in $H_0^1(\Omega)$ with

$$\partial M = \left\{u \in S : \int_\Omega |P_0 u|^2\,dx = \frac{1}{4}\right\}.$$

Lemma 3.3. *The following facts hold:*

(a) *if $\varrho_h \to 0$ and $u_h \to u$ strongly in $H_0^1(\Omega)$, then*

$$f_0(u) = \lim_h f_{\varrho_h}(u_h);$$

(b) *if $\varrho_h \to 0$ and $u_h \to u$ weakly in $H_0^1(\Omega)$, then*

$$f_0(u) \leq \liminf_h f_{\varrho_h}(u_h).$$

Proof. The assertions follow from [6, Theorem 2.2]. □

Lemma 3.4. *For each $\varepsilon > 0$ small enough, there exists $\varrho_0 > 0$ such that, for every $\varrho \in\,]0, \varrho_0]$, one has*

$$\operatorname{cat}_{\widetilde{f}_\varrho^{\mu+2\varepsilon}, \widetilde{f}_\varrho^{\mu-\varepsilon}} \widetilde{f}_\varrho^{\mu+\varepsilon} \geq 2.$$

Proof. If $\varepsilon > 0$ is small enough, there exist $0 < \varepsilon_1 < \varepsilon_0$ such that $M_0 \neq \emptyset$ and

$$M_0 \cap (V_- \oplus V_0) \subset \widetilde{f}_0^{\mu-3\varepsilon/2} \subset \widetilde{f}_0^{\mu-\varepsilon} \subset M_1,$$

where

$$M_0 = \left\{u \in M : \int_\Omega |P_- u|^2\,dx \geq \varepsilon_0^2\right\}, \quad M_1 = \left\{u \in M : \int_\Omega |P_- u|^2\,dx > \varepsilon_1^2\right\}.$$

It is easy to check that the inclusion map

$$i\colon (M \cap (V_- \oplus V_0), M_0 \cap (V_- \oplus V_0)) \to (M, M_1)$$

is a homotopy equivalence. Let π be a homotopy inverse.

We claim that, if $\varrho_0 > 0$ is small enough, then for every $\varrho \in]0, \varrho_0]$ we have

$$M_0 \cap (V_- \oplus V_0) \subset \widetilde{f}_\varrho^{\mu-\varepsilon} \subset M_1, \tag{3.1}$$

$$M \cap (V_- \oplus V_0) \subset \widetilde{f}_\varrho^{\mu+\varepsilon}. \tag{3.2}$$

Actually, since $M_0 \cap (V_- \oplus V_0)$ and $M \cap (V_- \oplus V_0)$ are compact, (3.2) and the first inclusion in (3.1) follow from (a) of Lemma 3.3.

To prove the second inclusion in (3.1), assume by contradiction that $\varrho_h \to 0$ and $u_h \in \widetilde{f}_{\varrho_h}^{\mu-\varepsilon} \setminus M_1$. Since M is bounded in $L^2(\Omega)$, from (g.2′) and (a.3′) we have that u_h is bounded also in $H_0^1(\Omega)$, hence weakly convergent, up to a subsequence, to some $u \in M \setminus M_1$. From (b) of Lemma 3.3 we deduce that $\widetilde{f}_0(u) \le \mu - \varepsilon$ and a contradiction follows.

Now, if we consider the inclusion maps

$$i_1\colon (M \cap (V_- \oplus V_0), M_0 \cap (V_- \oplus V_0)) \to (\widetilde{f}_\varrho^{\mu+2\varepsilon}, \widetilde{f}_\varrho^{\mu-\varepsilon}),$$
$$i_2\colon (\widetilde{f}_\varrho^{\mu+2\varepsilon}, \widetilde{f}_\varrho^{\mu-\varepsilon}) \to (M, M_1).$$

We have that $(\pi \circ i_2) \circ i_1$ is homotopic to the identity map of $(M \cap (V_- \oplus V_0), M_0 \cap (V_- \oplus V_0))$. Since $i_1^{-1}(\widetilde{f}_\varrho^{\mu+\varepsilon}) = M \cap (V_- \oplus V_0)$, from [7, Theorem 1.4.5] it follows

$$\mathrm{cat}_{\widetilde{f}_\varrho^{\mu+2\varepsilon}, \widetilde{f}_\varrho^{\mu-\varepsilon}} \widetilde{f}_\varrho^{\mu+\varepsilon} \ge \mathrm{cat}_{M \cap (V_- \oplus V_0), M_0 \cap (V_- \oplus V_0)} M \cap (V_- \oplus V_0).$$

On the other hand, the pair $(M \cap (V_- \oplus V_0), M_0 \cap (V_- \oplus V_0))$ is homotopically equivalent to the pair $(\mathbb{R}^m \times S^{n-1}, S^{m-1} \times S^{n-1})$, where $m = \dim V_-$ and $n = \dim V_0$.

If $n \ge 2$, it is well known that there exist

$$z_1 \in H_m(\mathbb{R}^m \times S^{n-1}, S^{m-1} \times S^{n-1}) \setminus \{0\},$$
$$z_2 \in H_{m+n-1}(\mathbb{R}^m \times S^{n-1}, S^{m-1} \times S^{n-1}),$$
$$\omega \in H^{n-1}(\mathbb{R}^m \times S^{n-1})$$

such that $z_1 = \omega \cap z_2$ (see e.g. [15, p. 347]). From [7, Theorem 1.4.8] we deduce that

$$\mathrm{cat}_{\mathbb{R}^m \times S^{n-1}, S^{m-1} \times S^{n-1}} \mathbb{R}^m \times S^{n-1} \ge 2 \tag{3.3}$$

and the assertion follows. By the way, equality holds in (3.3).

If $n = 1$, we have that $S^{n-1} = \{-1, 1\}$ is disconnected and the fact that

$$\mathrm{cat}_{\mathbb{R}^m \times S^{n-1}, S^{m-1} \times S^{n-1}} \mathbb{R}^m \times S^{n-1} = 2$$

can be seen directly. □

Lemma 3.5. *For every $u \in M$ with $|d\widetilde{f}_\varrho|(u) < \infty$, there exist $\lambda \in \mathbb{R}$, $\eta \geq 0$ and $w \in H^{-1}(\Omega)$ such that $\|w\| \leq |d\widetilde{f}_\varrho|(u)/2$ and*

$$\eta\bigg(\int_\Omega |P_0 u|^2\,dx - \frac{1}{4}\bigg) = 0, \tag{3.4}$$

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho u) D_i u D_j v\,dx + \frac{1}{2}\,\varrho \int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,\varrho u) D_i u D_j u\, v\,dx \\ - \frac{1}{\varrho}\int_\Omega g(x,\varrho u) v\,dx = \lambda \int_\Omega uv\,dx + \eta \int_\Omega P_0 u\, v\,dx + \langle w, v\rangle\end{aligned} \tag{3.5}$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. By [4, Theorem 6.1], for every $u \in H_0^1(\Omega)$, we have

$$f_\varrho^0(u;v) \leq \overline{f}_\varrho^0(u;v) < \infty \quad \text{for all } v \in C_c^\infty(\Omega), \tag{3.6}$$

and if $\partial f_\varrho(u) \neq \emptyset$ then

$$-\sum_{i,j=1}^n D_j\,(a_{ij}(x,\varrho u) D_i u) + \frac{1}{2}\,\varrho \sum_{i,j=1}^n D_s a_{ij}(x,\varrho u) D_i u D_j u - \frac{1}{\varrho}\, g(x,\varrho u) \in H^{-1}(\Omega)$$

in the sense of distributions. If $\partial f_\varrho(u) \neq \emptyset$ then

$$\begin{aligned}\partial f_\varrho(u) = \bigg\{ -2 \sum_{i,j=1}^n D_j(a_{ij}(x,\varrho u) D_i u) \\ + \varrho \sum_{i,j=1}^n D_s a_{ij}(x,\varrho u) D_i u D_j u - \frac{2}{\varrho}\, g(x,\varrho u)\bigg\}.\end{aligned} \tag{3.7}$$

Since, for every $u \in S$, there exist $v_-, v_+ \in C_c^\infty(\Omega)$ such that

$$\int_\Omega uv_-\,dx > 0 > \int_\Omega uv_+\,dx,$$

from (3.6) and (a) of Theorem 2.12 we deduce that

$$(f_\varrho + I_S)^0(u;v) \leq f_\varrho^0(u;v) \quad \text{for every } v \in H_0^1(\Omega) \text{ with } \int_\Omega uv\,dx = 0, \tag{3.8}$$

$$\partial(f_\varrho + I_S)(u) \subset \partial f_\varrho(u) + \{-\lambda u : \lambda \in \mathbb{R}\}. \tag{3.9}$$

Finally, if we set

$$U = \bigg\{u \in H_0^1(\Omega) : \int_\Omega |P_0 u|^2\,dx > \frac{1}{4}\bigg\},$$

for every $u \in \partial M$, the open sets

$$\left\{v \in H^1_0(\Omega) : \int_\Omega P_0 uv\,dx > 0,\ \int_\Omega uv\,dx > 0\right\},$$
$$\left\{v \in H^1_0(\Omega) : \int_\Omega P_0 uv\,dx > 0,\ \int_\Omega uv\,dx < 0\right\}$$

are not empty. Therefore, there exists $v_0 \in C^\infty_c(\Omega)$ such that

$$\int_\Omega P_0 uv_0\,dx > 0, \qquad \int_\Omega uv_0\,dx = 0.$$

From (3.6), (3.8) and (b) of Theorem 2.12 we deduce that, for every $u \in \partial M$,

$$\partial(f_\varrho + I_S + I_{\overline{U}})(u) \subset \partial(f_\varrho + I_S)(u) + \{-\eta P_0 u : \eta \geq 0\}. \tag{3.10}$$

Now let $u \in M$ with $|d\widetilde{f}_\varrho|(u) < \infty$. From Definition 2.1 it easily follows that $|d\widetilde{f}_\varrho|(u) = |d\left(f_\varrho + I_S + I_{\overline{U}}\right)|(u)$. By Theorem 2.10 there exists $w \in H^{-1}(\Omega)$ with $2w \in \partial\left(f_\varrho + I_S + I_{\overline{U}}\right)(u)$ and $\|2w\| \leq |d\widetilde{f}_\varrho|(u)$. If $u \in \partial M$, by (3.7), (3.9) and (3.10) we find $\lambda \in \mathbb{R}$ and $\eta \geq 0$ such that

$$\begin{aligned} w = &-\sum_{i,j=1}^n D_j(a_{ij}(x,\varrho u)D_i u) \\ &+ \frac{\varrho}{2}\sum_{i,j=1}^n D_s a_{ij}(x,\varrho u)D_i u D_j u - \frac{1}{\varrho}\,g(x,\varrho u) - \lambda u - \eta P_0 u. \end{aligned}$$

We deduce (3.4) and (3.5), provided that $v \in C^\infty_c(\Omega)$. An easy approximation argument then shows that (3.5) holds.

If $u \notin \partial M$, we have $\partial(f_\varrho + I_S + I_{\overline{U}})(u) = \partial(f_\varrho + I_S)(u)$, as the notion of subdifferential is local, and the assertion follows in a similar way. □

Lemma 3.6. *There exists $\delta > 0$ such that*

$$\text{for all } \varrho \in\,]0,\delta], \text{ for all } u \in \partial M: \text{ if } |\widetilde{f}_\varrho(u) - \mu| \leq \delta \text{ then } |d\widetilde{f}_\varrho|(u) \geq \delta. \tag{3.11}$$

Proof. By contradiction, let $\varrho_h \to 0$ and $u_h \in \partial M$ with $\widetilde{f}_{\varrho_h}(u_h) \to \mu$ and $|d\widetilde{f}_{\varrho_h}|(u_h) \to 0$. Since M is bounded in $L^2(\Omega)$, (g.2′) and (a.3′) imply that (u_h) is bounded in $H^1_0(\Omega)$. Up to a subsequence, (u_h) is convergent to some $u \in \partial M$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. By (b) of Lemma 3.3 we have $\widetilde{f}_0(u) \leq \mu$. It follows that $P_- u \neq 0$.

By Lemma 3.5 there exist $w_h \in H^{-1}(\Omega)$ with $w_h \to 0$, $\lambda_h \in \mathbb{R}$ and $\eta_h \geq 0$ satisfying

$$\begin{aligned}(3.12)\quad &\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho_h u_h) D_i u_h D_j v\,dx \\ &+ \frac{\varrho_h}{2}\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,\varrho_h u_h) D_i u_h D_j u_h\, v\,dx - \frac{1}{\varrho_h}\int_\Omega g(x,\varrho_h u_h) v\,dx \\ &= \lambda_h \int_\Omega u_h v\,dx + \eta_h \int_\Omega P_0 u_h\, v\,dx + \langle w_h, v\rangle\end{aligned}$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Since V_- is a finite dimensional subspace of $H_0^1(\Omega) \cap L^\infty(\Omega)$, we have that $(P_- u_h)$ is strongly convergent to $P_- u$ both in $H_0^1(\Omega)$ and in $L^\infty(\Omega)$. If we put $v = P_- u_h$ in (3.12), we get that (λ_h) is bounded. If we put $v = P_0 u_h$ in (3.12), we deduce in a similar way that also (η_h) is bounded. Up to a subsequence, we may assume that $\lambda_h \to \lambda$ and $\eta_h \to \eta \geq 0$.

By an easy adaptation of [8, Lemma 5.1] we have

$$(3.13)\qquad \lim_h \frac{1}{\varrho_h} g(x,\varrho_h u_h) = D_s g(x,0)u \qquad \text{strongly in } L^2(\Omega),$$

$$(3.14)\qquad \lim_h \frac{1}{\varrho_h^2} G(x,\varrho_h u_h) = \frac{1}{2} D_s g(x,0)u^2 \quad \text{strongly in } L^1(\Omega).$$

Passing to the limit in (3.12) as $h \to \infty$ and taking into account (3.13), we get

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega \sum_{i,j=1}^n a_{ij}(x,0) D_i u D_j v\,dx - \int_\Omega D_s g(x,0) uv\,dx \\ &= \lambda \int_\Omega uv\,dx + \eta \int_\Omega P_0 uv\,dx\end{aligned}$$

for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, hence by density for every $v \in H_0^1(\Omega)$. If we choose $v = P_0 u$, we obtain $\mu = \lambda + \eta$, while, if we choose $v = P_+ u$, we get

$$\overline{\mu} \int_\Omega |P_+ u|^2\,dx \leq \lambda \int_\Omega |P_+ u|^2\,dx,$$

where $\overline{\mu}$ is the minimal eigenvalue of A greater than μ. It follows that $P_+ u = 0$ and $\widetilde{f}_0(u) < \mu$.

By (a.4) and the result of [3], we can also put $v = u_h$ in (3.12). By (a.4), (3.13) and (3.14), it follows

$$\begin{aligned}\mu &= \lim_h \widetilde{f}_{\varrho_h}(u_h) \\ &= \lim_h \left[\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho_h u_h) D_i u_h D_j u_h\,dx - \frac{2}{\varrho_h^2}\int_\Omega G(x,\varrho_h u_h)\,dx\right] \\ &= \lim_h \left[\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho_h u_h) D_i u_h D_j u_h\,dx - \frac{1}{\varrho_h}\int_\Omega g(x,\varrho_h u_h) u_h\,dx\right]\end{aligned}$$

$$\le \lambda \int_\Omega u^2\,dx + \eta \int_\Omega |P_0 u|^2\,dx$$
$$= \int_\Omega \sum_{i,j=1}^n a_{ij}(x,0) D_i u D_j u\,dx - \int_\Omega D_s g(x,0) u^2\,dx = \widetilde{f}_0(u) < \mu,$$

whence a contradiction. □

Lemma 3.7. *There exists $\delta > 0$ such that*

$$\begin{cases} \text{for every } \varrho \in \,]0,\delta] \text{ and every } c \in [\mu-\delta, \mu+\delta], \\ \text{the functional } \widetilde{f}_\varrho \text{ satisfies } (\mathrm{PS})_c. \end{cases} \tag{3.15}$$

Proof. Let (u_h) be a sequence in M with $\widetilde{f}_\varrho(u_h) \to c$ and $|d\widetilde{f}_\varrho|(u_h) \to 0$. If δ is small enough, by Lemma 3.6 we have that $u_h \notin \partial M$ eventually as $h \to \infty$. As before, we have that (u_h) is bounded in $H_0^1(\Omega)$, hence convergent, up to a subsequence, to some $u \in M$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$.

By Lemma 3.5 there exist $w_h \in H^{-1}(\Omega)$ and $\lambda_h \in \mathbb{R}$ such that $w_h \to 0$ and

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho u_h) D_i u_h D_j v\,dx + \frac{\varrho}{2} \int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,\varrho u_h) D_i u_h D_j u_h\, v\,dx - \frac{1}{\varrho} \int_\Omega g(x,\varrho u_h) v\,dx = \lambda_h \int_\Omega u_h v\,dx + \langle w_h, v\rangle \tag{3.16}$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. If we put $v = P_0 u_h$ in (3.16), we find that (λ_h) is bounded. The assertion then follows from [5, Lemma 2.4]. □

Lemma 3.8. *Let (λ_h, u_h) be a sequence of nontrivial solutions of*

$$\begin{cases} (\lambda,u) \in \mathbb{R} \times H_0^1(\Omega), \\ \displaystyle\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j v\,dx + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u\, v\,dx \\ \qquad \displaystyle - \int_\Omega g(x,u) v\,dx = \lambda \int_\Omega uv\,dx \quad \text{for all } v \in H_0^1(\Omega)\cap L^\infty(\Omega), \end{cases} \tag{3.17}$$

with $u_h \to 0$ strongly in $H_0^1(\Omega)$. Then the following facts hold:

(a) *we have $u_h \in L^\infty(\Omega)$ and $u_h \to 0$ strongly in $L^\infty(\Omega)$;*
(b) *we have $\lambda_h \to \mu$ if and only if*

$$\lim_h \frac{f(u_h)}{\int_\Omega u_h^2\,dx} = \mu.$$

Proof. By (a.3′) and (a.4) we have

$$\nu \int_\Omega |DR_k(u_h)|^2\,dx \le \int_\Omega (g(x,u_h) + \lambda_h u_h) R_k(u_h)\,dx,$$

where $R_k\colon \mathbb{R} \to \mathbb{R}$ is the odd function such that $R_k(s) = (s-k)^+$ for $s \geq 0$. Taking into account (g.2′), by standard techniques of regularity theory (see e.g. [14]) assertion (a) follows.

If we set $\varrho_h = (\int_\Omega |u_h|^2\,dx)^{1/2}$ and $z_h = u_h/\varrho_h$, we have

$$\begin{aligned}(3.18)\quad &\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho_h z_h) D_i z_h D_j v\,dx \\ &+ \frac{\varrho_h}{2}\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,\varrho_h z_h) D_i z_h D_j z_h\, v\,dx - \frac{1}{\varrho_h}\int_\Omega g(x,\varrho_h z_h) v\,dx \\ &= \lambda_h \int_\Omega z_h v\,dx\end{aligned}$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Assume that $\lambda_h \to \mu$. If we put $v = z_h$ in (3.18) and take into account (a.3′), (a.4) and (g.2′), we find that (z_h) is bounded in $H_0^1(\Omega)$, hence weakly convergent, up to a subsequence, to some z. Combining this fact with (a.2′) and assertion (a), we deduce that

$$\lim_h \frac{\varrho_h}{2}\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,\varrho_h z_h) D_i z_h D_j z_h\, z_h\,dx = 0.$$

Coming back to (3.18) with $v = z_h$ and taking into account (3.13), (3.14), we deduce that

$$\begin{aligned}\lim_h \frac{f(u_h)}{\varrho_h^2} &= \lim_h \bigg[\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho_h z_h) D_i z_h D_j z_h\,dx - \frac{2}{\varrho_h^2}\int_\Omega G(x,\varrho_h z_h)\,dx\bigg] \\ &= \lim_h \bigg[\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho_h z_h) D_i z_h D_j z_h\,dx - \frac{1}{\varrho_h}\int_\Omega g(x,\varrho_h z_h) z_h\,dx\bigg] \\ &= \lim_h \lambda_h = \mu.\end{aligned}$$

Assume now that $f(u_h)/\varrho_h^2 \to \mu$, namely that

$$\lim_h \bigg[\int_\Omega \sum_{i,j=1}^n a_{ij}(x,\varrho_h z_h) D_i z_h D_j z_h\,dx - \frac{2}{\varrho_h^2}\int_\Omega G(x,\varrho_h z_h)\,dx\bigg] = \mu.$$

From (a.3′) and (g.2′) it follows that (z_h) is bounded in $H_0^1(\Omega)$. As before, we find that

$$\lim_h \lambda_h = \lim_h \frac{f(u_h)}{\varrho_h^2}$$

and the assertion follows. □

Proof of Theorem 3.2. First of all, by Lemma 3.8 the condition $\lambda_k(\varrho) \to \mu$ is equivalent to

$$\frac{f(u_k(\varrho))}{\int_\Omega |u_k(\varrho)|^2\,dx} \to \mu.$$

In turn, it is equivalent to prove that, for every $\varepsilon > 0$, there exists $\varrho_0 > 0$ such that, for every $\varrho \in]0, \varrho_0]$, there exist at least two solutions $(\lambda_k(\varrho), u_k(\varrho))$, $k = 1, 2$, of (3.17) with $u_1(\varrho) \neq u_2(\varrho)$ and

$$\int_\Omega |u_k(\varrho)|^2\,dx = \varrho^2, \quad \mu - \varepsilon \leq \frac{f(u_k(\varrho))}{\int_\Omega |u_k(\varrho)|^2\,dx} \leq \mu + \varepsilon.$$

In fact, by (a.3′) and (g.2′) it follows that $u_k(\varrho)/\varrho$ is bounded in $H^1_0(\Omega)$ as $\varrho \to 0$. Therefore $u_k(\varrho) \to 0$ strongly in $H^1_0(\Omega)$ as $\varrho \to 0$. From Lemma 3.8 it follows that $u_k(\varrho) \to 0$ also in $L^\infty(\Omega)$.

Now, let $\varepsilon > 0$ and let $\delta > 0$ be such that (3.11) and (3.15) hold. Without loss of generality, we may assume that $\varepsilon \leq \delta$ and that ε is small enough to apply Lemma 3.4. Let $\varrho_0 > 0$ be as in Lemma 3.4. Without loss of generality, we may also assume that $\varrho_0 \leq \delta$. Let $\varrho \in]0, \varrho_0]$.

If $u \in M$ and $\widetilde{f}_\varrho(u) < \mu + 2\varepsilon$, it is clear that the weak slope of $\widetilde{f}_\varrho|_{\widetilde{f}_\varrho^{\mu+2\varepsilon}}$ at u coincides with that of $\widetilde{f}_\varrho$ at u. Applying Theorem 2.5 to $\widetilde{f}_\varrho|_{\widetilde{f}_\varrho^{\mu+2\varepsilon}}$, we find two critical values $\mu - \varepsilon \leq c_1 \leq c_2 \leq \mu + \varepsilon$ of $\widetilde{f}_\varrho$. If $c_1 < c_2$, we immediately get two distinct critical points $z_1(\varrho), z_2(\varrho)$ of $\widetilde{f}_\varrho$ in $\widetilde{f}_\varrho^{-1}([\mu - \varepsilon, \mu + \varepsilon])$. If $c_1 = c_2$, we have that $\mathrm{cat}_{\widetilde{f}_\varrho^{\mu+2\varepsilon}} K_{c_1} \geq 2$. *A fortiori* we have $\mathrm{cat}_{\{\widetilde{f}_\varrho < \mu + 2\varepsilon\}} K_{c_1} \geq 2$. Being an open subset of a manifold, $\{\widetilde{f}_\varrho < \mu + 2\varepsilon\}$ is clearly weakly locally contractible. By Proposition 2.7 we find two distinct critical points $z_1(\varrho), z_2(\varrho)$ of $\widetilde{f}_\varrho$ in $\widetilde{f}_\varrho^{-1}([\mu - \varepsilon, \mu + \varepsilon])$ also in this case.

By (3.11) we have that $z_k(\varrho)$ does not belong to ∂M. From Lemma 3.5 it follows that there exist $\lambda_1(\varrho), \lambda_2(\varrho) \in \mathbb{R}$ such that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x, \varrho z_k(\varrho)) D_i z_k(\varrho) D_j v\,dx + \frac{1}{2}\varrho \int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x, \varrho z_k(\varrho)) D_i z_k(\varrho) D_j z_k(\varrho) v\,dx - \frac{1}{\varrho}\int_\Omega g(x, \varrho z_k(\varrho)) v\,dx = \lambda \int_\Omega z_k(\varrho) v\,dx$$

for all $v \in H^1_0(\Omega) \cap L^\infty(\Omega)$. If we set $u_k(\varrho) = \varrho z_k(\varrho)$, we have that $(\lambda_k(\varrho), u_k(\varrho))$ has the required properties. □

Proof of Proposition 3.1. Let (λ_h, u_h) be a sequence as in Definition 1.2. If we set $\varrho_h = (\int_\Omega |u_h|^2\,dx)^{\frac{1}{2}}$ and $z_h = u_h/\varrho_h$, by Lemma 3.8 we deduce that $f(u_h)/\varrho_h^2 \to \mu$. From (g.2′) and (a.3′) it follows that (z_h) is bounded in $H^1_0(\Omega)$, hence weakly convergent, up to a subsequence, to some $z \in H^1_0(\Omega) \setminus \{0\}$. Since (3.18) holds also in this case, passing to the limit as $h \to \infty$ and recalling (3.13), we find

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,0) D_i z D_j v \, dx - \int_\Omega D_s g(x,0) z v \, dx = \mu \int_\Omega z v \, dx$$

for all $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and the assertion follows. □

4. Proof of Proposition 1.3 and Theorem 1.4

Let $\vartheta \colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing smooth function such that $\vartheta(s) = s$ for $|s| \le 1$ and ϑ is constant on $]-\infty, -2]$ and on $[2, \infty[$.

If we set $\widehat{a}_{ij}(x,s) = a_{ij}(x, \vartheta(s))$ and $\widehat{g}(x,s) = g(x, \vartheta(s))$, it is readily seen that $\widehat{a}_{ij}$ and $\widehat{g}$ satisfy (a.1), (a.2′), (a.3′), (a.4), (g.1) and (g.2′).

On the other hand, if u is small enough in $L^\infty(\Omega)$, we have that (λ, u) is a solution of (1.2) with respect to $\widehat{a}_{ij}$ and $\widehat{g}$ if and only if it do it with respect to a_{ij} and g. Moreover, the linear operator A associated with $\widehat{a}_{ij}$ and $\widehat{g}$ coincides with that associated with a_{ij} and g.

If we apply Proposition 3.1 and Theorem 3.2 to $\widehat{a}_{ij}$ and $\widehat{g}$, the assertion follows. □

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